И. Е. Иродов

ЗАДАЧИ ПО ОБЩЕЙ ФИЗИКЕ

Издательство «Наука» Москва

I. E. Irodov

Problems in General Physics

Mir Publishers Moscow

Translated from the Russian by Yuri Atanov

First published 1981
Second printing 1983
Third printing 1988
Revised from the 1979 Russian edition

На английском языке

Printed in the Union of Soviet Socialist Republics

ISBN 5-03-000800-4

PREFACE

This book of problems is intended as a textbook for students at higher educational institutions studying advanced course in physics. Besides, because of the great number of simple problems it may be used by students studying a general course in physics.

The book contains about 1900 problems with hints for solving the most complicated ones.

For students' convenience each chapter opens with a time-saving summary of the principal formulas for the relevant area of physics. As a rule the formulas are given without detailed explanations since a student, starting solving a problem, is assumed to know the meaning of the quantities appearing in the formulas. Explanatory notes are only given in those cases when misunderstanding may arise.

All the formulas in the text and answers are in SI system, except in Part Six, where the Gaussian system is used. Quantitative data and answers are presented in accordance with the rules of approximation and numerical accuracy.

The main physical constants and tables are summarised at the end of the book.

The Periodic System of Elements is printed at the front end sheet and the Table of Elementary Particles at the back sheet of the book.

In the present edition, some misprints are corrected, and a number of problems are substituted by new ones, or the quantitative data in them are changed or refined (1.273, 1.361, 2.189, 3.249, 3.97, 4.194 and 5.78).

In conclusion, the author wants to express his deep gratitude to colleagues from MIPhI and to readers who sent their remarks on some problems , helping thereby to improve the book.

I.E. Irodov

CONTENTS

A FEW HINTS FOR SOLVING THE PROBLEMS

1. First of all, look through the tables in the Appendix, for many problems cannot be solved without them. Besides, the reference data quoted in the tables will make your work easier and save your time.

2. Begin the problem by recognizing its meaning and its formulation. Make sure that the data given are sufficient for solving the problem. Missing data can be found in the tables in the Appendix. Wherever possible, draw a diagram elucidating the essence of the problem; in many cases this simplifies both the search for a solution and the solution itself.

3. Solve each problem, as a rule, in the general form, that is in a letter notation, so that the quantity sought will be expressed in the same terms as the given data. A solution in the general form is particularly valuable since it makes clear the relationship between the sought quantity and the given data. What is more, an answer obtained in the general form allows one to make a fairly accurate judgement on the correctness of the solution itself (see the next item).

4. Having obtained the solution in the general form, check to see if it has the right dimensions. The wrong dimensions are an obvious indication of a wrong solution. If possible, investigate the behaviour of the solution in some extreme special cases. For example, whatever the form of the expression for the gravitational force between two extended bodies, it must turn into the well-known law of gravitational interaction of mass points as the distance between the bodies increases. Otherwise, it can be immediately inferred that the solution is wrong.

5. When starting calculations, remember that the numerical values of physical quantities are always known only approximately. Therefore, in calculations you should employ the rules for operating with approximate numbers. In particular, in presenting the quantitative data and answers strict attention should be paid to the rules of approximation and numerical accuracy.

6. Having obtained the numerical answer, evaluate its plausibility. In some cases such an evaluation may disclose an error in the result obtained. For example, a stone cannot be thrown by a man over the distance of the order of 1 km, the velocity of a body cannot surpass that of light in a vacuum, etc.

NOTATION

Vectors are written in boldface upright type, e.g., $\mathbf{r}$, $\mathbf{F}$; the same letters printed in lightface italic type (r, F) denote the modulus of a vector.

Unit vectors

$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors of the Cartesian coordinates x, y, z (sometimes the unit vectors are denoted as $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$),
$\mathbf{e}_\rho$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$ are the unit vectors of the cylindrical coordinates ρ, φ, z,
$\mathbf{n}$, $\boldsymbol{\tau}$ are the unit vectors of a normal and a tangent.

Mean values are taken in angle brackets $\langle\ \rangle$, e.g., $\langle \mathbf{v} \rangle$, $\langle P \rangle$.

Symbols Δ, d, and δ in front of quantities denote:
Δ, the finite increment of a quantity, e.g. $\Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$; $\Delta U = U_2 - U_1$,
d, the differential (infinitesimal increment), e.g. $d\mathbf{r}$, dU,
δ, the elementary value of a quantity, e.g. δA, the elementary work.

Time derivative of an arbitrary function f is denoted by df/dt, or by a dot over a letter, $\dot{f}$.

Vector operator ∇ ("nabla"). It is used to denote the following operations:
$\nabla\varphi$, the gradient of φ (grad φ).
$\nabla \cdot \mathbf{E}$, the divergence of $\mathbf{E}$ (div $\mathbf{E}$),
$\nabla \times \mathbf{E}$, the curl of $\mathbf{E}$ (curl $\mathbf{E}$).

Integrals of any multiplicity are denoted by a single sign $\int$ and differ only by the integration element: dV, a volume element, $d\mathbf{S}$, a surface element, and $d\mathbf{r}$, a line element. The sign $\oint$ denotes an integral over a closed surface, or around a closed loop.

PART ONE

PHYSICAL FUNDAMENTALS OF MECHANICS

1.1. KINEMATICS

- Average vectors of velocity and acceleration of a point:

$$\langle \mathbf{v} \rangle = \frac{\Delta \mathbf{r}}{\Delta t}, \qquad \langle \mathbf{w} \rangle = \frac{\Delta \mathbf{v}}{\Delta t}, \tag{1.1a}$$

where $\Delta \mathbf{r}$ is the displacement vector (an increment of a radius vector).

- Velocity and acceleration of a point:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \qquad \mathbf{w} = \frac{d\mathbf{v}}{dt}. \tag{1.1b}$$

- Acceleration of a point expressed in projections on the tangent and the normal to a trajectory:

$$w_\tau = \frac{dv_\tau}{dt}, \qquad w_n = \frac{v^2}{R}, \tag{1.1c}$$

where R is the radius of curvature of the trajectory at the given point.

- Distance covered by a point:

$$s = \int v\, dt, \tag{1.1d}$$

where v is the *modulus* of the velocity vector of a point.

- Angular velocity and angular acceleration of a solid body:

$$\omega = \frac{d\varphi}{dt}, \qquad \beta = \frac{d\omega}{dt}. \tag{1.1e}$$

- Relation between linear and angular quantities for a rotating solid body:

$$\mathbf{v} = [\boldsymbol{\omega}\mathbf{r}], \quad w_n = \omega^2 R, \quad |w_\tau| = \beta R, \tag{1.1.f}$$

where $\mathbf{r}$ is the radius vector of the considered point relative to an arbitrary point on the rotation axis, and R is the distance from the rotation axis.

1.1. A motorboat going downstream overcame a raft at a point A; $\tau = 60$ min later it turned back and after some time passed the raft at a distance $l = 6.0$ km from the point A. Find the flow velocity assuming the duty of the engine to be constant.

1.2. A point traversed half the distance with a velocity v_0. The remaining part of the distance was covered with velocity v_1 for half the time, and with velocity v_2 for the other half of the time. Find the mean velocity of the point averaged over the whole time of motion.

1.3. A car starts moving rectilinearly, first with acceleration $w = 5.0\ \text{m/s}^2$ (the initial velocity is equal to zero), then uniformly, and finally, decelerating at the same rate w, comes to a stop. The total time of motion equals $\tau = 25$ s. The average velocity during that time is equal to $\langle v \rangle = 72$ km per hour. How long does the car move uniformly?

1.4. A point moves rectilinearly in one direction. Fig. 1.1 shows

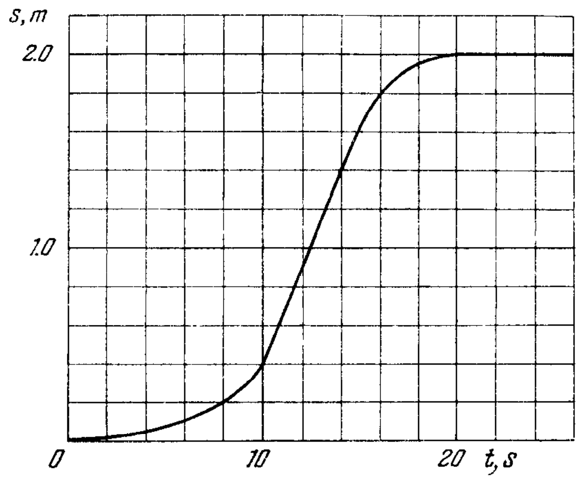


Fig. 1.1.

the distance s traversed by the point as a function of the time t. Using the plot find:

(a) the average velocity of the point during the time of motion;

(b) the maximum velocity;

(c) the time moment t_0 at which the instantaneous velocity is equal to the mean velocity averaged over the first t_0 seconds.

1.5. Two particles, *1* and *2*, move with constant velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. At the initial moment their radius vectors are equal to $\mathbf{r}_1$ and $\mathbf{r}_2$. How must these four vectors be interrelated for the particles to collide?

1.6. A ship moves along the equator to the east with velocity $v_0 = 30$ km/hour. The southeastern wind blows at an angle $\varphi = 60°$ to the equator with velocity $v = 15$ km/hour. Find the wind velocity v' relative to the ship and the angle φ' between the equator and the wind direction in the reference frame fixed to the ship.

1.7. Two swimmers leave point A on one bank of the river to reach point B lying right across on the other bank. One of them crosses the river along the straight line AB while the other swims at right angles to the stream and then walks the distance that he has been carried away by the stream to get to point B. What was the velocity u

of his walking if both swimmers reached the destination simultaneously? The stream velocity $v_0 = 2.0$ km/hour and the velocity v' of each swimmer with respect to water equals 2.5 km per hour.

1.8. Two boats, A and B, move away from a buoy anchored at the middle of a river along the mutually perpendicular straight lines: the boat A along the river, and the boat B across the river. Having moved off an equal distance from the buoy the boats returned. Find the ratio of times of motion of boats τ_A/τ_B if the velocity of each boat with respect to water is $\eta = 1.2$ times greater than the stream velocity.

1.9. A boat moves relative to water with a velocity which is $n = 2.0$ times less than the river flow velocity. At what angle to the stream direction must the boat move to minimize drifting?

1.10. Two bodies were thrown simultaneously from the same point: one, straight up, and the other, at an angle of $\theta = 60°$ to the horizontal. The initial velocity of each body is equal to $v_0 = 25$ m/s. Neglecting the air drag, find the distance between the bodies $t = 1.70$ s later.

1.11. Two particles move in a uniform gravitational field with an acceleration g. At the initial moment the particles were located at one point and moved with velocities $v_1 = 3.0$ m/s and $v_2 = 4.0$ m/s horizontally in opposite directions. Find the distance between the particles at the moment when their velocity vectors become mutually perpendicular.

1.12. Three points are located at the vertices of an equilateral triangle whose side equals a. They all start moving simultaneously with velocity v constant in modulus, with the first point heading continually for the second, the second for the third, and the third for the first. How soon will the points converge?

1.13. Point A moves uniformly with velocity v so that the vector $\mathbf{v}$ is continually "aimed" at point B which in its turn moves rectilinearly and uniformly with velocity $u < v$. At the initial moment of time $\mathbf{v} \perp \mathbf{u}$ and the points are separated by a distance l. How soon will the points converge?

1.14. A train of length $l = 350$ m starts moving rectilinearly with constant acceleration $w = 3.0 \cdot 10^{-2}\ \text{m/s}^2$; $t = 30$ s after the start the locomotive headlight is switched on (event *1*), and $\tau = 60$ s after that event the tail signal light is switched on (event *2*). Find the distance between these events in the reference frames fixed to the train and to the Earth. How and at what constant velocity V relative to the Earth must a certain reference frame K move for the two events to occur in it at the same point?

1.15. An elevator car whose floor-to-ceiling distance is equal to 2.7 m starts ascending with constant acceleration $1.2\ \text{m/s}^2$; 2.0 s after the start a bolt begins falling from the ceiling of the car. Find:

(a) the bolt's free fall time;

(b) the displacement and the distance covered by the bolt during the free fall in the reference frame fixed to the elevator shaft.

1.16. Two particles, *1* and *2*, move with constant velocities v_1 and v_2 along two mutually perpendicular straight lines toward the intersection point O. At the moment $t = 0$ the particles were located at the distances l_1 and l_2 from the point O. How soon will the distance between the particles become the smallest? What is it equal to?

1.17. From point A located on a highway (Fig. 1.2) one has to get by car as soon as possible to point B located in the field at a distance l from the highway. It is known that the car moves in the field η times slower than on the highway. At what distance from point D one must turn off the highway?

1.18. A point travels along the x axis with a velocity whose projection v_x is presented as a function of time by the plot in Fig. 1.3.

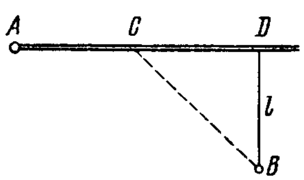


Fig. 1.2.

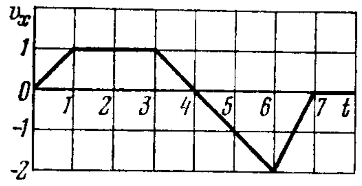


Fig. 1.3.

Assuming the coordinate of the point $x = 0$ at the moment $t = 0$, draw the approximate time dependence plots for the acceleration w_x, the x coordinate, and the distance covered s.

1.19. A point traversed half a circle of radius $R = 160$ cm during time interval $\tau = 10.0$ s. Calculate the following quantities averaged over that time:

(a) the mean velocity $\langle v \rangle$;

(b) the modulus of the mean velocity vector $|\langle \mathbf{v} \rangle|$;

(c) the modulus of the mean vector of the total acceleration $|\langle \mathbf{w} \rangle|$ if the point moved with constant tangent acceleration.

1.20. A radius vector of a particle varies with time t as $\mathbf{r} = \mathbf{a}t(1 - \alpha t)$, where $\mathbf{a}$ is a constant vector and α is a positive factor. Find:

(a) the velocity $\mathbf{v}$ and the acceleration $\mathbf{w}$ of the particle as functions of time;

(b) the time interval Δt taken by the particle to return to the initial points, and the distance s covered during that time.

1.21. At the moment $t = 0$ a particle leaves the origin and moves in the positive direction of the x axis. Its velocity varies with time as $\mathbf{v} = \mathbf{v}_0(1 - t/\tau)$, where $\mathbf{v}_0$ is the initial velocity vector whose modulus equals $v_0 = 10.0$ cm/s; $\tau = 5.0$ s. Find:

(a) the x coordinate of the particle at the moments of time 6.0, 10, and 20 s;

(b) the moments of time when the particle is at the distance 10.0 cm from the origin;

(c) the distance s covered by the particle during the first 4.0 and 8.0 s; draw the approximate plot $s(t)$.

1.22. The velocity of a particle moving in the positive direction of the x axis varies as $v = \alpha\sqrt{x}$, where α is a positive constant. Assuming that at the moment $t = 0$ the particle was located at the point $x = 0$, find:

(a) the time dependence of the velocity and the acceleration of the particle;

(b) the mean velocity of the particle averaged over the time that the particle takes to cover the first s metres of the path.

1.23. A point moves rectilinearly with deceleration whose modulus depends on the velocity v of the particle as $w = a\sqrt{v}$, where a is a positive constant. At the initial moment the velocity of the point is equal to v_0. What distance will it traverse before it stops? What time will it take to cover that distance?

1.24. A radius vector of a point A relative to the origin varies with time t as $\mathbf{r} = at\mathbf{i} - bt^2\mathbf{j}$, where a and b are positive constants, and $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes. Find:

(a) the equation of the point's trajectory $y(x)$; plot this function;

(b) the time dependence of the velocity $\mathbf{v}$ and acceleration $\mathbf{w}$ vectors, as well as of the moduli of these quantities;

(c) the time dependence of the angle α between the vectors $\mathbf{w}$ and $\mathbf{v}$;

(d) the mean velocity vector averaged over the first t seconds of motion, and the modulus of this vector.

1.25. A point moves in the plane xy according to the law $x = at$, $y = at(1 - \alpha t)$, where a and α are positive constants, and t is time. Find:

(a) the equation of the point's trajectory $y(x)$; plot this function;

(b) the velocity v and the acceleration w of the point as functions of time;

(c) the moment t_0 at which the velocity vector forms an angle $\pi/4$ with the acceleration vector.

1.26. A point moves in the plane xy according to the law $x = a \sin \omega t$, $y = a(1 - \cos \omega t)$, where a and ω are positive constants. Find:

(a) the distance s traversed by the point during the time τ;

(b) the angle between the point's velocity and acceleration vectors.

1.27. A particle moves in the plane xy with constant acceleration $\mathbf{w}$ directed along the negative y axis. The equation of motion of the particle has the form $y = ax - bx^2$, where a and b are positive constants. Find the velocity of the particle at the origin of coordinates.

1.28. A small body is thrown at an angle to the horizontal with the initial velocity $\mathbf{v}_0$. Neglecting the air drag, find:

(a) the displacement of the body as a function of time $\mathbf{r}(t)$;

(b) the mean velocity vector $\langle \mathbf{v} \rangle$ averaged over the first t seconds and over the total time of motion.

1.29. A body is thrown from the surface of the Earth at an angle α

to the horizontal with the initial velocity v_0. Assuming the air drag to be negligible, find:

(a) the time of motion;

(b) the maximum height of ascent and the horizontal range; at what value of the angle α they will be equal to each other;

(c) the equation of trajectory $y(x)$, where y and x are displacements of the body along the vertical and the horizontal respectively;

(d) the curvature radii of trajectory at its initial point and at its peak.

1.30. Using the conditions of the foregoing problem, draw the approximate time dependence of moduli of the normal w_n and tangent w_τ acceleration vectors, as well as of the projection of the total acceleration vector w_v on the velocity vector direction.

1.31. A ball starts falling with zero initial velocity on a smooth inclined plane forming an angle α with the horizontal. Having fallen the distance h, the ball rebounds elastically off the inclined plane. At what distance from the impact point will the ball rebound for the second time?

1.32. A cannon and a target are 5.10 km apart and located at the same level. How soon will the shell launched with the initial velocity 240 m/s reach the target in the absence of air drag?

1.33. A cannon fires successively two shells with velocity $v_0 = 250$ m/s; the first at the angle $\theta_1 = 60°$ and the second at the angle $\theta_2 = 45°$ to the horizontal, the azimuth being the same. Neglecting the air drag, find the time interval between firings leading to the collision of the shells.

1.34. A balloon starts rising from the surface of the Earth. The ascension rate is constant and equal to v_0. Due to the wind the balloon gathers the horizontal velocity component $v_x = ay$, where a is a constant and y is the height of ascent. Find how the following quantities depend on the height of ascent:

(a) the horizontal drift of the balloon $x(y)$;

(b) the total, tangential, and normal accelerations of the balloon.

1.35. A particle moves in the plane xy with velocity $\mathbf{v} = a\mathbf{i} + bx\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes, and a and b are constants. At the initial moment of time the particle was located at the point $x = y = 0$. Find:

(a) the equation of the particle's trajectory $y(x)$;

(b) the curvature radius of trajectory as a function of x.

1.36. A particle A moves in one direction along a given trajectory with a tangential acceleration $w_\tau = \mathbf{a}\boldsymbol{\tau}$, where $\mathbf{a}$ is a constant vector coinciding in direction with the x axis (Fig. 1.4), and $\boldsymbol{\tau}$ is a unit vector coinciding in direction with the velocity vector at a given point. Find how the velocity of the particle depends on x provided that its velocity is negligible at the point $x = 0$.

1.37. A point moves along a circle with a velocity $v = at$, where $a = 0.50$ m/s². Find the total acceleration of the point at the moment when it covered the n-th ($n = 0.10$) fraction of the circle after the beginning of motion.

1.38. A point moves with deceleration along the circle of radius R so that at any moment of time its tangential and normal accelerations

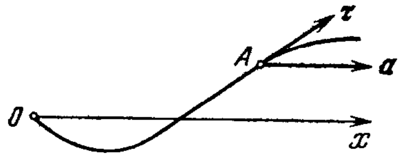


Fig. 1.4.

are equal in moduli. At the initial moment $t = 0$ the velocity of the point equals v_0. Find:

(a) the velocity of the point as a function of time and as a function of the distance covered s;

(b) the total acceleration of the point as a function of velocity and the distance covered.

1.39. A point moves along an arc of a circle of radius R. Its velocity depends on the distance covered s as $v = a\sqrt{s}$, where a is a constant. Find the angle α between the vector of the total acceleration and the vector of velocity as a function of s.

1.40. A particle moves along an arc of a circle of radius R according to the law $l = a \sin \omega t$, where l is the displacement from the initial position measured along the arc, and a and ω are constants. Assuming $R = 1.00$ m, $a = 0.80$ m, and $\omega = 2.00$ rad/s, find:

(a) the magnitude of the total acceleration of the particle at the points $l = 0$ and $l = \pm a$;

(b) the minimum value of the total acceleration w_{min} and the corresponding displacement l_m.

1.41. A point moves in the plane so that its tangential acceleration $w_\tau = a$, and its normal acceleration $w_n = bt^4$, where a and b are positive constants, and t is time. At the moment $t = 0$ the point was at rest. Find how the curvature radius R of the point's trajectory and the total acceleration w depend on the distance covered s.

1.42. A particle moves along the plane trajectory $y(x)$ with velocity v whose modulus is constant. Find the acceleration of the particle at the point $x = 0$ and the curvature radius of the trajectory at that point if the trajectory has the form

(a) of a parabola $y = ax^2$;

(b) of an ellipse $(x/a)^2 + (y/b)^2 = 1$; a and b are constants here.

1.43. A particle A moves along a circle of radius $R = 50$ cm so that its radius vector $\mathbf{r}$ relative to the point O (Fig. 1.5) rotates with the constant angular velocity $\omega = 0.40$ rad/s. Find the modulus of the velocity of the particle, and the modulus and direction of its total acceleration.

1.44. A wheel rotates around a stationary axis so that the rotation angle φ varies with time as $\varphi = at^2$, where $a = 0.20$ rad/s^2. Find the total acceleration w of the point A at the rim at the moment $t = 2.5$ s if the linear velocity of the point A at this moment $v = 0.65$ m/s.

1.45. A shell acquires the initial velocity $v = 320$ m/s, having made $n = 2.0$ turns inside the barrel whose length is equal to $l = 2.0$ m. Assuming that the shell moves inside the barrel with a uniform acceleration, find the angular velocity of its axial rotation at the moment when the shell escapes the barrel.

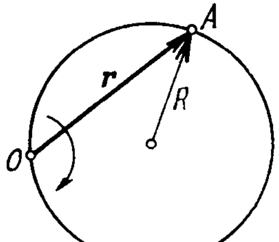


Fig. 1.5.

1.46. A solid body rotates about a stationary axis according to the law $\varphi = at - bt^3$, where $a = 6.0$ rad/s and $b = 2.0$ rad/s^3. Find:

(a) the mean values of the angular velocity and angular acceleration averaged over the time interval between $t = 0$ and the complete stop;

(b) the angular acceleration at the moment when the body stops.

1.47. A solid body starts rotating about a stationary axis with an angular acceleration $\beta = at$, where $a = 2.0 \cdot 10^{-2}$ rad/s^3. How soon after the beginning of rotation will the total acceleration vector of an arbitrary point of the body form an angle $\alpha = 60°$ with its velocity vector?

1.48. A solid body rotates with deceleration about a stationary axis with an angular deceleration $\beta \propto \sqrt{\omega}$, where ω is its angular velocity. Find the mean angular velocity of the body averaged over the whole time of rotation if at the initial moment of time its angular velocity was equal to ω_0.

1.49. A solid body rotates about a stationary axis so that its angular velocity depends on the rotation angle φ as $\omega = \omega_0 - a\varphi$, where ω_0 and a are positive constants. At the moment $t = 0$ the angle $\varphi = 0$. Find the time dependence of

(a) the rotation angle;

(b) the angular velocity.

1.50. A solid body starts rotating about a stationary axis with an angular acceleration $\boldsymbol{\beta} = \boldsymbol{\beta}_0 \cos\varphi$, where $\boldsymbol{\beta}_0$ is a constant vector and φ is an angle of rotation from the initial position. Find the angular velocity of the body as a function of the angle φ. Draw the plot of this dependence.

1.51. A rotating disc (Fig. 1.6) moves in the positive direction of the x axis. Find the equation $y(x)$ describing the position of the instantaneous axis of rotation, if at the initial moment the axis C of the disc was located at the point O after which it moved

(a) with a constant velocity v, while the disc started rotating counterclockwise with a constant angular acceleration β (the initial angular velocity is equal to zero);

(b) with a constant acceleration w (and the zero initial velocity), while the disc rotates counterclockwise with a constant angular velocity ω.

1.52. A point A is located on the rim of a wheel of radius $R = 0.50$ m which rolls without slipping along a horizontal surface with velocity $v = 1.00$ m/s. Find:

(a) the modulus and the direction of the acceleration vector of the point A;

(b) the total distance s traversed by the point A between the two successive moments at which it touches the surface.

1.53. A ball of radius $R = 10.0$ cm rolls without slipping down an inclined plane so that its centre moves with constant acceleration

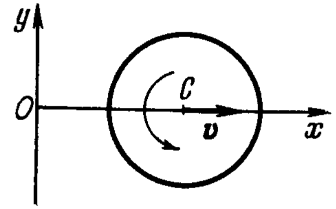


Fig. 1.6.

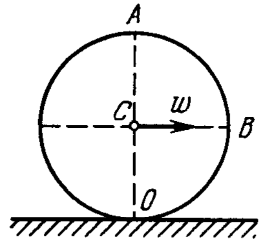


Fig. 1.7.

$w = 2.50$ cm/s^2; $t = 2.00$ s after the beginning of motion its position corresponds to that shown in Fig. 1.7. Find:

(a) the velocities of the points A, B, and O;

(b) the accelerations of these points.

1.54. A cylinder rolls without slipping over a horizontal plane. The radius of the cylinder is equal to r. Find the curvature radii of trajectories traced out by the points A and B (see Fig. 1.7).

1.55. Two solid bodies rotate about stationary mutually perpendicular intersecting axes with constant angular velocities $\omega_1 = 3.0$ rad/s and $\omega_2 = 4.0$ rad/s. Find the angular velocity and angular acceleration of one body relative to the other.

1.56. A solid body rotates with angular velocity $\boldsymbol{\omega} = at\mathbf{i} + bt^2\mathbf{j}$, where $a = 0.50$ rad/s^2, $b = 0.060$ rad/s^3, and $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes. Find:

(a) the moduli of the angular velocity and the angular acceleration at the moment $t = 10.0$ s;

(b) the angle between the vectors of the angular velocity and the angular acceleration at that moment.

1.57. A round cone with half-angle $\alpha = 30°$ and the radius of the base $R = 5.0$ cm rolls uniformly and without slipping over a horizontal plane as shown in Fig. 1.8. The cone apex is hinged at the point O which is on the same level with the point C, the cone base centre. The velocity of point C is $v = 10.0$ cm/s. Find the moduli of

(a) the vector of the angular velocity of the cone and the angle it forms with the vertical;

(b) the vector of the angular acceleration of the cone.

1.58. A solid body rotates with a constant angular velocity $\omega_0 = 0.50$ rad/s about a horizontal axis AB. At the moment $t = 0$

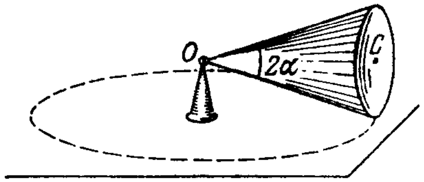


Fig. 1.8.

the axis AB starts turning about the vertical with a constant angular acceleration $\beta_0 = 0.10$ rad/s². Find the angular velocity and angular acceleration of the body after $t = 3.5$ s.

1.2. THE FUNDAMENTAL EQUATION OF DYNAMICS

• The fundamental equation of dynamics of a mass point (Newton's second law):

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}. \tag{1.2a}$$

• The same equation expressed in projections on the tangent and the normal of the point's trajectory:

$$m\frac{dv_\tau}{dt} = F_\tau, \quad m\frac{v^2}{R} = F_n. \tag{1.2b}$$

• The equation of dynamics of a point in the non-inertial reference frame K' which rotates with a constant angular velocity $\boldsymbol{\omega}$ about an axis translating with an acceleration $\mathbf{w}_0$:

$$m\mathbf{w}' = \mathbf{F} - m\mathbf{w}_0 + m\omega^2\mathbf{R} + 2m\,[\mathbf{v}'\boldsymbol{\omega}], \tag{1.2c}$$

where $\mathbf{R}$ is the radius vector of the point relative to the axis of rotation of the K' frame.

1.59. An aerostat of mass m starts coming down with a constant acceleration w. Determine the ballast mass to be dumped for the aerostat to reach the upward acceleration of the same magnitude. The air drag is to he neglected.

1.60. In the arrangement of Fig. 1.9 the masses m_0, m_1, and m_2 of bodies are equal, the masses of the pulley and the threads are negligible, and there is no friction in the pulley. Find the acceleration $\mathbf{w}$ with which the body m_0 comes down, and the tension of the thread binding together the bodies m_1 and m_2, if the coefficient of friction between these bodies and the horizontal surface is equal to k. Consider possible cases.

1.61. Two touching bars *1* and *2* are placed on an inclined plane forming an angle α with the horizontal (Fig. 1.10). The masses of the bars are equal to m_1 and m_2, and the coefficients of friction be-

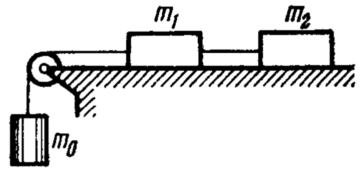


Fig. 1.9.

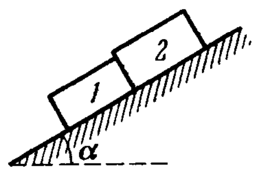


Fig. 1.10.

tween the inclined plane and these bars are equal to k_1 and k_2 respectively, with $k_1 > k_2$. Find:

(a) the force of interaction of the bars in the process of motion;

(b) the minimum value of the angle α at which the bars start sliding down.

1.62. A small body was launched up an inclined plane set at an angle $\alpha = 15°$ against the horizontal. Find the coefficient of friction, if the time of the ascent of the body is $\eta = 2.0$ times less than the time of its descent.

1.63. The following parameters of the arrangement of Fig. 1.11 are available: the angle α which the inclined plane forms with the horizontal, and the coefficient of friction k between the body m_1 and the inclined plane. The masses of the pulley and the threads, as well as the friction in the pulley, are negligible. Assuming both bodies to be motionless at the initial moment, find the mass ratio m_2/m_1 at which the body m_2

(a) starts coming down;

(b) starts going up;

(c) is at rest.

1.64. The inclined plane of Fig. 1.11 forms an angle $\alpha = 30°$ with the horizontal. The mass ratio $m_2/m_1 = \eta = 2/3$. The coefficient of friction between the body m_1 and the inclined plane is equal to $k = 0.10$. The masses of the pulley and the threads are negligible. Find the magnitude and the direction of acceleration of the body m_2 when the formerly stationary system of masses starts moving.

1.65. A plank of mass m_1 with a bar of mass m_2 placed on it lies on a smooth horizontal plane. A horizontal force growing with time t as $F = at$ (a is constant) is applied to the bar. Find how the accelerations of the plank w_1 and of the bar w_2 depend on t, if the coefficient of friction between the plank and the bar is equal to k. Draw the approximate plots of these dependences.

1.66. A small body A starts sliding down from the top of a wedge (Fig. 1.12) whose base is equal to $l = 2.10$ m. The coefficient of friction between the body and the wedge surface is $k = 0.140$. At

what value of the angle α will the time of sliding be the least? What will it be equal to?

1.67. A bar of mass m is pulled by means of a thread up an inclined plane forming an angle α with the horizontal (Fig. 1.13). The coef-

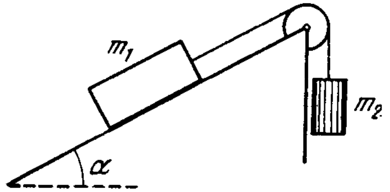


Fig. 1.11.

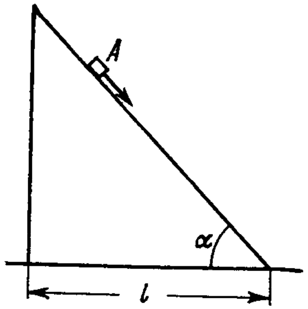


Fig. 1.12.

ficient of friction is equal to k. Find the angle β which the thread must form with the inclined plane for the tension of the thread to be minimum. What is it equal to?

1.68. At the moment $t = 0$ the force $F = at$ is applied to a small body of mass m resting on a smooth horizontal plane (a is a constant).

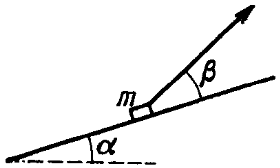


Fig. 1.13.

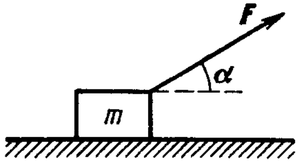


Fig. 1.14.

The permanent direction of this force forms an angle α with the horizontal (Fig. 1.14). Find:

(a) the velocity of the body at the moment of its breaking off the plane;

(b) the distance traversed by the body up to this moment.

1.69. A bar of mass m resting on a smooth horizontal plane starts moving due to the force $F = mg/3$ of constant magnitude. In the process of its rectilinear motion the angle α between the direction of this force and the horizontal varies as $\alpha = as$, where a is a constant, and s is the distance traversed by the bar from its initial position. Find the velocity of the bar as a function of the angle α.

1.70. A horizontal plane with the coefficient of friction k supports two bodies: a bar and an electric motor with a battery on a block. A thread attached to the bar is wound on the shaft of the electric motor. The distance between the bar and the electric motor is equal to l. When the motor is switched on, the bar, whose mass is twice

as great as that of the other body, starts moving with a constant acceleration w. How soon will the bodies collide?

1.71. A pulley fixed to the ceiling of an elevator car carries a thread whose ends are attached to the loads of masses m_1 and m_2. The car starts going up with an acceleration $\mathbf{w}_0$. Assuming the masses of the pulley and the thread, as well as the friction, to be negligible find:

(a) the acceleration of the load m_1 relative to the elevator shaft and relative to the car;

(b) the force exerted by the pulley on the ceiling of the car.

1.72. Find the acceleration $\mathbf{w}$ of body 2 in the arrangement shown in Fig. 1.15, if its mass is η times as great as the mass of bar 1 and

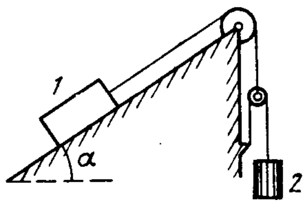


Fig. 1.15.

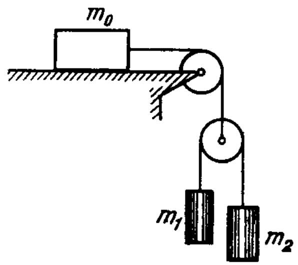


Fig. 1.16.

the angle that the inclined plane forms with the horizontal is equal to α. The masses of the pulleys and the threads, as well as the friction, are assumed to be negligible. Look into possible cases.

1.73. In the arrangement shown in Fig. 1.16 the bodies have masses m_0, m_1, m_2, the friction is absent, the masses of the pulleys and the threads are negligible. Find the acceleration of the body m_1. Look into possible cases.

1.74. In the arrangement shown in Fig. 1.17 the mass of the rod M exceeds the mass m of the ball. The ball has an opening permitting

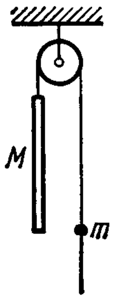


Fig. 1.17.

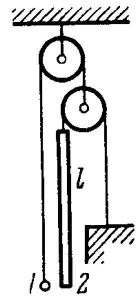


Fig. 1.18.

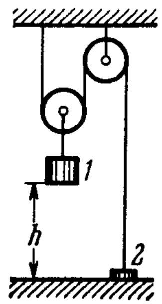


Fig. 1.19.

it to slide along the thread with some friction. The mass of the pulley and the friction in its axle are negligible. At the initial moment the ball was located opposite the lower end of the rod. When set free,

both bodies began moving with constant accelerations. Find the friction force between the ball and the thread if t seconds after the beginning of motion the ball got opposite the upper end of the rod. The rod length equals l.

1.75. In the arrangement shown in Fig. 1.18 the mass of ball *1* is $\eta = 1.8$ times as great as that of rod *2*. The length of the latter is $l = 100$ cm. The masses of the pulleys and the threads, as well as the friction, are negligible. The ball is set on the same level as the lower end of the rod and then released. How soon will the ball be opposite the upper end of the rod?

1.76. In the arrangement shown in Fig. 1.19 the mass of body *1* is $\eta = 4.0$ times as great as that of body *2*. The height $h = 20$ cm. The masses of the pulleys and the threads, as well as the friction, are negligible. At a certain moment body *2* is released and the arrangement set in motion. What is the maximum height that body *2* will go up to?

1.77. Find the accelerations of rod A and wedge B in the arrangement shown in Fig. 1.20 if the ratio of the mass of the wedge to that of the rod equals η, and the friction between all contact surfaces is negligible.

1.78. In the arrangement shown in Fig. 1.21 the masses of the wedge M and the body m are known. The appreciable friction exists

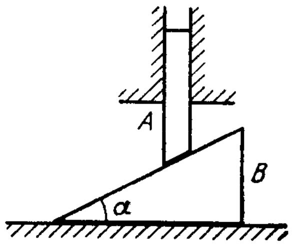


Fig. 1.20.

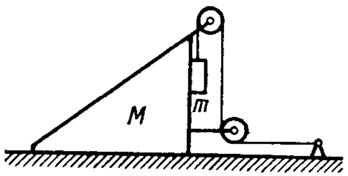


Fig. 1.21.

only between the wedge and the body m, the friction coefficient being equal to k. The masses of the pulley and the thread are negligible. Find the acceleration of the body m relative to the horizontal surface on which the wedge slides.

1.79. What is the minimum acceleration with which bar A (Fig. 1.22) should be shifted horizontally to keep bodies *1* and *2* stationary relative to the bar? The masses of the bodies are equal, and the coefficient of friction between the bar and the bodies is equal to k. The masses of the pulley and the threads are negligible, the friction in the pulley is absent.

1.80. Prism *1* with bar *2* of mass m placed on it gets a horizontal acceleration w directed to the left (Fig. 1.23). At what maximum value of this acceleration will the bar be still stationary relative to the prism, if the coefficient of friction between them $k < \cot \alpha$?

1.81. Prism *1* of mass m_1 and with angle α (see Fig. 1.23) rests on a horizontal surface. Bar *2* of mass m_2 is placed on the prism. Assuming the friction to be negligible, find the acceleration of the prism.

1.82. In the arrangement shown in Fig. 1.24 the masses m of the bar and M of the wedge, as well as the wedge angle α, are known.

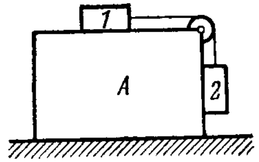


Fig. 1.22.

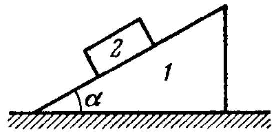


Fig. 1.23.

The masses of the pulley and the thread are negligible. The friction is absent. Find the acceleration of the wedge M.

1.83. A particle of mass m moves along a circle of radius R. Find the modulus of the average vector of the force acting on the particle over the distance equal to a quarter of the circle, if the particle moves

(a) uniformly with velocity v;

(b) with constant tangential acceleration w_τ, the initial velocity being equal to zero.

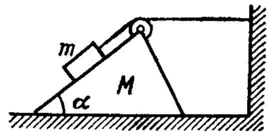


Fig. 1.24.

1.84. An aircraft loops the loop of radius $R = 500$ m with a constant velocity $v = 360$ km per hour. Find the weight of the flyer of mass $m = 70$ kg in the lower, upper, and middle points of the loop.

1.85. A small sphere of mass m suspended by a thread is first taken aside so that the thread forms the right angle with the vertical and then released. Find:

(a) the total acceleration of the sphere and the thread tension as a function of θ, the angle of deflection of the thread from the vertical;

(b) the thread tension at the moment when the vertical component of the sphere's velocity is maximum;

(c) the angle θ between the thread and the vertical at the moment when the total acceleration vector of the sphere is directed horizontally.

1.86. A ball suspended by a thread swings in a vertical plane so that its acceleration values in the extreme and the lowest position are equal. Find the thread deflection angle in the extreme position.

1.87. A small body A starts sliding off the top of a smooth sphere of radius R. Find the angle θ (Fig. 1.25) corresponding to the point at which the body breaks off the sphere, as well as the break-off velocity of the body.

1.88. A device (Fig. 1.26) consists of a smooth L-shaped rod located in a horizontal plane and a sleeve A of mass m attached by a weight-

less spring to a point B. The spring stiffness is equal to $\varkappa$. The whole system rotates with a constant angular velocity ω about a vertical axis passing through the point O. Find the elongation of the spring. How is the result affected by the rotation direction?

1.89. A cyclist rides along the circumference of a circular horizontal plane of radius R, the friction coefficient being dependent only on

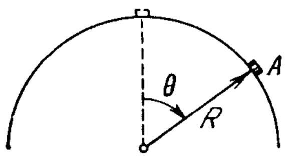


Fig. 1.25.

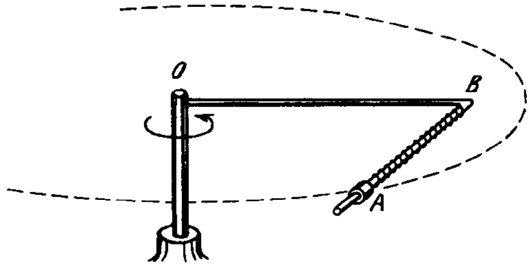


Fig. 1.26.

distance r from the centre O of the plane as $k = k_0(1 - r/R)$, where k_0 is a constant. Find the radius of the circle with the centre at the point along which the cyclist can ride with the maximum velocity. What is this velocity?

1.90. A car moves with a constant tangential acceleration $w_\tau = 0.62$ m/s^2 along a horizontal surface circumscribing a circle of radius $R = 40$ m. The coefficient of sliding friction between the wheels of the car and the surface is $k = 0.20$. What distance will the car ride without sliding if at the initial moment of time its velocity is equal to zero?

1.91. A car moves uniformly along a horizontal sine curve $y = a \sin(x/\alpha)$, where a and α are certain constants. The coefficient of friction between the wheels and the road is equal to k. At what velocity will the car ride without sliding?

1.92. A chain of mass m forming a circle of radius R is slipped on a smooth round cone with half-angle θ. Find the tension of the chain if it rotates with a constant angular velocity ω about a vertical axis coinciding with the symmetry axis of the cone.

1.93. A fixed pulley carries a weightless thread with masses m_1 and m_2 at its ends. There is friction between the thread and the pulley. It is such that the thread starts slipping when the ratio $m_2/m_1 = \eta_0$. Find:

(a) the friction coefficient;

(b) the acceleration of the masses when $m_2/m_1 = \eta > \eta_0$.

1.94. A particle of mass m moves along the internal smooth surface of a vertical cylinder of radius R. Find the force with which the particle acts on the cylinder wall if at the initial moment of time its velocity equals v_0 and forms an angle α with the horizontal.

1.95. Find the magnitude and direction of the force acting on the particle of mass m during its motion in the plane xy according to the law $x = a \sin \omega t$, $y = b \cos \omega t$, where a, b, and ω are constants.

1.96. A body of mass m is thrown at an angle to the horizontal with the initial velocity $\mathbf{v}_0$. Assuming the air drag to be negligible, find:

(a) the momentum increment $\Delta\mathbf{p}$ that the body acquires over the first t seconds of motion;

(b) the modulus of the momentum increment $\Delta\mathbf{p}$ during the total time of motion.

1.97. At the moment $t = 0$ a stationary particle of mass m experiences a time-dependent force $\mathbf{F} = \mathbf{a}t(\tau - t)$, where $\mathbf{a}$ is a constant vector, τ is the time during which the given force acts. Find:

(a) the momentum of the particle when the action of the force discontinued:

(b) the distance covered by the particle while the force acted.

1.98. At the moment $t = 0$ a particle of mass m starts moving due to a force $\mathbf{F} = \mathbf{F}_0 \sin \omega t$, where $\mathbf{F}_0$ and ω are constants. Find the distance covered by the particle as a function of t. Draw the approximate plot of this function.

1.99. At the moment $t = 0$ a particle of mass m starts moving due to a force $\mathbf{F} = \mathbf{F}_0 \cos \omega t$, where $\mathbf{F}_0$ and ω are constants. How long will it be moving until it stops for the first time? What distance will it traverse during that time? What is the maximum velocity of the particle over this distance?

1.100. A motorboat of mass m moves along a lake with velocity v_0. At the moment $t = 0$ the engine of the boat is shut down. Assuming the resistance of water to be proportional to the velocity of the boat $\mathbf{F} = -r\mathbf{v}$, find:

(a) how long the motorboat moved with the shutdown engine;

(b) the velocity of the motorboat as a function of the distance covered with the shutdown engine, as well as the total distance covered till the complete stop;

(c) the mean velocity of the motorboat over the time interval (beginning with the moment $t = 0$), during which its velocity decreases η times.

1.101. Having gone through a plank of thickness h, a bullet changed its velocity from v_0 to v. Find the time of motion of the bullet in the plank, assuming the resistance force to be proportional to the square of the velocity.

1.102. A small bar starts sliding down an inclined plane forming an angle α with the horizontal. The friction coefficient depends on the distance x covered as $k = ax$, where a is a constant. Find the distance covered by the bar till it stops, and its maximum velocity over this distance.

1.103. A body of mass m rests on a horizontal plane with the friction coefficient k. At the moment $t = 0$ a horizontal force is applied to it, which varies with time as $\mathbf{F} = \mathbf{a}t$, where $\mathbf{a}$ is a constant vector.

Find the distance traversed by the body during the first t seconds after the force action began.

1.104. A body of mass m is thrown straight up with velocity v_0. Find the velocity v' with which the body comes down if the air drag equals kv^2, where k is a constant and v is the velocity of the body.

1.105. A particle of mass m moves in a certain plane P due to a force F whose magnitude is constant and whose vector rotates in that plane with a constant angular velocity ω. Assuming the particle to be stationary at the moment $t = 0$, find:

(a) its velocity as a function of time;

(b) the distance covered by the particle between two successive stops, and the mean velocity over this time.

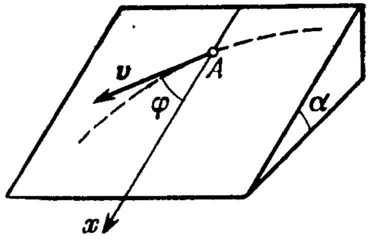


Fig. 1.27.

1.106. A small disc A is placed on an inclined plane forming an angle α with the horizontal (Fig. 1.27) and is imparted an initial velocity v_0. Find how the velocity of the disc depends on the angle φ if the friction coefficient $k = \tan\alpha$ and at the initial moment $\varphi_0 = \pi/2$.

1.107. A chain of length l is placed on a smooth spherical surface of radius R with one of its ends fixed at the top of the sphere. What will be the acceleration w of each element of the chain when its upper end is released? It is assumed that the length of the chain $l < \frac{1}{2}\pi R$.

1.108. A small body is placed on the top of a smooth sphere of radius R. Then the sphere is imparted a constant acceleration $\mathbf{w}_0$ in the horizontal direction and the body begins sliding down. Find:

(a) the velocity of the body relative to the sphere at the moment of break-off;

(b) the angle θ_0 between the vertical and the radius vector drawn from the centre of the sphere to the break-off point; calculate θ_0 for $w_0 = g$.

1.109. A particle moves in a plane under the action of a force which is always perpendicular to the particle's velocity and depends on a distance to a certain point on the plane as $1/r^n$, where n is a constant. At what value of n will the motion of the particle along the circle be *steady*?

1.110. A sleeve A can slide freely along a smooth rod bent in the shape of a half-circle of radius R (Fig. 1.28). The system is set in rotation with a constant angular velocity ω about a vertical axis OO'. Find the angle θ corresponding to the steady position of the sleeve.

1.111. A rifle was aimed at the vertical line on the target located precisely in the northern direction, and then fired. Assuming the air drag to be negligible, find how much off the line, and in what direction, will the bullet hit the target. The shot was fired in the horizontal

direction at the latitude $\varphi = 60°$, the bullet velocity $v = 900$ m/s, and the distance from the target equals $s = 1.0$ km.

1.112. A horizontal disc rotates with a constant angular velocity $\omega = 6.0$ rad/s about a vertical axis passing through its centre. A small body of mass $m = 0.50$ kg moves along a diameter of the disc with a velocity $v' = 50$ cm/s which is constant relative to the disc. Find the force that the disc exerts on the body at the moment when it is located at the distance $r = 30$ cm from the rotation axis.

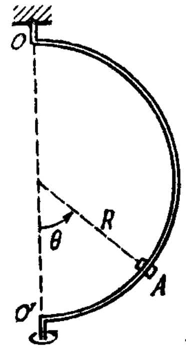


Fig. 1.28.

1.113. A horizontal smooth rod AB rotates with a constant angular velocity $\omega = 2.00$ rad/s about a vertical axis passing through its end A. A freely sliding sleeve of mass $m = 0.50$ kg moves along the rod from the point A with the initial velocity $v_0 = 1.00$ m/s. Find the Coriolis force acting on the sleeve (in the reference frame fixed to the rotating rod) at the moment when the sleeve is located at the distance $r = 50$ cm from the rotation axis.

1.114. A horizontal disc of radius R rotates with a constant angular velocity ω about a stationary vertical axis passing through its edge. Along the circumference of the disc a particle of mass m moves with a velocity that is constant relative to the disc. At the moment when the particle is at the maximum distance from the rotation axis, the resultant of the inertial forces F_{in} acting on the particle in the reference frame fixed to the disc turns into zero. Find:

(a) the acceleration w' of the particle relative to the disc;

(b) the dependence of F_{in} on the distance from the rotation axis.

1.115. A small body of mass $m = 0.30$ kg starts sliding down from the top of a smooth sphere of radius $R = 1.00$ m. The sphere rotates with a constant angular velocity $\omega = 6.0$ rad/s about a vertical axis passing through its centre. Find the centrifugal force of inertia and the Coriolis force at the moment when the body breaks off the surface of the sphere in the reference frame fixed to the sphere.

1.116. A train of mass $m = 2000$ tons moves in the latitude $\varphi = 60°$ North. Find:

(a) the magnitude and direction of the lateral force that the train exerts on the rails if it moves along a meridian with a velocity $v = 54$ km per hour;

(b) in what direction and with what velocity the train should move for the resultant of the inertial forces acting on the train in the reference frame fixed to the Earth to be equal to zero.

1.117. At the equator a stationary (relative to the Earth) body falls down from the height $h = 500$ m. Assuming the air drag to be negligible, find how much off the vertical, and in what direction, the body will deviate when it hits the ground.

1.3. LAWS OF CONSERVATION OF ENERGY, MOMENTUM, AND ANGULAR MOMENTUM

• Work and power of the force **F**:

$$A = \int \mathbf{F}\, d\mathbf{r} = \int F_s\, ds, \quad P = \mathbf{F}\mathbf{v}. \tag{1.3a}$$

• Increment of the kinetic energy of a particle:

$$T_2 - T_1 = A, \tag{1.3b}$$

where A is the work performed by the resultant of *all* the forces acting on the particle.

• Work performed by the forces of a field is equal to the decrease of the potential energy of a particle in the given field:

$$A = U_1 - U_2. \tag{1.3c}$$

• Relationship between the force of a field and the potential energy of a particle in the field:

$$\mathbf{F} = -\nabla U, \tag{1.3d}$$

i.e. the force is equal to the antigradient of the potential energy.

• Increment of the total mechanical energy of a particle in a given potential field:

$$E_2 - E_1 = A_{extr} \tag{1.3e}$$

where A_{extr} is the algebraic sum of works performed by all *extraneous* forces, that is, by the forces not belonging to those of the *given* field.

• Increment of the total mechanical energy of a system:

$$E_2 - E_1 = A_{ext} + A_{int}^{noncons}, \tag{1.3f}$$

where $E = T + U$, and U is the *inherent* potential energy of the system.

• Law of momentum variation of a system:

$$d\mathbf{p}/dt = \mathbf{F}, \tag{1.3g}$$

where **F** is the resultant of all *external* forces.

• Equation of motion of the system's centre of inertia:

$$m\frac{d\mathbf{v}_C}{dt} = \mathbf{F}, \tag{1.3h}$$

where **F** is the resultant of all *external* forces.

• Kinetic energy of a system

$$T = \widetilde{T} + \frac{mv_C^2}{2}, \tag{1.3i}$$

where $\widetilde{T}$ is its kinetic energy in the system of centre of inertia.

• Equation of dynamics of a body with variable mass:

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} + \frac{dm}{dt}\mathbf{u}, \tag{1.3j}$$

where **u** is the velocity of the separated (gained) substance relative to the body considered.

• Law of angular momentum variation of a system:

$$\frac{d\mathbf{M}}{dt} = \mathbf{N}, \tag{1.3k}$$

where **M** is the angular momentum of the system, and **N** is the total moment of all *external* forces.

• Angular momentum of a system:

$$\mathbf{M} = \widetilde{\mathbf{M}} + [\mathbf{r}_C \mathbf{p}], \tag{1.3l}$$

where $\widetilde{\mathbf{M}}$ is its angular momentum in the system of the centre of inertia, $\mathbf{r}_C$ is the radius vector of the centre of inertia, and **p** is the momentum of the system.

1.118. A particle has shifted along some trajectory in the plane xy from point *1* whose radius vector $\mathbf{r}_1 = \mathbf{i} + 2\mathbf{j}$ to point *2* with the radius vector $\mathbf{r}_2 = 2\mathbf{i} - 3\mathbf{j}$. During that time the particle experienced the action of certain forces, one of which being $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j}$. Find the work performed by the force **F**. (Here r_1, r_2, and F are given in SI units).

1.119. A locomotive of mass m starts moving so that its velocity varies according to the law $v = a\sqrt{s}$, where a is a constant, and s is the distance covered. Find the total work performed by all the forces which are acting on the locomotive during the first t seconds after the beginning of motion.

1.120. The kinetic energy of a particle moving along a circle of radius R depends on the distance covered s as $T = as^2$, where a is a constant. Find the force acting on the particle as a function of s.

1.121. A body of mass m was slowly hauled up the hill (Fig. 1.29) by a force **F** which at each point was directed along a tangent to the trajectory. Find the work performed by this force, if the height of the hill is h, the length of its base l, and the coefficient of friction k.

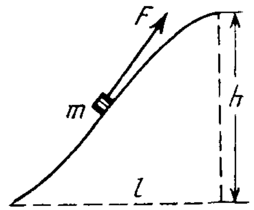


Fig. 1.29.

1.122. A disc of mass $m = 50$ g slides with the zero initial velocity down an inclined plane set at an angle $\alpha = 30°$ to the horizontal; having traversed the distance $l = 50$ cm along the horizontal plane, the disc stops. Find the work performed by the friction forces over the whole distance, assuming the friction coefficient $k = 0.15$ for both inclined and horizontal planes.

1.123. Two bars of masses m_1 and m_2 connected by a non-deformed light spring rest on a horizontal plane. The coefficient of friction between the bars and the surface is equal to k. What minimum constant force has to be applied in the horizontal direction to the bar of mass m_1 in order to shift the other bar?

1.124. A chain of mass $m = 0.80$ kg and length $l = 1.5$ m rests on a rough-surfaced table so that one of its ends hangs over the edge. The chain starts sliding off the table all by itself provided the overhanging part equals $\eta = 1/3$ of the chain length. What will be the

total work performed by the friction forces acting on the chain by the moment it slides completely off the table?

1.125. A body of mass m is thrown at an angle α to the horizontal with the initial velocity v_0. Find the mean power developed by gravity over the whole time of motion of the body, and the instantaneous power of gravity as a function of time.

1.126. A particle of mass m moves along a circle of radius R with a normal acceleration varying with time as $w_n = at^2$, where a is a constant. Find the time dependence of the power developed by all the forces acting on the particle, and the mean value of this power averaged over the first t seconds after the beginning of motion.

1.127. A small body of mass m is located on a horizontal plane at the point O. The body acquires a horizontal velocity v_0. Find:

(a) the mean power developed by the friction force during the whole time of motion, if the friction coefficient $k = 0.27$, $m = 1.0$ kg, and $v_0 = 1.5$ m/s;

(b) the maximum instantaneous power developed by the friction force, if the friction coefficient varies as $k = \alpha x$, where α is a constant, and x is the distance from the point O.

1.128. A small body of mass $m = 0.10$ kg moves in the reference frame rotating about a stationary axis with a constant angular velocity $\omega = 5.0$ rad/s. What work does the centrifugal force of inertia perform during the transfer of this body along an arbitrary path from point *1* to point *2* which are located at the distances $r_1 = 30$ cm and $r_2 = 50$ cm from the rotation axis?

1.129. A system consists of two springs connected in series and having the stiffness coefficients k_1 and k_2. Find the minimum work to be performed in order to stretch this system by Δl.

1.130. A body of mass m is hauled from the Earth's surface by applying a force **F** varying with the height of ascent y as $\mathbf{F} = 2(ay - 1)m\mathbf{g}$, where a is a positive constant. Find the work performed by this force and the increment of the body's potential energy in the gravitational field of the Earth over the first half of the ascent.

1.131. The potential energy of a particle in a certain field has the form $U = a/r^2 - b/r$, where a and b are positive constants, r is the distance from the centre of the field. Find:

(a) the value of r_0 corresponding to the equilibrium position of the particle; examine whether this position is steady;

(b) the maximum magnitude of the attraction force; draw the plots $U(r)$ and $F_r(r)$ (the projections of the force on the radius vector **r**).

1.132. In a certain two-dimensional field of force the potential energy of a particle has the form $U = \alpha x^2 + \beta y^2$, where α and β are positive constants whose magnitudes are different. Find out:

(a) whether this field is central;

(b) what is the shape of the equipotential surfaces and also of the surfaces for which the magnitude of the vector of force $F = \text{const}$.

1.133. There are two stationary fields of force $\mathbf{F} = ay\mathbf{i}$ and $\mathbf{F} = ax\mathbf{i} + by\mathbf{j}$, where **i** and **j** are the unit vectors of the x and y axes, and a and b are constants. Find out whether these fields are potential.

1.134. A body of mass m is pushed with the initial velocity v_0 up an inclined plane set at an angle α to the horizontal. The friction coefficient is equal to k. What distance will the body cover before it stops and what work do the friction forces perform over this distance?

1.135. A small disc A slides down with initial velocity equal to zero from the top of a smooth hill of height H having a horizontal portion (Fig. 1.30). What must be the height of the horizontal portion h to ensure the maximum distance s covered by the disc? What is it equal to?

1.136. A small body A starts sliding from the height h down an inclined groove passing into a half-circle of radius $h/2$ (Fig. 1.31).

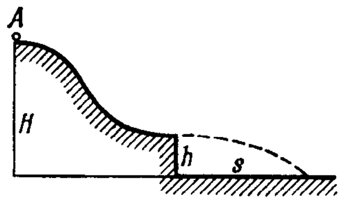


Fig. 1.30.

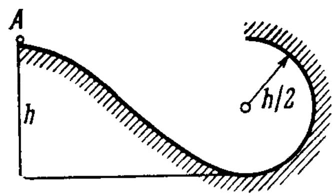


Fig. 1.31.

Assuming the friction to be negligible, find the velocity of the body at the highest point of its trajectory (after breaking off the groove).

1.137. A ball of mass m is suspended by a thread of length l. With what minimum velocity has the point of suspension to be shifted in the horizontal direction for the ball to move along the circle about that point? What will be the tension of the thread at the moment it will be passing the horizontal position?

1.138. A horizontal plane supports a stationary vertical cylinder of radius R and a disc A attached to the cylinder by a horizontal thread AB of length l_0 (Fig. 1.32, top view). An initial velocity v_0

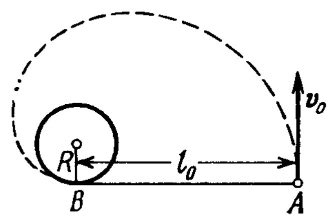


Fig. 1.32.

Fig. 1.33.

is imparted to the disc as shown in the figure. How long will it move along the plane until it strikes against the cylinder? The friction is assumed to be absent.

1.139. A smooth rubber cord of length l whose coefficient of elasticity is k is suspended by one end from the point O (Fig. 1.33). The other end is fitted with a catch B. A small sleeve A of mass m starts falling from the point O. Neglecting the masses of the thread and the catch, find the maximum elongation of the cord.

1.140. A small bar A resting on a smooth horizontal plane is attached by threads to a point P (Fig. 1.34) and, by means of a weightless pulley, to a weight B possessing the same mass as the bar itself.

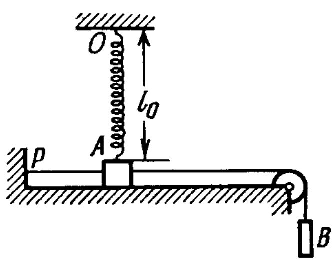


Fig. 1.34.

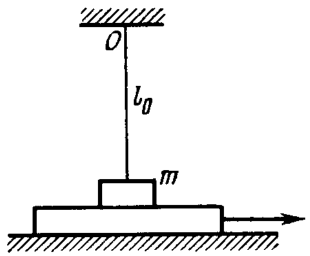


Fig. 1.35.

Besides, the bar is also attached to a point O by means of a light non-deformed spring of length $l_0 = 50$ cm and stiffness $\varkappa = 5\ mg/l_0$, where m is the mass of the bar. The thread PA having been burned, the bar starts moving. Find its velocity at the moment when it is breaking off the plane.

1.141. A horizontal plane supports a plank with a bar of mass $m = 1.0$ kg placed on it and attached by a light elastic non-deformed cord of length $l_0 = 40$ cm to a point O (Fig. 1.35). The coefficient of friction between the bar and the plank equals $k = 0.20$. The plank is slowly shifted to the right until the bar starts sliding over it. It occurs at the moment when the cord deviates from the vertical by an angle $\theta = 30°$. Find the work that has been performed by that moment by the friction force acting on the bar in the reference frame fixed to the plane.

1.142. A smooth light horizontal rod AB can rotate about a vertical axis passing through its end A. The rod is fitted with a small sleeve of mass m attached to the end A by a weightless spring of length l_0 and stiffness $\varkappa$. What work must be performed to slowly get this system going and reaching the angular velocity ω?

1.143. A pulley fixed to the ceiling carries a thread with bodies of masses m_1 and m_2 attached to its ends. The masses of the pulley and the thread are negligible, friction is absent. Find the acceleration $\mathbf{w}_C$ of the centre of inertia of this system.

1.144. Two interacting particles form a closed system whose centre of inertia is at rest. Fig. 1.36 illustrates the positions of both particles at a certain moment and the trajectory of the particle of mass m_1. Draw the trajectory of the particle of mass m_2 if $m_2 = m_1/2$.

1.145. A closed chain A of mass $m = 0.36$ kg is attached to a vertical rotating shaft by means of a thread (Fig. 1.37), and rotates with a constant angular velocity $\omega = 35$ rad/s. The thread forms an angle $\theta = 45°$ with the vertical. Find the distance between the chain's centre of gravity and the rotation axis, and the tension of the thread.

Fig. 1.36.

1.146. A round cone A of mass $m = 3.2$ kg and half-angle $\alpha = 10°$ rolls uniformly and without slipping along a round conical surface B so that its apex O remains stationary (Fig. 1.38). The centre of gravity of the cone A is at the same level as the point O and at a distance $l = 17$ cm from it. The cone's axis moves with angular velocity ω. Find:

(a) the static friction force acting on the cone A, if $\omega = 1.0$ rad/s;

(b) at what values of ω the cone A will roll without sliding, if the coefficient of friction between the surfaces is equal to $k = 0.25$.

1.147. In the reference frame K two particles travel along the x axis, one of mass m_1 with velocity $\mathbf{v}_1$, and the other of mass m_2 with velocity $\mathbf{v}_2$. Find:

(a) the velocity $\mathbf{V}$ of the reference frame K' in which the cumulative kinetic energy of these particles is minimum;

(b) the cumulative kinetic energy of these particles in the K' frame.

1.148. The reference frame, in which the centre of inertia of a given system of particles is at rest, translates with a velocity $\mathbf{V}$ relative

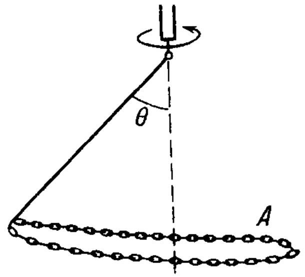


Fig. 1.37.

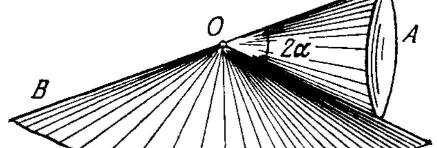


Fig. 1.38.

to an inertial reference frame K. The mass of the system of particles equals m, and the total energy of the system in the frame of the centre of inertia is equal to $\tilde{E}$. Find the total energy E of this system of particles in the reference frame K.

1.149. Two small discs of masses m_1 and m_2 interconnected by a weightless spring rest on a smooth horizontal plane. The discs are set in motion with initial velocities v_1 and v_2 whose directions are

mutually perpendicular and lie in a horizontal plane. Find the total energy $\tilde{E}$ of this system in the frame of the centre of inertia.

1.150. A system consists of two small spheres of masses m_1 and m_2 interconnected by a weightless spring. At the moment $t = 0$ the spheres are set in motion with the initial velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ after which the system starts moving in the Earth's uniform gravitational field. Neglecting the air drag, find the time dependence of the total momentum of this system in the process of motion and of the radius vector of its centre of inertia relative to the initial position of the centre.

1.151. Two bars of masses m_1 and m_2 connected by a weightless spring of stiffness $\varkappa$ (Fig. 1.39) rest on a smooth horizontal plane.

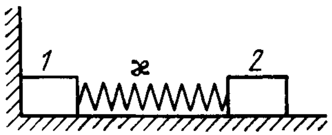


Fig. 1.39.

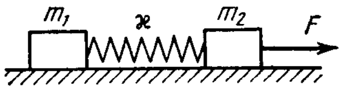


Fig. 1.40.

Bar *2* is shifted a small distance x to the left and then released. Find the velocity of the centre of inertia of the system after bar *1* breaks off the wall.

1.152. Two bars connected by a weightless spring of stiffness $\varkappa$ and length (in the non-deformed state) l_0 rest on a horizontal plane. A constant horizontal force F starts acting on one of the bars as shown in Fig. 1.40. Find the maximum and minimum distances between the bars during the subsequent motion of the system, if the masses of the bars are:

(a) equal;

(b) equal to m_1 and m_2, and the force F is applied to the bar of mass m_2.

1.153. A system consists of two identical cubes, each of mass m, linked together by the compressed weightless spring of stiffness $\varkappa$ (Fig. 1.41). The cubes are also connected by a thread which is burned through at a certain moment. Find:

(a) at what values of Δl, the initial compression of the spring, the lower cube will bounce up after the thread has been burned through;

(b) to what height h the centre of gravity of this system will rise if the initial compression of the spring $\Delta l = 7\ mg/\varkappa$.

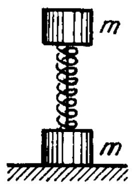


Fig. 1.41.

1.154. Two identical buggies *1* and *2* with one man in each move without friction due to inertia along the parallel rails toward each other. When the buggies get opposite each other, the men exchange their places by jumping in the direction perpendicular to the motion direction. As a consequence, buggy *1* stops and buggy *2* keeps moving in the same direction, with its velocity becoming equal to $\mathbf{v}$. Find the initial velocities of the buggies $\mathbf{v}_1$ and $\mathbf{v}_2$ if the mass of each buggy (without a man) equals M and the mass of each man m.

1.155. Two identical buggies move one after the other due to inertia (without friction) with the same velocity $\mathbf{v}_0$. A man of mass m rides the rear buggy. At a certain moment the man jumps into the front buggy with a velocity $\mathbf{u}$ relative to his buggy. Knowing that the mass of each buggy is equal to M, find the velocities with which the buggies will move after that.

1.156. Two men, each of mass m, stand on the edge of a stationary buggy of mass M. Assuming the friction to be negligible, find the velocity of the buggy after both men jump off with the same horizontal velocity $\mathbf{u}$ relative to the buggy: (1) simultaneously; (2) one after the other. In what case will the velocity of the buggy be greater and how many times?

1.157. A chain hangs on a thread and touches the surface of a table by its lower end. Show that after the thread has been burned through, the force exerted on the table by the falling part of the chain at any moment is twice as great as the force of pressure exerted by the part already resting on the table.

1.158. A steel ball of mass $m = 50$ g falls from the height $h = 1.0$ m on the horizontal surface of a massive slab. Find the cumulative momentum that the ball imparts to the slab after numerous bounces, if every impact decreases the velocity of the ball $\eta = 1.25$ times.

1.159. A raft of mass M with a man of mass m aboard stays motionless on the surface of a lake. The man moves a distance $\mathbf{l}'$ relative to the raft with velocity $\mathbf{v}'(t)$ and then stops. Assuming the water resistance to be negligible, find:

(a) the displacement of the raft $\mathbf{l}$ relative to the shore;

(b) the horizontal component of the force with which the man acted on the raft during the motion.

1.160. A stationary pulley carries a rope whose one end supports a ladder with a man and the other end the counterweight of mass M. The man of mass m climbs up a distance $\mathbf{l}'$ with respect to the ladder and then stops. Neglecting the mass of the rope and the friction in the pulley axle, find the displacement $\mathbf{l}$ of the centre of inertia of this system.

1.161. A cannon of mass M starts sliding freely down a smooth inclined plane at an angle α to the horizontal. After the cannon covered the distance l, a shot was fired, the shell leaving the cannon in the horizontal direction with a momentum $\mathbf{p}$. As a consequence, the cannon stopped. Assuming the mass of the shell to be negligible, as compared to that of the cannon, determine the duration of the shot.

1.162. A horizontally flying bullet of mass m gets stuck in a body of mass M suspended by two identical threads of length l (Fig. 1.42).

As a result, the threads swerve through an angle θ. Assuming $m \ll M$, find:

(a) the velocity of the bullet before striking the body;

(b) the fraction of the bullet's initial kinetic energy that turned into heat.

1.163. A body of mass M (Fig. 1.43) with a small disc of mass m placed on it rests on a smooth horizontal plane. The disc is set in

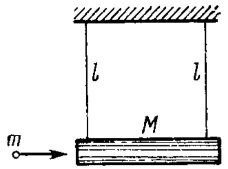


Fig. 1.42.

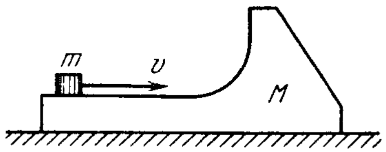


Fig. 1.43.

motion in the horizontal direction with velocity v. To what height (relative to the initial level) will the disc rise after breaking off the body M? The friction is assumed to be absent.

1.164. A small disc of mass m slides down a smooth hill of height h without initial velocity and gets onto a plank of mass M lying on

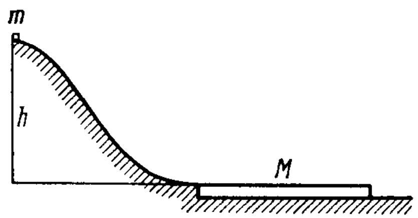


Fig. 1.44.

the horizontal plane at the base of the hill (Fig. 1.44). Due to friction between the disc and the plank the disc slows down and, beginning with a certain moment, moves in one piece with the plank.

(1) Find the total work performed by the friction forces in this process.

(2) Can it be stated that the result obtained does not depend on the choice of the reference frame?

1.165. A stone falls down without initial velocity from a height h onto the Earth's surface. The air drag assumed to be negligible, the stone hits the ground with velocity $v_0 = \sqrt{2gh}$ relative to the Earth. Obtain the same formula in terms of the reference frame "falling" to the Earth with a constant velocity v_0.

1.166. A particle of mass 1.0 g moving with velocity $\mathbf{v}_1 = 3.0\mathbf{i} - 2.0\mathbf{j}$ experiences a perfectly inelastic collision with another particle of mass 2.0 g and velocity $\mathbf{v}_2 = 4.0\mathbf{j} - 6.0\mathbf{k}$. Find the velocity of the formed particle (both the vector $\mathbf{v}$ and its modulus), if the components of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are given in the SI units.

1.167. Find the increment of the kinetic energy of the closed system comprising two spheres of masses m_1 and m_2 due to their perfectly inelastic collision, if the initial velocities of the spheres were equal to $\mathbf{v}_1$ and $\mathbf{v}_2$.

1.168. A particle of mass m_1 experienced a perfectly elastic collision with a stationary particle of mass m_2. What fraction of the kinetic energy does the striking particle lose, if

(a) it recoils at right angles to its original motion direction;

(b) the collision is a head-on one?

1.169. Particle *1* experiences a perfectly elastic collision with a stationary particle *2*. Determine their mass ratio, if

(a) after a head-on collision the particles fly apart in the opposite directions with equal velocities;

(b) the particles fly apart symmetrically relative to the initial motion direction of particle *1* with the angle of divergence $\theta = 60°$.

1.170. A ball moving translationally collides elastically with another, stationary, ball of the same mass. At the moment of impact the angle between the straight line passing through the centres of the balls and the direction of the initial motion of the striking ball is equal to $\alpha = 45°$. Assuming the balls to be smooth, find the fraction η of the kinetic energy of the striking ball that turned into potential energy at the moment of the maximum deformation.

1.171. A shell flying with velocity $v = 500$ m/s bursts into three identical fragments so that the kinetic energy of the system increases $\eta = 1.5$ times. What maximum velocity can one of the fragments obtain?

1.172. Particle *1* moving with velocity $v = 10$ m/s experienced a head-on collision with a stationary particle *2* of the same mass. As a result of the collision, the kinetic energy of the system decreased by $\eta = 1.0\%$. Find the magnitude and direction of the velocity of particle *1* after the collision.

1.173. A particle of mass m having collided with a stationary particle of mass M deviated by an angle $\pi/2$ whereas the particle M recoiled at an angle $\theta = 30°$ to the direction of the initial motion of the particle m. How much (in per cent) and in what way has the kinetic energy of this system changed after the collision, if $M/m = 5.0$?

1.174. A closed system consists of two particles of masses m_1 and m_2 which move at right angles to each other with velocities v_1 and v_2. Find:

(a) the momentum of each particle and

(b) the total kinetic energy of the two particles in the reference frame fixed to their centre of inertia.

1.175. A particle of mass m_1 collides elastically with a stationary particle of mass m_2 ($m_1 > m_2$). Find the maximum angle through which the striking particle may deviate as a result of the collision.

1.176. Three identical discs A, B, and C (Fig. 1.45) rest on a smooth horizontal plane. The disc A is set in motion with velocity $\mathbf{v}$ after

which it experiences an elastic collision simultaneously with the discs B and C. The distance between the centres of the latter discs prior to the collision is η times greater than the diameter of each disc. Find the velocity of the disc A after the collision. At what value of η will the disc A recoil after the collision; stop; move on?

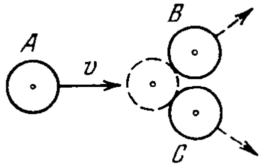


Fig. 1.45.

1.177. A molecule collides with another, stationary, molecule of the same mass. Demonstrate that the angle of divergence

(a) equals 90° when the collision is ideally elastic;

(b) differs from 90° when the collision is inelastic.

1.178. A rocket ejects a steady jet whose velocity is equal to $\mathbf{u}$ relative to the rocket. The gas discharge rate equals μ kg/s. Demonstrate that the rocket motion equation in this case takes the form

$$m\mathbf{w} = \mathbf{F} - \mu\mathbf{u},$$

where m is the mass of the rocket at a given moment, $\mathbf{w}$ is its acceleration, and $\mathbf{F}$ is the external force.

1.179. A rocket moves in the absence of external forces by ejecting a steady jet with velocity $\mathbf{u}$ constant relative to the rocket. Find the velocity $\mathbf{v}$ of the rocket at the moment when its mass is equal to m, if at the initial moment it possessed the mass m_0 and its velocity was equal to zero. Make use of the formula given in the foregoing problem.

1.180. Find the law according to which the mass of the rocket varies with time, when the rocket moves with a constant acceleration w, the external forces are absent, the gas escapes with a constant velocity u relative to the rocket, and its mass at the initial moment equals m_0.

1.181. A spaceship of mass m_0 moves in the absence of external forces with a constant velocity $\mathbf{v}_0$. To change the motion direction, a jet engine is switched on. It starts ejecting a gas jet with velocity u which is constant relative to the spaceship and directed at right angles to the spaceship motion. The engine is shut down when the mass of the spaceship decreases to m. Through what angle α did the motion direction of the spaceship deviate due to the jet engine operation?

1.182. A cart loaded with sand moves along a horizontal plane due to a constant force $\mathbf{F}$ coinciding in direction with the cart's velocity vector. In the process, sand spills through a hole in the bottom with a constant velocity μ kg/s. Find the acceleration and the velocity of the cart at the moment t, if at the initial moment $t = 0$ the cart with loaded sand had the mass m_0 and its velocity was equal to zero. The friction is to be neglected.

1.183. A flatcar of mass m_0 starts moving to the right due to a constant horizontal force $\mathbf{F}$ (Fig. 1.46). Sand spills on the flatcar from a stationary hopper. The velocity of loading is constant and equal to μ kg/s. Find the time dependence of the velocity and the acceleration of the flatcar in the process of loading. The friction is negligibly small.

1.184. A chain AB of length l is located in a smooth horizontal tube so that its fraction of length h hangs freely and touches the surface of the table with its end B (Fig. 1.47). At a certain moment the end A of the chain is set free. With what velocity will this end of the chain slip out of the tube?

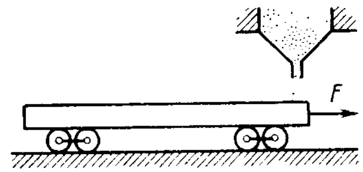


Fig. 1.46.

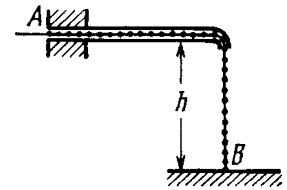


Fig. 1.47.

1.185. The angular momentum of a particle relative to a certain point O varies with time as $\mathbf{M} = \mathbf{a} + \mathbf{b}t^2$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, with $\mathbf{a} \perp \mathbf{b}$. Find the force moment $\mathbf{N}$ relative to the point O acting on the particle when the angle between the vectors $\mathbf{N}$ and $\mathbf{M}$ equals 45°.

1.186. A ball of mass m is thrown at an angle α to the horizontal with the initial velocity v_0. Find the time dependence of the magnitude of the ball's angular momentum vector relative to the point from which the ball is thrown. Find the angular momentum M at the highest point of the trajectory if $m = 130$ g, $\alpha = 45°$, and $v_0 = 25$ m/s. The air drag is to be neglected.

1.187. A disc A of mass m sliding over a smooth horizontal surface with velocity v experiences a perfectly elastic collision with a smooth stationary wall at a point O (Fig. 1.48). The angle between the motion direction of the disc and the normal of the wall is equal to α. Find:

(a) the points relative to which the angular momentum $\mathbf{M}$ of the disc remains constant in this process;

(b) the magnitude of the increment of the vector of the disc's angular momentum relative to the point O' which is located in the plane of the disc's motion at the distance l from the point O.

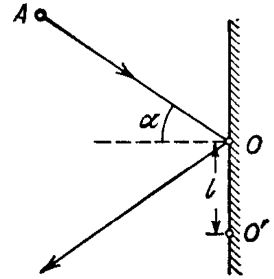


Fig. 1.48.

1.188. A small ball of mass m suspended from the ceiling at a point O by a thread of length l moves along a horizontal circle with a constant angular velocity ω. Relative to which points does the angular momentum $\mathbf{M}$ of the ball remain constant? Find the magnitude of the increment

of the vector of the ball's angular momentum relative to the point O picked up during half a revolution.

1.189. A ball of mass m falls down without initial velocity from a height h over the Earth's surface. Find the increment of the ball's angular momentum vector picked up during the time of falling (relative to the point O of the reference frame moving translationally in a horizontal direction with a velocity V). The ball starts falling from the point O. The air drag is to be neglected.

1.190. A smooth horizontal disc rotates with a constant angular velocity ω about a stationary vertical axis passing through its centre, the point O. At a moment $t = 0$ a disc is set in motion from that

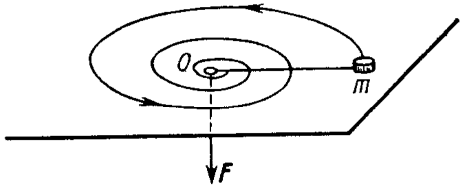


Fig. 1.49.

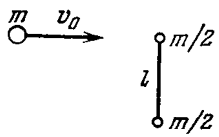


Fig. 1.50.

point with velocity v_0. Find the angular momentum $M(t)$ of the disc relative to the point O in the reference frame fixed to the disc. Make sure that this angular momentum is caused by the Coriolis force.

1.191. A particle moves along a closed trajectory in a central field of force where the particle's potential energy $U = kr^2$ (k is a positive constant, r is the distance of the particle from the centre O of the field). Find the mass of the particle if its minimum distance from the point O equals r_1 and its velocity at the point farthest from O equals v_2.

1.192. A small ball is suspended from a point O by a light thread of length l. Then the ball is drawn aside so that the thread deviates through an angle θ from the vertical and set in motion in a horizontal direction at right angles to the vertical plane in which the thread is located. What is the initial velocity that has to be imparted to the ball so that it could deviate through the maximum angle $\pi/2$ in the process of motion?

1.193. A small body of mass m tied to a non-stretchable thread moves over a smooth horizontal plane. The other end of the thread is being drawn into a hole O (Fig. 1.49) with a constant velocity. Find the thread tension as a function of the distance r between the body and the hole if at $r = r_0$ the angular velocity of the thread is equal to ω_0.

1.194. A light non-stretchable thread is wound on a massive fixed pulley of radius R. A small body of mass m is tied to the free end of the thread. At a moment $t = 0$ the system is released and starts moving. Find its angular momentum relative to the pulley axle as a function of time t.

1.195. A uniform sphere of mass m and radius R starts rolling without slipping down an inclined plane at an angle α to the horizontal. Find the time dependence of the angular momentum of the sphere relative to the point of contact at the initial moment. How will the obtained result change in the case of a perfectly smooth inclined plane?

1.196. A certain system of particles possesses a total momentum $\mathbf{p}$ and an angular momentum $\mathbf{M}$ relative to a point O. Find its angular momentum $\mathbf{M}'$ relative to a point O' whose position with respect to the point O is determined by the radius vector $\mathbf{r}_0$. Find out when the angular momentum of the system of particles does not depend on the choice of the point O.

1.197. Demonstrate that the angular momentum $\mathbf{M}$ of the system of particles relative to a point O of the reference frame K can be represented as

$$\mathbf{M} = \tilde{\mathbf{M}} + [\mathbf{r}_C \mathbf{p}],$$

where $\tilde{\mathbf{M}}$ is its proper angular momentum (in the reference frame moving translationally and fixed to the centre of inertia), $\mathbf{r}_C$ is the radius vector of the centre of inertia relative to the point O, $\mathbf{p}$ is the total momentum of the system of particles in the reference frame K.

1.198. A ball of mass m moving with velocity v_0 experiences a head-on elastic collision with one of the spheres of a stationary rigid dumbbell as whown in Fig. 1.50. The mass of each sphere equals $m/2$, and the distance between them is l. Disregarding the size of the spheres, find the proper angular momentum $\tilde{M}$ of the dumbbell after the collision, i.e. the angular momentum in the reference frame moving translationally and fixed to the dumbbell's centre of inertia.

1.199. Two small identical discs, each of mass m, lie on a smooth horizontal plane. The discs are interconnected by a light non-deformed spring of length l_0 and stiffness $\varkappa$. At a certain moment one of the discs is set in motion in a horizontal direction perpendicular to the spring with velocity v_0. Find the maximum elongation of the spring in the process of motion, if it is known to be considerably less than unity.

1.4. UNIVERSAL GRAVITATION

- Universal gravitation law

$$F = \gamma \frac{m_1 m_2}{r^2}. \tag{1.4a}$$

- The squares of the periods of revolution of any two planets around the Sun are proportional to the cubes of the major semiaxes of their orbits (Kepler):

$$T^2 \propto a^3. \tag{1.4b}$$

- Strength $\mathbf{G}$ and potential φ of the gravitational field of a mass point:

$$\mathbf{G} = -\gamma \frac{m}{r^3}\mathbf{r}, \quad \varphi = -\gamma \frac{m}{r}. \tag{1.4c}$$

- Orbital and escape velocities:

$$v_1 = \sqrt{gR}, \quad v_2 = \sqrt{2}\, v_1. \tag{1.4d}$$

1.200. A planet of mass M moves along a circle around the Sun with velocity $v = 34.9$ km/s (relative to the heliocentric reference frame). Find the period of revolution of this planet around the Sun.

1.201. The Jupiter's period of revolution around the Sun is 12 times that of the Earth. Assuming the planetary orbits to be circular, find:

(a) how many times the distance between the Jupiter and the Sun exceeds that between the Earth and the Sun;

(b) the velocity and the acceleration of Jupiter in the heliocentric reference frame.

1.202. A planet of mass M moves around the Sun along an ellipse so that its minimum distance from the Sun is equal to r and the maximum distance to R. Making use of Kepler's laws, find its period of revolution around the Sun.

1.203. A small body starts falling onto the Sun from a distance equal to the radius of the Earth's orbit. The initial velocity of the body is equal to zero in the heliocentric reference frame. Making use of Kepler's laws, find how long the body will be falling.

1.204. Suppose we have made a model of the Solar system scaled down in the ratio η but of materials of the same mean density as the actual materials of the planets and the Sun. How will the orbital periods of revolution of planetary models change in this case?

1.205. A double star is a system of two stars moving around the centre of inertia of the system due to gravitation. Find the distance between the components of the double star, if its total mass equals M and the period of revolution T.

1.206. Find the potential energy of the gravitational interaction

(a) of two mass points of masses m_1 and m_2 located at a distance r from each other;

(b) of a mass point of mass m and a thin uniform rod of mass M and length l, if they are located along a straight line at a distance a from each other; also find the force of their interaction.

1.207. A planet of mass m moves along an ellipse around the Sun so that its maximum and minimum distances from the Sun are equal to r_1 and r_2 respectively. Find the angular momentum M of this planet relative to the centre of the Sun.

1.208. Using the conservation laws, demonstrate that the total mechanical energy of a planet of mass m moving around the Sun along an ellipse depends only on its semi-major axis a. Find this energy as a function of a.

1.209. A planet A moves along an elliptical orbit around the Sun. At the moment when it was at the distance r_0 from the Sun its velocity was equal to v_0 and the angle between the radius vector $\mathbf{r}_0$ and the velocity vector $\mathbf{v}_0$ was equal to α. Find the maximum and minimum distances that will separate this planet from the Sun during its orbital motion.

1.210. A cosmic body A moves to the Sun with velocity v_0 (when far from the Sun) and aiming parameter l the arm of the vector $\mathbf{v}_0$ relative to the centre of the Sun (Fig. 1.51). Find the minimum distance by which this body will get to the Sun.

1.211. A particle of mass m is located outside a uniform sphere of mass M at a distance r from its centre. Find:

(a) the potential energy of gravitational interaction of the particle and the sphere;

(b) the gravitational force which the sphere exerts on the particle.

1.212. Demonstrate that the gravitational force acting on a particle A inside a uniform spherical layer of matter is equal to zero.

1.213. A particle of mass m was transferred from the centre of the base of a uniform hemisphere of mass M and radius R into infinity. What work was performed in the process by the gravitational force exerted on the particle by the hemisphere?

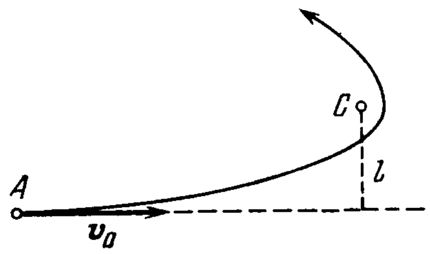


Fig. 1.51.

1.214. There is a uniform sphere of mass M and radius R. Find the strength $\mathbf{G}$ and the potential φ of the gravitational field of this sphere as a function of the distance r from its centre (with $r < R$ and $r > R$). Draw the approximate plots of the functions $G(r)$ and $\varphi(r)$.

1.215. Inside a uniform sphere of density ρ there is a spherical cavity whose centre is at a distance $\mathbf{l}$ from the centre of the sphere. Find the strength $\mathbf{G}$ of the gravitational field inside the cavity.

1.216. A uniform sphere has a mass M and radius R. Find the pressure p inside the sphere, caused by gravitational compression, as a function of the distance r from its centre. Evaluate p at the centre of the Earth, assuming it to be a uniform sphere.

1.217. Find the proper potential energy of gravitational interaction of matter forming

(a) a thin uniform spherical layer of mass m and radius R;

(b) a uniform sphere of mass m and radius R (make use of the answer to Problem 1.214).

1.218. Two Earth's satellites move in a common plane along circular orbits. The orbital radius of one satellite $r = 7000$ km while that of the other satellite is $\Delta r = 70$ km less. What time interval separates the periodic approaches of the satellites to each other over the minimum distance?

1.219. Calculate the ratios of the following accelerations: the acceleration w_1 due to the gravitational force on the Earth's surface,

the acceleration w_2 due to the centrifugal force of inertia on the Earth's equator, and the acceleration w_3 caused by the Sun to the bodies on the Earth.

1.220. At what height over the Earth's pole the free-fall acceleration decreases by one per cent; by half?

1.221. On the pole of the Earth a body is imparted velocity v_0 directed vertically up. Knowing the radius of the Earth and the free-fall acceleration on its surface, find the height to which the body will ascend. The air drag is to be neglected.

1.222. An artificial satellite is launched into a circular orbit around the Earth with velocity v relative to the reference frame moving translationally and fixed to the Earth's rotation axis. Find the distance from the satellite to the Earth's surface. The radius of the Earth and the free-fall acceleration on its surface are supposed to be known.

1.223. Calculate the radius of the circular orbit of a stationary Earth's satellite, which remains motionless with respect to its surface. What are its velocity and acceleration in the inertial reference frame fixed at a given moment to the centre of the Earth?

1.224. A satellite revolving in a circular equatorial orbit of radius $R = 2.00 \cdot 10^4$ km from west to east appears over a certain point at the equator every $\tau = 11.6$ hours. Using these data, calculate the mass of the Earth. The gravitational constant is supposed to be known.

1.225. A satellite revolves from east to west in a circular equatorial orbit of radius $R = 1.00 \cdot 10^4$ km around the Earth. Find the velocity and the acceleration of the satellite in the reference frame fixed to the Earth.

1.226. A satellite must move in the equatorial plane of the Earth close to its surface either in the Earth's rotation direction or against it. Find how many times the kinetic energy of the satellite in the latter case exceeds that in the former case (in the reference frame fixed to the Earth).

1.227. An artificial satellite of the Moon revolves in a circular orbit whose radius exceeds the radius of the Moon η times. In the process of motion the satellite experiences a slight resistance due to cosmic dust. Assuming the resistance force to depend on the velocity of the satellite as $F = \alpha v^2$, where α is a constant, find how long the satellite will stay in orbit until it falls onto the Moon's surface.

1.228. Calculate the orbital and escape velocities for the Moon. Compare the results obtained with the corresponding velocities for the Earth.

1.229. A spaceship approaches the Moon along a parabolic trajectory which is almost tangent to the Moon's surface. At the moment of the maximum approach the brake rocket was fired for a short time interval, and the spaceship was transferred into a circular orbit of a Moon satellite. Find how the spaceship velocity modulus increased in the process of braking.

1.230. A spaceship is launched into a circular orbit close to the Earth's surface. What additional velocity has to be imparted to the spaceship to overcome the gravitational pull?

1.231. At what distance from the centre of the Moon is the point at which the strength of the resultant of the Earth's and Moon's gravitational fields is equal to zero? The Earth's mass is assumed to be $\eta = 81$ times that of the Moon, and the distance between the centres of these planets $n = 60$ times greater than the radius of the Earth R.

1.232. What is the minimum work that has to be performed to bring a spaceship of mass $m = 2.0 \cdot 10^3$ kg from the surface of the Earth to the Moon?

1.233. Find approximately the third cosmic velocity v_3, i.e. the minimum velocity that has to be imparted to a body relative to the Earth's surface to drive it out of the Solar system. The rotation of the Earth about its own axis is to be neglected.

1.5. DYNAMICS OF A SOLID BODY

• Equation of dynamics of a solid body rotating about a stationary axis z:

$$I\beta_z = N_z, \tag{1.5a}$$

where N_z is the algebraic sum of the moments of external forces relative to the z axis.

• According to Steiner's theorem:

$$I = I_C + ma^2. \tag{1.5b}$$

• Kinetic energy of a solid body rotating about a stationary axis:

$$T = \frac{1}{2} I\omega^2. \tag{1.5c}$$

• Work performed by external forces during the rotation of a solid body about a stationary axis:

$$A = \int N_z \, d\varphi. \tag{1.5d}$$

• Kinetic energy of a solid body in plane motion:

$$T = \frac{I_C\omega^2}{2} + \frac{mv_C^2}{2}. \tag{1.5e}$$

• Relationship between the angular velocity ω' of gyroscope precession, its angular momentum $\mathbf{M}$ equal to $I\omega$, and the moment $\mathbf{N}$ of the external forces:

$$[\boldsymbol{\omega}'\mathbf{M}] = \mathbf{N}. \tag{1.5f}$$

1.234. A thin uniform rod AB of mass $m = 1.0$ kg moves translationally with acceleration $w = 2.0$ m/s² due to two antiparallel forces $\mathbf{F}_1$ and $\mathbf{F}_2$ (Fig. 1.52). The distance between the points at which these forces are applied is equal to $a = 20$ cm. Besides, it is known that $F_2 = 5.0$ N. Find the length of the rod.

1.235. A force $\mathbf{F} = A\mathbf{i} + B\mathbf{j}$ is applied to a point whose radius vector relative to the origin of coordinates O is equal to $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$, where a, b, A, B are constants, and $\mathbf{i}$, $\mathbf{j}$ are the unit vectors of

the x and y axes. Find the moment $\mathbf{N}$ and the arm l of the force $\mathbf{F}$ relative to the point O.

1.236. A force $\mathbf{F}_1 = A\mathbf{j}$ is applied to a point whose radius vector $\mathbf{r}_1 = a\mathbf{i}$, while a force $\mathbf{F}_2 = B\mathbf{i}$ is applied to the point whose radius vector $\mathbf{r}_2 = b\mathbf{j}$. Both radius vectors are determined relative to the origin of coordinates O, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes, a, b, A, B are constants. Find the arm l of the resultant force relative to the point O.

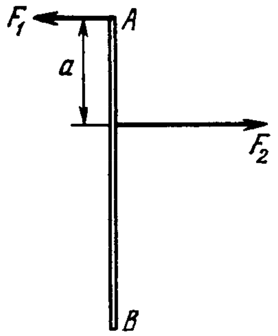


Fig. 1.52.

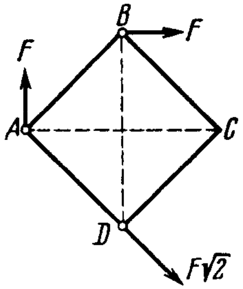


Fig. 1.53.

1.237. Three forces are applied to a square plate as shown in Fig. 1.53. Find the modulus, direction, and the point of application of the resultant force, if this point is taken on the side BC.

1.238. Find the moment of inertia

(a) of a thin uniform rod relative to the axis which is perpendicular to the rod and passes through its end, if the mass of the rod is m and its length l;

(b) of a thin uniform rectangular plate relative to the axis passing perpendicular to the plane of the plate through one of its vertices, if the sides of the plate are equal to a and b, and its mass is m.

1.239. Calculate the moment of inertia

(a) of a copper uniform disc relative to the symmetry axis perpendicular to the plane of the disc, if its thickness is equal to $b=2.0$ mm and its radius to $R = 100$ mm;

(b) of a uniform solid cone relative to its symmetry axis, if the mass of the cone is equal to m and the radius of its base to R.

1.240. Demonstrate that in the case of a thin plate of arbitrary shape there is the following relationship between the moments of inertia: $I_1 + I_2 = I_3$, where subindices 1, 2, and 3 define three mutually perpendicular axes passing through one point, with axes 1 and 2 lying in the plane of the plate. Using this relationship, find the moment of inertia of a thin uniform round disc of radius R and mass m relative to the axis coinciding with one of its diameters.

1.241. A uniform disc of radius $R = 20$ cm has a round cut as shown in Fig. 1.54. The mass of the remaining (shaded) portion of the disc equals $m = 7.3$ kg. Find the moment of inertia of such a disc relative to the axis passing through its centre of inertia and perpendicular to the plane of the disc.

1.242. Using the formula for the moment of inertia of a uniform sphere, find the moment of inertia of a thin spherical layer of mass m and radius R relative to the axis passing through its centre.

1.243. A light thread with a body of mass m tied to its end is wound on a uniform solid cylinder of mass M and radius R (Fig. 1.55). At a moment $t = 0$ the system is set in motion. Assuming the friction in the axle of the cylinder to be negligible, find the time dependence of

(a) the angular velocity of the cylinder;

(b) the kinetic energy of the whole system.

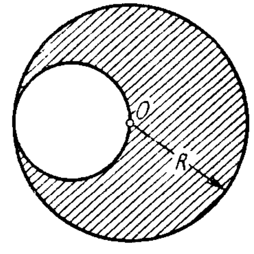


Fig. 1.54.

1.244. The ends of thin threads tightly wound on the axle of radius r of the Maxwell disc are attached to a horizontal bar. When the disc unwinds, the bar is raised to keep the disc at the same height. The mass of the disc with the axle is equal to m, the moment of inertia of the arrangement relative to its axis is I. Find the tension of each thread and the acceleration of the bar.

1.245. A thin horizontal uniform rod AB of mass m and length l can rotate freely about a vertical axis passing through its end A. At a certain moment the end B starts experiencing a constant force F which is always perpendicular to the original position of the stationary rod and directed in a horizontal plane. Find the angular velocity of the rod as a function of its rotation angle φ counted relative to the initial position.

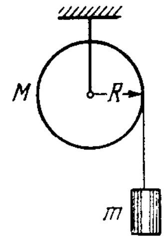


Fig. 1.55.

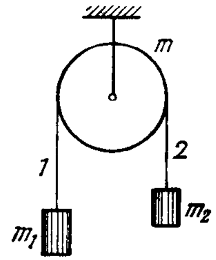


Fig. 1.56.

1.246. In the arrangement shown in Fig. 1.56 the mass of the uniform solid cylinder of radius R is equal to m and the masses of two bodies are equal to m_1 and m_2. The thread slipping and the friction in the axle of the cylinder are supposed to be absent. Find the angular acceleration of the cylinder and the ratio of tensions T_1/T_2 of the vertical sections of the thread in the process of motion.

1.247. In the system shown in Fig. 1.57 the masses of the bodies are known to be m_1 and m_2, the coefficient of friction between the body m_1 and the horizontal plane is equal to k, and a pulley of mass m is assumed to be a uniform disc. The thread does not slip over the pulley. At the moment $t = 0$ the body m_2 starts descending. Assuming the mass of the thread and the friction in the axle of the pulley to be negligible, find the work performed by the friction forces acting on the body m_1 over the first t seconds after the beginning of motion.

1.248. A uniform cylinder of radius R is spinned about its axis to the angular velocity ω_0 and then placed into a corner (Fig. 1.58).

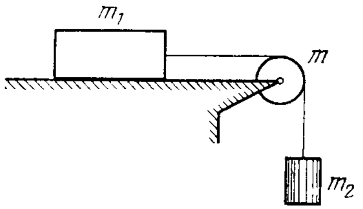


Fig. 1.57.

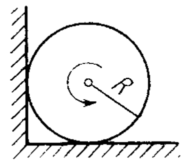


Fig. 1.58.

The coefficient of friction between the corner walls and the cylinder is equal to k. How many turns will the cylinder accomplish before it stops?

1.249. A uniform disc of radius R is spinned to the angular velocity ω and then carefully placed on a horizontal surface. How long will the disc be rotating on the surface if the friction coefficient is equal to k? The pressure exerted by the disc on the surface can be regarded as uniform.

1.250. A flywheel with the initial angular velocity ω_0 decelerates due to the forces whose moment relative to the axis is proportional to the square root of its angular velocity. Find the mean angular velocity of the flywheel averaged over the total deceleration time.

1.251. A uniform cylinder of radius R and mass M can rotate freely about a stationary horizontal axis O (Fig. 1.59). A thin cord of length l and mass m is wound on the cylinder in a single layer. Find the angular acceleration of the cylinder as a function of the length x of the hanging part of the cord. The wound part of the cord is supposed to have its centre of gravity on the cylinder axis.

1.252. A uniform sphere of mass m and radius R rolls without slipping down an inclined plane set at an angle α to the horizontal. Find:

(a) the magnitudes of the friction coefficient at which slipping is absent;

(b) the kinetic energy of the sphere t seconds after the beginning of motion.

1.253. A uniform cylinder of mass $m = 8.0$ kg and radius $R = 1.3$ cm (Fig. 1.60) starts descending at a moment $t = 0$ due to gravity. Neglecting the mass of the thread, find:

(a) the tension of each thread and the angular acceleration of the cylinder;

(b) the time dependence of the instantaneous power developed by the gravitational force.

1.254. Thin threads are tightly wound on the ends of a uniform solid cylinder of mass m. The free ends of the threads are attached to

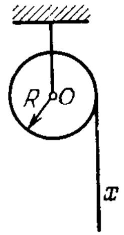


Fig. 1.59.

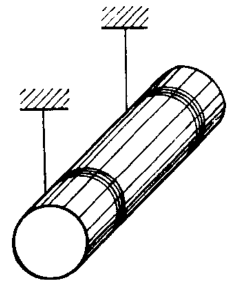


Fig. 1.60.

the ceiling of an elevator car. The car starts going up with an acceleration $\mathbf{w}_0$. Find the acceleration $\mathbf{w}'$ of the cylinder relative to the car and the force $\mathbf{F}$ exerted by the cylinder on the ceiling (through the threads).

1.255. A spool with a thread wound on it is placed on an inclined smooth plane set at an angle $\alpha = 30°$ to the horizontal. The free end of the thread is attached to the wall as shown in Fig. 1.61. The mass of the spool is $m = 200$ g, its moment of inertia relative to its own axis $I = 0.45\ \text{g}\cdot\text{m}^2$, the radius of the wound thread layer $r = 3.0$ cm. Find the acceleration of the spool axis.

1.256. A uniform solid cylinder of mass m rests on two horizontal planks. A thread is wound on the cylinder. The hanging end of the thread is pulled vertically down with a constant force F (Fig. 1.62).

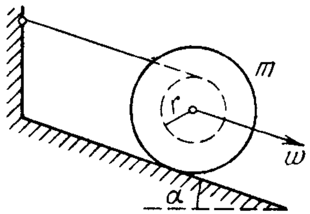


Fig. 1.61.

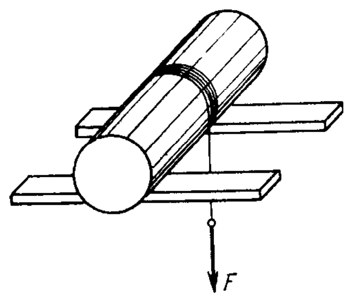


Fig. 1.62.

Find the maximum magnitude of the force F which still does not bring about any sliding of the cylinder, if the coefficient of friction between the cylinder and the planks is equal to k. What is the ac-

celeration w_{max} of the axis of the cylinder rolling down the inclined plane?

1.257. A spool with thread wound on it, of mass m, rests on a rough horizontal surface. Its moment of inertia relative to its own axis is equal to $I = \gamma m R^2$, where γ is a numerical factor, and R is the outside radius of the spool. The radius of the wound thread layer is equal

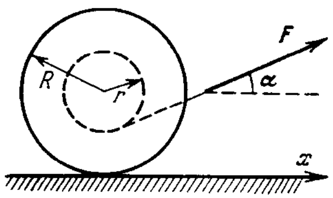


Fig. 1.63.

to r. The spool is pulled without sliding by the thread with a constant force **F** directed at an angle α to the horizontal (Fig. 1.63). Find:

(a) the projection of the acceleration vector of the spool axis on the x-axis;

(b) the work performed by the force **F** during the first t seconds after the beginning of motion.

1.258. The arrangement shown in Fig. 1.64 consists of two identical uniform solid cylinders, each of mass m, on which two light threads

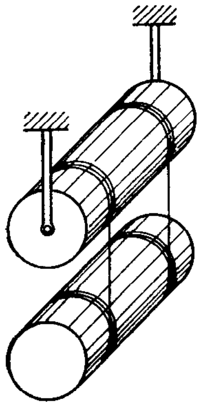


Fig. 1.64.

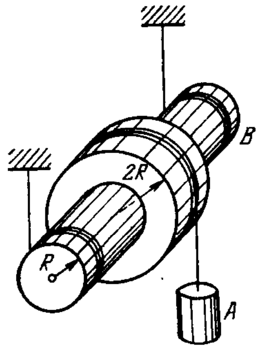


Fig. 1.65.

are wound symmetrically. Find the tension of each thread in the process of motion. The friction in the axle of the upper cylinder is assumed to be absent.

1.259. In the arrangement shown in Fig. 1.65 a weight A possesses mass m. a pulley B possesses mass M. Also known are the moment of inertia I of the pulley relative to its axis and the radii of the pulley

R and $2R$. The mass of the threads is negligible. Find the acceleration of the weight A after the system is set free.

1.260. A uniform solid cylinder A of mass m_1 can freely rotate about a horizontal axis fixed to a mount B of mass m_2 (Fig. 1.66). A constant horizontal force F is applied to the end K of a light thread tightly wound on the cylinder. The friction between the mount and the supporting horizontal plane is assumed to be absent. Find:

(a) the acceleration of the point K;

(b) the kinetic energy of this system t seconds after the beginning of motion.

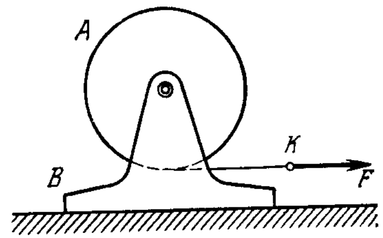


Fig. 1.66.

1.261. A plank of mass m_1 with a uniform sphere of mass m_2 placed on it rests on a smooth horizontal plane. A constant horizontal force F is applied to the plank. With what accelerations will the plank and the centre of the sphere move provided there is no sliding between the plank and the sphere?

1.262. A uniform solid cylinder of mass m and radius R is set in rotation about its axis with an angular velocity ω_0, then lowered with its lateral surface onto a horizontal plane and released. The coefficient of friction between the cylinder and the plane is equal to k. Find:

(a) how long the cylinder will move with sliding;

(b) the total work performed by the sliding friction force acting on the cylinder.

1.263. A uniform ball of radius r rolls without slipping down from the top of a sphere of radius R. Find the angular velocity of the ball at the moment it breaks off the sphere. The initial velocity of the ball is negligible.

1.264. A uniform solid cylinder of radius $R = 15$ cm rolls over a horizontal plane passing into an inclined plane forming an angle

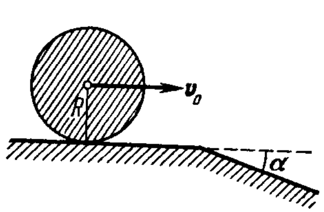


Fig. 1.67.

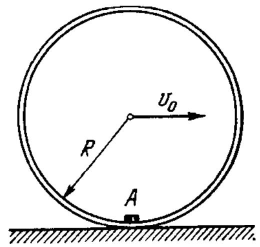


Fig. 1.68.

$\alpha = 30°$ with the horizontal (Fig. 1.67). Find the maximum value of the velocity v_0 which still permits the cylinder to roll onto the inclined plane section without a jump. The sliding is assumed to be absent.

1.265. A small body A is fixed to the inside of a thin rigid hoop of radius R and mass equal to that of the body A. The hoop rolls without slipping over a horizontal plane; at the moments when the body A gets into the lower position, the centre of the hoop moves with velocity

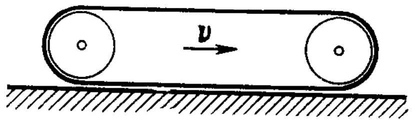


Fig. 1.69.

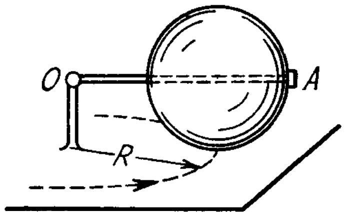


Fig. 1.70.

v_0 (Fig. 1.68). At what values of v_0 will the hoop move without bouncing?

1.266. Determine the kinetic energy of a tractor crawler belt of mass m if the tractor moves with velocity v (Fig. 1.69).

1.267. A uniform sphere of mass m and radius r rolls without sliding over a horizontal plane, rotating about a horizontal axle OA (Fig. 1.70). In the process, the centre of the sphere moves with velocity v along a circle of radius R. Find the kinetic energy of the sphere.

1.268. Demonstrate that in the reference frame rotating with a constant angular velocity ω about a stationary axis a body of mass m experiences the resultant

(a) centrifugal force of inertia $\mathbf{F}_{cf} = m\omega^2\mathbf{R}_C$, where $\mathbf{R}_C$ is the radius vector of the body's centre of inertia relative to the rotation axis;

(b) Coriolis force $\mathbf{F}_{cor} = 2m\,[\mathbf{v}_C'\boldsymbol{\omega}]$, where $\mathbf{v}_C'$ is the velocity of the body's centre of inertia in the rotating reference frame.

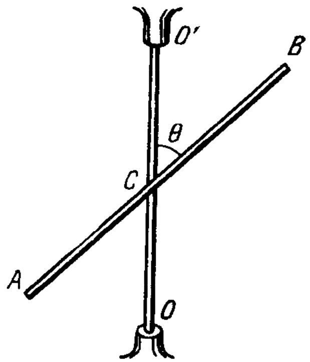


Fig. 1.71.

1.269. A midpoint of a thin uniform rod AB of mass m and length l is rigidly fixed to a rotation axle OO' as shown in Fig. 1.71. The rod is set into rotation with a constant angular velocity ω. Find the resultant moment of the centrifugal forces of inertia relative to the point C in the reference frame fixed to the axle OO' and to the rod.

1.270. A conical pendulum, a thin uniform rod of length l and mass m, rotates uniformly about a vertical axis with angular velocity ω (the upper end of the rod is hinged). Find the angle θ between the rod and the vertical.

1.271. A uniform cube with edge a rests on a horizontal plane whose friction coefficient equals k. The cube is set in motion with an initial velocity, travels some distance over the plane and comes to a stand-

still. Explain the disappearance of the angular momentum of the cube relative to the axis lying in the plane at right angles to the cube's motion direction. Find the distance between the resultants of gravitational forces and the reaction forces exerted by the supporting plane.

1.272. A smooth uniform rod AB of mass M and length l rotates freely with an angular velocity ω_0 in a horizontal plane about a stationary vertical axis passing through its end A. A small sleeve of mass m starts sliding along the rod from the point A. Find the velocity v' of the sleeve relative to the rod at the moment it reaches its other end B.

1.273. A uniform rod of mass $m = 5.0$ kg and length $l = 90$ cm rests on a smooth horizontal surface. One of the ends of the rod is struck with the impulse $J = 3.0$ N·s in a horizontal direction perpendicular to the rod. As a result, the rod obtains the momentum $p = 3.0$ N·s. Find the force with which one half of the rod will act on the other in the process of motion.

1.274. A thin uniform square plate with side l and mass M can rotate freely about a stationary vertical axis coinciding with one of its sides. A small ball of mass m flying with velocity $\mathbf{v}$ at right angles to the plate strikes elastically the centre of it. Find:

(a) the velocity of the ball $\mathbf{v}'$ after the impact;

(b) the horizontal component of the resultant force which the axis will exert on the plate after the impact.

1.275. A vertically oriented uniform rod of mass M and length l can rotate about its upper end. A horizontally flying bullet of mass m strikes the lower end of the rod and gets stuck in it; as a result, the rod swings through an angle α. Assuming that $m \ll M$, find:

(a) the velocity of the flying bullet;

(b) the momentum increment in the system "bullet-rod" during the impact; what causes the change of that momentum;

(c) at what distance x from the upper end of the rod the bullet must strike for the momentum of the system "bullet-rod" to remain constant during the impact.

1.276. A horizontally oriented uniform disc of mass M and radius R rotates freely about a stationary vertical axis passing through its centre. The disc has a radial guide along which can slide without friction a small body of mass m. A light thread running down through the hollow axle of the disc is tied to the body. Initially the body was located at the edge of the disc and the whole system rotated with an angular velocity ω_0. Then by means of a force F applied to the lower end of the thread the body was slowly pulled to the rotation axis. Find:

(a) the angular velocity of the system in its final state;

(b) the work performed by the force F.

1.277. A man of mass m_1 stands on the edge of a horizontal uniform disc of mass m_2 and radius R which is capable of rotating freely about a stationary vertical axis passing through its centre. At a cer-

tain moment the man starts moving along the edge of the disc; he shifts over an angle φ' relative to the disc and then stops. In the process of motion the velocity of the man varies with time as $v'(t)$. Assuming the dimensions of the man to be negligible, find:

(a) the angle through which the disc had turned by the moment the man stopped;

(b) the force moment (relative to the rotation axis) with which the man acted on the disc in the process of motion.

1.278. Two horizontal discs rotate freely about a vertical axis passing through their centres. The moments of inertia of the discs relative to this axis are equal to I_1 and I_2, and the angular velocities to ω_1 and ω_2. When the upper disc fell on the lower one, both discs began rotating, after some time, as a single whole (due to friction). Find:

(a) the steady-state angular rotation velocity of the discs;

(b) the work performed by the friction forces in this process.

1.279. A small disc and a thin uniform rod of length l, whose mass is η times greater than the mass of the disc, lie on a smooth horizontal plane. The disc is set in motion, in horizontal direction and perpendicular to the rod, with velocity $\mathbf{v}$, after which it elastically collides with the end of the rod. Find the velocity of the disc and the angular velocity of the rod after the collision. At what value of η will the velocity of the disc after the collision be equal to zero? reverse its direction?

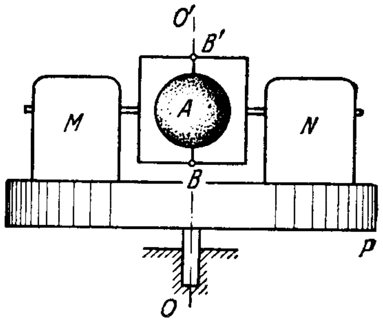


Fig. 1.72.

1.280. A stationary platform P which can rotate freely about a vertical axis (Fig. 1.72) supports a motor M and a balance weight N. The moment of inertia of the platform with the motor and the balance weight relative to this axis is equal to I. A light frame is fixed to the motor's shaft with a uniform sphere A rotating freely with an angular velocity ω_0 about a shaft BB' coinciding with the axis OO'. The moment of inertia of the sphere relative to the rotation axis is equal to I_0. Find:

(a) the work performed by the motor in turning the shaft BB' through 90°; through 180°;

(b) the moment of external forces which maintains the axis of the arrangement in the vertical position after the motor turns the shaft BB' through 90°.

1.281. A horizontally oriented uniform rod AB of mass $m = 1.40$ kg and length $l_0 = 100$ cm rotates freely about a stationary vertical axis OO' passing through its end A. The point A is located at the middle of the axis OO' whose length is equal to $l = 55$ cm. At what angular velocity of the rod the horizontal component of the force acting on the lower end of the axis OO' is equal to zero? What is in this case the horizontal component of the force acting on the upper end of the axis?

1.282. The middle of a uniform rod of mass m and length l is rigidly fixed to a vertical axis OO' so that the angle between the rod and the axis is equal to θ (see Fig. 1.71). The ends of the axis OO' are provided with bearings. The system rotates without friction with an angular velocity ω. Find:

(a) the magnitude and direction of the rod's angular momentum $\mathbf{M}$ relative to the point C, as well as its angular momentum relative to the rotation axis;

(b) how much the modulus of the vector $\mathbf{M}$ relative to the point C increases during a half-turn;

(c) the moment of external forces N acting on the axle OO' in the process of rotation.

1.283. A top of mass $m = 0.50$ kg, whose axis is tilted by an angle $\theta = 30°$ to the vertical, precesses due to gravity. The moment of inertia of the top relative to its symmetry axis is equal to $I = 2.0\ \text{g}\cdot\text{m}^2$, the angular velocity of rotation about that axis is equal to $\omega = 350$ rad/s, the distance from the point of rest to the centre of inertia of the top is $l = 10$ cm. Find:

(a) the angular velocity of the top's precession;

(b) the magnitude and direction of the horizontal component of the reaction force acting on the top at the point of rest.

1.284. A gyroscope, a uniform disc of radius $R = 5.0$ cm at the end of a rod of length $l = 10$ cm (Fig. 1.73), is mounted on the floor of an elevator car going up with a constant acceleration $w = 2.0\ \text{m/s}^2$. The other end of the rod is hinged at the point O. The gyroscope precesses with an angular velocity $n = 0.5$ rps. Neglecting the friction and the mass of the rod, find the proper angular velocity of the disc.

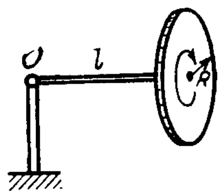


Fig. 1.73.

1.285. A top of mass $m = 1.0$ kg and moment of inertia relative to its own axis $I = 4.0\ \text{g}\cdot\text{m}^2$ spins with an angular velocity $\omega = 310$ rad/s. Its point of rest is located on a block which is shifted in a horizontal direction with a constant acceleration $w = 1.0\ \text{m/s}^2$. The distance between the point of rest and the centre of inertia of the top equals $l = 10$ cm. Find the magnitude and direction of the angular velocity of precession $\boldsymbol{\omega}'$.

1.286. A uniform sphere of mass $m = 5.0$ kg and radius $R = 6.0$ cm rotates with an angular velocity $\omega = 1250$ rad/s about a horizontal axle passing through its centre and fixed on the mounting base by means of bearings. The distance between the bearings equals $l = 15$ cm. The base is set in rotation about a vertical axis with an angular velocity $\omega' = 5.0$ rad/s. Find the modulus and direction of the gyroscopic forces.

1.287. A cylindrical disc of a gyroscope of mass $m = 15$ kg and radius $r = 5.0$ cm spins with an angular velocity $\omega = 330$ rad/s.

The distance between the bearings in which the axle of the disc is mounted is equal to $l = 15$ cm. The axle is forced to oscillate about a horizontal axis with a period $T = 1.0$ s and amplitude $\varphi_m = 20°$. Find the maximum value of the gyroscopic forces exerted by the axle on the bearings.

1.288. A ship moves with velocity $v = 36$ km per hour along an arc of a circle of radius $R = 200$ m. Find the moment of the gyroscopic forces exerted on the bearings by the shaft with a flywheel whose moment of inertia relative to the rotation axis equals $I = 3.8 \cdot 10^3$ kg·m² and whose rotation velocity $n = 300$ rpm. The rotation axis is oriented along the length of the ship.

1.289. A locomotive is propelled by a turbine whose axle is parallel to the axes of wheels. The turbine's rotation direction coincides with that of wheels. The moment of inertia of the turbine rotor relative to its own axis is equal to $I = 240$ kg·m². Find the additional force exerted by the gyroscopic forces on the rails when the locomotive moves along a circle of radius $R = 250$ m with velocity $v = 50$ km per hour. The gauge is equal to $l = 1.5$ m. The angular velocity of the turbine equals $n = 1500$ rpm.

1.6. ELASTIC DEFORMATIONS OF A SOLID BODY

• Relation between tensile (compressive) strain ε and stress σ:

$$\varepsilon = \sigma/E, \tag{1.6a}$$

where E is Young's modulus.

• Relation between lateral compressive (tensile) strain ε' and longitudinal tensile (compressive) strain ε:

$$\varepsilon' = -\mu\varepsilon, \tag{1.6b}$$

where μ is Poisson's ratio.

• Relation between shear strain γ and tangential stress τ:

$$\gamma = \tau/G, \tag{1.6c}$$

where G is shear modulus.

• Compressibility:

$$\beta = -\frac{1}{V}\frac{dV}{dp}. \tag{1.6d}$$

• Volume density of elastic strain energy:

$$u = E\varepsilon^2/2, \quad u = G\gamma^2/2. \tag{1.6e}$$

1.290. What pressure has to be applied to the ends of a steel cylinder to keep its length constant on raising its temperature by 100 °C?

1.291. What internal pressure (in the absence of an external pressure) can be sustained

(a) by a glass tube; (b) by a glass spherical flask, if in both cases the wall thickness is equal to $\Delta r = 1.0$ mm and the radius of the tube and the flask equals $r = 25$ mm?

1.292. A horizontally oriented copper rod of length $l = 1.0$ m is rotated about a vertical axis passing through its middle. What is the number of rps at which this rod ruptures?

1.293. A ring of radius $r = 25$ cm made of lead wire is rotated about a stationary vertical axis passing through its centre and perpendicular to the plane of the ring. What is the number of rps at which the ring ruptures?

1.294. A steel wire of diameter $d = 1.0$ mm is stretched horizontally between two clamps located at the distance $l = 2.0$ m from each other. A weight of mass $m = 0.25$ kg is suspended from the midpoint O of the wire. What will the resulting descent of the point O be in centimetres?

1.295. A uniform elastic plank moves over a smooth horizontal plane due to a constant force F_0 distributed uniformly over the end face. The surface of the end face is equal to S, and Young's modulus of the material to E. Find the compressive strain of the plank in the direction of the acting force.

1.296. A thin uniform copper rod of length l and mass m rotates uniformly with an angular velocity ω in a horizontal plane about a vertical axis passing through one of its ends. Determine the tension in the rod as a function of the distance r from the rotation axis. Find the elongation of the rod.

1.297. A solid copper cylinder of length $l = 65$ cm is placed on a horizontal surface and subjected to a vertical compressive force $F = 1000$ N directed downward and distributed uniformly over the end face. What will be the resulting change of the volume of the cylinder in cuhic millimetres?

1.298. A copper rod of length l is suspended from the ceiling by one of its ends. Find:

(a) the elongation Δl of the rod due to its own weight;

(b) the relative increment of its volume $\Delta V/V$.

1.299. A bar made of material whose Young's modulus is equal to E and Poisson's ratio to μ is subjected to the hydrostatic pressure p. Find:

(a) the fractional decrement of its volume;

(b) the relationship between the compressibility β and the elastic constants E and μ.

Show that Poisson's ratio μ cannot exceed 1/2.

1.300. One end of a steel rectangular girder is embedded into a wall (Fig. 1.74). Due to gravity it sags slightly. Find the radius of curvature of the neutral layer (see the dotted line in the figure) in

the vicinity of the point O if the length of the protruding section of

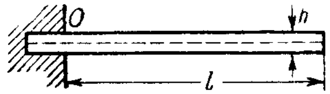


Fig. 1.74.

the girder is equal to $l = 6.0$ m and the thickness of the girder equals $h = 10$ cm.

1.301. The bending of an elastic rod is described by the *elastic curve* passing through centres of gravity of rod's cross-sections. At small bendings the equation of this curve takes the form

$$N(x) = EI\,\frac{d^2y}{dx^2},$$

where $N(x)$ is the bending moment of the elastic forces in the cross-section corresponding to the x coordinate, E is Young's modulus, I is the *moment of inertia* of the cross-section relative to the axis passing through the neutral layer ($I = \int z^2 dS$, Fig. 1.75).

Suppose one end of a steel rod of a square cross-section with side a is embedded into a wall, the protruding section being of length l

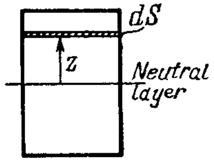


Fig. 1.75.

Fig. 1.76.

(Fig. 1.76). Assuming the mass of the rod to be negligible, find the shape of the elastic curve and the deflection of the rod λ, if its end A experiences

(a) the bending moment of the couple N_0;
(b) a force F oriented along the y axis.

1.302. A steel girder of length l rests freely on two supports (Fig. 1.77). The moment of inertia of its cross-section is equal to I (see the foregoing problem). Neglecting the mass of the girder and assuming the sagging to be slight, find the deflection λ due to the force F applied to the middle of the girder.

1.303. The thickness of a rectangular steel girder equals h. Using the equation of Problem 1.301, find the deflection λ caused by the weight of the girder in two cases:

(a) one end of the girder is embedded into a wall with the length of the protruding section being equal to l (Fig. 1.78a);
(b) the girder of length $2l$ rests freely on two supports (Fig. 1.78b).

1.304. A steel plate of thickness h has the shape of a square whose side equals l, with $h \ll l$. The plate is rigidly fixed to a vertical axle

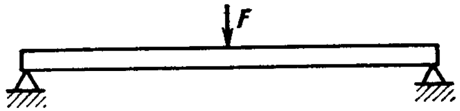


Fig. 1.77.

OO which is rotated with a constant angular acceleration β (Fig. 1.79). Find the deflection λ, assuming the sagging to be small.

1.305. Determine the relationship between the torque N and the torsion angle φ for

(a) the tube whose wall thickness Δr is considerably less than the tube radius;
(b) for the solid rod of circular cross-section. Their length l, radius r, and shear modulus G are supposed to be known.

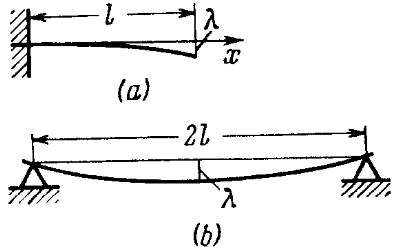


Fig. 1.78.

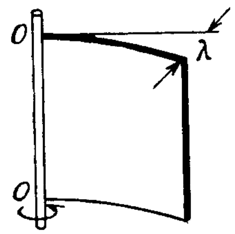


Fig. 1.79.

1.306. Calculate the torque N twisting a steel tube of length $l = 3.0$ m through an angle $\varphi = 2.0°$ about its axis, if the inside and outside diameters of the tube are equal to $d_1 = 30$ mm and $d_2 = 50$ mm.

1.307. Find the maximum power which can be transmitted by means of a steel shaft rotating about its axis with an angular velocity $\omega = 120$ rad/s, if its length $l = 200$ cm, radius $r = 1.50$ cm, and the permissible torsion angle $\varphi = 2.5°$.

1.308. A uniform ring of mass m, with the outside radius r_2, is fitted tightly on a shaft of radius r_1. The shaft is rotated about its axis with a constant angular acceleration β. Find the moment of elastic forces in the ring as a function of the distance r from the rotation axis.

1.309. Find the elastic deformation energy of a steel rod of mass $m = 3.1$ kg stretched to a tensile strain $\varepsilon = 1.0 \cdot 10^{-3}$.

1.310. A steel cylindrical rod of length l and radius r is suspended by its end from the ceiling.

(a) Find the elastic deformation energy U of the rod.
(b) Define U in terms of tensile strain $\Delta l/l$ of the rod.

1.311. What work has to be performed to make a hoop out of a steel band of length $l = 2.0$ m, width $h = 6.0$ cm, and thickness $\delta = 2.0$ mm? The process is assumed to proceed within the elasticity range of the material.

1.312. Find the elastic deformation energy of a steel rod whose one end is fixed and the other is twisted through an angle $\varphi = 6.0°$. The length of the rod is equal to $l = 1.0$ m, and the radius to $r = 10$ mm.

1.313. Find how the volume density of the elastic deformation energy is distributed in a steel rod depending on the distance r from its axis. The length of the rod is equal to l, the torsion angle to φ.

1.314. Find the volume density of the elastic deformation energy in fresh water at the depth of $h = 1000$ m.

1.7. HYDRODYNAMICS

• The fundamental equation of hydrodynamics of ideal fluid (Eulerian equation):

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{f} - \nabla p, \tag{1.7a}$$

where ρ is the fluid density, $\mathbf{f}$ is the volume density of mass forces ($\mathbf{f} = \rho \mathbf{g}$ in the case of gravity), ∇p is the pressure gradient.

• Bernoulli's equation. In the steady flow of an ideal fluid

$$\frac{\rho v^2}{2} + \rho g h + p = \text{const} \tag{1.7b}$$

along any streamline.

• Reynolds number defining the flow pattern of a viscous fluid:

$$\mathrm{Re} = \rho v l/\eta, \tag{1.7c}$$

where l is a characteristic length, η is the fluid viscosity.

• Poiseuille's law. The volume of liquid flowing through a circular tube (in m³/s):

$$Q = \frac{\pi R^4}{8\eta} \frac{p_1 - p_2}{l}, \tag{1.7d}$$

where R and l are the tube's radius and length, $p_1 - p_2$ is the pressure difference between the ends of the tube.

• Stokes' law. The friction force on the sphere of radius r moving through a viscous fluid:

$$F = 6\pi\eta r v. \tag{1.7e}$$

1.315. Ideal fluid flows along a flat tube of constant cross-section, located in a horizontal plane and bent as shown in Fig. 1.80 (top view). The flow is steady. Are the pressures and velocities of the fluid equal at points *1* and *2*? What is the shape of the streamlines?

1.316. Two manometric tubes are mounted on a horizontal pipe of varying cross-section at the sections S_1 and S_2 (Fig. 1.81). Find

the volume of water flowing across the pipe's section per unit time if the difference in water columns is equal to Δh.

1.317. A Pitot tube (Fig. 1.82) is mounted along the axis of a gas pipeline whose cross-sectional area is equal to S. Assuming the viscosity to be negligible, find the volume of gas flowing across the

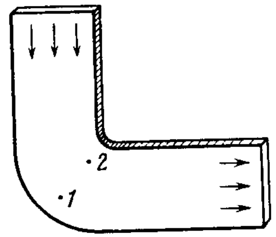


Fig. 1.80.

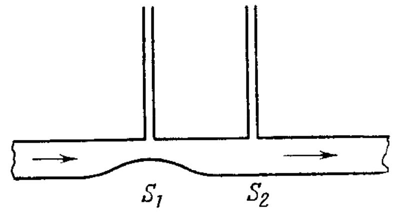


Fig. 1.81.

section of the pipe per unit time, if the difference in the liquid columns is equal to Δh, and the densities of the liquid and the gas are ρ_0 and ρ respectively.

1.318. A wide vessel with a small hole in the bottom is filled with water and kerosene. Neglecting the viscosity, find the velocity of the water flow, if the thickness of the water layer is equal to $h_1 = 30$ cm and that of the kerosene layer to $h_2 = 20$ cm.

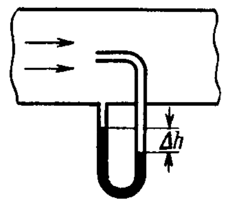


Fig. 1.82.

1.319. A wide cylindrical vessel 50 cm in height is filled with water and rests on a table. Assuming the viscosity to be negligible, find at what height from the bottom of the vessel a small hole should be perforated for the water jet coming out of it to hit the surface of the table at the maximum distance l_{max} from the vessel. Find l_{max}.

1.320. A bent tube is lowered into a water stream as shown in Fig. 1.83. The velocity of the stream relative to the tube is equal to $v = 2.5$ m/s. The closed upper end of the tube located at the height $h_0 = 12$ cm has a small orifice. To what height h will the water jet spurt?

1.321. The horizontal bottom of a wide vessel with an ideal fluid has a round orifice of radius R_1 over which a round closed cylinder is mounted, whose radius $R_2 > R_1$ (Fig. 1.84). The clearance between the cylinder and the bottom of the vessel is very small, the fluid density is ρ. Find the static pressure of the fluid in the clearance as a function of the distance r from the axis of the orifice (and the cylinder), if the height of the fluid is equal to h.

1.322. What work should be done in order to squeeze all water from a horizontally located cylinder (Fig. 1.85) during the time t by means of a constant force acting on the piston? The volume of water in the cylinder is equal to V, the cross-sectional area of the ori-

fice to s, with s being considerably less than the piston area. The friction and viscosity are negligibly small.

1.323. A cylindrical vessel of height h and base area S is filled with water. An orifice of area $s \ll S$ is opened in the bottom of the vessel. Neglecting the viscosity of water, determine how soon all the water will pour out of the vessel.

1.324. A horizontally oriented tube AB of length l rotates with a constant angular velocity ω about a stationary vertical axis OO' passing through the end A (Fig. 1.86). The tube is filled with an ideal fluid. The end A of the tube is open, the closed end B has a very small orifice. Find the velocity of the fluid relative to the tube as a function of the column "height" h.

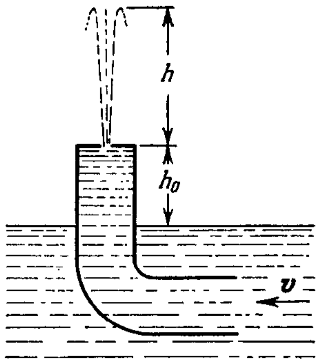


Fig. 1.83.

1.325. Demonstrate that in the case of a steady flow of an ideal fluid Eq. (1.7a) turns into Bernoulli equation.

1.326. On the opposite sides of a wide vertical vessel filled with water two identical holes are opened, each having the cross-sectional

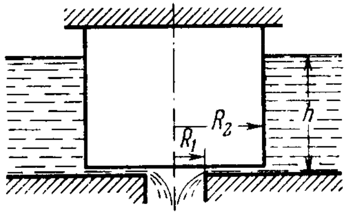


Fig. 1.84.

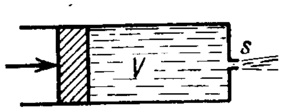


Fig. 1.85.

area $S = 0.50$ cm^2. The height difference between them is equal to $\Delta h = 51$ cm. Find the resultant force of reaction of the water flowing out of the vessel.

1.327. The side wall of a wide vertical cylindrical vessel of height $h = 75$ cm has a narrow vertical slit running all the way down to the bottom of the vessel. The length of the slit is $l = 50$ cm and the width $b = 1.0$ mm. With the slit closed, the vessel is filled with water. Find the resultant force of reaction of the water flowing out of the vessel immediately after the slit is opened.

1.328. Water flows out of a big tank along a tube bent at right angles; the inside radius of the tube is equal to $r = 0.50$ cm (Fig. 1.87). The length of the horizontal section of the tube is equal to $l = 22$ cm. The water flow rate is $Q = 0.50$ litres per second. Find the moment of reaction forces of flowing water, acting on the tube's walls, relative to the point O.

1.329. A side wall of a wide open tank is provided with a narrowing tube (Fig. 1.88) through which water flows out. The cross-sectional area of the tube decreases from $S = 3.0$ cm^2 to $s = 1.0$ cm^2. The water level in the tank is $h = 4.6$ m higher than that in the tube.

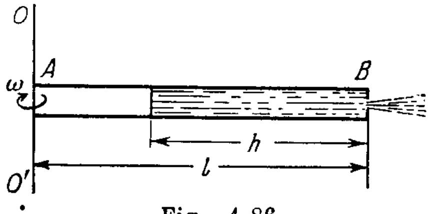


Fig. 1.86.

Neglecting the viscosity of the water, find the horizontal component of the force tending to pull the tube out of the tank.

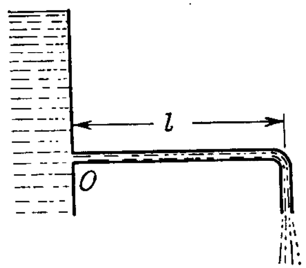


Fig. 1.87.

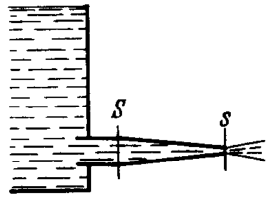


Fig. 1.88.

1.330. A cylindrical vessel with water is rotated about its vertical axis with a constant angular velocity ω. Find:

(a) the shape of the free surface of the water;

(b) the water pressure distribution over the bottom of the vessel along its radius provided the pressure at the central point is equal to p_0.

1.331. A thin horizontal disc of radius $R = 10$ cm is located within a cylindrical cavity filled with oil whose viscosity $\eta = 0.08$ P (Fig. 1.89). The clearance between the disc and the horizontal planes

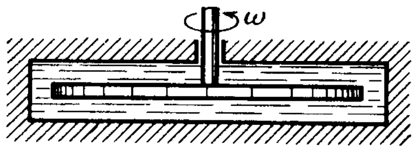


Fig. 1.89.

of the cavity is equal to $h = 1.0$ mm. Find the power developed by the viscous forces acting on the disc when it rotates with the angular velocity $\omega = 60$ rad/s. The end effects are to be neglected.

1.332. A long cylinder of radius R_1 is displaced along its axis with a constant velocity v_0 inside a stationary co-axial cylinder of radius R_2. The space between the cylinders is filled with viscous liquid. Find the velocity of the liquid as a function of the distance r from the axis of the cylinders. The flow is laminar.

1.333. A fluid with viscosity η fills the space between two long co-axial cylinders of radii R_1 and R_2, with $R_1 < R_2$. The inner cylinder is stationary while the outer one is rotated with a constant angular velocity ω_2. The fluid flow is laminar. Taking into account that the friction force acting on a unit area of a cylindrical surface of radius r is defined by the formula $\sigma = \eta r\,(\partial\omega/\partial r)$, find:

(a) the angular velocity of the rotating fluid as a function of radius r;

(b) the moment of the friction forces acting on a unit length of the outer cylinder.

1.334. A tube of length l and radius R carries a steady flow of fluid whose density is ρ and viscosity η. The fluid flow velocity depends on the distance r from the axis of the tube as $v = v_0(1 - r^2/R^2)$. Find:

(a) the volume of the fluid flowing across the section of the tube per unit time;

(b) the kinetic energy of the fluid within the tube's volume;

(c) the friction force exerted on the tube by the fluid;

(d) the pressure difference at the ends of the tube.

1.335. In the arrangement shown in Fig. 1.90 a viscous liquid whose density is $\rho = 1.0\ \text{g/cm}^3$ flows along a tube out of a wide tank

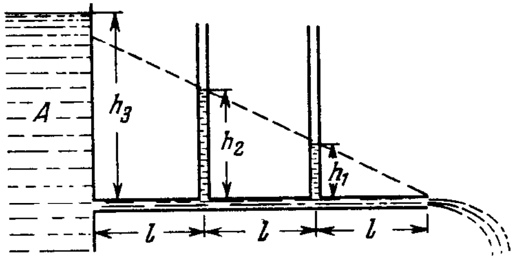


Fig. 1.90.

A. Find the velocity of the liquid flow, if $h_1 = 10$ cm, $h_2 = 20$ cm, and $h_3 = 35$ cm. All the distances l are equal.

1.336. The cross-sectional radius of a pipeline decreases gradually as $r = r_0 e^{-\alpha x}$, where $\alpha = 0.50\ \text{m}^{-1}$, x is the distance from the pipeline inlet. Find the ratio of Reynolds numbers for two cross-sections separated by $\Delta x = 3.2$ m.

1.337. When a sphere of radius $r_1 = 1.2$ mm moves in glycerin, the laminar flow is observed if the velocity of the sphere does not exceed $v_1 = 23$ cm/s. At what minimum velocity v_2 of a sphere of radius $r_2 = 5.5$ cm will the flow in water become turbulent? The viscosities of glycerin and water are equal to $\eta_1 = 13.9$ P and $\eta_2 = 0.011$ P respectively.

1.338. A lead sphere is steadily sinking in glycerin whose viscosity is equal to $\eta = 13.9$ P. What is the maximum diameter of the sphere at which the flow around that sphere still remains laminar? It is known that the transition to the turbulent flow corresponds to Reynolds number Re $= 0.5$. (Here the characteristic length is taken to be the sphere diameter.)

1.339. A steel ball of diameter $d = 3.0$ mm starts sinking with zero initial velocity in olive oil whose viscosity is $\eta = 0.90$ P. How soon after the beginning of motion will the velocity of the ball differ from the steady-state velocity by $n = 1.0\%$?

1.8. RELATIVISTIC MECHANICS

- Lorentz contraction of length and slowing of a moving clock:

$$l = l_0\sqrt{1-(v/c)^2}, \qquad \Delta t = \frac{\Delta t_0}{\sqrt{1-(v/c)^2}}, \tag{1.8a}$$

where l_0 is the proper length and Δt_0 is the proper time of the moving clock.

- Lorentz transformation*:

$$x' = \frac{x - Vt}{\sqrt{1-(V/c)^2}}, \qquad y' = y, \qquad t' = \frac{t - xV/c^2}{1-(V/c)^2}. \tag{1.8b}$$

- Interval s_{12} is an invariant:

$$s_{12}^2 = c^2 t_{12}^2 - l_{12}^2 = \text{inv}, \tag{1.8c}$$

where t_{12} is the time interval between events 1 and 2, l_{12} is the distance between the points at which these events occurred.

- Transformation of velocity*:

$$v'_x = \frac{v_x - V}{1 - v_x V/c^2}, \qquad v'_y = \frac{v_y\sqrt{1-(V/c)^2}}{1 - v_x V/c^2}. \tag{1.8d}$$

- Relativistic mass and relativistic momentum:

$$m = \frac{m_0}{\sqrt{1-(v/c)^2}}, \qquad \mathbf{p} = m\mathbf{v} = \frac{m_0\mathbf{v}}{\sqrt{1-(v/c)^2}}, \tag{1.8e}$$

where m_0 is the rest mass, or, simply, the mass.

- Relativistic equation of dynamics for a particle:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{1.8f}$$

where $\mathbf{p}$ is the relativistic momentum of the particle.

- Total and kinetic energies of a relativistic particle:

$$E = mc^2 = m_0c^2 + T, \quad T = (m - m_0)\,c^2. \tag{1.8g}$$

---

* The reference frame K' is assumed to move with a velocity V in the positive direction of the x axis of the frame K, with the x' and x axes coinciding and the y' and y axes parallel.

• Relationship between the energy and momentum of a relativistic particle

$$E^2-p^2c^2=m_0^2c^4, \qquad pc=\sqrt{T\,(T+2m_0c^2)}. \tag{1.8h}$$

• When considering the collisions of particles it helps to use the following invariant quantity:

$$E^2-p^2c^2=m_0^2c^4, \tag{1.8i}$$

where E and p are the total energy and momentum of the system prior to the collision, and m_0 is the rest mass of the particle (or the system) formed.

1.340. A rod moves lengthwise with a constant velocity v relative to the inertial reference frame K. At what value of v will the length of the rod in this frame be $\eta = 0.5\%$ less than its proper length?

1.341. In a triangle the proper length of each side equals a. Find the perimeter of this triangle in the reference frame moving relative to it with a constant velocity V along one of its

(a) bisectors; (b) sides.

Investigate the results obtained at $V \ll c$ and $V \to c$, where c is the velocity of light.

1.342. Find the proper length of a rod if in the laboratory frame of reference its velocity is $v = c/2$, the length $l = 1.00$ m, and the angle between the rod and its direction of motion is $\theta = 45°$.

1.343. A stationary upright cone has a taper angle $\theta = 45°$, and the area of the lateral surface $S_0 = 4.0$ m^2. Find: (a) its taper angle; (b) its lateral surface area, in the reference frame moving with a velocity $v = (4/5)c$ along the axis of the cone.

1.344. With what velocity (relative to the reference frame K) did the clock move, if during the time interval $t = 5.0$ s, measured by the clock of the frame K, it became slow by $\Delta t = 0.10$ s?

1.345. A rod flies with constant velocity past a mark which is stationary in the reference frame K. In the frame K it takes $\Delta t = 20$ ns for the rod to fly past the mark. In the reference frame fixed to the rod the mark moves past the rod for $\Delta t' = 25$ ns. Find the proper length of the rod.

1.346. The proper lifetime of an unstable particle is equal to $\Delta t_0 = 10$ ns. Find the distance this particle will traverse till its decay in the laboratory frame of reference, where its lifetime is equal to $\Delta t = 20$ ns.

1.347. In the reference frame K a muon moving with a velocity $v = 0.990c$ travelled a distance $l = 3.0$ km from its birthplace to the point where it decayed. Find:

(a) the proper lifetime of this muon;

(b) the distance travelled by the muon in the frame K "from the muon's standpoint".

1.348. Two particles moving in a laboratory frame of reference along the same straight line with the same velocity $v = (3/4)c$ strike against a stationary target with the time interval $\Delta t = 50$ ns. Find the proper distance between the particles prior to their hitting the target.

1.349. A rod moves along a ruler with a constant velocity. When the positions of both ends of the rod are marked simultaneously in the reference frame fixed to the ruler, the difference of readings on the ruler is equal to $\Delta x_1 = 4.0$ m. But when the positions of the rod's ends are marked simultaneously in the reference frame fixed to the rod, the difference of readings on the same ruler is equal to $\Delta x_2 = 9.0$ m. Find the proper length of the rod and its velocity relative to the ruler.

1.350. Two rods of the same proper length l_0 move toward each other parallel to a common horizontal axis. In the reference frame fixed to one of the rods the time interval between the moments, when the right and left ends of the rods coincide, is equal to Δt. What is the velocity of one rod relative to the other?

1.351. Two unstable particles move in the reference frame K along a straight line in the same direction with a velocity $v = 0.990c$. The distance between them in this reference frame is equal to $l = 120$ m. At a certain moment both particles decay simultaneously in the reference frame fixed to them. What time interval between the moments of decay of the two particles will be observed in the frame K? Which particle decays later in the frame K?

1.352. A rod AB oriented along the x axis of the reference frame K moves in the positive direction of the x axis with a constant velocity v. The point A is the forward end of the rod, and the point B its rear end. Find:

(a) the proper length of the rod, if at the moment t_A the coordinate of the point A is equal to x_A, and at the moment t_B the coordinate of the point B is equal to x_B;

(b) what time interval should separate the markings of coordinates of the rod's ends in the frame K for the difference of coordinates to become equal to the proper length of the rod.

1.353. The rod $A'B'$ moves with a constant velocity v relative to the rod AB (Fig. 1.91). Both rods have the same proper length l_0 and

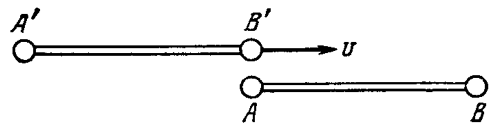


Fig. 1.91.

at the ends of each of them clocks are mounted, which are synchronized pairwise: A with B and A' with B'. Suppose the moment when the clock B' gets opposite the clock A is taken for the beginning of the time count in the reference frames fixed to each of the rods. Determine:

(a) the readings of the clocks B and B' at the moment when they are opposite each other;

(b) the same for the clocks A and A'.

1.354. There are two groups of mutually synchronized clocks K and K' moving relative to each other with a velocity v as shown in Fig. 1.92. The moment when the clock A' gets opposite the clock A

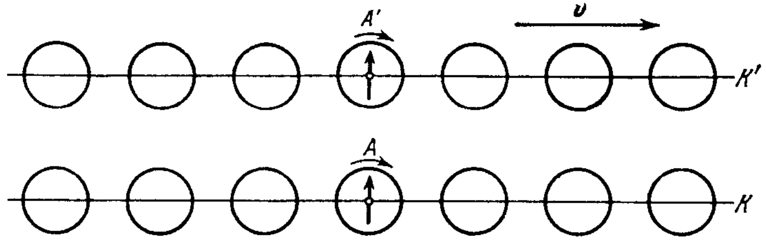


Fig. 1.92.

is taken for the beginning of the time count. Draw the approximate position of hands of all the clocks at this moment "in terms of the K clocks"; "in terms of the K' clocks".

1.355. The reference frame K' moves in the positive direction of the x axis of the frame K with a relative velocity V. Suppose that at the moment when the origins of coordinates O and O' coincide, the clock readings at these points are equal to zero in both frames. Find the displacement velocity $\dot{x}$ of the point (in the frame K) at which the readings of the clocks of both reference frames will be permanently identical. Demonstrate that $\dot{x} < V$.

1.356. At two points of the reference frame K two events occurred separated by a time interval Δt. Demonstrate that if these events obey the cause-and-effect relationship in the frame K (e.g. a shot fired and a bullet hitting a target), they obey that relationship in any other inertial reference frame K'.

1.357. The space-time diagram of Fig. 1.93 shows three events A, B, and C which occurred on the x axis of some inertial reference frame. Find:

(a) the time interval between the events A and B in the reference frame where the two events occurred at the same point;

(b) the distance between the points at which the events A and C occurred in the reference frame where these two events are simultaneous.

1.358. The velocity components of a particle moving in the xy plane of the reference frame K are equal to v_x and v_y. Find the velocity v' of this particle in the frame K' which moves with the velocity V relative to the frame K in the positive direction of its x axis.

1.359. Two particles move toward each other with velocities $v_1 = 0.50c$ and $v_2 = 0.75c$ relative to a laboratory frame of reference. Find:

(a) the approach velocity of the particles in the laboratory frame of reference;

(b) their relative velocity.

1.360. Two rods having the same proper length l_0 move lengthwise toward each other parallel to a common axis with the same velocity

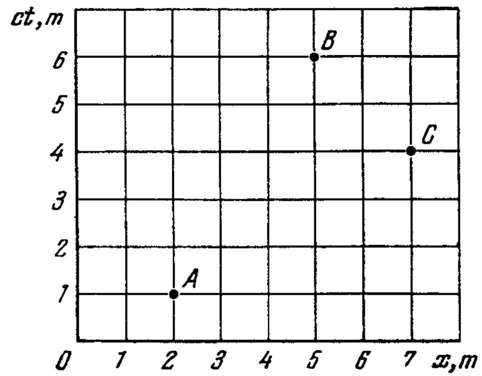


Fig. 1.93.

v relative to the laboratory frame of reference. What is the length of each rod in the reference frame fixed to the other rod?

1.361. Two relativistic particles move at right angles to each other in a laboratory frame of reference, one with the velocity v_1 and the other with the velocity v_2. Find their relative velocity.

1.362. An unstable particle moves in the reference frame K' along its y' axis with a velocity v'. In its turn, the frame K' moves relative to the frame K in the positive direction of its x axis with a velocity V. The x' and x axes of the two reference frames coincide, the y' and y axes are parallel. Find the distance which the particle traverses in the frame K, if its proper lifetime is equal to Δt_0.

1.363. A particle moves in the frame K with a velocity v at an angle θ to the x axis. Find the corresponding angle in the frame K' moving with a velocity V relative to the frame K in the positive direction of its x axis, if the x and x' axes of the two frames coincide.

1.364. The rod AB oriented parallel to the x' axis of the reference frame K' moves in this frame with a velocity v' along its y' axis. In its turn, the frame K' moves with a velocity V relative to the frame K as shown in Fig. 1.94. Find the angle θ between the rod and the x axis in the frame K.

1.365. The frame K' moves with a constant velocity $\mathbf{V}$ relative to the frame K. Find the acceleration w' of a particle in the frame K',

if in the frame K this particle moves with a velocity v and acceleration w along a straight line

(a) in the direction of the vector $\mathbf{V}$;

(b) perpendicular to the vector $\mathbf{V}$.

1.366. An imaginary space rocket launched from the Earth moves with an acceleration $w' = 10g$ which is the same in every instantaneous co-moving inertial reference frame. The boost stage lasted

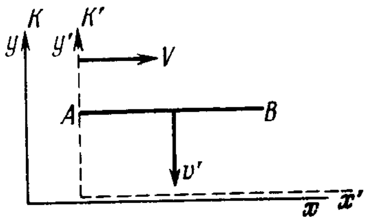


Fig. 1.94.

$\tau = 1.0$ year of terrestrial time. Find how much (in per cent) does the rocket velocity differ from the velocity of light at the end of the boost stage. What distance does the rocket cover by that moment?

1.367. From the conditions of the foregoing problem determine the boost time τ_0 in the reference frame fixed to the rocket. Remember that this time is defined by the formula

$$\tau_0 = \int_0^{\tau} \sqrt{1-(v/c)^2}\, dt,$$

where dt is the time in the geocentric reference frame.

1.368. How many times does the relativistic mass of a particle whose velocity differs from the velocity of light by 0.010% exceed its rest mass?

1.369. The density of a stationary body is equal to ρ_0. Find the velocity (relative to the body) of the reference frame in which the density of the body is $\eta = 25\%$ greater than ρ_0.

1.370. A proton moves with a momentum $p = 10.0$ GeV/c, where c is the velocity of light. How much (in per cent) does the proton velocity differ from the velocity of light?

1.371. Find the velocity at which the relativistic momentum of a particle exceeds its Newtonian momentum $\eta = 2$ times.

1.372. What work has to be performed in order to increase the velocity of a particle of rest mass m_0 from $0.60\,c$ to $0.80\,c$? Compare the result obtained with the value calculated from the classical formula.

1.373. Find the velocity at which the kinetic energy of a particle equals its rest energy.

1.374. At what values of the ratio of the kinetic energy to rest energy can the velocity of a particle be calculated from the classical formula with the relative error less than $\varepsilon = 0.010$?

1.375. Find how the momentum of a particle of rest mass m_0 depends on its kinetic energy. Calculate the momentum of a proton whose kinetic energy equals 500 MeV.

1.376. A beam of relativistic particles with kinetic energy T strikes against an absorbing target. The beam current equals I, the charge and rest mass of each particle are equal to e and m_0 respectively. Find the pressure developed by the beam on the target surface, and the power liberated there.

1.377. A sphere moves with a relativistic velocity v through a gas whose unit volume contains n slowly moving particles, each of mass m. Find the pressure p exerted by the gas on a spherical surface element perpendicular to the velocity of the sphere, provided that the particles scatter elastically. Show that the pressure is the same both in the reference frame fixed to the sphere and in the reference frame fixed to the gas.

1.378. A particle of rest mass m_0 starts moving at a moment $t = 0$ due to a constant force $\mathbf{F}$. Find the time dependence of the particle's velocity and of the distance covered.

1.379. A particle of rest mass m_0 moves along the x axis of the frame K in accordance with the law $x = \sqrt{a^2 + c^2t^2}$, where a is a constant, c is the velocity of light, and t is time. Find the force acting on the particle in this reference frame.

1.380. Proceeding from the fundamental equation of relativistic dynamics, find:

(a) under what circumstances the acceleration of a particle coincides in direction with the force $\mathbf{F}$ acting on it;

(b) the proportionality factors relating the force $\mathbf{F}$ and the acceleration $\mathbf{w}$ in the cases when $\mathbf{F} \perp \mathbf{v}$ and $\mathbf{F} \parallel \mathbf{v}$, where $\mathbf{v}$ is the velocity of the particle.

1.381. A relativistic particle with momentum p and total energy E moves along the x axis of the frame K. Demonstrate that in the frame K' moving with a constant velocity V relative to the frame K in the positive direction of its axis x the momentum and the total energy of the given particle are defined by the formulas:

$$p'_x = \frac{p_x - EV/c^2}{\sqrt{1-\beta^2}}, \qquad E' = \frac{E - p_xV}{\sqrt{1-\beta^2}}$$

where $\beta = V/c$.

1.382. The photon energy in the frame K is equal to ε. Making use of the transformation formulas cited in the foregoing problem, find the energy ε' of this photon in the frame K' moving with a velocity V relative to the frame K in the photon's motion direction. At what value of V is the energy of the photon equal to $\varepsilon' = \varepsilon/2$?

1.383. Demonstrate that the quantity $E^2 - p^2c^2$ for a particle is an invariant, i.e. it has the same magnitude in all inertial reference frames. What is the magnitude of this invariant?

1.384. A neutron with kinetic energy $T = 2m_0c^2$, where m_0 is its rest mass, strikes another, stationary, neutron. Determine:

(a) the combined kinetic energy $\widetilde{T}$ of both neutrons in the frame of their centre of inertia and the momentum $\widetilde{p}$ of each neutron in that frame;

(b) the velocity of the centre of inertia of this system of particles.

Instruction. Make use of the invariant $E^2 - p^2c^2$ remaining constant on transition from one inertial reference frame to another (E is the total energy of the system, p is its composite momentum).

1.385. A particle of rest mass m_0 with kinetic energy T strikes a stationary particle of the same rest mass. Find the rest mass and the velocity of the compound particle formed as a result of the collision.

1.386. How high must be the kinetic energy of a proton striking another, stationary, proton for their combined kinetic energy in the frame of the centre of inertia to be equal to the total kinetic energy of two protons moving toward each other with individual kinetic energies $T = 25.0$ GeV?

1.387. A stationary particle of rest mass m_0 disintegrates into three particles with rest masses m_1, m_2, and m_3. Find the maximum total energy that, for example, the particle m_1 may possess.

1.388. A relativistic rocket emits a gas jet with non-relativistic velocity $\mathbf{u}$ constant relative to the rocket. Find how the velocity v of the rocket depends on its rest mass m if the initial rest mass of the rocket equals m_0.

PART TWO

THERMODYNAMICS AND MOLECULAR PHYSICS

2.1. EQUATION OF THE GAS STATE. PROCESSES

- Ideal gas law:

$$pV = \frac{m}{M} RT, \tag{2.1a}$$

where M is the molar mass.

- Barometric formula:

$$p = p_0 e^{-Mgh/RT}, \tag{2.1b}$$

where p_0 is the pressure at the height $h = 0$.

- Van der Waals equation of gas state (for a mole):

$$\left(p + \frac{a}{V_M^2}\right)(V_M - b) = RT, \tag{2.1c}$$

where V_M is the molar volume under given p and T.

2.1. A vessel of volume $V = 30$ l contains ideal gas at the temperature 0 °C. After a portion of the gas has been let out, the pressure in the vessel decreased by $\Delta p = 0.78$ atm (the temperature remaining constant). Find the mass of the released gas. The gas density under the normal conditions $\rho = 1.3$ g/l.

2.2. Two identical vessels are connected by a tube with a valve letting the gas pass from one vessel into the other if the pressure difference $\Delta p \geqslant 1.10$ atm. Initially there was a vacuum in one vessel while the other contained ideal gas at a temperature $t_1 = 27$ °C and pressure $p_1 = 1.00$ atm. Then both vessels were heated to a temperature $t_2 = 107$ °C. Up to what value will the pressure in the first vessel (which had vacuum initially) increase?

2.3. A vessel of volume $V = 20$ l contains a mixture of hydrogen and helium at a temperature $t = 20$ °C and pressure $p = 2.0$ atm. The mass of the mixture is equal to $m = 5.0$ g. Find the ratio of the mass of hydrogen to that of helium in the given mixture.

2.4. A vessel contains a mixture of nitrogen ($m_1 = 7.0$ g) and carbon dioxide ($m_2 = 11$ g) at a temperature $T = 290$ K and pressure $p_0 = 1.0$ atm. Find the density of this mixture, assuming the gases to be ideal.

2.5. A vessel of volume $V = 7.5$ l contains a mixture of ideal gases at a temperature $T = 300$ K: $\nu_1 = 0.10$ mole of oxygen, $\nu_2 = 0.20$ mole of nitrogen, and $\nu_3 = 0.30$ mole of carbon dioxide. Assuming the gases to be ideal, find:

(a) the pressure of the mixture;

(b) the mean molar mass M of the given mixture which enters its equation of state $pV = (m/M)\,RT$, where m is the mass of the mixture.

2.6. A vertical cylinder closed from both ends is equipped with an easily moving piston dividing the volume into two parts, each containing one mole of air. In equilibrium at $T_0 = 300$ K the volume of the upper part is $\eta = 4.0$ times greater than that of the lower part. At what temperature will the ratio of these volumes be equal to $\eta' = 3.0$?

2.7. A vessel of volume V is evacuated by means of a piston air pump. One piston stroke captures the volume ΔV. How many strokes are needed to reduce the pressure in the vessel η times? The process is assumed to be isothermal, and the gas ideal.

2.8. Find the pressure of air in a vessel being evacuated as a function of evacuation time t. The vessel volume is V, the initial pressure is p_0. The process is assumed to be isothermal, and the evacuation rate equal to C and independent of pressure.

Note. The evacuation rate is the gas volume being evacuated per unit time, with that volume being measured under the gas pressure attained by that moment.

2.9. A chamber of volume $V = 87$ l is evacuated by a pump whose evacuation rate (see Note to the foregoing problem) equals $C = 10$ l/s. How soon will the pressure in the chamber decrease by $\eta = 1000$ times?

2.10. A smooth vertical tube having two different sections is open from both ends and equipped with two pistons of different areas (Fig. 2.1). Each piston slides within a respective tube section. One mole of ideal gas is enclosed between the pistons tied with a non-stretchable thread. The cross-sectional area of the upper piston is $\Delta S = 10\ \text{cm}^2$ greater than that of the lower one. The combined mass of the two pistons is equal to $m = 5.0$ kg. The outside air pressure is $p_0 = 1.0$ atm. By how many kelvins must the gas between the pistons be heated to shift the pistons through $l = 5.0$ cm?

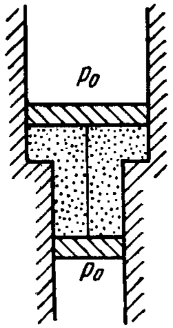


Fig. 2.1.

2.11. Find the maximum attainable temperature of ideal gas in each of the following processes:

(a) $p = p_0 - \alpha V^2$; (b) $p = p_0 e^{-\beta V}$,

where p_0, α and β are positive constants, and V is the volume of one mole of gas.

2.12. Find the minimum attainable pressure of ideal gas in the process $T = T_0 + \alpha V^2$, where T_0 and α are positive constants, and V is the volume of one mole of gas. Draw the approximate p vs V plot of this process.

2.13. A tall cylindrical vessel with gaseous nitrogen is located in a uniform gravitational field in which the free-fall acceleration is equal to g. The temperature of the nitrogen varies along the height h so that its density is the same throughout the volume. Find the temperature gradient dT/dh.

2.14. Suppose the pressure p and the density ρ of air are related as $p/\rho^n = \text{const}$ regardless of height (n is a constant here). Find the corresponding temperature gradient.

2.15. Let us assume that air is under standard conditions close to the Earth's surface. Presuming that the temperature and the molar mass of air are independent of height, find the air pressure at the height 5.0 km over the surface and in a mine at the depth 5.0 km below the surface.

2.16. Assuming the temperature and the molar mass of air, as well as the free-fall acceleration, to be independent of the height, find the difference in heights at which the air densities at the temperature 0 °C differ

(a) e times; (b) by $\eta = 1.0\%$.

2.17. An ideal gas of molar mass M is contained in a tall vertical cylindrical vessel whose base area is S and height h. The temperature of the gas is T, its pressure on the bottom base is p_0. Assuming the temperature and the free-fall acceleration g to be independent of the height, find the mass of gas in the vessel.

2.18. An ideal gas of molar mass M is contained in a very tall vertical cylindrical vessel in the uniform gravitational field in which the free-fall acceleration equals g. Assuming the gas temperature to be the same and equal to T, find the height at which the centre of gravity of the gas is located.

2.19. An ideal gas of molar mass M is located in the uniform gravitational field in which the free-fall acceleration is equal to g. Find the gas pressure as a function of height h, if $p = p_0$ at $h = 0$, and the temperature varies with height as

(a) $T = T_0(1 - ah)$; (b) $T = T_0(1 + ah)$,

where a is a positive constant.

2.20. A horizontal cylinder closed from one end is rotated with a constant angular velocity ω about a vertical axis passing through the open end of the cylinder. The outside air pressure is equal to p_0, the temperature to T, and the molar mass of air to M. Find the air pressure as a function of the distance r from the rotation axis. The molar mass is assumed to be independent of r.

2.21. Under what pressure will carbon dioxide have the density $\rho = 500$ g/l at the temperature $T = 300$ K? Carry out the calculations both for an ideal and for a Van der Waals gas.

2.22. One mole of nitrogen is contained in a vessel of volume $V = 1.00$ l. Find:

(a) the temperature of the nitrogen at which the pressure can be calculated from an ideal gas law with an error $\eta = 10\%$ (as compared with the pressure calculated from the Van der Waals equation of state);

(b) the gas pressure at this temperature.

2.23. One mole of a certain gas is contained in a vessel of volume $V = 0.250$ l. At a temperature $T_1 = 300$ K the gas pressure is $p_1 =$

$= 90$ atm, and at a temperature $T_2 = 350$ K the pressure is $p_2 =$ $= 110$ atm. Find the Van der Waals parameters for this gas.

2.24. Find the isothermal compressibility $\varkappa$ of a Van der Waals gas as a function of volume V at temperature T.

Note. By definition, $\varkappa = -\frac{1}{V}\frac{\partial V}{\partial p}$.

2.25. Making use of the result obtained in the foregoing problem, find at what temperature the isothermal compressibility $\varkappa$ of a Van der Waals gas is greater than that of an ideal gas. Examine the case when the molar volume is much greater than the parameter b.

2.2. THE FIRST LAW OF THERMODYNAMICS. HEAT CAPACITY

- The first law of thermodynamics:

$$Q = \Delta U + A, \tag{2.2a}$$

where ΔU is the increment of the internal energy of the system.

- Work performed by gas:

$$A = \int p\, dV. \tag{2.2b}$$

- Internal energy of an ideal gas:

$$U = \frac{m}{M} C_V T = \frac{m}{M}\frac{RT}{\gamma - 1} = \frac{pV}{\gamma - 1}. \tag{2.2c}$$

- Molar heat capacity in a polytropic process ($pV^n = \text{const}$):

$$C = \frac{R}{\gamma - 1} - \frac{R}{n - 1} = \frac{(n - \gamma) R}{(n - 1)(\gamma - 1)}. \tag{2.2d}$$

- Internal energy of one mole of a Van der Waals gas:

$$U = C_V T - \frac{a}{V_M}. \tag{2.2e}$$

2.26. Demonstrate that the interval energy U of the air in a room is independent of temperature provided the outside pressure p is constant. Calculate U, if p is equal to the normal atmospheric pressure and the room's volume is equal to $V = 40$ m^3.

2.27. A thermally insulated vessel containing a gas whose molar mass is equal to M and the ratio of specific heats $C_p/C_V = \gamma$ moves with a velocity v. Find the gas temperature increment resulting from the sudden stoppage of the vessel.

2.28. Two thermally insulated vessels *1* and *2* are filled with air and connected by a short tube equipped with a valve. The volumes of the vessels, the pressures and temperatures of air in them are known (V_1, p_1, T_1 and V_2, p_2, T_2). Find the air temperature and pressure established after the opening of the valve.

2.29. Gaseous hydrogen contained initially under standard conditions in a sealed vessel of volume $V = 5.0$ l was cooled by $\Delta T =$

$= 55$ K. Find how much the internal energy of the gas will change and what amount of heat will be lost by the gas.

2.30. What amount of heat is to be transferred to nitrogen in the isobaric heating process for that gas to perform the work $A = 2.0$ J?

2.31. As a result of the isobaric heating by $\Delta T = 72$ K one mole of a certain ideal gas obtains an amount of heat $Q = 1.60$ kJ. Find the work performed by the gas, the increment of its internal energy, and the value of $\gamma = C_p/C_V$.

2.32. Two moles of a certain ideal gas at a temperature $T_0 = 300$ K were cooled isochorically so that the gas pressure reduced $n = 2.0$ times. Then, as a result of the isobaric process, the gas expanded till its temperature got back to the initial value. Find the total amount of heat absorbed by the gas in this process.

2.33. Calculate the value of $\gamma = C_p/C_V$ for a gaseous mixture consisting of $\nu_1 = 2.0$ moles of oxygen and $\nu_2 = 3.0$ moles of carbon dioxide. The gases are assumed to be ideal.

2.34. Find the specific heat capacities c_V and c_p for a gaseous mixture consisting of 7.0 g of nitrogen and 20 g of argon. The gases are assumed to be ideal.

2.35. One mole of a certain ideal gas is contained under a weightless piston of a vertical cylinder at a temperature T. The space over the piston opens into the atmosphere. What work has to be performed in order to increase isothermally the gas volume under the piston n times by slowly raising the piston? The friction of the piston against the cylinder walls is negligibly small.

2.36. A piston can freely move inside a horizontal cylinder closed from both ends. Initially, the piston separates the inside space of the cylinder into two equal parts each of volume V_0, in which an ideal gas is contained under the same pressure p_0 and at the same temperature. What work has to be performed in order to increase isothermally the volume of one part of gas η times compared to that of the other by slowly moving the piston?

2.37. Three moles of an ideal gas being initially at a temperature $T_0 = 273$ K were isothermally expanded $n = 5.0$ times its initial volume and then isochorically heated so that the pressure in the final state became equal to that in the initial state. The total amount of heat transferred to the gas during the process equals $Q = 80$ kJ. Find the ratio $\gamma = C_p/C_V$ for this gas.

2.38. Draw the approximate plots of isochoric, isobaric, isothermal, and adiabatic processes for the case of an ideal gas, using the following variables:

(a) p, T; (b) V, T.

2.39. One mole of oxygen being initially at a temperature $T_0 =$ $= 290$ K is adiabatically compressed to increase its pressure $\eta =$ $= 10.0$ times. Find:

(a) the gas temperature after the compression;

(b) the work that has been performed on the gas.

2.40. A certain mass of nitrogen was compressed $\eta = 5.0$ times

(in terms of volume), first adiabatically, and then isothermally. In both cases the initial state of the gas was the same. Find the ratio of the respective works expended in each compression.

2.41. A heat-conducting piston can freely move inside a closed thermally insulated cylinder with an ideal gas. In equilibrium the piston divides the cylinder into two equal parts, the gas temperature being equal to T_0. The piston is slowly displaced. Find the gas temperature as a function of the ratio η of the volumes of the greater and smaller sections. The adiabatic exponent of the gas is equal to γ.

2.42. Find the rate v with which helium flows out of a thermally insulated vessel into vacuum through a small hole. The flow rate of the gas inside the vessel is assumed to be negligible under these conditions. The temperature of helium in the vessel is $T = 1{,}000$ K.

2.43. The volume of one mole of an ideal gas with the adiabatic exponent γ is varied according to the law $V = a/T$, where a is a constant. Find the amount of heat obtained by the gas in this process if the gas temperature increased by ΔT.

2.44. Demonstrate that the process in which the work performed by an ideal gas is proportional to the corresponding increment of its internal energy is described by the equation $pV^n = \text{const}$, where n is a constant.

2.45. Find the molar heat capacity of an ideal gas in a polytropic process $pV^n = \text{const}$ if the adiabatic exponent of the gas is equal to γ. At what values of the polytropic constant n will the heat capacity of the gas be negative?

2.46. In a certain polytropic process the volume of argon was increased $\alpha = 4.0$ times. Simultaneously, the pressure decreased $\beta = 8.0$ times. Find the molar heat capacity of argon in this process, assuming the gas to be ideal.

2.47. One mole of argon is expanded polytropically, the polytropic constant being $n = 1.50$. In the process, the gas temperature changes by $\Delta T = -26$ K. Find:

(a) the amount of heat obtained by the gas;

(b) the work performed by the gas.

2.48. An ideal gas whose adiabatic exponent equals γ is expanded according to the law $p = \alpha V$, where α is a constant. The initial volume of the gas is equal to V_0. As a result of expansion the volume increases η times. Find:

(a) the increment of the internal energy of the gas;

(b) the work performed by the gas;

(c) the molar heat capacity of the gas in the process.

2.49. An ideal gas whose adiabatic exponent equals γ is expanded so that the amount of heat transferred to the gas is equal to the decrease of its internal energy. Find:

(a) the molar heat capacity of the gas in this process;

(b) the equation of the process in the variables T, V;

(c) the work performed by one mole of the gas when its volume increases η times if the initial temperature of the gas is T_0.

2.50. One mole of an ideal gas whose adiabatic exponent equals γ undergoes a process in which the gas pressure relates to the temperature as $p = aT^\alpha$, where a and α are constants. Find:

(a) the work performed by the gas if its temperature gets an increment ΔT;

(b) the molar heat capacity of the gas in this process; at what value of α will the heat capacity be negative?

2.51. An ideal gas with the adiabatic exponent γ undergoes a process in which its internal energy relates to the volume as $U = aV^\alpha$, where a and α are constants. Find:

(a) the work performed by the gas and the amount of heat to be transferred to this gas to increase its internal energy by ΔU;

(b) the molar heat capacity of the gas in this process.

2.52. An ideal gas has a molar heat capacity C_V at constant volume. Find the molar heat capacity of this gas as a function of its volume V, if the gas undergoes the following process:

(a) $T = T_0 e^{\alpha V}$; (b) $p = p_0 e^{\alpha V}$,

where T_0, p_0, and α are constants.

2.53. One mole of an ideal gas whose adiabatic exponent equals γ undergoes a process $p = p_0 + \alpha/V$, where p_0 and α are positive constants. Find:

(a) heat capacity of the gas as a function of its volume;

(b) the internal energy increment of the gas, the work performed by it, and the amount of heat transferred to the gas, if its volume increased from V_1 to V_2.

2.54. One mole of an ideal gas with heat capacity at constant pressure C_p undergoes the process $T = T_0 + \alpha V$, where T_0 and α are constants. Find:

(a) heat capacity of the gas as a function of its volume;

(b) the amount of heat transferred to the gas, if its volume increased from V_1 to V_2.

2.55. For the case of an ideal gas find the equation of the process (in the variables T, V) in which the molar heat capacity varies as:

(a) $C = C_V + \alpha T$; (b) $C = C_V + \beta V$; (c) $C = C_V + ap$,

where α, β, and a are constants.

2.56. An ideal gas has an adiabatic exponent γ. In some process its molar heat capacity varies as $C = \alpha/T$, where α is a constant. Find:

(a) the work performed by one mole of the gas during its heating from the temperature T_0 to the temperature η times higher;

(b) the equation of the process in the variables p, V.

2.57. Find the work performed by one mole of a Van der Waals gas during its isothermal expansion from the volume V_1 to V_2 at a temperature T.

2.58. One mole of oxygen is expanded from a volume $V_1 = 1.00$ l to $V_2 = 5.0$ l at a constant temperature $T = 280$ K. Calculate:

(a) the increment of the internal energy of the gas:

(b) the amount of the absorbed heat.

The gas is assumed to be a Van der Waals gas.

2.59. For a Van der Waals gas find:

(a) the equation of the adiabatic curve in the variables T, V;

(b) the difference of the molar heat capacities $C_p - C_V$ as a function of T and V.

2.60. Two thermally insulated vessels are interconnected by a tube equipped with a valve. One vessel of volume $V_1 = 10$ l contains $\nu = 2.5$ moles of carbon dioxide. The other vessel of volume $V_2 = 100$ l is evacuated. The valve having been opened, the gas adiabatically expanded. Assuming the gas to obey the Van der Waals equation, find its temperature change accompanying the expansion.

2.61. What amount of heat has to be transferred to $\nu = 3.0$ moles of carbon dioxide to keep its temperature constant while it expands into vacuum from the volume $V_1 = 5.0$ l to $V_2 = 10$ l? The gas is assumed to be a Van der Waals gas.

2.3. KINETIC THEORY OF GASES. BOLTZMANN'S LAW AND MAXWELL'S DISTRIBUTION

• Number of collisions exercised by gas molecules on a unit area of the wall surface per unit time:

$$\nu = \frac{1}{4} n \langle v \rangle, \tag{2.3a}$$

where n is the concentration of molecules, and $\langle v \rangle$ is their mean velocity.

• Equation of an ideal gas state:

$$p = nkT. \tag{2.3b}$$

• Mean energy of molecules:

$$\langle \varepsilon \rangle = \frac{i}{2} kT, \tag{2.3c}$$

where i is the sum of translational, rotational, and the double number of vibrational degrees of freedom.

• Maxwellian distribution:

$$dN(v_x) = N \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-mv_x^2/2kT} dv_x, \tag{2.3d}$$

$$dN(v) = N \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} 4\pi v^2 dv. \tag{2.3e}$$

• Maxwellian distribution in a reduced form:

$$dN(u) = N \frac{4}{\sqrt{\pi}} e^{-u^2} u^2 du, \tag{2.3f}$$

where $u = v/v_p$, v_p is the most probable velocity.

• The most probable, the mean, and the root mean square velocities of molecules:

$$v_p = \sqrt{2\frac{kT}{m}} \qquad \langle v \rangle = \sqrt{\frac{8}{\pi}\frac{kT}{m}}, \qquad v_{sq} = \sqrt{3\frac{kT}{m}}. \tag{2.3g}$$

• Boltzmann's formula:

$$n = n_0 e^{-(U-U_0)/kT}, \tag{2.3h}$$

where U is the potential energy of a molecule.

2.62. Modern vacuum pumps permit the pressures down to $p = 4 \cdot 10^{-15}$ atm to be reached at room temperatures. Assuming that the gas exhausted is nitrogen, find the number of its molecules per 1 cm^3 and the mean distance between them at this pressure.

2.63. A vessel of volume $V = 5.0$ l contains $m = 1.4$ g of nitrogen at a temperature $T = 1800$ K. Find the gas pressure, taking into account that $\eta = 30\%$ of molecules are disassociated into atoms at this temperature.

2.64. Under standard conditions the density of the helium and nitrogen mixture equals $\rho = 0.60$ g/l. Find the concentration of helium atoms in the given mixture.

2.65. A parallel beam of nitrogen molecules moving with velocity $v = 400$ m/s impinges on a wall at an angle $\theta = 30°$ to its normal. The concentration of molecules in the beam $n = 0.9 \cdot 10^{19}$ cm^{-3}. Find the pressure exerted by the beam on the wall assuming the molecules to scatter in accordance with the perfectly elastic collision law.

2.66. How many degrees of freedom have the gas molecules, if under standard conditions the gas density is $\rho = 1.3$ mg/cm^3 and the velocity of sound propagation in it is $v = 330$ m/s.

2.67. Determine the ratio of the sonic velocity v in a gas to the root mean square velocity of molecules of this gas, if the molecules are

(a) monatomic; (b) rigid diatomic.

2.68. A gas consisting of N-atomic molecules has the temperature T at which all degrees of freedom (translational, rotational, and vibrational) are excited. Find the mean energy of molecules in such a gas. What fraction of this energy corresponds to that of translational motion?

2.69. Suppose a gas is heated up to a temperature at which all degrees of freedom (translational, rotational, and vibrational) of its molecules are excited. Find the molar heat capacity of such a gas in the isochoric process, as well as the adiabatic exponent γ, if the gas consists of

(a) diatomic;

(b) linear N-atomic;

(c) network N-atomic

molecules.

2.70. An ideal gas consisting of N-atomic molecules is expanded isobarically. Assuming that all degrees of freedom (translational, rotational, and vibrational) of the molecules are excited, find what fraction of heat transferred to the gas in this process is spent to perform the work of expansion. How high is this fraction in the case of a monatomic gas?

2.71. Find the molar mass and the number of degrees of freedom of molecules in a gas if its heat capacities are known: $c_V = 0.65$ J/(g·K) and $c_p = 0.91$ J/(g·K).

2.72. Find the number of degrees of freedom of molecules in a gas whose molar heat capacity

(a) at constant pressure is equal to $C_p = 29$ J/(mol·K);

(b) is equal to $C = 29$ J/(mol·K) in the process $pT =$ const.

2.73. Find the adiabatic exponent γ for a mixture consisting of ν_1 moles of a monatomic gas and ν_2 moles of gas of rigid diatomic molecules.

2.74. A thermally insulated vessel with gaseous nitrogen at a temperature $t = 27$ °C moves with velocity $v = 100$ m/s. How much (in per cent) and in what way will the gas pressure change on a sudden stoppage of the vessel?

2.75. Calculate at the temperature $t = 17$ °C:

(a) the root mean square velocity and the mean kinetic energy of an oxygen molecule in the process of translational motion;

(b) the root mean square velocity of a water droplet of diameter $d = 0.10$ μm suspended in the air.

2.76. A gas consisting of rigid diatomic molecules is expanded adiabatically. How many times has the gas to be expanded to reduce the root mean square velocity of the molecules $\eta = 1.50$ times?

2.77. The mass $m = 15$ g of nitrogen is enclosed in a vessel at a temperature $T = 300$ K. What amount of heat has to be transferred to the gas to increase the root mean square velocity of its molecules $\eta = 2.0$ times?

2.78. The temperature of a gas consisting of rigid diatomic molecules is $T = 300$ K. Calculate the angular root mean square velocity of a rotating molecule if its moment of inertia is equal to $I = 2.1 \cdot 10^{-39}$ g·cm^2.

2.79. A gas consisting of rigid diatomic molecules was initially under standard conditions. Then the gas was compressed adiabatically $\eta = 5.0$ times. Find the mean kinetic energy of a rotating molecule in the final state.

2.80. How will the rate of collisions of rigid diatomic molecules against the vessel's wall change, if the gas is expanded adiabatically η times?

2.81. The volume of gas consisting of rigid diatomic molecules was increased $\eta = 2.0$ times in a polytropic process with the molar heat capacity $C = R$. How many times will the rate of collisions of molecules against a vessel's wall be reduced as a result of this process?

2.82. A gas consisting of rigid diatomic molecules was expanded in a polytropic process so that the rate of collisions of the molecules against the vessel's wall did not change. Find the molar heat capacity of the gas in this process.

2.83. Calculate the most probable, the mean, and the root mean square velocities of a molecule of a gas whose density under standard atmospheric pressure is equal to $\rho = 1.00$ g/l.

2.84. Find the fraction of gas molecules whose velocities differ by less than $\delta\eta = 1.00\%$ from the value of

(a) the most probable velocity;

(b) the root mean square velocity.

2.85. Determine the gas temperature at which

(a) the root mean square velocity of hydrogen molecules exceeds their most probable velocity by $\Delta v = 400$ m/s;

(b) the velocity distribution function $F(v)$ for the oxygen molecules will have the maximum value at the velocity $v = 420$ m/s.

2.86. In the case of gaseous nitrogen find:

(a) the temperature at which the velocities of the molecules $v_1 = 300$ m/s and $v_2 = 600$ m/s are associated with equal values of the Maxwell distribution function $F(v)$;

(b) the velocity of the molecules v at which the value of the Maxwell distribution function $F(v)$ for the temperature T_0 will be the same as that for the temperature η times higher.

2.87. At what temperature of a nitrogen and oxygen mixture do the most probable velocities of nitrogen and oxygen molecules differ by $\Delta v = 30$ m/s?

2.88. The temperature of a hydrogen and helium mixture is $T = 300$ K. At what value of the molecular velocity v will the Maxwell distribution function $F(v)$ yield the same magnitude for both gases?

2.89. At what temperature of a gas will the number of molecules, whose velocities fall within the given interval from v to $v + dv$, be the greatest? The mass of each molecule is equal to m.

2.90. Find the fraction of molecules whose velocity projections on the x axis fall within the interval from v_x to $v_x + dv_x$, while the moduli of perpendicular velocity components fall within the interval from $v_\perp$ to $v_\perp + dv_\perp$. The mass of each molecule is m, and the temperature is T.

2.91. Using the Maxwell distribution function, calculate the mean velocity projection $\langle v_x \rangle$ and the mean value of the modulus of this projection $\langle |v_x| \rangle$ if the mass of each molecule is equal to m and the gas temperature is T.

2.92. From the Maxwell distribution function find $\langle v_x^2 \rangle$, the mean value of the squared v_x projection of the molecular velocity in a gas at a temperature T. The mass of each molecule is equal to m.

2.93. Making use of the Maxwell distribution function, calculate the number ν of gas molecules reaching a unit area of a wall per unit time, if the concentration of molecules is equal to n, the temperature to T, and the mass of each molecule is m.

2.94. Using the Maxwell distribution function, determine the pressure exerted by gas on a wall, if the gas temperature is T and the concentration of molecules is n.

2.95. Making use of the Maxwell distribution function, find $\langle 1/v \rangle$, the mean value of the reciprocal of the velocity of molecules

in an ideal gas at a temperature T, if the mass of each molecule is equal to m. Compare the value obtained with the reciprocal of the mean velocity.

2.96. A gas consists of molecules of mass m and is at a temperature T. Making use of the Maxwell velocity distribution function, find the corresponding distribution of the molecules over the kinetic energies ε. Determine the most probable value of the kinetic energy ε_p. Does ε_p correspond to the most probable velocity?

2.97. What fraction of monatomic molecules of a gas in a thermal equilibrium possesses kinetic energies differing from the mean value by $\delta\eta = 1.0$ % and less?

2.98. What fraction of molecules in a gas at a temperature T has the kinetic energy of translational motion exceeding ε_0 if $\varepsilon_0 \gg kT$?

2.99. The velocity distribution of molecules in a beam coming out of a hole in a vessel is described by the function $F(v) = Av^3 e^{-mv^2/2kT}$, where T is the temperature of the gas in the vessel. Find the most probable values of

(a) the velocity of the molecules in the beam; compare the result obtained with the most probable velocity of the molecules in the vessel;

(b) the kinetic energy of the molecules in the beam.

2.100. An ideal gas consisting of molecules of mass m with concentration n has a temperature T. Using the Maxwell distribution function, find the number of molecules reaching a unit area of a wall at the angles between θ and $\theta + d\theta$ to its normal per unit time.

2.101. From the conditions of the foregoing problem find the number of molecules reaching a unit area of a wall with the velocities in the interval from v to $v + dv$ per unit time.

2.102. Find the force exerted on a particle by a uniform field if the concentrations of these particles at two levels separated by the distance $\Delta h = 3.0$ cm (along the field) differ by $\eta = 2.0$ times. The temperature of the system is equal to $T = 280$ K.

2.103. When examining the suspended gamboge droplets under a microscope, their average numbers in the layers separated by the distance $h = 40$ μm were found to differ by $\eta = 2.0$ times. The environmental temperature is equal to $T = 290$ K. The diameter of the droplets is $d = 0.40$ μm, and their density exceeds that of the surrounding fluid by $\Delta\rho = 0.20$ g/cm³. Find Avogadro's number from these data.

2.104. Suppose that η_0 is the ratio of the molecular concentration of hydrogen to that of nitrogen at the Earth's surface, while η is the corresponding ratio at the height $h = 3000$ m. Find the ratio η/η_0 at the temperature $T = 280$ K, assuming that the temperature and the free fall acceleration are independent of the height.

2.105. A tall vertical vessel contains a gas composed of two kinds of molecules of masses m_1 and m_2, with $m_2 > m_1$. The concentrations of these molecules at the bottom of the vessel are equal to n_1 and n_2

respectively, with $n_2 > n_1$. Assuming the temperature T and the free-fall acceleration g to be independent of the height, find the height at which the concentrations of these kinds of molecules are equal.

2.106. A very tall vertical cylinder contains carbon dioxide at a certain temperature T. Assuming the gravitational field to be uniform, find how the gas pressure on the bottom of the vessel will change when the gas temperature increases η times.

2.107. A very tall vertical cylinder contains a gas at a temperature T. Assuming the gravitational field to be uniform, find the mean value of the potential energy of the gas molecules. Does this value depend on whether the gas consists of one kind of molecules or of several kinds?

2.108. A horizontal tube of length $l = 100$ cm closed from both ends is displaced lengthwise with a constant acceleration w. The tube contains argon at a temperature $T = 330$ K. At what value of w will the argon concentrations at the tube's ends differ by $\eta = 1.0\%$?

2.109. Find the mass of a mole of colloid particles if during their centrifuging with an angular velocity ω about a vertical axis the concentration of the particles at the distance r_2 from the rotation axis is η times greater than that at the distance r_1 (in the same horizontal plane). The densities of the particles and the solvent are equal to ρ and to ρ_0 respectively.

2.110. A horizontal tube with closed ends is rotated with a constant angular velocity ω about a vertical axis passing through one of its ends. The tube contains carbon dioxide at a temperature $T = 300$ K. The length of the tube is $l = 100$ cm. Find the value ω at which the ratio of molecular concentrations at the opposite ends of the tube is equal to $\eta = 2.0$.

2.111. The potential energy of gas molecules in a certain central field depends on the distance r from the field's centre as $U(r) = ar^2$, where a is a positive constant. The gas temperature is T, the concentration of molecules at the centre of the field is n_0. Find:

(a) the number of molecules located at the distances between r and $r + dr$ from the centre of the field;

(b) the most probable distance separating the molecules from the centre of the field;

(c) the fraction of molecules located in the spherical layer between r and $r + dr$;

(d) how many times the concentration of molecules in the centre of the field will change if the temperature decreases η times.

2.112. From the conditions of the foregoing problem find:

(a) the number of molecules whose potential energy lies within the interval from U to $U + dU$;

(b) the most probable value of the potential energy of a molecule; compare this value with the potential energy of a molecule located at its most probable distance from the centre of the field.

2.4. THE SECOND LAW OF THERMODYNAMICS. ENTROPY

- Heat engine efficiency:

$$\eta = \frac{A}{Q_1} = 1 - \frac{Q_2'}{Q_1}, \tag{2.4a}$$

where Q_1 is the heat obtained by the working substance, Q_2' is the heat released by the working substance.

- Efficiency of a Carnot cycle:

$$\eta = \frac{T_1 - T_2}{T_1}, \tag{2.4b}$$

where T_1 and T_2 are the temperatures of the hot and cold bodies respectively.

- Clausius inequality:

$$\oint \frac{\delta Q}{T} \leqslant 0, \tag{2.4c}$$

where δQ is the elementary amount of heat transferred to the system (δQ is an algebraic quantity).

- Entropy increment of a system:

$$\Delta S \geqslant \int \frac{\delta Q}{T}. \tag{2.4d}$$

- Fundamental relation of thermodynamics:

$$T\,dS \geqslant dU + p\,dV. \tag{2.4e}$$

- Relation between the entropy and the statistical weight Ω (the thermodynamic probability):

$$S = k \ln \Omega, \tag{2.4f}$$

where k is the Boltzmann constant.

2.113. In which case will the efficiency of a Carnot cycle be higher: when the hot body temperature is increased by ΔT, or when the cold body temperature is decreased by the same magnitude?

2.114. Hydrogen is used in a Carnot cycle as a working substance. Find the efficiency of the cycle, if as a result of an adiabatic expansion
(a) the gas volume increases $n = 2.0$ times;
(b) the pressure decreases $n = 2.0$ times.

2.115. A heat engine employing a Carnot cycle with an efficiency of $\eta = 10\%$ is used as a refrigerating machine, the thermal reservoirs being the same. Find its refrigerating efficiency ε.

2.116. An ideal gas goes through a cycle consisting of alternate isothermal and adiabatic curves (Fig. 2.2). The isothermal processes proceed at the temperatures T_1, T_2, and T_3. Find the efficiency of such a cycle, if in each isothermal expansion the gas volume increases in the same proportion.

2.117. Find the efficiency of a cycle consisting of two isochoric and two adiabatic lines, if the volume of the ideal gas changes $n = 10$ times within the cycle. The working substance is nitrogen.

2.118. Find the efficiency of a cycle consisting of two isobaric and two adiabatic lines, if the pressure changes n times within the cycle. The working substance is an ideal gas whose adiabatic exponent is equal to γ.

2.119. An ideal gas whose adiabatic exponent equals γ goes through a cycle consisting of two isochoric and two isobaric lines. Find the efficiency of such a cycle, if the absolute temperature of the gas rises n times both in the isochoric heating and in the isobaric expansion.

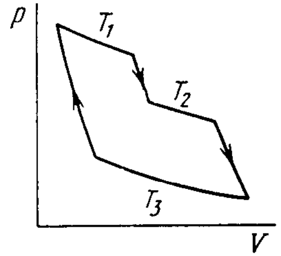


Fig. 2.2.

2.120. An ideal gas goes through a cycle consisting of
(a) isochoric, adiabatic, and isothermal lines;
(b) isobaric, adiabatic, and isothermal lines,
with the isothermal process proceeding at the *minimum* temperature of the whole cycle. Find the efficiency of each cycle if the absolute temperature varies n-fold within the cycle.

2.121. The conditions are the same as in the foregoing problem with the exception that the isothermal process proceeds at the *maximum* temperature of the whole cycle.

2.122. An ideal gas goes through a cycle consisting of isothermal, polytropic, and adiabatic lines, with the isothermal process proceeding at the *maximum* temperature of the whole cycle. Find the efficiency of such a cycle if the absolute temperature varies n-fold within the cycle.

2.123. An ideal gas with the adiabatic exponent γ goes through a direct (clockwise) cycle consisting of adiabatic, isobaric, and isochoric lines. Find the efficiency of the cycle if in the adiabatic process the volume of the ideal gas
(a) increases n-fold; (b) decreases n-fold.

2.124. Calculate the efficiency of a cycle consisting of isothermal, isobaric, and isochoric lines, if in the isothermal process the volume of the ideal gas with the adiabatic exponent γ
(a) increases n-fold; (b) decreases n-fold.

2.125. Find the efficiency of a cycle consisting of two isochoric and two isothermal lines if the volume varies ν-fold and the absolute temperature τ-fold within the cycle. The working substance is an ideal gas with the adiabatic exponent γ.

2.126. Find the efficiency of a cycle consisting of two isobaric and two isothermal lines if the pressure varies n-fold and the absolute temperature τ-fold within the cycle. The working substance is an ideal gas with the adiabatic exponent γ.

2.127. An ideal gas with the adiabatic exponent γ goes through a cycle (Fig. 2.3) within which the absolute temperature varies τ-fold. Find the efficiency of this cycle.

2.128. Making use of the Clausius inequality, demonstrate that

all cycles having the same maximum temperature T_{max} and the same minimum temperature T_{min} are less efficient compared to the Carnot cycle with the same T_{max} and T_{min}.

2.129. Making use of the Carnot theorem, show that in the case of a physically uniform substance whose state is defined by the parameters T and V

$$(\partial U/\partial V)_T = T\,(\partial p/\partial T)_V - p,$$

where $U(T, V)$ is the internal energy of the substance.

Instruction. Consider the infinitesimal Carnot cycle in the variables p, V.

2.130. Find the entropy increment of one mole of carbon dioxide when its absolute temperature increases $n = 2.0$ times if the process of heating is

(a) isochoric; (b) isobaric.

The gas is to be regarded as ideal.

2.131. The entropy of $\nu = 4.0$ moles of an ideal gas increases by $\Delta S = 23$ J/K due to the isothermal expansion. How many times should the volume $\nu = 4.0$ moles of the gas be increased?

2.132. Two moles of an ideal gas are cooled isochorically and then expanded isobarically to lower the gas temperature back to the initial value. Find the entropy increment of the gas if in this process the gas pressure changed $n = 3.3$ times.

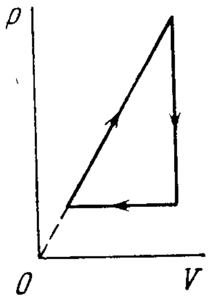


Fig. 2.3.

2.133. Helium of mass $m = 1.7$ g is expanded adiabatically $n = 3.0$ times and then compressed isobarically down to the initial volume. Find the entropy increment of the gas in this process.

2.134. Find the entropy increment of $\nu = 2.0$ moles of an ideal gas whose adiabatic exponent $\gamma = 1.30$ if, as a result of a certain process, the gas volume increased $\alpha = 2.0$ times while the pressure dropped $\beta = 3.0$ times.

2.135. Vessels *1* and *2* contain $\nu = 1.2$ moles of gaseous helium. The ratio of the vessels' volumes $V_2/V_1 = \alpha = 2.0$, and the ratio of the absolute temperatures of helium in them $T_1/T_2 = \beta = 1.5$. Assuming the gas to be ideal, find the difference of gas entropies in these vessels, $S_2 - S_1$.

2.136. One mole of an ideal gas with the adiabatic exponent γ goes through a polytropic process as a result of which the absolute temperature of the gas increases τ-fold. The polytropic constant equals n. Find the entropy increment of the gas in this process.

2.137. The expansion process of $\nu = 2.0$ moles of argon proceeds so that the gas pressure increases in direct proportion to its volume. Find the entropy increment of the gas in this process provided its volume increases $\alpha = 2.0$ times.

2.138. An ideal gas with the adiabatic exponent γ goes through a process $p = p_0 - \alpha V$, where p_0 and α are positive constants, and V is the volume. At what volume will the gas entropy have the maximum value?

2.139. One mole of an ideal gas goes through a process in which the entropy of the gas changes with temperature T as $S = aT + C_V \ln T$, where a is a positive constant, C_V is the molar heat capacity of this gas at constant volume. Find the volume dependence of the gas temperature in this process if $T = T_0$ at $V = V_0$.

2.140. Find the entropy increment of one mole of a Van der Waals gas due to the isothermal variation of volume from V_1 to V_2. The Van der Waals corrections are assumed to be known.

2.141. One mole of a Van der Waals gas which had initially the volume V_1 and the temperature T_1 was transferred to the state with the volume V_2 and the temperature T_2. Find the corresponding entropy increment of the gas, assuming its molar heat capacity C_V to be known.

2.142. At very low temperatures the heat capacity of crystals is equal to $C = aT^3$, where a is a constant. Find the entropy of a crystal as a function of temperature in this temperature interval.

2.143. Find the entropy increment of an aluminum bar of mass $m = 3.0$ kg on its heating from the temperature $T_1 = 300$ K up to $T_2 = 600$ K if in this temperature interval the specific heat capacity of aluminum varies as $c = a + bT$, where $a = 0.77$ J/(g·K), $b = 0.46$ mJ/(g·K²).

2.144. In some process the temperature of a substance depends on its entropy S as $T = aS^n$, where a and n are constants. Find the corresponding heat capacity C of the substance as a function of S. At what condition is $C < 0$?

2.145. Find the temperature T as a function of the entropy S of a substance for a polytropic process in which the heat capacity of the substance equals C. The entropy of the substance is known to be equal to S_0 at the temperature T_0. Draw the approximate plots $T(S)$ for $C > 0$ and $C < 0$.

2.146. One mole of an ideal gas with heat capacity C_V goes through a process in which its entropy S depends on T as $S = \alpha/T$, where α is a constant. The gas temperature varies from T_1 to T_2. Find:

(a) the molar heat capacity of the gas as a function of its temperature;

(b) the amount of heat transferred to the gas;

(c) the work performed by the gas.

2.147. A working substance goes through a cycle within which the absolute temperature varies n-fold, and the shape of the cycle is shown in (a) Fig. 2.4*a*; (b) Fig. 2.4*b*, where T is the absolute temperature, and S the entropy. Find the efficiency of each cycle.

2.148. One of the two thermally insulated vessels interconnected by a tube with a valve contains $\nu = 2.2$ moles of an ideal gas. The other vessel is evacuated. The valve having been opened, the gas increased its volume $n = 3.0$ times. Find the entropy increment of the gas.

2.149. A weightless piston divides a thermally insulated cylinder into two equal parts. One part contains one mole of an ideal gas with adiabatic exponent γ, the other is evacuated. The initial gas temperature is T_0. The piston is released and the gas fills the whole

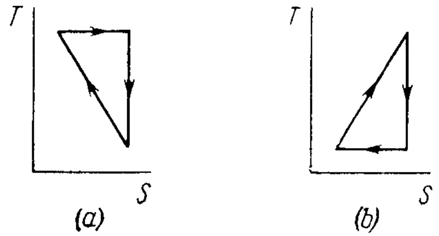


Fig. 2.4.

volume of the cylinder. Then the piston is slowly displaced back to the initial position. Find the increment of the internal energy and the entropy of the gas resulting from these two processes.

2.150. An ideal gas was expanded from the initial state to the volume V without any heat exchange with the surrounding bodies. Will the final gas pressure be the same in the case of (a) a fast and in the case of (b) a very slow expansion process?

2.151. A thermally insulated vessel is partitioned into two parts so that the volume of one part is $n = 2.0$ times greater than that of the other. The smaller part contains $\nu_1 = 0.30$ mole of nitrogen, and the greater one $\nu_2 = 0.70$ mole of oxygen. The temperature of the gases is the same. A hole is punctured in the partition and the gases are mixed. Find the corresponding increment of the system's entropy, assuming the gases to be ideal.

2.152. A piece of copper of mass $m_1 = 300$ g with initial temperature $t_1 = 97$ °C is placed into a calorimeter in which the water of mass $m_2 = 100$ g is at a temperature $t_2 = 7$ °C. Find the entropy increment of the system by the moment the temperatures equalize. The heat capacity of the calorimeter itself is negligibly small.

2.153. Two identical thermally insulated vessels interconnected by a tube with a valve contain one mole of the same ideal gas each. The gas temperature in one vessel is equal to T_1 and in the other, T_2. The molar heat capacity of the gas of constant volume equals C_V. The valve having been opened, the gas comes to a new equilibrium state. Find the entropy increment ΔS of the gas. Demonstrate that $\Delta S > 0$.

2.154. N atoms of gaseous helium are enclosed in a cubic vessel of volume 1.0 cm^3 at room temperature. Find:

(a) the probability of atoms gathering in one half of the vessel;

(b) the approximate numerical value of N ensuring the occurrence of this event within the time interval $t \approx 10^{10}$ years (the age of the Universe).

2.155. Find the statistical weight of the most probable distribution of $N = 10$ identical molecules over two halves of the cylinder's volume. Find also the probability of such a distribution.

2.156. A vessel contains N molecules of an ideal gas. Dividing mentally the vessel into two halves A and B, find the probability that the half A contains n molecules. Consider the cases when $N = 5$ and $n = 0, 1, 2, 3, 4, 5$.

2.157. A vessel of volume V_0 contains N molecules of an ideal gas. Find the probability of n molecules getting into a certain separated part of the vessel of volume V. Examine, in particular, the case $V = V_0/2$.

2.158. An ideal gas is under standard conditions. Find the diameter of the sphere within whose volume the relative fluctuation of the number of molecules is equal to $\eta = 1.0 \cdot 10^{-3}$. What is the average number of molecules inside such a sphere?

2.159. One mole of an ideal gas consisting of monatomic molecules is enclosed in a vessel at a temperature $T_0 = 300$ K. How many times and in what way will the statistical weight of this system (gas) vary if it is heated isochorically by $\Delta T = 1.0$ K?

2.5. LIQUIDS. CAPILLARY EFFECTS

• Additional (capillary) pressure in a liquid under an arbitrary surface (Laplace's formula):

$$\Delta p = \alpha \left(\frac{1}{R_1} + \frac{1}{R_2} \right), \qquad (2.5a)$$

where α is the surface tension of a given liquid.

• Free energy increment of the surface layer of a liquid:

$$dF = \alpha\, dS, \qquad (2.5b)$$

where dS is the area increment of the surface layer.

• Amount of heat required to form a unit area of the liquid surface layer during the isothermal increase of its surface:

$$q = -T \frac{d\alpha}{dT}. \qquad (2.5c)$$

2.160. Find the capillary pressure

(a) in mercury droplets of diameter $d = 1.5$ μm;

(b) inside a soap bubble of diameter $d = 3.0$ mm if the surface tension of the soap water solution is $\alpha = 45$ mN/m.

2.161. In the bottom of a vessel with mercury there is a round hole of diameter $d = 70$ μm. At what maximum thickness of the mercury layer will the liquid still not flow out through this hole?

2.162. A vessel filled with air under pressure p_0 contains a soap bubble of diameter d. The air pressure having been reduced isothermally n-fold, the bubble diameter increased η-fold. Find the surface tension of the soap water solution.

2.163. Find the pressure in an air bubble of diameter $d = 4.0$ μm, located in water at a depth $h = 5.0$ m. The atmospheric pressure has the standard value p_0.

2.164. The diameter of a gas bubble formed at the bottom of a pond is $d = 4.0$ μm. When the bubble rises to the surface its diameter increases $n = 1.1$ times. Find how deep is the pond at that spot. The atmospheric pressure is standard, the gas expansion is assumed to be isothermal.

2.165. Find the difference in height of mercury columns in two communicating vertical capillaries whose diameters are $d_1 = 0.50$ mm and $d_2 = 1.00$ mm, if the contact angle $\theta = 138°$.

2.166. A vertical capillary with inside diameter 0.50 mm is submerged into water so that the length of its part protruding over the water surface is equal to $h = 25$ mm. Find the curvature radius of the meniscus.

2.167. A glass capillary of length $l = 110$ mm and inside diameter $d = 20$ μm is submerged vertically into water. The upper end of the capillary is sealed. The outside pressure is standard. To what length x has the capillary to be submorged to make the water levels inside and outside the capillary coincide?

2.168. When a vertical capillary of length l with the sealed upper end was brought in contact with the surface of a liquid, the level of this liquid rose to the height h. The liquid density is ρ, the inside diameter of the capillary is d, the contact angle is θ, the atmospheric pressure is p_0. Find the surface tension of the liquid.

2.169. A glass rod of diameter $d_1 = 1.5$ mm is inserted symmetrically into a glass capillary with inside diameter $d_2 = 2.0$ mm. Then the whole arrangement is vertically oriented and brought in contact with the surface of water. To what height will the water rise in the capillary?

2.170. Two vertical plates submerged partially in a wetting liquid form a wedge with a very small angle $\delta\varphi$. The edge of this wedge is vertical. The density of the liquid is ρ, its surface tension is α, the contact angle is θ. Find the height h, to which the liquid rises, as a function of the distance x from the edge.

2.171. A vertical water jet flows out of a round hole. One of the horizontal sections of the jet has the diameter $d = 2.0$ mm while the other section located $l = 20$ mm lower has the diameter which is $n = 1.5$ times less. Find the volume of the water flowing from the hole each second.

2.172. A water drop falls in air with a uniform velocity. Find the difference between the curvature radii of the drop's surface at the upper and lower points of the drop separated by the distance $h = 2.3$ mm.

2.173. A mercury drop shaped as a round tablet of radius R and thickness h is located between two horizontal glass plates. Assuming that $h \ll R$, find the mass m of a weight which has to be placed on the upper plate to diminish the distance between the plates n-times. The contact angle equals θ. Calculate m if $R = 2.0$ cm, $h = 0.38$ mm, $n = 2.0$, and $\theta = 135°$.

2.174. Find the attraction force between two parallel glass plates, separated by a distance $h = 0.10$ mm, after a water drop of mass $m = 70$ mg was introduced between them. The wetting is assumed to be complete.

2.175. Two glass discs of radius $R = 5.0$ cm were wetted with water and put together so that the thickness of the water layer between them was $h = 1.9$ μm. Assuming the wetting to be complete, find the force that has to be applied at right angles to the plates in order to pull them apart.

2.176. Two vertical parallel glass plates are partially submerged in water. The distance between the plates is $d = 0.10$ mm, and their width is $l = 12$ cm. Assuming that the water between the plates does not reach the upper edges of the plates and that the wetting is complete, find the force of their mutual attraction.

2.177. Find the lifetime of a soap bubble of radius R connected with the atmosphere through a capillary of length l and inside radius r. The surface tension is α, the viscosity coefficient of the gas is η.

2.178. A vertical capillary is brought in contact with the water surface. What amount of heat is liberated while the water rises along the capillary? The wetting is assumed to be complete, the surface tension equals α.

2.179. Find the free energy of the surface layer of

(a) a mercury droplet of diameter $d = 1.4$ mm;

(b) a soap bubble of diameter $d = 6.0$ mm if the surface tension of the soap water solution is equal to $\alpha = 45$ mN/m.

2.180. Find the increment of the free energy of the surface layer when two identical mercury droplets, each of diameter $d = 1.5$ mm, merge isothermally.

2.181. Find the work to be performed in order to blow a soap bubble of radius R if the outside air pressure is equal to p_0 and the surface tension of the soap water solution is equal to α.

2.182. A soap bubble of radius r is inflated with an ideal gas. The atmospheric pressure is p_0, the surface tension of the soap water solution is α. Find the difference between the molar heat capacity of the gas during its heating inside the bubble and the molar heat capacity of the gas under constant pressure, $C - C_p$.

2.183. Considering the Carnot cycle as applied to a liquid film, show that in an isothermal process the amount of heat required for the formation of a unit area of the surface layer is equal to $q = -T \cdot d\alpha/dT$, where $d\alpha/dT$ is the temperature derivative of the surface tension.

2.184. The surface of a soap film was increased isothermally by $\Delta\sigma$ at a temperature T. Knowing the surface tension of the soap water solution α and the temperature coefficient $d\alpha/dT$, find the increment

(a) of the entropy of the film's surface layer;

(b) of the internal energy of the surface layer.

2.6. PHASE TRANSFORMATIONS

• Relations between Van der Waals constants and the parameters of the critical state of a substance:

$$V_{M\,cr} = 3b, \quad p_{cr} = \frac{a}{27b^2}, \quad T_{cr} = \frac{8a}{27Rb}. \tag{2.6a}$$

• Relation between the critical parameters for a mole of substance:

$$p_{cr} V_{M\,cr} = (3/8)\, R T_{cr}. \tag{2.6b}$$

• Clausius-Clapeyron equation:

$$\frac{dp}{dT} = \frac{q_{12}}{T\,(V_2' - V_1')}, \tag{2.6c}$$

where q_{12} is the specific heat absorbed in the transformation $1 \to 2$, V_1' and V_2' are the specific volumes of phases 1 and 2.

2.185. A saturated water vapour is contained in a cylindrical vessel under a weightless piston at a temperature $t = 100$ °C. As a result of a slow introduction of the piston a small fraction of the vapour $\Delta m = 0.70$ g gets condensed. What amount of work was performed over the gas? The vapour is assumed to be ideal, the volume of the liquid is to be neglected.

2.186. A vessel of volume $V = 6.0$ l contains water together with its saturated vapour under a pressure of 40 atm and at a temperature of 250 °C. The specific volume of the vapour is equal to $V_v' = 50$ l/kg under these conditions. The total mass of the system water-vapour equals $m = 5.0$ kg. Find the mass and the volume of the vapour.

2.187. The saturated water vapour is enclosed in a cylinder under a piston and occupies a volume $V_0 = 5.0$ l at the temperature $t = 100$ °C. Find the mass of the liquid phase formed after the volume under the piston decreased isothermally to $V = 1.6$ l. The saturated vapour is assumed to be ideal.

2.188. A volume occupied by a saturated vapour is reduced isothermally n-fold. Find what fraction η of the final volume is occupied by the liquid phase if the specific volumes of the saturated vapour and the liquid phase differ by N times ($N > n$). Solve the same problem under the condition that the final volume of the substance corresponds to the midpoint of a horizontal portion of the isothermal line in the diagram p, V.

2.189. An amount of water of mass $m = 1.00$ kg, boiling at standard atmospheric pressure, turns completely into saturated vapour. Assuming the saturated vapour to be an ideal gas find the increment of entropy and internal energy of the system.

2.190. Water of mass $m = 20$ g is enclosed in a thermally insulated cylinder at the temperature of 0 °C under a weightless piston whose area is $S = 410$ cm². The outside pressure is equal to standard atmospheric pressure. To what height will the piston rise when the water absorbs $Q = 20.0$ kJ of heat?

2.191. One gram of saturated water vapour is enclosed in a thermally insulated cylinder under a weightless piston. The outside pressure being standard, $m = 1.0$ g of water is introduced into the cylinder at a temperature $t_0 = 22$ °C. Neglecting the heat capacity of the cylinder and the friction of the piston against the cylinder's walls, find the work performed by the force of the atmospheric pressure during the lowering of the piston.

2.192. If an additional pressure Δp of a saturated vapour over a convex spherical surface of a liquid is considerably less than the vapour pressure over a plane surface, then $\Delta p = (\rho_v/\rho_l)\, 2\alpha/r$, where ρ_v and ρ_l are the densities of the vapour and the liquid, α is the surface tension, and r is the radius of curvature of the surface. Using this formula, find the diameter of water droplets at which the saturated vapour pressure exceeds the vapour pressure over the plane surface by $\eta = 1.0\%$ at a temperature $t = 27$ °C. The vapour is assumed to be an ideal gas.

2.193. Find the mass of all molecules leaving one square centimetre of water surface per second into a saturated water vapour above it at a temperature $t = 100$ °C. It is assumed that $\eta = 3.6\%$ of all water vapour molecules falling on the water surface are retained in the liquid phase.

2.194. Find the pressure of saturated tungsten vapour at a temperature $T = 2000$ K if a tungsten filament is known to lose a mass $\mu = 1.2 \cdot 10^{-13}$ g/(s·cm²) from a unit area per unit time when evaporating into high vacuum at this temperature.

2.195. By what magnitude would the pressure exerted by water on the walls of the vessel have increased if the intermolecular attraction forces had vanished?

2.196. Find the internal pressure p_i of a liquid if its density ρ and specific latent heat of vaporization q are known. The heat q is assumed to be equal to the work performed against the forces of the internal pressure, and the liquid obeys the Van der Waals equation. Calculate p_i in water.

2.197. Demonstrate that Eqs. (2.6a) and (2.6b) are valid for a substance, obeying the Van der Waals equation, in critical state.

Instruction. Make use of the fact that the critical state corresponds to the point of inflection in the isothermal curve $p(V)$.

2.198. Calculate the Van der Waals constants for carbon dioxide if its critical temperature $T_{cr} = 304$ K and critical pressure $p_{cr} = 73$ atm.

2.199. Find the specific volume of benzene (C_6H_6) in critical state if its critical temperature $T_{cr} = 562$ K and critical pressure $p_{cr} = 47$ atm.

2.200. Write the Van der Waals equation via the reduced parameters π, ν, and τ, having taken the corresponding critical values for the units of pressure, volume, and temperature. Using the equation obtained, find how many times the gas temperature exceeds its critical temperature if the gas pressure is 12 times as high as critical pressure, and the volume of gas is equal to half the critical volume.

2.201. Knowing the Van der Waals constants, find:

(a) the maximum volume which water of mass $m = 1.00$ kg can occupy in liquid state;

(b) the maximum pressure of the saturated water vapour.

2.202. Calculate the temperature and density of carbon dioxide in critical state, assuming the gas to be a Van der Waals one.

2.203. What fraction of the volume of a vessel must liquid ether occupy at room temperature in order to pass into critical state when critical temperature is reached? Ether has $T_{cr} = 467$ K, $p_{cr} = 35.5$ atm, $M = 74$ g/mol.

2.204. Demonstrate that the straight line *1-5* corresponding to the isothermal-isobaric phase transition cuts the Van der Waals isotherm so that areas *I* and *II* are equal (Fig. 2.5).

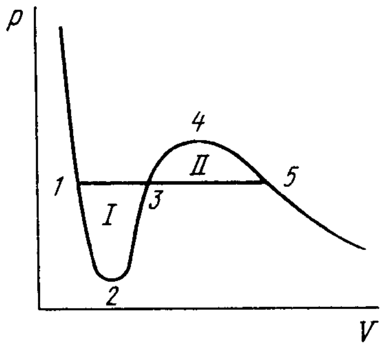


Fig. 2.5.

2.205. What fraction of water supercooled down to the temperature $t = -20$ °C under standard pressure turns into ice when the system passes into the equilibrium state? At what temperature of the supercooled water does it turn into ice completely?

2.206. Find the increment of the ice melting temperature in the vicinity of 0 °C when the pressure is increased by $\Delta p = 1.00$ atm. The specific volume of ice exceeds that of water by $\Delta V' = 0.091$ cm^3/g.

2.207. Find the specific volume of saturated water vapour under standard pressure if a decrease of pressure by $\Delta p \doteq 3.2$ kPa is known to decrease the water boiling temperature by $\Delta T = 0.9$ K.

2.208. Assuming the saturated water vapour to be ideal, find its pressure at the temperature 101.1 °C.

2.209. A small amount of water and its saturated vapour are enclosed in a vessel at a temperature $t = 100$ °C. How much (in per cent) will the mass of the saturated vapour increase if the temperature of the system goes up by $\Delta T = 1.5$ K? Assume that the vapour is an ideal gas and the specific volume of water is negligible as compared to that of vapour.

2.210. Find the pressure of saturated vapour as a function of temperature $p(T)$ if at a temperature T_0 its pressure equals p_0.

Assume that: the specific latent heat of vaporization q is independent of T, the specific volume of liquid is negligible as compared to that of vapour, saturated vapour obeys the equation of state for an ideal gas. Investigate under what conditions these assumptions are permissible.

2.211. An ice which was initially under standard conditions was compressed up to the pressure $p = 640$ atm. Assuming the lowering of the ice melting temperature to be a linear function of pressure under the given conditions, find what fraction of the ice melted. The specific volume of water is less than that of ice by $\Delta V' = 0.09$ cm^3/g.

2.212. In the vicinity of the triple point the saturated vapour pressure p of carbon dioxide depends on temperature T as $\log p = a - b/T$, where a and b are constants. If p is expressed in atmospheres, then for the sublimation process $a = 9.05$ and $b = 1.80$ kK, and for the vaporization process $a = 6.78$ and $b = 1.31$ kK. Find:

(a) temperature and pressure at the triple point;

(b) the values of the specific latent heats of sublimation, vaporization, and melting in the vicinity of the triple point.

2.213. Water of mass $m = 1.00$ kg is heated from the temperature $t_1 = 10$ °C up to $t_2 = 100$ °C at which it evaporates completely. Find the entropy increment of the system.

2.214. The ice with the initial temperature $t_1 = 0$ °C was first melted, then heated to the temperature $t_2 = 100$ °C and evaporated. Find the increment of the system's specific entropy.

2.215. A piece of copper of mass $m = 90$ g at a temperature $t_1 = 90$ °C was placed in a calorimeter in which ice of mass 50 g was at a temperature -3 °C. Find the entropy increment of the piece of copper by the moment the thermal equilibrium is reached.

2.216. A chunk of ice of mass $m_1 = 100$ g at a temperature $t_1 = 0$ °C was placed in a calorimeter in which water of mass $m_2 = 100$ g was at a temperature t_2. Assuming the heat capacity of the calorimeter to be negligible, find the entropy increment of the system by the moment the thermal equilibrium is reached. Consider two cases: (a) $t_2 = 60$ °C; (b) $t_2 = 94$ °C.

2.217. Molten lead of mass $m = 5.0$ g at a temperature $t_2 = 327$ °C (the melting temperature of lead) was poured into a calorimeter packed with a large amount of ice at a temperature $t_1 = 0$ °C. Find the entropy increment of the system lead-ice by the moment the thermal equilibrium is reached. The specific latent heat of melting of lead is equal to $q = 22.5$ J/g and its specific heat capacity is equal to $c = 0.125$ J/(g·K).

2.218. A water vapour filling the space under the piston of a cylinder, is compressed (or expanded) so that it remains saturated all the time, being just on the verge of condensation. Find the molar heat capacity C of the vapour in this process as a function of temperature T, assuming the vapour to be an ideal gas and neglecting the specific volume of water in comparison with that of vapour. Calculate C at a temperature $t = 100$ °C.

2.219. One mole of water being in equilibrium with a negligible amount of its saturated vapour at a temperature T_1 was completely converted into saturated vapour at a temperature T_2. Find the entropy increment of the system. The vapour is assumed to be an ideal gas, the specific volume of the liquid is negligible in comparison with that of the vapour.

2.7. TRANSPORT PHENOMENA

• Relative number of gas molecules traversing the distance s without collisions:

$$N/N_0 = e^{-s/\lambda}, \tag{2.7a}$$

where λ is the mean free path.

• Mean free path of a gas molecule:

$$\lambda = \frac{1}{\sqrt{2}\pi d^2 n}, \tag{2.7b}$$

where d is the effective diameter of a molecule, and n is the number of molecules per unit volume.

• Coefficients of diffusion D, viscosity η, and heat conductivity $\varkappa$ of gases:

$$D = \frac{1}{3}\langle v\rangle\lambda, \quad \eta = \frac{1}{3}\langle v\rangle\lambda\rho, \quad \varkappa = \frac{1}{3}\langle v\rangle\lambda\rho c_V, \tag{2.7c}$$

where ρ is the gas density, and c_V is its specific heat capacity at constant volume.

• Friction force acting on a unit area of plates during their motion parallel to each other in a highly rarefied gas:

$$F = \frac{1}{6}\langle v\rangle\rho\,|u_1 - u_2|, \tag{2.7d}$$

where u_1 and u_2 are the velocities of the plates.

• Density of a thermal flux transferred between two walls by highly rarefied gas:

$$q = \frac{1}{6}\langle v\rangle\rho c_V\,|T_1 - T_2|, \tag{2.7e}$$

where T_1 and T_2 are the temperatures of the walls.

2.220. Calculate what fraction of gas molecules
(a) traverses without collisions the distances exceeding the mean free path λ;
(b) has the free path values lying within the interval from λ to 2λ.

2.221. A narrow molecular beam makes its way into a vessel filled with gas under low pressure. Find the mean free path of molecules if the beam intensity decreases η-fold over the distance Δl.

2.222. Let αdt be the probability of a gas molecule experiencing a collision during the time interval dt; α is a constant. Find:
(a) the probability of a molecule experiencing no collisions during the time interval t;
(b) the mean time interval between successive collisions.

2.223. Find the mean free path and the mean time interval between successive collisions of gaseous nitrogen molecules
(a) under standard conditions;

(b) at temperature $t = 0$ °C and pressure $p = 1.0$ nPa (such a pressure can be reached by means of contemporary vacuum pumps).

2.224. How many times does the mean free path of nitrogen molecules exceed the mean distance between the molecules under standard conditions?

2.225. Find the mean free path of gas molecules under standard conditions if the Van der Waals constant of this gas is equal to $b = 40$ ml/mol.

2.226. An acoustic wave propagates through nitrogen under standard conditions. At what frequency will the wavelength be equal to the mean free path of the gas molecules?

2.227. Oxygen is enclosed at the temperature 0 °C in a vessel with the characteristic dimension $l = 10$ mm (this is the linear dimension determining the character of a physical process in question). Find:
(a) the gas pressure below which the mean free path of the molecules $\lambda > l$;
(b) the corresponding molecular concentration and the mean distance between the molecules.

2.228. For the case of nitrogen under standard conditions find:
(a) the mean number of collisions experienced by each molecule per second;
(b) the total number of collisions occurring between the molecules within 1 cm³ of nitrogen per second.

2.229. How does the mean free path λ and the number of collisions of each molecule per unit time ν depend on the absolute temperature of an ideal gas undergoing
(a) an isochoric process;
(b) an isobaric process?

2.230. As a result of some process the pressure of an ideal gas increases n-fold. How many times have the mean free path λ and the number of collisions of each molecule per unit time ν changed and how, if the process is
(a) isochoric; (b) isothermal?

2.231. An ideal gas consisting of rigid diatomic molecules goes through an adiabatic process. How do the mean free path λ and the number of collisions of each molecule per second ν depend in this process on
(a) the volume V; (b) the pressure p; (c) the temperature T?

2.232. An ideal gas goes through a polytropic process with exponent n. Find the mean free path λ and the number of collisions of each molecule per second ν as a function of
(a) the volume V; (b) the pressure p; (c) the temperature T.

2.233. Determine the molar heat capacity of a polytropic process through which an ideal gas consisting of rigid diatomic molecules goes and in which the number of collisions between the molecules remains constant
(a) in a unit volume; (b) in the total volume of the gas.

2.234. An ideal gas of molar mass M is enclosed in a vessel of volume V whose thin walls are kept at a constant temperature T. At a moment $t = 0$ a small hole of area S is opened, and the gas starts escaping into vacuum. Find the gas concentration n as a function of time t if at the initial moment $n(0) = n_0$.

2.235. A vessel filled with gas is divided into two equal parts *1* and *2* by a thin heat-insulating partition with two holes. One hole has a small diameter, and the other has a very large diameter (in comparison with the mean free path of molecules). In part *2* the gas is kept at a temperature η times higher than that of part *1*. How will the concentration of molecules in part *2* change and how many times after the large hole is closed?

2.236. As a result of a certain process the viscosity coefficient of an ideal gas increases $\alpha = 2.0$ times and its diffusion coefficient $\beta = 4.0$ times. How does the gas pressure change and how many times?

2.237. How will a diffusion coefficient D and the viscosity coefficient η of an ideal gas change if its volume increases n times:
(a) isothermally; (b) isobarically?

2.238. An ideal gas consists of rigid diatomic molecules. How will a diffusion coefficient D and viscosity coefficient η change and how many times if the gas volume is decreased adiabatically $n = 10$ times?

2.239. An ideal gas goes through a polytropic process. Find the polytropic exponent n if in this process the coefficient
(a) of diffusion; (b) of viscosity; (c) of heat conductivity remains constant.

2.240. Knowing the viscosity coefficient of helium under standard conditions, calculate the effective diameter of the helium atom.

2.241. The heat conductivity of helium is 8.7 times that of argon (under standard conditions). Find the ratio of effective diameters of argon and helium atoms.

2.242. Under standard conditions helium fills up the space between two long coaxial cylinders. The mean radius of the cylinders is equal to R, the gap between them is equal to ΔR, with $\Delta R \ll R$. The outer cylinder rotates with a fairly low angular velocity ω about the stationary inner cylinder. Find the moment of friction forces acting on a unit length of the inner cylinder. Down to what magnitude should the helium pressure be lowered (keeping the temperature constant) to decrease the sought moment of friction forces $n = 10$ times if $\Delta R = 6$ mm?

2.243. A gas fills up the space between two long coaxial cylinders of radii R_1 and R_2, with $R_1 < R_2$. The outer cylinder rotates with a fairly low angular velocity ω about the stationary inner cylinder. The moment of friction forces acting on a unit length of the inner cylinder is equal to N_1. Find the viscosity coefficient η of the gas taking into account that the friction force acting on a unit area of the cylindrical surface of radius r is determined by the formula $\sigma = \eta r (\partial\omega/\partial r)$.

2.244. Two identical parallel discs have a common axis and are located at a distance h from each other. The radius of each disc is equal to a, with $a \gg h$. One disc is rotated with a low angular velocity ω relative to the other, stationary, disc. Find the moment of friction forces acting on the stationary disc if the viscosity coefficient of the gas between the discs is equal to η.

2.245. Solve the foregoing problem, assuming that the discs are located in an ultra-rarefied gas of molar mass M, at temperature T and under pressure p.

2.246. Making use of Poiseuille's equation (1.7d), find the mass μ of gas flowing per unit time through the pipe of length l and radius a if constant pressures p_1 and p_2 are maintained at its ends.

2.247. One end of a rod, enclosed in a thermally insulating sheath, is kept at a temperature T_1 while the other, at T_2. The rod is composed of two sections whose lengths are l_1 and l_2 and heat conductivity coefficients $\varkappa_1$ and $\varkappa_2$. Find the temperature of the interface.

2.248. Two rods whose lengths are l_1 and l_2 and heat conductivity coefficients $\varkappa_1$ and $\varkappa_2$ are placed end to end. Find the heat conductivity coefficient of a uniform rod of length $l_1 + l_2$ whose conductivity is the same as that of the system of these two rods. The lateral surfaces of the rods are assumed to be thermally insulated.

2.249. A rod of length l with thermally insulated lateral surface consists of material whose heat conductivity coefficient varies with temperature as $\varkappa = \alpha/T$, where α is a constant. The ends of the rod are kept at temperatures T_1 and T_2. Find the function $T(x)$, where x is the distance from the end whose temperature is T_1, and the heat flow density.

2.250. Two chunks of metal with heat capacities C_1 and C_2 are interconnected by a rod of length l and cross-sectional area S and fairly low heat conductivity $\varkappa$. The whole system is thermally insulated from the environment. At a moment $t = 0$ the temperature difference between the two chunks of metal equals $(\Delta T)_0$. Assuming the heat capacity of the rod to be negligible, find the temperature difference between the chunks as a function of time.

2.251. Find the temperature distribution in a substance placed between two parallel plates kept at temperatures T_1 and T_2. The plate separation is equal to l, the heat conductivity coefficient of the substance $\varkappa \propto \sqrt{T}$.

2.252. The space between two large horizontal plates is filled with helium. The plate separation equals $l = 50$ mm. The lower plate is kept at a temperature $T_1 = 290$ K, the upper, at $T_2 = 330$ K. Find the heat flow density if the gas pressure is close to standard.

2.253. The space between two large parallel plates separated by a distance $l = 5.0$ mm is filled with helium under a pressure $p = 1.0$ Pa. One plate is kept at a temperature $t_1 = 17$ °C and the other, at a temperature $t_2 = 37$ °C. Find the mean free path of helium atoms and the heat flow density.

2.254. Find the temperature distribution in the space between two coaxial cylinders of radii R_1 and R_2 filled with a uniform heat conducting substance if the temperatures of the cylinders are constant and are equal to T_1 and T_2 respectively.

2.255. Solve the foregoing problem for the case of two concentric spheres of radii R_1 and R_2 and temperatures T_1 and T_2.

2.256. A constant electric current flows along a uniform wire with cross-sectional radius R and heat conductivity coefficient $\varkappa$. A unit volume of the wire generates a thermal power w. Find the temperature distribution across the wire provided the steady-state temperature at the wire surface is equal to T_0.

2.257. The thermal power of density w is generated uniformly inside a uniform sphere of radius R and heat conductivity coefficient $\varkappa$. Find the temperature distribution in the sphere provided the steady-state temperature at its surface is equal to T_0.

PART THREE

ELECTRODYNAMICS

3.1. CONSTANT ELECTRIC FIELD IN VACUUM

• Strength and potential of the field of a point charge q:

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\mathbf{r}, \qquad \varphi=\frac{1}{4\pi\varepsilon_0}\frac{q}{r}. \tag{3.1a}$$

• Relation between field strength and potential:

$$\mathbf{E}=-\nabla\varphi, \tag{3.1b}$$

i.e. field strength is equal to the antigradient of the potential.

• Gauss's theorem and circulation of the vector $\mathbf{E}$:

$$\oint \mathbf{E}\,d\mathbf{S}=q/\varepsilon_0, \qquad \oint \mathbf{E}\,d\mathbf{r}=0. \tag{3.1c}$$

• Potential and strength of the field of a point dipole with electric moment $\mathbf{p}$:

$$\varphi=\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{pr}}{r^3}, \qquad E=\frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{1+3\cos^2\theta}, \tag{3.1d}$$

where θ is the angle between the vectors $\mathbf{r}$ and $\mathbf{p}$.

• Energy W of the dipole $\mathbf{p}$ in an external electric field, and the moment $\mathbf{N}$ of forces acting on the dipole:

$$W=-\mathbf{pE}, \quad \mathbf{N}=[\mathbf{pE}]. \tag{3.1e}$$

• Force $\mathbf{F}$ acting on a dipole, and its projection F_x:

$$\mathbf{F}=p\frac{\partial\mathbf{E}}{\partial l}, \qquad F_x=\mathbf{p}\cdot\nabla E_x, \tag{3.1f}$$

where $\partial\mathbf{E}/\partial l$ is the derivative of the vector $\mathbf{E}$ with respect to the dipole direction, ∇E_x is the gradient of the function E_x.

3.1. Calculate the ratio of the electrostatic to gravitational interaction forces between two electrons, between two protons. At what value of the specific charge q/m of a particle would these forces become equal (in their absolute values) in the case of interaction of identical particles?

3.2. What would be the interaction force between two copper spheres, each of mass 1 g, separated by the distance 1 m, if the total electronic charge in them differed from the total charge of the nuclei by one per cent?

3.3. Two small equally charged spheres, each of mass m, are suspended from the same point by silk threads of length l. The distance between the spheres $x \ll l$. Find the rate dq/dt with which

the charge leaks off each sphere if their approach velocity varies as $v = a/\sqrt{x}$, where a is a constant.

3.4. Two positive charges q_1 and q_2 are located at the points with radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Find a negative charge q_3 and a radius vector $\mathbf{r}_3$ of the point at which it has to be placed for the force acting on each of the three charges to be equal to zero.

3.5. A thin wire ring of radius r has an electric charge q. What will be the increment of the force stretching the wire if a point charge q_0 is placed at the ring's centre?

3.6. A positive point charge 50 μC is located in the plane xy at the point with radius vector $\mathbf{r}_0 = 2\mathbf{i} + 3\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes. Find the vector of the electric field strength $\mathbf{E}$ and its magnitude at the point with radius vector $\mathbf{r} = 8\mathbf{i} - 5\mathbf{j}$. Here r_0 and r are expressed in metres.

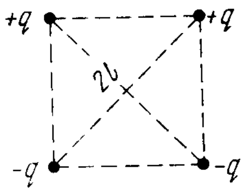


Fig. 3.1.

3.7. Point charges q and $-q$ are located at the vertices of a square with diagonals $2l$ as shown in Fig. 3.1. Find the magnitude of the electric field strength at a point located symmetrically with respect to the vertices of the square at a distance x from its centre.

3.8. A thin half-ring of radius $R = 20$ cm is uniformly charged with a total charge $q = 0.70$ nC. Find the magnitude of the electric field strength at the curvature centre of this half-ring.

3.9. A thin wire ring of radius r carries a charge q. Find the magnitude of the electric field strength on the axis of the ring as a function of distance l from its centre. Investigate the obtained function at $l \gg r$. Find the maximum strength magnitude and the corresponding distance l. Draw the approximate plot of the function $E(l)$.

3.10. A point charge q is located at the centre of a thin ring of radius R with uniformly distributed charge $-q$. Find the magnitude of the electric field strength vector at the point lying on the axis of the ring at a distance x from its centre, if $x \gg R$.

3.11. A system consists of a thin charged wire ring of radius R and a very long uniformly charged thread oriented along the axis of the ring, with one of its ends coinciding with the centre of the ring. The total charge of the ring is equal to q. The charge of the thread (per unit length) is equal to λ. Find the interaction force between the ring and the thread.

3.12. A thin nonconducting ring of radius R has a linear charge density $\lambda = \lambda_0 \cos\varphi$, where λ_0 is a constant, φ is the azimuthal angle. Find the magnitude of the electric field strength

(a) at the centre of the ring;

(b) on the axis of the ring as a function of the distance x from its centre. Investigate the obtained function at $x \gg R$.

3.13. A thin straight rod of length $2a$ carrying a uniformly distributed charge q is located in vacuum. Find the magnitude of the electric field strength as a function of the distance r from the rod's centre along the straight line

(a) perpendicular to the rod and passing through its centre;

(b) coinciding with the rod's direction (at the points lying outside the rod).

Investigate the obtained expressions at $r \gg a$.

3.14. A very long straight uniformly charged thread carries a charge λ per unit length. Find the magnitude and direction of the electric field strength at a point which is at a distance y from the thread and lies on the perpendicular passing through one of the thread's ends.

3.15. A thread carrying a uniform charge λ per unit length has the configurations shown in Fig. 3.2 a and b. Assuming a curvature

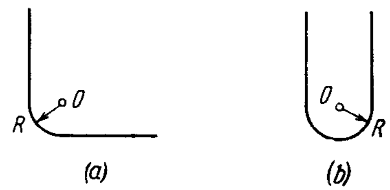


Fig. 3.2.

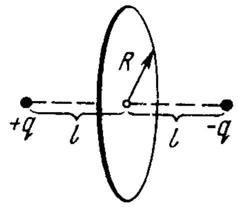


Fig. 3.3.

radius R to be considerably less than the length of the thread, find the magnitude of the electric field strength at the point O.

3.16. A sphere of radius r carries a surface charge of density $\sigma = \mathbf{ar}$, where $\mathbf{a}$ is a constant vector, and $\mathbf{r}$ is the radius vector of a point of the sphere relative to its centre. Find the electric field strength vector at the centre of the sphere.

3.17. Suppose the surface charge density over a sphere of radius R depends on a polar angle θ as $\sigma = \sigma_0 \cos\theta$, where σ_0 is a positive constant. Show that such a charge distribution can be represented as a result of a small relative shift of two uniformly charged *balls* of radius R whose charges are equal in magnitude and opposite in sign. Resorting to this representation, find the electric field strength vector inside the given sphere.

3.18. Find the electric field strength vector at the centre of a ball of radius R with volume charge density $\rho = \mathbf{ar}$, where $\mathbf{a}$ is a constant vector, and $\mathbf{r}$ is a radius vector drawn from the ball's centre.

3.19. A very long uniformly charged thread oriented along the axis of a circle of radius R rests on its centre with one of the ends. The charge of the thread per unit length is equal to λ. Find the flux of the vector $\mathbf{E}$ across the circle area.

3.20. Two point charges q and $-q$ are separated by the distance $2l$ (Fig. 3.3). Find the flux of the electric field strength vector across a circle of radius R.

3.21. A ball of radius R is uniformly charged with the volume density ρ. Find the flux of the electric field strength vector across

the ball's section formed by the plane located at a distance $r_0 < R$ from the centre of the ball.

3.22. Each of the two long parallel threads carries a uniform charge λ per unit length. The threads are separated by a distance l. Find the maximum magnitude of the electric field strength in the symmetry plane of this system located between the threads.

3.23. An infinitely long cylindrical surface of circular cross-section is uniformly charged lengthwise with the surface density $\sigma = \sigma_0 \cos \varphi$, where φ is the polar angle of the cylindrical coordinate system whose z axis coincides with the axis of the given surface. Find the magnitude and direction of the electric field strength vector on the z axis.

3.24. The electric field strength depends only on the x and y coordinates according to the law $\mathbf{E} = a(x\mathbf{i} + y\mathbf{j})/(x^2 + y^2)$, where a is a constant, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes. Find the flux of the vector $\mathbf{E}$ through a sphere of radius R with its centre at the origin of coordinates.

3.25. A ball of radius R carries a positive charge whose volume density depends only on a separation r from the ball's centre as $\rho = \rho_0 (1 - r/R)$, where ρ_0 is a constant. Assuming the permittivities of the ball and the environment to be equal to unity, find:

(a) the magnitude of the electric field strength as a function of the distance r both inside and outside the ball;

(b) the maximum intensity E_{max} and the corresponding distance r_m.

3.26. A system consists of a ball of radius R carrying a spherically symmetric charge and the surrounding space filled with a charge of volume density $\rho = \alpha/r$, where α is a constant, r is the distance from the centre of the ball. Find the ball's charge at which the magnitude of the electric field strength vector is independent of r outside the ball. How high is this strength? The permittivities of the ball and the surrounding space are assumed to be equal to unity.

3.27. A space is filled up with a charge with volume density $\rho = \rho_0 e^{-\alpha r^3}$, where ρ_0 and α are positive constants, r is the distance from the centre of this system. Find the magnitude of the electric field strength vector as a function of r. Investigate the obtained expression for the small and large values of r, i.e. at $\alpha r^3 \ll 1$ and $\alpha r^3 \gg 1$.

3.28. Inside a ball charged uniformly with volume density ρ there is a spherical cavity. The centre of the cavity is displaced with respect to the centre of the ball by a distance $\mathbf{a}$. Find the field strength $\mathbf{E}$ inside the cavity, assuming the permittivity equal to unity.

3.29. Inside an infinitely long circular cylinder charged uniformly with volume density ρ there is a circular cylindrical cavity. The distance between the axes of the cylinder and the cavity is equal to a. Find the electric field strength $\mathbf{E}$ inside the cavity. The permittivity is assumed to be equal to unity.

3.30. There are two thin wire rings, each of radius R, whose axes coincide. The charges of the rings are q and $-q$. Find the potential difference between the centres of the rings separated by a distance a.

3.31. There is an infinitely long straight thread carrying a charge with linear density $\lambda = 0.40\ \mu\text{C/m}$. Calculate the potential difference between points *1* and *2* if point *2* is removed $\eta = 2.0$ times farther from the thread than point *1*.

3.32. Find the electric field potential and strength at the centre of a hemisphere of radius R charged uniformly with the surface density σ.

3.33. A very thin round plate of radius R carrying a uniform surface charge density σ is located in vacuum. Find the electric field potential and strength along the plate's axis as a function of a distance l from its centre. Investigate the obtained expression at $l \to 0$ and $l \gg R$.

3.34. Find the potential φ at the edge of a thin disc of radius R carrying the uniformly distributed charge with surface density σ.

3.35. Find the electric field strength vector if the potential of this field has the form $\varphi = \mathbf{ar}$, where $\mathbf{a}$ is a constant vector, and $\mathbf{r}$ is the radius vector of a point of the field.

3.36. Determine the electric field strength vector if the potential of this field depends on x, y coordinates as

a) $\varphi = a(x^2 - y^2)$; (b) $\varphi = axy$,

where a is a constant. Draw the approximate shape of these fields using lines of force (in the x, y plane).

3.37. The potential of a certain electrostatic field has the form $\varphi = a(x^2 + y^2) + bz^2$, where a and b are constants. Find the magnitude and direction of the electric field strength vector. What shape have the equipotential surfaces in the following cases:

(a) $a > 0$, $b > 0$; (b) $a > 0$, $b < 0$?

3.38. A charge q is uniformly distributed over the volume of a sphere of radius R. Assuming the permittivity to be equal to unity throughout, find the potential

(a) at the centre of the sphere;

(b) inside the sphere as a function of the distance r from its centre.

3.39. Demonstrate that the potential of the field generated by a dipole with the electric moment $\mathbf{p}$ (Fig. 3.4) may be represented as $\varphi = \mathbf{pr}/4\pi\varepsilon_0 r^3$, where $\mathbf{r}$ is the radius vector. Using this expression, find the magnitude of the electric field strength vector as a function of r and θ.

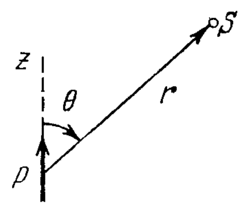


Fig. 3.4.

3.40. A point dipole with an electric moment p oriented in the positive direction of the z axis is located at the origin of coordinates. Find the projections E_z and $E_\perp$ of the electric field strength vector (on the plane perpendicular to the z axis at the point S (see Fig. 3.4)). At which points is $\mathbf{E}$ perpendicular to $\mathbf{p}$?

3.41. A point electric dipole with a moment $\mathbf{p}$ is placed in the external uniform electric field whose strength equals $\mathbf{E}_0$, with

$\mathbf{p} \uparrow\uparrow \mathbf{E}_0$. In this case one of the equipotential surfaces enclosing the dipole forms a sphere. Find the radius of this sphere.

3.42. Two thin parallel threads carry a uniform charge with linear densities λ and $-\lambda$. The distance between the threads is equal to l. Find the potential of the electric field and the magnitude of its strength vector at the distance $r \gg l$ at the angle θ to the vector $\mathbf{l}$ (Fig. 3.5).

3.43. Two coaxial rings, each of radius R, made of thin wire are separated by a small distance l ($l \ll R$) and carry the charges q and $-q$. Find the electric field potential and strength at the axis of the system as a function of the x coordinate (Fig. 3.6). Show in the same drawing the approximate plots of the functions obtained. Investigate these functions at $|x| \gg R$.

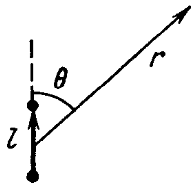


Fig. 3.5.

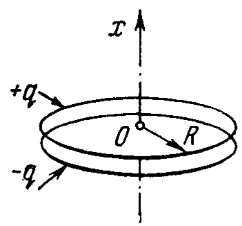


Fig. 3.6.

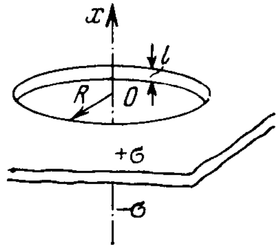


Fig. 3.7.

3.44. Two infinite planes separated by a distance l carry a uniform surface charge of densities σ and $-\sigma$ (Fig. 3.7). The planes have round coaxial holes of radius R, with $l \ll R$. Taking the origin O and the x coordinate axis as shown in the figure, find the potential of the electric field and the projection of its strength vector E_x on the axes of the system as functions of the x coordinate. Draw the approximate plot $\varphi(x)$.

3.45. An electric capacitor consists of thin round parallel plates, each of radius R, separated by a distance l ($l \ll R$) and uniformly charged with surface densities σ and $-\sigma$. Find the potential of the electric field and the magnitude of its strength vector at the axes of the capacitor as functions of a distance x from the plates if $x \gg l$. Investigate the obtained expressions at $x \gg R$.

3.46. A dipole with an electric moment $\mathbf{p}$ is located at a distance r from a long thread charged uniformly with a linear density λ. Find the force $\mathbf{F}$ acting on the dipole if the vector $\mathbf{p}$ is oriented

(a) along the thread;

(b) along the radius vector $\mathbf{r}$;

(c) at right angles to the thread and the radius vector $\mathbf{r}$.

3.47. Find the interaction force between two water molecules separated by a distance $l = 10$ nm if their electric moments are oriented along the same straight line. The moment of each molecule equals $p = 0.62 \cdot 10^{-29}$ C·m.

3.48. Find the potential $\varphi(x, y)$ of an electrostatic field $\mathbf{E} = a(y\mathbf{i} + x\mathbf{j})$, where a is a constant, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes.

3.49. Find the potential $\varphi(x, y)$ of an electrostatic field $\mathbf{E} = 2axy\mathbf{i} + a(x^2 - y^2)\mathbf{j}$, where a is a constant, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes.

3.50. Determine the potential $\varphi(x, y, z)$ of an electrostatic field $\mathbf{E} = ay\mathbf{i} + (ax + bz)\mathbf{j} + by\mathbf{k}$, where a and b are constants, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors of the axes x, y, z.

3.51. The field potential in a certain region of space depends only on the x coordinate as $\varphi = -ax^3 + b$, where a and b are constants. Find the distribution of the space charge $\rho(x)$.

3.52. A uniformly distributed space charge fills up the space between two large parallel plates separated by a distance d. The potential difference between the plates is equal to $\Delta\varphi$. At what value of charge density ρ is the field strength in the vicinity of one of the plates equal to zero? What will then be the field strength near the other plate?

3.53. The field potential inside a charged ball depends only on the distance from its centre as $\varphi = ar^2 + b$, where a and b are constants. Find the space charge distribution $\rho(r)$ inside the ball.

3.2. CONDUCTORS AND DIELECTRICS IN AN ELECTRIC FIELD

- Electric field strength near the surface of a conductor in vacuum:

$$E_n = \sigma/\varepsilon_0. \qquad (3.2a)$$

- Flux of polarization $\mathbf{P}$ across a closed surface:

$$\oint \mathbf{P}\, d\mathbf{S} = -q', \qquad (3.2b)$$

where q' is the algebraic sum of *bound* charges enclosed by this surface.

- Vector $\mathbf{D}$ and Gauss's theorem for it:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \qquad \oint \mathbf{D}\, d\mathbf{S} = q, \qquad (3.2c)$$

where q is the algebraic sum of *extraneous* charges inside a closed surface.

- Relations at the boundary between two dielectrics:

$$P_{2n} - P_{1n} = -\sigma', \quad D_{2n} - D_{1n} = \sigma, \quad E_{2\tau} = E_{1\tau}, \qquad (3.2d)$$

where σ' and σ are the surface densities of bound and extraneous charges, and the unit vector $\mathbf{n}$ of the normal is directed from medium 1 to medium 2.

- In isotropic dielectrics:

$$\mathbf{P} = \varkappa\varepsilon_0 \mathbf{E}, \quad \mathbf{D} = \varepsilon\varepsilon_0 \mathbf{E}, \quad \varepsilon = 1 + \varkappa. \qquad (3.2e)$$

- In the case of an isotropic uniform dielectric filling up all the space between the equipotential surfaces:

$$\mathbf{E} = \mathbf{E}_0/\varepsilon. \qquad (3.2f)$$

3.54. A small ball is suspended over an infinite horizontal conducting plane by means of an insulating elastic thread of stiffness k. As soon as the ball was charged, it descended by x cm and its separation from the plane became equal to l. Find the charge of the ball.

3.55. A point charge q is located at a distance l from the infinite conducting plane. What amount of work has to be performed in order to slowly remove this charge very far from the plane.

3.56. Two point charges, q and $-q$, are separated by a distance l, both being located at a distance $l/2$ from the infinite conducting plane. Find:

(a) the modulus of the vector of the electric force acting on each charge;

(b) the magnitude of the electric field strength vector at the mid-point between these charges.

3.57. A point charge q is located between two mutually perpendicular conducting half-planes. Its distance from each half-plane is equal to l. Find the modulus of the vector of the force acting on the charge.

3.58. A point dipole with an electric moment **p** is located at a distance l from an infinite conducting plane. Find the modulus of the vector of the force acting on the dipole if the vector **p** is perpendicular to the plane.

3.59. A point charge q is located at a distance l from an infinite conducting plane. Determine the surface density of charges induced on the plane as a function of separation r from the base of the perpendicular drawn to the plane from the charge.

3.60. A thin infinitely long thread carrying a charge λ per unit length is oriented parallel to the infinite conducting plane. The distance between the thread and the plane is equal to l. Find:

(a) the modulus of the vector of the force acting on a unit length of the thread;

(b) the distribution of surface charge density $\sigma(x)$ over the plane, where x is the distance from the plane perpendicular to the conducting surface and passing through the thread.

3.61. A very long straight thread is oriented at right angles to an infinite conducting plane; its end is separated from the plane by a distance l. The thread carries a uniform charge of linear density λ. Suppose the point O is the trace of the thread on the plane. Find the surface density of the induced charge on the plane

(a) at the point O;

(b) as a function of a distance r from the point O.

3.62. A thin wire ring of radius R carries a charge q. The ring is oriented parallel to an infinite conducting plane and is separated by a distance l from it. Find:

(a) the surface charge density at the point of the plane symmetrical with respect to the ring;

(b) the strength and the potential of the electric field at the centre of the ring.

3.63. Find the potential φ of an uncharged conducting sphere outside of which a point charge q is located at a distance l from the sphere's centre.

3.64. A point charge q is located at a distance r from the centre O of an uncharged conducting spherical layer whose inside and outside radii are equal to R_1 and R_2 respectively. Find the potential at the point O if $r < R_1$.

3.65. A system consists of two concentric conducting spheres, with the inside sphere of radius a carrying a positive charge q_1. What charge q_2 has to be deposited on the outside sphere of radius b to reduce the potential of the inside sphere to zero? How does the potential φ depend in this case on a distance r from the centre of the system? Draw the approximate plot of this dependence.

3.66. Four large metal plates are located at a small distance d from one another as shown in Fig. 3.8. The extreme plates are inter-

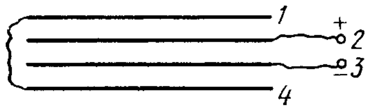


Fig. 3.8.

connected by means of a conductor while a potential difference $\Delta\varphi$ is applied to internal plates. Find:

(a) the values of the electric field strength between neighbouring plates;

(b) the total charge per unit area of each plate.

3.67. Two infinite conducting plates *1* and *2* are separated by a distance l. A point charge q is located between the plates at a distance x from plate *1*. Find the charges induced on each plate.

3.68. Find the electric force experienced by a charge reduced to a unit area of an arbitrary conductor if the surface density of the charge equals σ.

3.69. A metal ball of radius $R = 1.5$ cm has a charge $q = 10$ μC. Find the modulus of the vector of the resultant force acting on a charge located on one half of the ball.

3.70. When an uncharged conducting ball of radius R is placed in an external uniform electric field, a surface charge density $\sigma = \sigma_0 \cos\theta$ is induced on the ball's surface (here σ_0 is a constant, θ is a polar angle). Find the magnitude of the resultant electric force acting on an induced charge of the same sign.

3.71. An electric field of strength $E = 1.0$ kV/cm produces polarization in water equivalent to the correct orientation of only one out of N molecules. Find N. The electric moment of a water molecule equals $p = 0.62 \cdot 10^{-29}$ C·m.

3.72. A non-polar molecule with polarizability β is located at a great distance l from a polar molecule with electric moment **p**. Find the magnitude of the interaction force between the molecules if the vector **p** is oriented along a straight line passing through both molecules.

3.73. A non-polar molecule is located at the axis of a thin uniformly charged ring of radius R. At what distance x from the ring's centre is the magnitude of the force F acting on the given molecule

(a) equal to zero; (b) maximum?

Draw the approximate plot $F_x(x)$.

3.74. A point charge q is located at the centre of a ball made of uniform isotropic dielectric with permittivity ε. Find the polarization $\mathbf{P}$ as a function of the radius vector $\mathbf{r}$ relative to the centre of the system, as well as the charge q' inside a sphere whose radius is less than the radius of the ball.

3.75. Demonstrate that at a dielectric-conductor interface the surface density of the dielectric's bound charge $\sigma' = -\sigma(\varepsilon - 1)/\varepsilon$, where ε is the permittivity, σ is the surface density of the charge on the conductor.

3.76. A conductor of arbitrary shape, carrying a charge q, is surrounded with uniform dielectric of permittivity ε (Fig. 3.9).

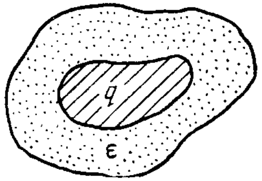


Fig. 3.9.

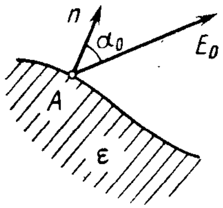


Fig. 3.10.

Find the total bound charges at the inner and outer surfaces of the dielectric.

3.77. A uniform isotropic dielectric is shaped as a spherical layer with radii a and b. Draw the approximate plots of the electric field strength E and the potential φ vs the distance r from the centre of the layer if the dielectric has a certain positive extraneous charge distributed uniformly:

(a) over the internal surface of the layer; (b) over the volume of the layer.

3.78. Near the point A (Fig. 3.10) lying on the boundary between glass and vacuum the electric field strength in vacuum is equal to $E_0 = 10.0$ V/m, the angle between the vector $\mathbf{E}_0$ and the normal $\mathbf{n}$ of the boundary line being equal to $\alpha_0 = 30°$. Find the field strength E in glass near the point A, the angle α between the vector $\mathbf{E}$ and $\mathbf{n}$, as well as the surface density of the bound charges at the point A.

3.79. Near the plane surface of a uniform isotropic dielectric with permittivity ε the electric field strength in vacuum is equal to E_0, the vector $\mathbf{E}_0$ forming an angle θ with the normal of the dielectric's surface (Fig. 3.11). Assuming the field to be uniform both inside and outside the dielectric, find:

(a) the flux of the vector $\mathbf{E}$ through a sphere of radius R with centre located at the surface of the dielectric;

(b) the circulation of the vector $\mathbf{D}$ around the closed path Γ of length l (see Fig. 3.11) whose plane is perpendicular to the surface of the dielectric and parallel to the vector $\mathbf{E}_0$.

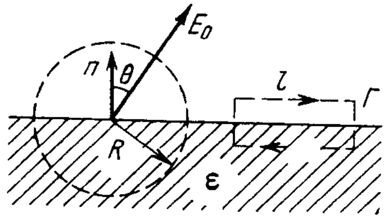


Fig. 3.11.

3.80. An infinite plane of uniform dielectric with permittivity ε is uniformly charged with extraneous charge of space density ρ. The thickness of the plate is equal to $2d$. Find:

(a) the magnitude of the electric field strength and the potential as functions of distance l from the middle point of the plane (where the potential is assumed to be equal to zero); having chosen the x coordinate axis perpendicular to the plate, draw the approximate plots of the projection $E_x(x)$ of the vector $\mathbf{E}$ and the potential $\varphi(x)$;

(b) the surface and space densities of the bound charge.

3.81. Extraneous charges are uniformly distributed with space density $\rho > 0$ over a ball of radius R made of uniform isotropic dielectric with permittivity ε. Find:

(a) the magnitude of the electric field strength as a function of distance r from the centre of the ball; draw the approximate plots $E(r)$ and $\varphi(r)$;

(b) the space and surface densities of the bound charges.

3.82. A round dielectric disc of radius R and thickness d is *statically* polarized so that it gains the uniform polarization $\mathbf{P}$, with the vector $\mathbf{P}$ lying in the plane of the disc. Find the strength $\mathbf{E}$ of the electric field at the centre of the disc if $d \ll R$.

3.83. Under certain conditions the polarization of an infinite uncharged dielectric plate takes the form $\mathbf{P} = \mathbf{P}_0(1 - x^2/d^2)$, where $\mathbf{P}_0$ is a vector perpendicular to the plate, x is the distance from the middle of the plate, d is its half-thickness. Find the strength $\mathbf{E}$ of the electric field inside the plate and the potential difference between its surfaces.

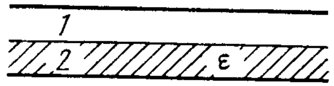


Fig. 3.12.

3.84. Initially the space between the plates of the capacitor is filled with air, and the field strength in the gap is equal to E_0. Then half the gap is filled with uniform isotropic dielectric with permittivity ε as shown in Fig. 3.12. Find the moduli of the vectors $\mathbf{E}$ and $\mathbf{D}$ in both parts of the gap (*1* and *2*) if the introduction of the dielectric

(a) does not change the voltage across the plates;
(b) leaves the charges at the plates constant.

3.85. Solve the foregoing problem for the case when half the gap is filled with the dielectric in the way shown in Fig. 3.13.

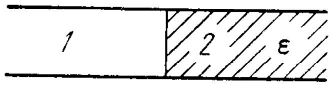


Fig. 3.13.

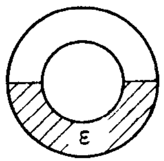


Fig. 3.14.

3.86. Half the space between two concentric electrodes of a spherical capacitor is filled, as shown in Fig. 3.14, with uniform isotropic dielectric with permittivity ε. The charge of the capacitor is q. Find the magnitude of the electric field strength between the electrodes as a function of distance r from the curvature centre of the electrodes.

3.87. Two small identical balls carrying the charges of the same sign are suspended from the same point by insulating threads of equal length. When the surrounding space was filled with kerosene the divergence angle between the threads remained constant. What is the density of the material of which the balls are made?

3.88. A uniform electric field of strength $E = 100$ V/m is generated inside a ball made of uniform isotropic dielectric with permittivity $\varepsilon = 5.00$. The radius of the ball is $R = 3.0$ cm. Find the maximum surface density of the bound charges and the total bound charge of one sign.

3.89. A point charge q is located in vacuum at a distance l from the plane surface of a uniform isotropic dielectric filling up all the half-space. The permittivity of the dielectric equals ε. Find:
(a) the surface density of the bound charges as a function of distance r from the point charge q; analyse the obtained result at $l \to 0$;
(b) the total bound charge on the surface of the dielectric.

3.90. Making use of the formulation and the solution of the foregoing problem, find the magnitude of the force exerted by the charges bound on the surface of the dielectric on the point charge q.

3.91. A point charge q is located on the plane dividing vacuum and infinite uniform isotropic dielectric with permittivity ε. Find the moduli of the vectors **D** and **E** as well as the potential φ as functions of distance r from the charge q.

3.92. A small conducting ball carrying a charge q is located in a uniform isotropic dielectric with permittivity ε at a distance l from an infinite boundary plane between the dielectric and vacuum. Find the surface density of the bound charges on the boundary plane as a function of distance r from the ball. Analyse the obtained result for $l \to 0$.

3.93. A half-space filled with uniform isotropic dielectric with permittivity ε has the conducting boundary plane. Inside the dielectric, at a distance l from this plane, there is a small metal ball possessing a charge q. Find the surface density of the bound charges at the boundary plane as a function of distance r from the ball.

3.94. A plate of thickness h made of uniform *statically* polarized dielectric is placed inside a capacitor whose parallel plates are interconnected by a conductor. The polarization of the dielectric is equal

Fig. 3.15.

to **P** (Fig. 3.15). The separation between the capacitor plates is d. Find the strength and induction vectors for the electric field both inside and outside the plates.

3.95. A long round dielectric cylinder is polarized so that the vector $\mathbf{P} = \alpha \mathbf{r}$, where α is a positive constant and **r** is the distance from the axis. Find the space density ρ' of bound charges as a function of distance r from the axis.

3.96. A dielectric ball is polarized uniformly and *statically*. Its polarization equals **P**. Taking into account that a ball polarized *in this way* may be represented as a result of a small shift of all positive charges of the dielectric relative to all negative charges,
(a) find the electric field strength **E** inside the ball;
(b) demonstrate that the field outside the ball is that of a dipole located at the centre of the ball, the potential of that field being equal to $\varphi = \mathbf{p}_0\mathbf{r}/4\pi\varepsilon_0$, where $\mathbf{p}_0$ is the electric moment of the ball, and **r** is the distance from its centre.

3.97. Utilizing the solution of the foregoing problem, find the electric field strength $\mathbf{E}_0$ in a spherical cavity in an infinite statically polarized uniform dielectric if the dielectric's polarization is **P**, and far from the cavity the field strength is **E**.

3.98. A uniform dielectric ball is placed in a uniform electric field of strength $\mathbf{E}_0$. Under these conditions the dielectric becomes polarized uniformly. Find the electric field strength **E** inside the ball and the polarization **P** of the dielectric whose permittivity equals ε. Make use of the result obtained in Problem 3.96.

3.99. An infinitely long round dielectric cylinder is polarized uniformly and statically, the polarization **P** being perpendicular to the axis of the cylinder. Find the electric field strength **E** inside the dielectric.

3.100. A long round cylinder made of uniform dielectric is placed in a uniform electric field of strength $\mathbf{E}_0$. The axis of the cylinder is perpendicular to vector $\mathbf{E}_0$. Under these conditions the dielectric becomes polarized uniformly. Making use of the result

obtained in the foregoing problem, find the electric field strength $\mathbf{E}$ in the cylinder and the polarization $\mathbf{P}$ of the dielectric whose permittivity is equal to ε.

3.3. ELECTRIC CAPACITANCE. ENERGY OF AN ELECTRIC FIELD

- Capacitance of a parallel-plate capacitor:

$$C = \varepsilon\varepsilon_0 S/d. \tag{3.3a}$$

- Interaction energy of a system of point charges:

$$W = \frac{1}{2}\sum q_i\varphi_i. \tag{3.3b}$$

- Total electric energy of a system with continuous charge distribution:

$$W = \frac{1}{2}\int \varphi\rho\, dV. \tag{3.3c}$$

- Total electric energy of two charged bodies 1 and 2:

$$W = W_1 + W_2 + W_{12}, \tag{3.3d}$$

where W_1 and W_2 are the self-energies of the bodies, and W_{12} is the interaction energy.

- Energy of a charged capacitor:

$$W = \frac{qV}{2} = \frac{q^2}{2C} = \frac{CV^2}{2} \tag{3.3e}$$

- Volume density of electric field energy:

$$w = \frac{\mathbf{ED}}{2} = \frac{\varepsilon\varepsilon_0 E^2}{2} \tag{3.3f}$$

3.101. Find the capacitance of an isolated ball-shaped conductor of radius R_1 surrounded by an adjacent concentric layer of dielectric with permittivity ε and outside radius R_2.

3.102. Two parallel-plate air capacitors, each of capacitance C, were connected in series to a battery with emf $\mathscr{E}$. Then one of the capacitors was filled up with uniform dielectric with permittivity ε. How many times did the electric field strength in that capacitor decrease? What amount of charge flows through the battery?

3.103. The space between the plates of a parallel-plate capacitor is filled consecutively with two dielectric layers *1* and *2* having the thicknesses d_1 and d_2 and the permittivities ε_1 and ε_2 respectively. The area of each plate is equal to S. Find:

(a) the capacitance of the capacitor;

(b) the density σ' of the bound charges on the boundary plane if the voltage across the capacitor equals V and the electric field is directed from layer *1* to layer *2*.

3.104. The gap between the plates of a parallel-plate capacitor is filled with isotropic dielectric whose permittivity ε varies linearly from ε_1 to ε_2 ($\varepsilon_2 > \varepsilon_1$) in the direction perpendicular to the plates. The area of each plate equals S, the separation between the plates is equal to d. Find:

(a) the capacitance of the capacitor;

(b) the space density of the bound charges as a function of ε if the charge of the capacitor is q and the field $\mathbf{E}$ in it is directed toward the growing ε values.

3.105. Find the capacitance of a spherical capacitor whose electrodes have radii R_1 and $R_2 > R_1$ and which is filled with isotropic dielectric whose permittivity varies as $\varepsilon = a/r$, where a is a constant, and r is the distance from the centre of the capacitor.

3.106. A cylindrical capacitor is filled with two cylindrical layers of dielectric with permittivities ε_1 and ε_2. The inside radii of the layers are equal to R_1 and $R_2 > R_1$. The maximum permissible values of electric field strength are equal to E_{1m} and E_{2m} for these dielectrics. At what relationship between ε, R, and E_m will the voltage increase result in the field strength reaching the breakdown value for both dielectrics simultaneously?

3.107. There is a double-layer cylindrical capacitor whose parameters are shown in Fig. 3.16. The breakdown field strength values for these dielectrics are equal to E_1 and E_2 respectively. What is the breakdown voltage of this capacitor if $\varepsilon_1 R_1 E_1 < \varepsilon_2 R_2 E_2$?

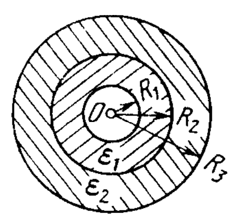


Fig. 3.16.

3.108. Two long straight wires with equal cross-sectional radii a are located parallel to each other in air. The distance between their axes equals b. Find the mutual capacitance of the wires per unit length under the condition $b \gg a$.

3.109. A long straight wire is located parallel to an infinite conducting plate. The wire cross-sectional radius is equal to a, the distance between the axis of the wire and the plane equals b. Find the mutual capacitance of this system per unit length of the wire under the condition $a \ll b$.

3.110. Find the capacitance of a system of two identical metal balls of radius a if the distance between their centres is equal to b, with $b \gg a$. The system is located in a uniform dielectric with permittivity ε.

3.111. Determine the capacitance of a system consisting of a metal ball of radius a and an infinite conducting plane separated from the centre of the ball by the distance l if $l \gg a$.

3.112. Find the capacitance of a system of identical capacitors between points A and B shown in

(a) Fig. 3.17*a*; (b) Fig. 3.17*b*.

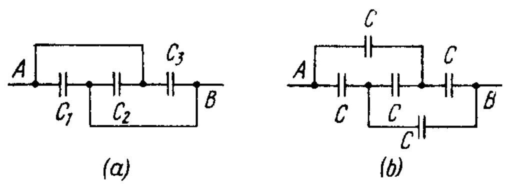


Fig. 3.17.

3.113. Four identical metal plates are located in air at equal distances d from one another. The area of each plate is equal to S. Find the capacitance of the system between points A and B if the plates are interconnected as shown (a) in Fig. 3.18a; (b) in Fig. 3.18b.

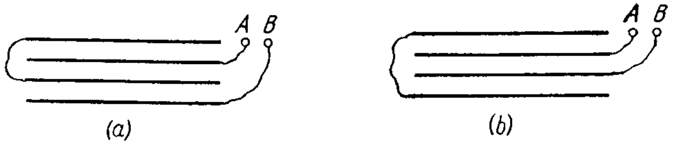


Fig. 3.18.

3.114. A capacitor of capacitance $C_1 = 1.0$ μF withstands the maximum voltage $V_1 = 6.0$ kV while a capacitor of capacitance $C_2 = 2.0$ μF, the maximum voltage $V_2 = 4.0$ kV. What voltage will the system of these two capacitors withstand if they are connected in series?

3.115. Find the potential difference between points A and B of the system shown in Fig. 3.19 if the emf is equal to $\mathscr{E} = 110$ V and the capacitance ratio $C_2/C_1 = \eta = 2.0$.

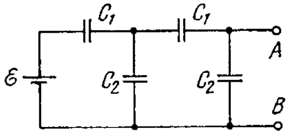


Fig. 3.19.

3.116. Find the capacitance of an infinite circuit formed by the repetition of the same link consisting of two identical capacitors, each with capacitance C (Fig. 3.20).

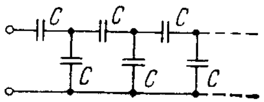


Fig. 3.20.

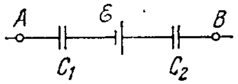


Fig. 3.21.

3.117. A circuit has a section AB shown in Fig. 3.21. The emf of the source equals $\mathscr{E} = 10$ V, the capacitor capacitances are equal to $C_1 = 1.0$ μF and $C_2 = 2.0$ μF, and the potential difference $\varphi_A - \varphi_B = 5.0$ V. Find the voltage across each capacitor.

3.118. In a circuit shown in Fig. 3.22 find the potential difference between the left and right plates of each capacitor.

3.119. Find the charge of each capacitor in the circuit shown in Fig. 3.22.

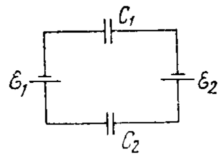


Fig. 3.22.

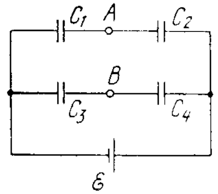


Fig. 3.23.

3.120. Determine the potential difference $\varphi_A - \varphi_B$ between points A and B of the circuit shown in Fig. 3.23. Under what condition is it equal to zero?

3.121. A capacitor of capacitance $C_1 = 1.0$ μF charged up to a voltage $V = 110$ V is connected in parallel to the terminals of a circuit consisting of two uncharged capacitors connected in series and possessing the capacitances $C_2 = 2.0$ μF and $C_3 = 3.0$ μF. What charge will flow through the connecting wires?

3.122. What charges will flow after the shorting of the switch Sw in the circuit illustrated in Fig. 3.24 through sections 1 and 2 in the directions indicated by the arrows?

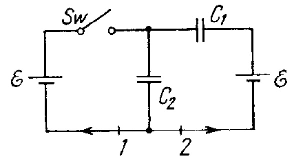


Fig. 3.24.

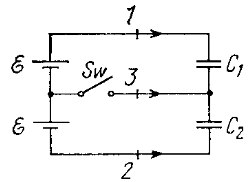


Fig. 3.25.

3.123. In the circuit shown in Fig. 3.25 the emf of each battery is equal to $\mathscr{E} = 60$ V, and the capacitor capacitances are equal to $C_1 = 2.0$ μF and $C_2 = 3.0$ μF. Find the charges which will flow after the shorting of the switch Sw through sections 1, 2 and 3 in the directions indicated by the arrows.

3.124. Find the potential difference $\varphi_A - \varphi_B$ between points A and B of the circuit shown in Fig. 3.26.

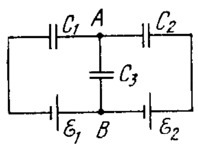


Fig. 3.26.

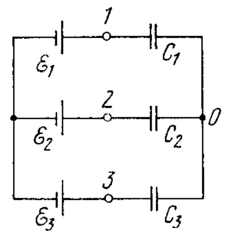


Fig. 3.27.

3.125. Determine the potential at point 1 of the circuit shown in Fig. 3.27, assuming the potential at the point O to be equal to zero.

Using the symmetry of the formula obtained, write the expressions for the potentials at points 2 and 3.

3.126. Find the capacitance of the circuit shown in Fig. 3.28 between points A and B.

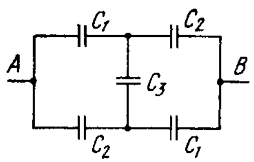


Fig. 3.28.

3.127. Determine the interaction energy of the point charges located at the corners of a square with the side a in the circuits shown in Fig. 3.29.

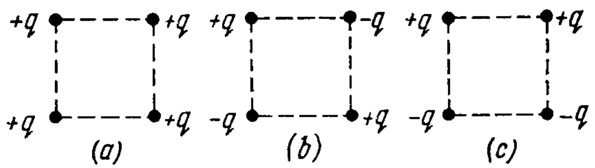


Fig. 3.29.

3.128. There is an infinite straight chain of alternating charges q and $-q$. The distance between the neighbouring charges is equal to a. Find the interaction energy of each charge with all the others.

Instruction. Make use of the expansion of $\ln(1+\alpha)$ in a power series in α.

3.129. A point charge q is located at a distance l from an infinite tonducting plane. Find the interaction energy of that charge with chose induced on the plane.

3.130. Calculate the interaction energy of two balls whose charges q_1 and q_2 are spherically symmetrical. The distance between the centres of the balls is equal to l.

Instruction. Start with finding the interaction energy of a ball and a thin spherical layer.

3.131. A capacitor of capacitance $C_1 = 1.0\ \mu F$ carrying initially a voltage $V = 300$ V is connected in parallel with an uncharged capacitor of capacitance $C_2 = 2.0\ \mu F$. Find the increment of the electric energy of this system by the moment equilibrium is reached. Explain the result obtained.

3.132. What amount of heat will be generated in the circuit shown in Fig. 3.30 after the switch Sw is shifted from position 1 to position 2?

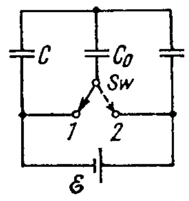


Fig. 3.30.

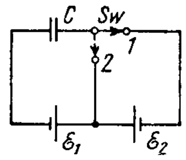


Fig. 3.31.

3.133. What amount of heat will be generated in the circuit shown in Fig. 3.31 after the switch Sw is shifted from position 1 to position 2?

3.134. A system consists of two thin concentric metal shells of radii R_1 and R_2 with corresponding charges q_1 and q_2. Find the self-energy values W_1 and W_2 of each shell, the interaction energy of the shells W_{12}, and the total electric energy of the system.

3.135. A charge q is distributed uniformly over the volume of a ball of radius R. Assuming the permittivity to be equal to unity, find:

(a) the electrostatic self-energy of the ball;

(b) the ratio of the energy W_1 stored in the ball to the energy W_2 pervading the surrounding space.

3.136. A point charge $q = 3.0\ \mu C$ is located at the centre of a spherical layer of uniform isotropic dielectric with permittivity $\varepsilon = 3.0$. The inside radius of the layer is equal to $a = 250$ mm, the outside radius is $b = 500$ mm. Find the electrostatic energy inside the dielectric layer.

3.137. A spherical shell of radius R_1 with uniform charge q is expanded to a radius R_2. Find the work performed by the electric forces in this process.

3.138. A spherical shell of radius R_1 with a uniform charge q has a point charge q_0 at its centre. Find the work performed by the electric forces during the shell expansion from radius R_1 to radius R_2.

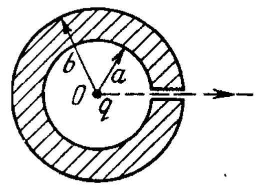


Fig. 3.32.

3.139. A spherical shell is uniformly charged with the surface density σ. Using the energy conservation law, find the magnitude of the electric force acting on a unit area of the shell.

3.140. A point charge q is located at the centre O of a spherical uncharged conducting layer provided with a small orifice (Fig. 3.32). The inside and outside radii of the layer are equal to a and b respectively. What amount of work has to be performed to slowly transfer the charge q from the point O through the orifice and into infinity?

3.141. Each plate of a parallel-plate air capacitor has an area S. What amount of work has to be performed to slowly increase the distance between the plates from x_1 to x_2 if

(a) the capacitance of the capacitor, which is equal to q, or (b) the voltage across the capacitor, which is equal to V, is kept constant in the process?

3.142. Inside a parallel-plate capacitor there is a plate parallel to the outer plates, whose thickness is equal to $\eta = 0.60$ of the gap width. When the plate is absent the capacitor capacitance equals $c = 20$ nF. First, the capacitor was connected in parallel to a constant voltage source producing $V = 200$ V, then it was disconnected from it, after which the plate was slowly removed from the gap. Find the work performed during the removal, if the plate is

(a) made of metal; (b) made of glass.

3.143. A parallel-plate capacitor was lowered into water in a horizontal position, with water filling up the gap between the plates $d = 1.0$ mm wide. Then a constant voltage $V = 500$ V was applied to the capacitor. Find the water pressure increment in the gap.

3.144. A parallel-plate capacitor is located horizontally so that one of its plates is submerged into liquid while the other is over its surface (Fig. 3.33). The permittivity of the liquid is equal to ε, its density is equal to ρ. To what height will the level of the liquid in the capacitor rise after its plates get a charge of surface density σ?

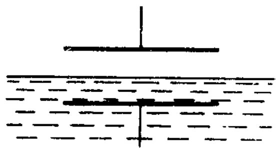


Fig. 3.33

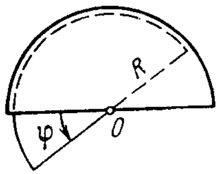


Fig. 3.34.

3.145. A cylindrical layer of dielectric with permittivity ε is inserted into a cylindrical capacitor to fill up all the space between the electrodes. The mean radius of the electrodes equals R, the gap between them is equal to d, with $d \ll R$. The constant voltage V is applied across the electrodes of the capacitor. Find the magnitude of the electric force pulling the dielectric into the capacitor.

3.146. A capacitor consists of two stationary plates shaped as a semi-circle of radius R and a movable plate made of dielectric with permittivity ε and capable of rotating about an axis O between the stationary plates (Fig. 3.34). The thickness of the movable plate is equal to d which is practically the separation between the stationary plates. A potential difference V is applied to the capacitor. Find the magnitude of the moment of forces relative to the axis O acting on the movable plate in the position shown in the figure.

3.4. ELECTRIC CURRENT

- Ohm's law for an inhomogeneous segment of a circuit:

$$I = \frac{V_{12}}{R} = \frac{\varphi_1 - \varphi_2 + \mathscr{E}_{12}}{R}, \tag{3.4a}$$

where V_{12} is the voltage drop across the segment.

- Differential form of Ohm's law:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{E}^*), \tag{3.4b}$$

where $\mathbf{E}^*$ is the strength of a field produced by extraneous forces.

- Kirchhoff's laws (for an electric circuit):

$$\sum I_k = 0, \quad \sum I_k R_k = \sum \mathscr{E}_k. \tag{3.4c}$$

- Power P of current and thermal power Q:

$$P = VI = (\varphi_1 - \varphi_2 + \mathscr{E}_{12}) I, \quad Q = RI^2. \tag{3.4d}$$

- Specific power P_{sp} of current and specific thermal power Q_{sp}:

$$P_{sp} = \mathbf{j}(\mathbf{E} + \mathbf{E}^*), \quad Q_{sp} = \rho \mathbf{j}^2 \tag{3.4e}$$

- Current density in a metal:

$$\mathbf{j} = en\mathbf{u}, \tag{3.4f}$$

where $\mathbf{u}$ is the average velocity of carriers.

- Number of ions recombining per unit volume of gas per unit time:

$$\dot{n}_r = rn^2, \tag{3.4g}$$

where r is the recombination coefficient.

3.147. A long cylinder with uniformly charged surface and cross-sectional radius $a = 1.0$ cm moves with a constant velocity $v = 10$ m/s along its axis. An electric field strength at the surface of the cylinder is equal to $E = 0.9$ kV/cm. Find the resulting convection current, that is, the current caused by mechanical transfer of a charge.

3.148. An air cylindrical capacitor with a dc voltage $V = 200$ V applied across it is being submerged vertically into a vessel filled with water at a velocity $v = 5.0$ mm/s. The electrodes of the capacitor are separated by a distance $d = 2.0$ mm, the mean curvature radius of the electrodes is equal to $r = 50$ mm. Find the current flowing in this case along lead wires, if $d \ll r$.

3.149. At the temperature 0 °C the electric resistance of conductor *2* is η times that of conductor *1*. Their temperature coefficients of resistance are equal to α_2 and α_1 respectively. Find the temperature coefficient of resistance of a circuit segment consisting of these two conductors when they are connected

(a) in series; (b) in parallel.

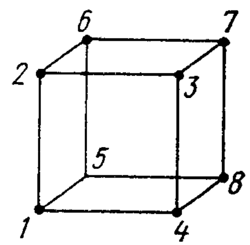


Fig. 3.35.

3.150. Find the resistance of a wire frame shaped as a cube (Fig. 3.35) when measured between points

(a) *1-7*; (b) *1-2*; (c) *1-3*.

The resistance of each edge of the frame is R

3.151. At what value of the resistance R_x in the circuit shown in Fig. 3.36 will the total resistance between points A and B be independent of the number of cells?

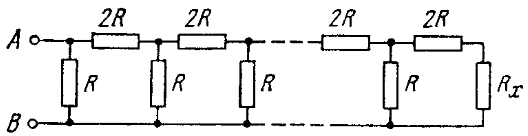


Fig. 3.36.

3.152. Fig. 3.37 shows an infinite circuit formed by the repetition of the same link, consisting of resistance $R_1 = 4.0\ \Omega$ and $R_2 = 3.0\ \Omega$. Find the resistance of this circuit between points A and B.

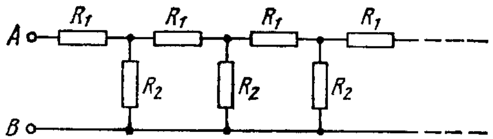


Fig. 3.37.

3.153. There is an infinite wire grid with square cells (Fig. 3.38). The resistance of each wire between neighbouring joint connections is equal to R_0. Find the resistance R of the whole grid between points A and B.

Instruction. Make use of principles of symmetry and superposition.

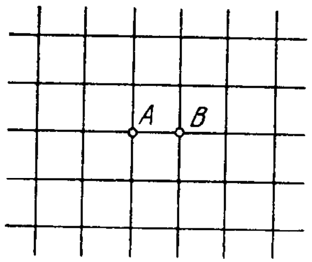


Fig. 3.38.

3.154. A homogeneous poorly conducting medium of resistivity ρ fills up the space between two thin coaxial ideally conducting cylinders. The radii of the cylinders are equal to a and b, with $a < b$, the length of each cylinder is l. Neglecting the edge effects, find the resistance of the medium between the cylinders.

3.155. A metal ball of radius a is surrounded by a thin concentric metal shell of radius b. The space between these electrodes is filled up with a poorly conducting homogeneous medium of resistivity ρ. Find the resistance of the interelectrode gap. Analyse the obtained solution at $b \to \infty$.

3.156. The space between two conducting concentric spheres of radii a and b ($a < b$) is filled up with homogeneous poorly conducting medium. The capacitance of such a system equals C. Find the resistivity of the medium if the potential difference between the spheres, when they are disconnected from an external voltage, decreases η-fold during the time interval Δt.

3.157. Two metal balls of the same radius a are located in a homogeneous poorly conducting medium with resistivity ρ. Find the resistance of the medium between the balls provided that the separation between them is much greater than the radius of the ball.

3.158. A metal ball of radius a is located at a distance l from an infinite ideally conducting plane. The space around the ball is filled with a homogeneous poorly conducting medium with resistivity ρ. In the case of $a \ll l$ find:

(a) the current density at the conducting plane as a function of distance r from the ball if the potential difference between the ball and the plane is equal to V;

(b) the electric resistance of the medium between the ball and the plane.

3.159. Two long parallel wires are located in a poorly conducting medium with resistivity ρ. The distance between the axes of the wires is equal to l, the cross-section radius of each wire equals a. In the case $a \ll l$ find:

(a) the current density at the point equally removed from the axes of the wires by a distance r if the potential difference between the wires is equal to V;

(b) the electric resistance of the medium per unit length of the wires.

3.160. The gap between the plates of a parallel-plate capacitor is filled with glass of resistivity $\rho = 100\ \text{G}\Omega\cdot\text{m}$. The capacitance of the capacitor equals $C = 4.0$ nF. Find the leakage current of the capacitor when a voltage $V = 2.0$ kV is applied to it.

3.161. Two conductors of arbitrary shape are embedded into an infinite homogeneous poorly conducting medium with resistivity ρ and permittivity ε. Find the value of a product RC for this system, where R is the resistance of the medium between the conductors, and C is the mutual capacitance of the wires in the presence of the medium.

3.162. A conductor with resistivity ρ bounds on a dielectric with permittivity ε. At a certain point A at the conductor's surface the electric displacement equals D, the vector $\mathbf{D}$ being directed away from the conductor and forming an angle α with the normal of the surface. Find the surface density of charges on the conductor at the point A and the current density in the conductor in the vicinity of the same point.

3.163. The gap between the plates of a parallel-plate capacitor is filled up with an inhomogeneous poorly conducting medium whose conductivity varies linearly in the direction perpendicular to the plates from $\sigma_1 = 1.0$ pS/m to $\sigma_2 = 2.0$ pS/m. Each plate has an area $S = 230\ \text{cm}^2$, and the separation between the plates is $d = 2.0$ mm. Find the current flowing through the capacitor due to a voltage $V = 300$ V.

3.164. Demonstrate that the law of refraction of direct current lines at the boundary between two conducting media has the form

$\tan \alpha_2/\tan \alpha_1 = \sigma_2/\sigma_1$, where σ_1 and σ_2 are the conductivities of the media, α_2 and α_1 are the angles between the current lines and the normal of the boundary surface.

3.165. Two cylindrical conductors with equal cross-sections and different resistivities ρ_1 and ρ_2 are put end to end. Find the charge at the boundary of the conductors if a current I flows from conductor *1* to conductor *2*.

3.166. The gap between the plates of a parallel-plate capacitor is filled up with two dielectric layers *1* and *2* with thicknesses d_1 and d_2, permittivities ε_1 and ε_2, and resistivities ρ_1 and ρ_2. A dc voltage V is applied to the capacitor, with electric field directed from layer *1* to layer *2*. Find σ, the surface density of extraneous charges at the boundary between the dielectric layers, and the condition under which $\sigma = 0$.

3.167. An inhomogeneous poorly conducting medium fills up the space between plates *1* and *2* of a parallel-plate capacitor. Its permittivity and resistivity vary from values ε_1, ρ_1 at plate *1* to values ε_2, ρ_2 at plate *2*. A dc voltage is applied to the capacitor through which a steady current I flows from plate *1* to plate *2*. Find the total extraneous charge in the given medium.

3.168. The space between the plates of a parallel-plate capacitor is filled up with inhomogeneous poorly conducting medium whose resistivity varies linearly in the direction perpendicular to the plates. The ratio of the maximum value of resistivity to the minimum one is equal to η The gap width equals d. Find the volume density of the charge in the gap if a voltage V is applied to the capacitor. ε is assumed to be 1 everywhere.

3.169. A long round conductor of cross-sectional area S is made of material whose resistivity depends only on a distance r from the axis of the conductor as $\rho = \alpha/r^2$, where α is a constant. Find:

(a) the resistance per unit length of such a conductor;

(b) the electric field strength in the conductor due to which a current I flows through it.

3.170. A capacitor with capacitance $C = 400$ pF is connected via a resistance $R = 650\ \Omega$ to a source of constant voltage V_0. How soon will the voltage developed across the capacitor reach a value $V = 0.90\ V_0$?

3.171. A capacitor filled with dielectric of permittivity $\varepsilon = 2.1$ loses half the charge acquired during a time interval $\tau = 3.0$ min. Assuming the charge to leak only through the dielectric filler, calculate its resistivity.

3.172. A circuit consists of a source of a constant emf $\mathscr{E}$ and a resistance R and a capacitor with capacitance C connected in series. The internal resistance of the source is negligible. At a moment $t = 0$ the capacitance of the capacitor is abruptly decreased η-fold. Find the current flowing through the circuit as a function of time t.

3.173. An ammeter and a voltmeter are connected in series to a battery with an emf $\mathscr{E} = 6.0$ V. When a certain resistance is connected

in parallel with the voltmeter, the readings of the latter decrease $\eta = 2.0$ times, whereas the readings of the ammeter increase the same number of times. Find the voltmeter readings after the connection of the resistance.

3.174. Find a potential difference $\varphi_1 - \varphi_2$ between points *1* and *2* of the circuit shown in Fig. 3.39 if $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $\mathscr{E}_1 = 5.0$ V, and $\mathscr{E}_2 = 2.0$ V. The internal resistances of the current sources are negligible.

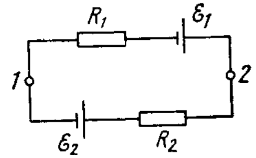


Fig. 3.39.

3.175. Two sources of current of equal emf are connected in series and have different internal resistances R_1 and R_2 $(R_2 > R_1)$. Find the external resistance R at which the potential difference across the terminals of one of the sources (which one in particular?) becomes equal to zero.

3.176. N sources of current with different emf's are connected as shown in Fig. 3.40. The emf's of the sources are proportional to

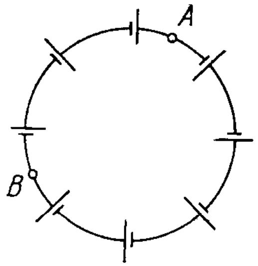


Fig. 3.40.

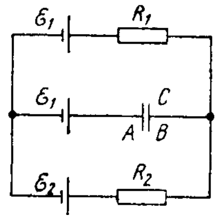


Fig. 3.41.

their internal resistances, i.e. $\mathscr{E} = \alpha R$, where α is an assigned constant. The lead wire resistance is negligible. Find:

(a) the current in the circuit;

(b) the potential difference between points A and B dividing the circuit in n and $N - n$ links.

3.177. In the circuit shown in Fig. 3.41 the sources have emf's $\mathscr{E}_1 = 1.0$ V and $\mathscr{E}_2 = 2.5$ V and the resistances have the values $R_1 = 10\ \Omega$ and $R_2 = 20\ \Omega$. The internal resistances of the sources are negligible. Find a potential difference $\varphi_A - \varphi_B$ between the plates A and B of the capacitor C.

3.178. In the circuit shown in Fig. 3.42 the emf of the source is equal to $\mathscr{E} = 5.0$ V and the resistances are equal to $R_1 = 4.0\ \Omega$ and $R_2 = 6.0\ \Omega$. The internal resistance of the source equals $R = 0.10\ \Omega$. Find the currents flowing through the resistances R_1 and R_2.

3.179. Fig. 3.43 illustrates a potentiometric circuit by means of which we can vary a voltage V applied to a certain device possessing a resistance R. The potentiometer has a length l and a resistance

R_0, and voltage V_0 is applied to its terminals. Find the voltage V fed to the device as a function of distance x. Analyse separately the case $R \gg R_0$.

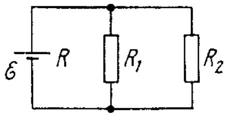


Fig. 3.42.

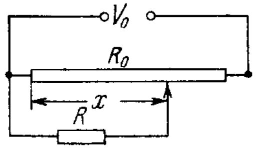


Fig. 3.43.

3.180. Find the emf and the internal resistance of a source which is equivalent to two batteries connected in parallel whose emf's are equal to $\mathscr{E}_1$ and $\mathscr{E}_2$ and internal resistances to R_1 and R_2.

3.181. Find the magnitude and direction of the current flowing through the resistance R in the circuit shown in Fig. 3.44 if the

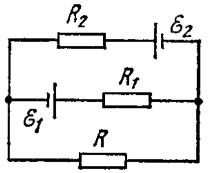


Fig. 3.44.

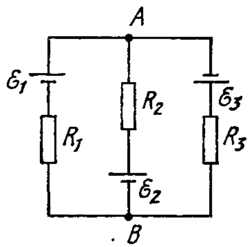


Fig. 3.45.

emf's of the sources are equal to $\mathscr{E}_1 = 1.5$ V and $\mathscr{E}_2 = 3.7$ V and the resistances are equal to $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $R = 5.0\ \Omega$. The internal resistances of the sources are negligible.

3.182. In the circuit shown in Fig. 3.45 the sources have emf's $\mathscr{E}_1 = 1.5$ V, $\mathscr{E}_2 = 2.0$ V, $\mathscr{E}_3 = 2.5$ V, and the resistances are equal to $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $R_3 = 30\ \Omega$. The internal resistances of the sources are negligible. Find:

(a) the current flowing through the resistance R_1;

(b) a potential difference $\varphi_A - \varphi_B$ between the points A and B.

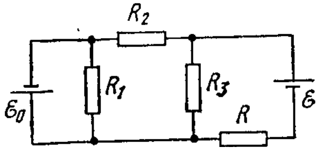


Fig. 3.46.

3.183. Find the current flowing through the resistance R in the circuit shown in Fig. 3.46. The internal resistances of the batteries are negligible.

3.184. Find a potential difference $\varphi_A - \varphi_B$ between the plates of a capacitor C in the circuit shown in Fig. 3.47 if the sources have emf's $\mathscr{E}_1 = 4.0$ V and $\mathscr{E}_2 = 1.0$ V and the resistances are equal to $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, and $R_3 = 30\ \Omega$. The internal resistances of the sources are negligible.

3.185. Find the current flowing through the resistance R_1 of the circuit shown in Fig. 3.48 if the resistances are equal to $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, and $R_3 = 30\ \Omega$, and the potentials of points *1*, *2*, and *3* are equal to $\varphi_1 = 10$ V, $\varphi_2 = 6$ V, and $\varphi_3 = 5$ V

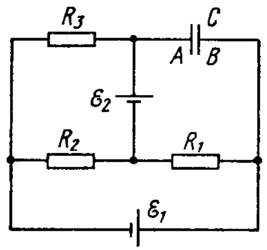


Fig. 3.47.

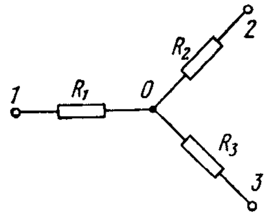


Fig. 3.48.

3.186. A constant voltage $V = 25$ V is maintained between points A and B of the circuit (Fig. 3.49). Find the magnitude and

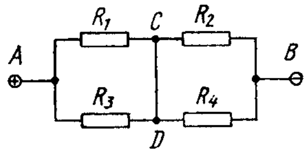


Fig. 3.49.

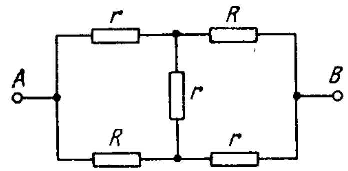


Fig. 3.50.

direction of the current flowing through the segment CD if the resistances are equal to $R_1 = 1.0\ \Omega$, $R_2 = 2.0\ \Omega$, $R_3 = 3.0\ \Omega$, and $R_4 = 4.0\ \Omega$.

3.187. Find the resistance between points A and B of the circuit shown in Fig. 3.50.

3.188. Find how the voltage across the capacitor C varies with time t (Fig. 3.51) after the shorting of the switch Sw at the moment $t = 0$.

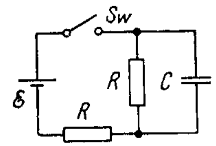


Fig. 3.51.

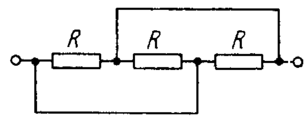


Fig. 3.52.

3.189. What amount of heat will be generated in a coil of resistance R due to a charge q passing through it if the current in the coil

(a) decreases down to zero uniformly during a time interval Δt;

(b) decreases down to zero halving its value every Δt seconds?

3.190. A dc source with internal resistance R_0 is loaded with three identical resistances R interconnected as shown in Fig. 3.52.

At what value of R will the thermal power generated in this circuit be the highest?

3.191. Make sure that the current distribution over two resistances R_1 and R_2 connected in parallel corresponds to the minimum thermal power generated in this circuit.

3.192. A storage battery with emf $\mathscr{E} = 2.6$ V loaded with an external resistance produces a current $I = 1.0$ A. In this case the potential difference between the terminals of the storage battery equals $V = 2.0$ V. Find the thermal power generated in the battery and the power developed in it by electric forces.

3.193. A voltage V is applied to a dc electric motor. The armature winding resistance is equal to R. At what value of current flowing through the winding will the useful power of the motor be the highest? What is it equal to? What is the motor efficiency in this case?

3.194. How much (in per cent) has a filament diameter decreased due to evaporation if the maintenance of the previous temperature required an increase of the voltage by $\eta = 1.0\%$? The amount of heat transferred from the filament into surrounding space is assumed to be proportional to the filament surface area.

3.195. A conductor has a temperature-independent resistance R and a total heat capacity C. At the moment $t = 0$ it is connected to a dc voltage V. Find the time dependence of a conductor's temperature T assuming the thermal power dissipated into surrounding space to vary as $q = k(T - T_0)$, where k is a constant, T_0 is the environmental temperature (equal to the conductor's temperature at the initial moment).

3.196. A circuit shown in Fig. 3.53 has resistances $R_1 = 20\ \Omega$ and $R_2 = 30\ \Omega$. At what value of the resistance R_x will the thermal

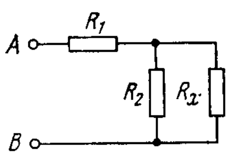


Fig. 3.53.

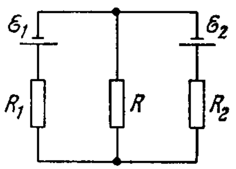


Fig. 3.54.

power generated in it be practically independent of small variations of that resistance? The voltage between the points A and B is supposed to be constant in this case.

3.197. In a circuit shown in Fig. 3.54 resistances R_1 and R_2 are known, as well as emf's $\mathscr{E}_1$ and $\mathscr{E}_2$. The internal resistances of the sources are negligible. At what value of the resistance R will the thermal power generated in it be the highest? What is it equal to?

3.198. A series-parallel combination battery consisting of a large number $N = 300$ of identical cells, each with an internal resistance

$r = 0.3\ \Omega$, is loaded with an external resistance $R = 10\ \Omega$. Find the number n of parallel groups consisting of an equal number of cells connected in series, at which the external resistance generates the highest thermal power.

3.199. A capacitor of capacitance $C = 5.00\ \mu\text{F}$ is connected to a source of constant emf $\mathscr{E} = 200$ V (Fig. 3.55). Then the switch *Sw* was thrown over from contact *1* to contact *2*. Find the amount of heat generated in a resistance $R_1 = 500\ \Omega$ if $R_2 = 330\ \Omega$.

3.200. Between the plates of a parallel-plate capacitor there is a metallic plate whose thickness takes up $\eta = 0.60$ of the capacitor

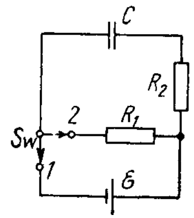


Fig. 3.55.

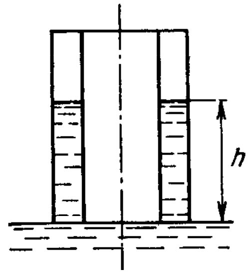


Fig. 3.56.

gap. When that plate is absent the capacitor has a capacity $C = 20$ nF. The capacitor is connected to a dc voltage source $V = 100$ V. The metallic plate is slowly extracted from the gap. Find:
(a) the energy increment of the capacitor;
(b) the mechanical work performed in the process of plate extraction.

3.201. A glass plate totally fills up the gap between the electrodes of a parallel-plate capacitor whose capacitance in the absence of that glass plate is equal to $C = 20$ nF. The capacitor is connected to a dc voltage source $V = 100$ V. The plate is slowly, and without friction, extracted from the gap. Find the capacitor energy increment and the mechanical work performed in the process of plate extraction.

3.202. A cylindrical capacitor connected to a dc voltage source V touches the surface of water with its end (Fig. 3.56). The separation d between the capacitor electrodes is substantially less than their mean radius. Find a height h to which the water level in the gap will rise. The capillary effects are to be neglected.

3.203. The radii of spherical capacitor electrodes are equal to a and b, with $a < b$. The interelectrode space is filled with homogeneous substance of permittivity ε and resistivity ρ. Initially the capacitor is not charged. At the moment $t = 0$ the internal electrode gets a charge q_0. Find:
(a) the time variation of the charge on the internal electrode;
(b) the amount of heat generated during the spreading of the charge.

3.204. The electrodes of a capacitor of capacitance $C = 2.00\ \mu F$ carry opposite charges $q_0 = 1.00$ mC. Then the electrodes are interconnected through a resistance $R = 5.0\ M\Omega$. Find:

(a) the charge flowing through that resistance during a time interval $\tau = 2.00$ s;

(b) the amount of heat generated in the resistance during the same interval.

3.205. In a circuit shown in Fig. 3.57 the capacitance of each capacitor is equal to C and the resistance, to R. One of the capacitors was connected to a voltage V_0 and then at the moment $t = 0$ was shorted by means of the switch *Sw*. Find:

(a) a current I in the circuit as a function of time t;

(b) the amount of generated heat provided a dependence $I(t)$ is known.

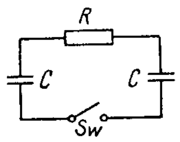


Fig. 3.57.

3.206. A coil of radius $r = 25$ cm wound of a thin copper wire of length $l = 500$ m rotates with an angular velocity $\omega = 300$ rad/s about its axis. The coil is connected to a ballistic galvanometer by means of sliding contacts. The total resistance of the circuit is equal to $R = 21\ \Omega$. Find the specific charge of current carriers in copper if a sudden stoppage of the coil makes a charge $q = 10$ nC flow through the galvanometer.

3.207. Find the total momentum of electrons in a straight wire of length $l = 1000$ m carrying a current $I = 70$ A.

3.208. A copper wire carries a current of density $j = 1.0\ A/mm^2$. Assuming that one free electron corresponds to each copper atom, evaluate the distance which will be covered by an electron during its displacement $l = 10$ mm along the wire.

3.209. A straight copper wire of length $l = 1000$ m and cross-sectional area $S = 1.0\ mm^2$ carries a current $I = 4.5$ A. Assuming that one free electron corresponds to each copper atom, find:

(a) the time it takes an electron to displace from one end of the wire to the other;

(b) the sum of electric forces acting on all free electrons in the given wire.

3.210. A homogeneous proton beam accelerated by a potential difference $V = 600$ kV has a round cross-section of radius $r = 5.0$ mm. Find the electric field strength on the surface of the beam and the potential difference between the surface and the axis of the beam if the beam current is equal to $I = 50$ mA.

3.211. Two large parallel plates are located in vacuum. One of them serves as a cathode, a source of electrons whose initial velocity is negligible. An electron flow directed toward the opposite plate produces a space charge causing the potential in the gap between the plates to vary as $\varphi = ax^{4/3}$, where a is a positive constant, and x is the distance from the cathode. Find:

(a) the volume density of the space charge as a function of x;

(b) the current density.

3.212. The air between two parallel plates separated by a distance $d = 20$ mm is ionized by X-ray radiation. Each plate has an area $S = 500\ cm^2$. Find the concentration of positive ions if at a voltage $V = 100$ V a current $I = 3.0\ \mu A$ flows between the plates, which is well below the saturation current. The air ion mobilities are $u_0^+ = 1.37\ cm^2/(V \cdot s)$ and $u_0^- = 1.91\ cm^2/(V \cdot s)$.

3.213. A gas is ionized in the immediate vicinity of the surface of plane electrode *1* (Fig. 3.58) separated from electrode *2* by a distance l. An alternating voltage varying with time t as $V = V_0 \sin \omega t$ is applied to the electrodes. On decreasing the frequency ω it was observed that the galvanometer G indicates a current only at $\omega < \omega_0$, where ω_0 is a certain cut-off frequency. Find the mobility of ions reaching electrode *2* under these conditions.

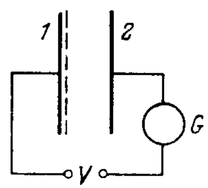


Fig. 3.58.

3.214. The air between two closely located plates is uniformly ionized by ultraviolet radiation. The air volume between the plates is equal to $V = 500\ cm^3$, the observed saturation current is equal to $I_{sat} = 0.48\ \mu A$. Find:

(a) the number of ion pairs produced in a unit volume per unit time;

(b) the equilibrium concentration of ion pairs if the recombination coefficient for air ions is equal to $r = 1.67 \cdot 10^{-6}\ cm^3/s$.

3.215. Having been operated long enough, the ionizer producing $\dot{n}_i = 3.5 \cdot 10^9\ cm^{-3} \cdot s^{-1}$ of ion pairs per unit volume of air per unit time was switched off. Assuming that the only process tending to reduce the number of ions in air is their recombination with coefficient $r = 1.67 \cdot 10^{-6}\ cm^3/s$, find how soon after the ionizer's switching off the ion concentration decreases $\eta = 2.0$ times.

3.216. A parallel-plate air capacitor whose plates are separated by a distance $d = 5.0$ mm is first charged to a potential difference $V = 90$ V and then disconnected from a dc voltage source. Find the time interval during which the voltage across the capacitor decreases by $\eta = 1.0\%$, taking into account that the average number of ion pairs formed in air under standard conditions per unit volume per unit time is equal to $\dot{n}_i = 5.0\ cm^{-3} \cdot s^{-1}$ and that the given voltage corresponds to the saturation current.

3.217. The gap between two plane plates of a capacitor equal to d is filled with a gas. One of the plates emits ν_0 electrons per second which, moving in an electric field, ionize gas molecules; this way each electron produces α new electrons (and ions) along a unit length of its path. Find the electronic current at the opposite plate, neglecting the ionization of gas molecules by formed ions.

3.218. The gas between the capacitor plates separated by a distance d is uniformly ionized by ultraviolet radiation so that n_i electrons per unit volume per second are formed. These electrons moving in the electric field of the capacitor ionize gas molecules, each electron producing α new electrons (and ions) per unit length of its path. Neglecting the ionization by ions, find the electronic current density at the plate possessing a higher potential.

3.5. CONSTANT MAGNETIC FIELD. MAGNETICS

• Magnetic field of a point charge q moving with non-relativistic velocity $\mathbf{v}$:

$$\mathbf{B}=\frac{\mu_0}{4\pi}\frac{q\,[\mathbf{vr}]}{r^3}. \tag{3.5a}$$

• Biot-Savart law:

$$d\mathbf{B}=\frac{\mu_0}{4\pi}\frac{[\mathbf{jr}]}{r^3}\,dV, \quad d\mathbf{B}=\frac{\mu_0}{4\pi}\frac{I\,[d\mathbf{l},\,\mathbf{r}]}{r^3}. \tag{3.5b}$$

• Circulation of a vector $\mathbf{B}$ and Gauss's theorem for it:

$$\oint \mathbf{B}\,d\mathbf{r}=\mu_0 I, \quad \oint \mathbf{B}\,d\mathbf{S}=0. \tag{3.5c}$$

• Lorentz force:

$$\mathbf{F}=q\mathbf{E}+q\,[\mathbf{vB}]. \tag{3.5d}$$

• Ampere force:

$$d\mathbf{F}=[\mathbf{jB}]\,dV, \quad d\mathbf{F}=I\,[d\mathbf{l},\,\mathbf{B}]. \tag{3.5e}$$

• Force and moment of forces acting on a magnetic dipole $\mathbf{p}_m=IS\mathbf{n}$:

$$\mathbf{F}=p_m\frac{\partial\mathbf{B}}{\partial n}, \quad \mathbf{N}=[\mathbf{p}_m\mathbf{B}], \tag{3.5f}$$

where $\partial\mathbf{B}/\partial n$ is the derivative of a vector $\mathbf{B}$ with respect to the dipole direction.

• Circulation of magnetization $\mathbf{J}$:

$$\oint \mathbf{J}\,d\mathbf{r}=I', \tag{3.5g}$$

where I' is the total molecular current.

• Vector $\mathbf{H}$ and its circulation:

$$\mathbf{H}=\frac{\mathbf{B}}{\mu_0}-\mathbf{J}, \quad \oint \mathbf{H}\,d\mathbf{r}=I, \tag{3.5h}$$

where I is the algebraic sum of macroscopic currents.

• Relations at the boundary between two magnetics:

$$B_{1n}=B_{2n}, \quad H_{1\tau}=H_{2\tau}. \tag{3.5i}$$

• For the case of magnetics in which $\mathbf{J}=\chi\mathbf{H}$:

$$\mathbf{B}=\mu\mu_0\mathbf{H}, \quad \mu=1+\chi. \tag{3.5j}$$

3.219. A current $I=1.00$ A circulates in a round thin-wire loop of radius $R=100$ mm. Find the magnetic induction

(a) at the centre of the loop;

(b) at the point lying on the axis of the loop at a distance $x=100$ mm from its centre.

3.220. A current I flows along a thin wire shaped as a regular polygon with n sides which can be inscribed into a circle of radius R. Find the magnetic induction at the centre of the polygon. Analyse the obtained expression at $n\to\infty$.

3.221. Find the magnetic induction at the centre of a rectangular wire frame whose diagonal is equal to $d=16$ cm and the angle between the diagonals is equal to $\varphi=30°$; the current flowing in the frame equals $I=5.0$ A.

3.222. A current $I=5.0$ A flows along a thin wire shaped as shown in Fig. 3.59. The radius of a curved part of the wire is equal to $R=120$ mm, the angle $2\varphi=90°$. Find the magnetic induction of the field at the point O.

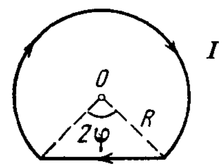


Fig. 3.59.

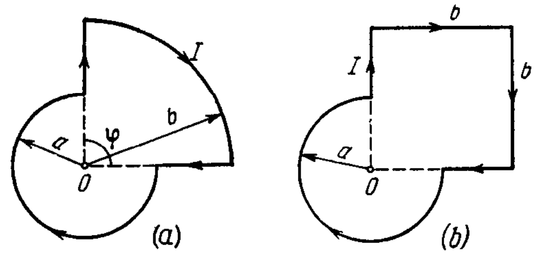


Fig. 3.60.

3.223. Find the magnetic induction of the field at the point O of a loop with current I, whose shape is illustrated

(a) in Fig. 3.60a, the radii a and b, as well as the angle φ are known;

(b) in Fig. 3.60b, the radius a and the side b are known.

3.224. A current I flows along a lengthy thin-walled tube of radius R with longitudinal slit of width h. Find the induction of the magnetic field inside the tube under the condition $h\ll R$.

3.225. A current I flows in a long straight wire with cross-section having the form of a thin half-ring of radius R (Fig. 3.61). Find the induction of the magnetic field at the point O.

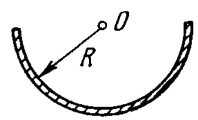


Fig. 3.61.

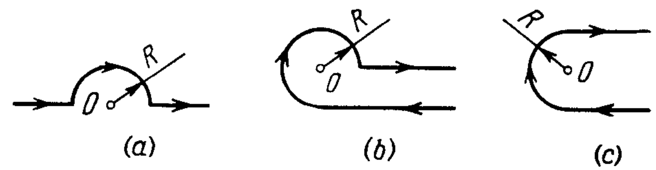


Fig. 3.62.

3.226. Find the magnetic induction of the field at the point O if a current-carrying wire has the shape shown in Fig. 3.62 a, b, c. The radius of the curved part of the wire is R, the linear parts are assumed to be very long.

3.227. A very long wire carrying a current $I = 5.0$ A is bent at right angles. Find the magnetic induction at a point lying on a perpendicular to the wire, drawn through the point of bending, at a distance $l = 35$ cm from it.

3.228. Find the magnetic induction at the point O if the wire carrying a current $I = 8.0$ A has the shape shown in Fig. 3.63 *a*, *b*, *c*.

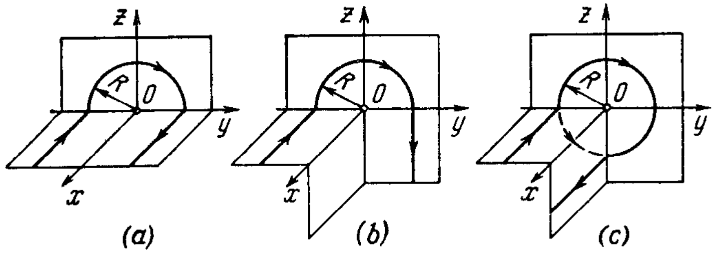


Fig. 3.63.

The radius of the curved part of the wire is $R = 100$ mm, the linear parts of the wire are very long.

3.229. Find the magnitude and direction of the magnetic induction vector $\mathbf{B}$

(a) of an infinite plane carrying a current of linear density $\mathbf{i}$; the vector $\mathbf{i}$ is the same at all points of the plane;

(b) of two parallel infinite planes carrying currents of linear densities $\mathbf{i}$ and $-\mathbf{i}$; the vectors $\mathbf{i}$ and $-\mathbf{i}$ are constant at all points of the corresponding planes.

3.230. A uniform current of density j flows inside an infinite plate of thickness $2d$ parallel to its surface. Find the magnetic induction induced by this current as a function of the distance x from the median plane of the plate. The magnetic permeability is assumed to be equal to unity both inside and outside the plate.

3.231. A direct current I flows along a lengthy straight wire. From the point O (Fig. 3.64) the current spreads radially all over an infinite conducting plane perpendicular to the wire. Find the magnetic induction at all points of space.

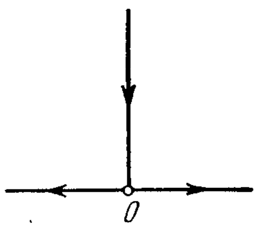


Fig. 3.64.

3.232. A current I flows along a round loop. Find the integral $\int \mathbf{B}\, d\mathbf{r}$ along the axis of the loop within the range from $-\infty$ to $+\infty$. Explain the result obtained.

3.233. A direct current of density $\mathbf{j}$ flows along a round uniform straight wire with cross-section radius R. Find the magnetic induction vector of this current at the point whose position relative to the axis of the wire is defined by a radius vector $\mathbf{r}$. The magnetic permeability is assumed to be equal to unity throughout all the space.

3.234. Inside a long straight uniform wire of round cross-section there is a long round cylindrical cavity whose axis is parallel to the axis of the wire and displaced from the latter by a distance $\mathbf{l}$. A direct current of density $\mathbf{j}$ flows along the wire. Find the magnetic induction inside the cavity. Consider, in particular, the case $\mathbf{l} = 0$.

3.235. Find the current density as a function of distance r from the axis of a radially symmetrical parallel stream of electrons if the magnetic induction inside the stream varies as $B = br^{\alpha}$, where b and α are positive constants.

3.236. A single-layer coil (solenoid) has length l and cross-section radius R. A number of turns per unit length is equal to n. Find the magnetic induction at the centre of the coil when a current I flows through it.

3.237. A very long straight solenoid has a cross-section radius R and n turns per unit length. A direct current I flows through the solenoid. Suppose that x is the distance from the end of the solenoid, measured along its axis. Find:

(a) the magnetic induction B on the axis as a function of x; draw an approximate plot of B vs the ratio x/R;

(b) the distance x_0 to the point on the axis at which the value of B differs by $\eta = 1\%$ from that in the middle section of the solenoid.

3.238. A thin conducting strip of width $h = 2.0$ cm is tightly wound in the shape of a very long coil with cross-section radius $R = 2.5$ cm to make a single-layer straight solenoid. A direct current $I = 5.0$ A flows through the strip. Find the magnetic induction inside and outside the solenoid as a function of the distance r from its axis.

3.239. $N = 2.5 \cdot 10^3$ wire turns are uniformly wound on a wooden toroidal core of very small cross-section. A current I flows through the wire. Find the ratio η of the magnetic induction inside the core to that at the centre of the toroid.

3.240. A direct current $I = 10$ A flows in a long straight round conductor. Find the magnetic flux through a half of wire's cross-section per one metre of its length.

3.241. A very long straight solenoid carries a current I. The cross-sectional area of the solenoid is equal to S, the number of turns per unit length is equal to n. Find the flux of the vector $\mathbf{B}$ through the end plane of the solenoid.

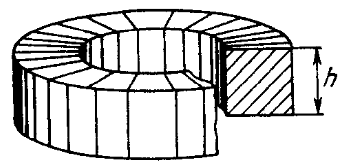


Fig. 3.65.

3.242. Fig. 3.65 shows a toroidal solenoid whose cross-section is rectangular. Find the magnetic flux through this cross-section if the current through the winding equals $I = 1.7$ A, the total number of turns is $N = 1000$, the ratio of the outside diameter to the inside one is $\eta = 1.6$, and the height is equal to $h = 5.0$ cm.

3.243. Find the magnetic moment of a thin round loop with current if the radius of the loop is equal to $R = 100$ mm and the magnetic induction at its centre is equal to $B = 6.0$ μT.

3.244. Calculate the magnetic moment of a thin wire with a current $I = 0.8$ A, wound tightly on half a tore (Fig. 3.66). The diameter of the cross-section of the tore is equal to $d = 5.0$ cm, the number of turns is $N = 500$.

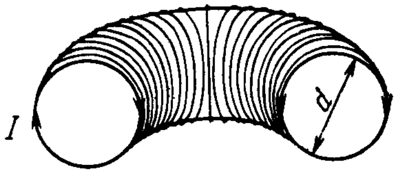


Fig. 3.66.

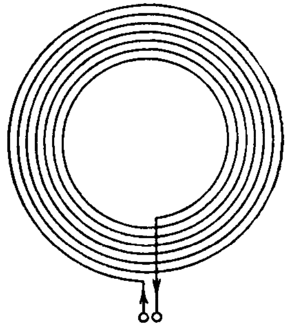


Fig. 3.67.

3.245. A thin insulated wire forms a plane spiral of $N = 100$ tight turns carrying a current $I = 8$ mA. The radii of inside and outside turns (Fig. 3.67) are equal to $a = 50$ mm and $b = 100$ mm. Find:

(a) the magnetic induction at the centre of the spiral;

(b) the magnetic moment of the spiral with a given current.

3.246. A non-conducting thin disc of radius R charged uniformly over one side with surface density σ rotates about its axis with an angular velocity ω. Find:

(a) the magnetic induction at the centre of the disc;

(b) the magnetic moment of the disc.

3.247. A non-conducting sphere of radius $R = 50$ mm charged uniformly with surface density $\sigma = 10.0$ μC/m^2 rotates with an angular velocity $\omega = 70$ rad/s about the axis passing through its centre. Find the magnetic induction at the centre of the sphere.

3.248. A charge q is uniformly distributed over the volume of a uniform ball of mass m and radius R which rotates with an angular velocity ω about the axis passing through its centre. Find the respective magnetic moment and its ratio to the mechanical moment.

3.249. A long dielectric cylinder of radius R is statically polarized so that at all its points the polarization is equal to $\mathbf{P} = \alpha\mathbf{r}$, where α is a positive constant, and $\mathbf{r}$ is the distance from the axis. The cylinder is set into rotation about its axis with an angular velocity $\boldsymbol{\omega}$. Find the magnetic induction $\mathbf{B}$ at the centre of the cylinder.

3.250. Two protons move parallel to each other with an equal velocity $v = 300$ km/s. Find the ratio of forces of magnetic and electrical interaction of the protons.

3.251. Find the magnitude and direction of a force vector acting on a unit length of a thin wire, carrying a current $I = 8.0$ A, at a point O, if the wire is bent as shown in

(a) Fig. 3.68*a*, with curvature radius $R = 10$ cm;

(b) Fig. 3.68*b*, the distance between the long parallel segments of the wire being equal to $l = 20$ cm.

3.252. A coil carrying a current $I = 10$ mA is placed in a uniform magnetic field so that its axis coincides with the field direction. The single-layer winding of the coil is made of copper wire with

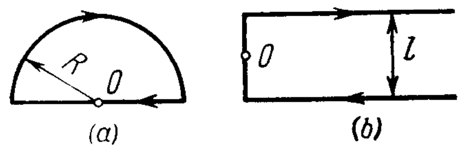


Fig. 3.68.

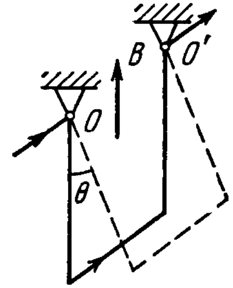


Fig. 3.69.

diameter $d = 0.10$ mm, radius of turns is equal to $R = 30$ mm. At what value of the induction of the external magnetic field can the coil winding be ruptured?

3.253. A copper wire with cross-sectional area $S = 2.5$ mm^2 bent to make three sides of a square can turn about a horizontal axis OO' (Fig. 3.69). The wire is located in uniform vertical magnetic field. Find the magnetic induction if on passing a current $I = 16$ A through the wire the latter deflects by an angle $\theta = 20°$.

3.254. A small coil C with $N = 200$ turns is mounted on one end of a balance beam and introduced between the poles of an electromagnet as shown in Fig. 3.70. The cross-sectional area of the coil

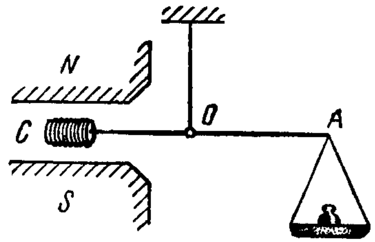


Fig. 3.70.

is $S = 1.0$ cm^2, the length of the arm OA of the balance beam is $l = 30$ cm. When there is no current in the coil the balance is in equilibrium. On passing a current $I = 22$ mA through the coil the equilibrium is restored by putting the additional counterweight of

mass $\Delta m = 60$ mg on the balance pan. Find the magnetic induction at the spot where the coil is located.

3.255. A square frame carrying a current $I = 0.90$ A is located in the same plane as a long straight wire carrying a current $I_0 = 5.0$ A. The frame side has a length $a = 8.0$ cm. The axis of the frame passing through the midpoints of opposite sides is parallel to the wire and is separated from it by the distance which is $\eta = 1.5$ times greater than the side of the frame. Find:

(a) Ampere force acting on the frame;

(b) the mechanical work to be performed in order to turn the frame through 180° about its axis, with the currents maintained constant.

3.256. Two long parallel wires of negligible resistance are connected at one end to a resistance R and at the other end to a dc voltage source. The distance between the axes of the wires is $\eta = 20$ times greater than the cross-sectional radius of each wire. At what value of resistance R does the resultant force of interaction between the wires turn into zero?

3.257. A direct current I flows in a long straight conductor whose cross-section has the form of a thin half-ring of radius R. The same current flows in the opposite direction along a thin conductor located on the "axis" of the first conductor (point O in Fig. 3.61). Find the magnetic interaction force between the given conductors reduced to a unit of their length.

3.258. Two long thin parallel conductors of the shape shown in Fig. 3.71 carry direct currents I_1 and I_2. The separation between the conductors is a, the width of the right-hand conductor is equal to b. With both conductors lying in one plane, find the magnetic interaction force between them reduced to a unit of their length.

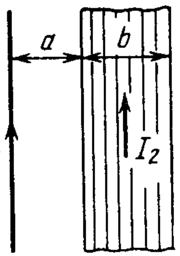


Fig. 3.71.

3.259. A system consists of two parallel planes carrying currents producing a uniform magnetic field of induction B between the planes. Outside this space there is no magnetic field. Find the magnetic force acting per unit area of each plane.

3.260. A conducting current-carrying plane is placed in an external uniform magnetic field. As a result, the magnetic induction becomes equal to B_1 on one side of the plane and to B_2, on the other. Find the magnetic force acting per unit area of the plane in the cases illustrated in Fig. 3.72. Determine the direction of the current in the plane in each case.

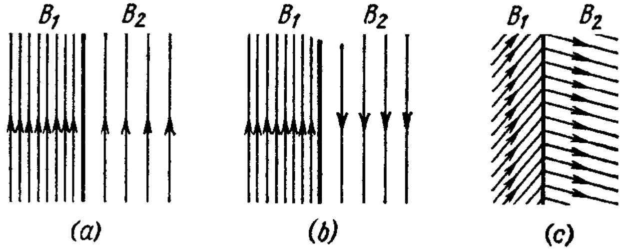


Fig. 3.72.

3.261. In an electromagnetic pump designed for transferring molten metals a pipe section with metal is located in a uniform magnetic field of induction B (Fig. 3.73). A current I is made to flow across this pipe section in the direction perpendicular both to the vector $\mathbf{B}$ and to the axis of the pipe. Find the gauge pressure produced by the pump if $B = 0.10$ T, $I = 100$ A, and $a = 2.0$ cm.

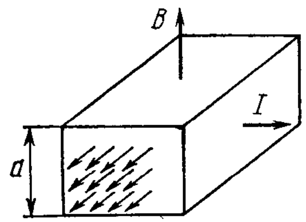


Fig. 3.73.

3.262. A current I flows in a long thin-walled cylinder of radius R. What pressure do the walls of the cylinder experience?

3.263. What pressure does the lateral surface of a long straight solenoid with n turns per unit length experience when a current I flows through it?

3.264. A current I flows in a long single-layer solenoid with cross-sectional radius R. The number of turns per unit length of the solenoid equals n. Find the limiting current at which the winding may rupture if the tensile strength of the wire is equal to F_{lim}.

3.265. A parallel-plate capacitor with area of each plate equal to S and the separation between them to d is put into a stream of conducting liquid with resistivity ρ. The liquid moves parallel to the plates with a constant velocity v. The whole system is located in a uniform magnetic field of induction B, vector $\mathbf{B}$ being parallel to the plates and perpendicular to the stream direction. The capacitor plates are interconnected by means of an external resistance R. What amount of power is generated in that resistance? At what value of R is the generated power the highest? What is this highest power equal to?

3.266. A straight round copper conductor of radius $R = 5.0$ mm carries a current $I = 50$ A. Find the potential difference between the axis of the conductor and its surface. The concentration of the conduction electrons in copper is equal to $n = 0.9 \cdot 10^{23}$ cm^{-3}.

3.267. In Hall effect measurements in a sodium conductor the strength of a transverse field was found to be equal to $E = 5.0$ μV/cm with a current density $j = 200$ A/cm^2 and magnetic induction $B = 1.00$ T. Find the concentration of the conduction electrons and its ratio to the total number of atoms in the given conductor.

3.268. Find the mobility of the conduction electrons in a copper conductor if in Hall effect measurements performed in the magnetic field of induction $B = 100$ mT the transverse electric field strength of the given conductor turned out to be $\eta = 3.1 \cdot 10^3$ times less than that of the longitudinal electric field.

3.269. A small current-carrying loop is located at a distance r from a long straight conductor with current I. The magnetic moment

of the loop is equal to $\mathbf{p}_m$. Find the magnitude and direction of the force vector applied to the loop if the vector $\mathbf{p}_m$

(a) is parallel to the straight conductor;

(b) is oriented along the radius vector $\mathbf{r}$;

(c) coincides in direction with the magnetic field produced by the current I at the point where the loop is located.

3.270. A small current-carrying coil having a magnetic moment $\mathbf{p}_m$ is located at the axis of a round loop of radius R with current I flowing through it. Find the magnitude of the vector force applied to the coil if its distance from the centre of the loop is equal to x and the vector $\mathbf{p}_m$ coincides in direction with the axis of the loop.

3.271. Find the interaction force of two coils with magnetic moments $p_{1m} = 4.0\ \text{mA}\cdot\text{m}^2$ and $p_{2m} = 6.0\ \text{mA}\cdot\text{m}^2$ and collinear axes if the separation between the coils is equal to $l = 20$ cm which exceeds considerably their linear dimensions.

3.272. A permanent magnet has the shape of a sufficiently thin disc magnetized along its axis. The radius of the disc is $R = 1.0$ cm. Evaluate the magnitude of a molecular current I' flowing along the rim of the disc if the magnetic induction at the point on the axis of the disc, lying at a distance $x = 10$ cm from its centre, is equal to $B = 30\ \mu\text{T}$.

3.273. The magnetic induction in vacuum at a plane surface of a uniform isotropic magnetic is equal to B, the vector $\mathbf{B}$ forming an angle α with the normal of the surface. The permeability of the magnetic is equal to μ. Find the magnitude of the magnetic induction B' in the magnetic in the vicinity of its surface.

3.274. The magnetic induction in vacuum at a plane surface of a magnetic is equal to B and the vector $\mathbf{B}$ forms an angle θ with the

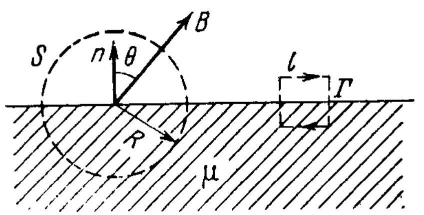


Fig. 3.74.

normal $\mathbf{n}$ of the surface (Fig. 3.74). The permeability of the magnetic is equal to μ. Find:

(a) the flux of the vector $\mathbf{H}$ through the spherical surface S of radius R, whose centre lies on the surface of the magnetic;

(b) the circulation of the vector $\mathbf{B}$ around the square path Γ with side l located as shown in the figure.

3.275. A direct current I flows in a long round uniform cylindrical wire made of paramagnetic with susceptibility χ. Find:

(a) the surface molecular current I'_s;

(b) the volume molecular current I'_v.

How are these currents directed toward each other?

3.276. Half of an infinitely long straight current-carrying solenoid is filled with magnetic substance as shown in Fig. 3.75. Draw the

Fig. 3.75.

approximate plots of magnetic induction B, strength H, and magnetization J on the axis as functions of x.

3.277. An infinitely long wire with a current I flowing in it is located in the boundary plane between two non-conducting media with permeabilities μ_1 and μ_2. Find the modulus of the magnetic induction vector throughout the space as a function of the distance r from the wire. It should be borne in mind that the lines of the vector $\mathbf{B}$ are circles whose centres lie on the axis of the wire.

3.278. A round current-carrying loop lies in the plane boundary between magnetic and vacuum. The permeability of the magnetic is equal to μ. Find the magnetic induction $\mathbf{B}$ at an arbitrary point on the axis of the loop if in the absence of the magnetic the magnetic induction at the same point becomes equal to $\mathbf{B}_0$. Generalize the obtained result to all points of the field.

3.279. When a ball made of uniform magnetic is introduced into an external uniform magnetic field with induction $\mathbf{B}_0$, it gets uniformly magnetized. Find the magnetic induction $\mathbf{B}$ inside the ball with permeability μ; recall that the magnetic field inside a uniformly magnetized ball is uniform and its strength is equal to $\mathbf{H}' = -\mathbf{J}/3$, where $\mathbf{J}$ is the magnetization.

3.280. $N = 300$ turns of thin wire are uniformly wound on a permanent magnet shaped as a cylinder whose length is equal to $l = 15$ cm. When a current $I = 3.0$ A was passed through the wiring the field outside the magnet disappeared. Find the coercive force H_c of the material from which the magnet was manufactured.

3.281. A permanent magnet is shaped as a ring with a narrow gap between the poles. The mean diameter of the ring equals $d = 20$ cm. The width of the gap is equal to $b = 2.0$ mm and the magnetic induction in the gap is equal to $B = 40$ mT. Assuming that the scattering of the magnetic flux at the gap edges is negligible, find the modulus of the magnetic field strength vector inside the magnet.

3.282. An iron core shaped as a tore with mean radius $R = 250$ mm supports a winding with the total number of turns $N = 1000$. The core has a cross-cut of width $b = 1.00$ mm. With a current $I = 0.85$ A flowing through the winding, the magnetic induction in the gap is equal to $B = 0.75$ T. Assuming the scattering of the magnetic flux at the gap edges to be negligible, find the permeability of iron under these conditions.

3.283. Fig. 3.76 illustrates a basic magnetization curve of iron (commercial purity grade). Using this plot, draw the permeability

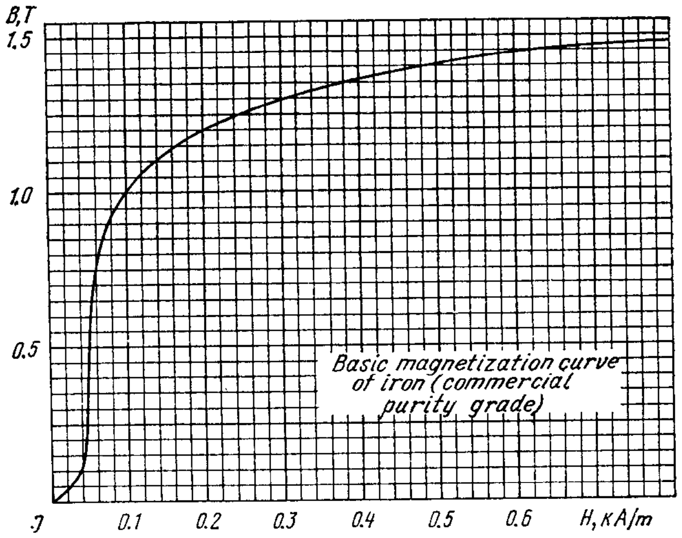


Fig. 3.76.

μ as a function of the magnetic field strength H. At what value of H is the permeability the greatest? What is μ_{max} equal to?

3.284. A thin iron ring with mean diameter $d = 50$ cm supports a winding consisting of $N = 800$ turns carrying current $I = 3.0$ A. The ring has a cross-cut of width $b = 2.0$ mm. Neglecting the scattering of the magnetic flux at the gap edges, and using the plot shown in Fig. 3.76, find the permeability of iron under these conditions.

3.285. A long thin cylindrical rod made of paramagnetic with magnetic susceptibility χ and having a cross-sectional area S is located along the axis of a current-carrying coil. One end of the rod is located at the coil centre where the magnetic induction is equal to B whereas the other end is located in the region where the magnetic field is practically absent. What is the force that the coil exerts on the rod?

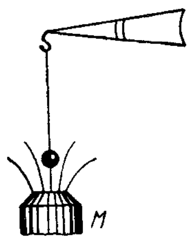


Fig. 3.77.

3.286. In the arrangement shown in Fig. 3.77 it is possible to measure (by means of a balance) the force with which a paramagnetic ball of volume $V = 41$ mm^3 is attrabted to a pole of the electromagnet M. The magnetic induction at the axis of the poleshoe depends on the height x as $B = B_0 \exp(-ax^2)$, where $B_0 = 1.50$ T, $a = 100$ m^{-2}. Find:

(a) at what height x_m the ball experiences the maximum attraction;

(b) the magnetic susceptibility of the paramagnetic if the maximum attraction force equals $F_{max} = 160$ μN.

3.287. A small ball of volume V made of paramagnetic with susceptibility χ was slowly displaced along the axis of a current-carrying coil from the point where the magnetic induction equals B out to the region where the magnetic field is practically absent. What amount of work was performed during this process?

3.6. ELECTROMAGNETIC INDUCTION. MAXWELL'S EQUATIONS

• Faraday's law of electromagnetic induction:

$$\mathscr{E}_i = -\frac{d\Phi}{dt} \tag{3.6a}$$

• In the case of a solenoid and doughnut coil:

$$\Phi = N\Phi_1, \tag{3.6b}$$

where N is the number of turns, Φ_1 is the magnetic flux through each turn.

• Inductance of a solenoid:

$$L = \mu\mu_0 n^2 V. \tag{3.6c}$$

• Intrinsic energy of a current and interaction energy of two currents:

$$W = \frac{LI^2}{2}, \qquad W_{12} = L_{12} I_1 I_2. \tag{3.6d}$$

• Volume density of magnetic field energy:

$$w = \frac{B^2}{2\mu\mu_0} = \frac{\mathbf{BH}}{2}. \tag{3.6e}$$

• Displacement current density:

$$\mathbf{j}_{dis} = \frac{\partial \mathbf{B}}{\partial t}. \tag{3.6f}$$

• Maxwell's equations in differential form:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \cdot \mathbf{B} = 0, \\ \nabla \times \mathbf{H} &= \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}, \qquad \nabla \cdot \mathbf{D} = \rho, \end{aligned} \tag{3.6g}$$

where $\nabla \times \equiv$ rot (the rotor) and $\nabla \cdot \equiv$ div (the divergence).

• Field transformation formulas for transition from a reference frame K to a reference frame K' moving with the velocity $\mathbf{v}_0$ relative to it.

In the case $v_0 \ll c$

$$\mathbf{E}' = \mathbf{E} + [\mathbf{v}_0\mathbf{B}], \quad \mathbf{B}' = \mathbf{B} - [\mathbf{v}_0\mathbf{E}]/c^2 \tag{3.6h}$$

In the general case

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel}, & \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel}, \\ \mathbf{E}'_{\perp} &= \frac{\mathbf{E}_{\perp} + [\mathbf{v}_0\mathbf{B}]}{\sqrt{1-(v_0/c)^2}}, & \mathbf{B}'_{\perp} &= \frac{\mathbf{B}_{\perp} - [\mathbf{v}_0\mathbf{E}]/c^2}{\sqrt{1-(v_0/c)^2}}, \end{aligned} \tag{3.6i}$$

where the symbols $\parallel$ and $\perp$ denote the field components, respectively parallel and perpendicular to the vector $\mathbf{v}_0$.

3.288. A wire bent as a parabola $y = ax^2$ is located in a uniform magnetic field of induction B, the vector **B** being perpendicular to the plane x, y. At the moment $t = 0$ a connector starts sliding translationwise from the parabola apex with a constant acceleration w (Fig. 3.78). Find the emf of electromagnetic induction in the loop thus formed as a function of y.

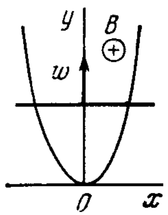


Fig. 3.78.

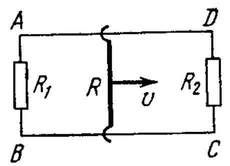


Fig. 3.79.

3.289. A rectangular loop with a sliding connector of length l is located in a uniform magnetic field perpendicular to the loop plane (Fig. 3.79). The magnetic induction is equal to B. The connector has an electric resistance R, the sides AB and CD have resistances R_1 and R_2 respectively. Neglecting the self-inductance of the loop, find the current flowing in the connector during its motion with a constant velocity v.

3.290. A metal disc of radius $a = 25$ cm rotates with a constant angular velocity $\omega = 130$ rad/s about its axis. Find the potential difference between the centre and the rim of the disc if

(a) the external magnetic field is absent;

(b) the external uniform magnetic field of induction $B = 5.0$ mT is directed perpendicular to the disc.

3.291. A thin wire AC shaped as a semi-circle of diameter $d = 20$ cm rotates with a constant angular velocity $\omega = 100$ rad/s in a uniform magnetic field of induction $B = 5.0$ mT, with $\boldsymbol{\omega} \uparrow\uparrow \mathbf{B}$. The rotation axis passes through the end A of the wire and is perpendicular to the diameter AC. Find the value of a line integral $\int \mathbf{E}\, d\mathbf{r}$ along the wire from point A to point C. Generalize the obtained result.

3.292. A wire loop enclosing a semi-circle of radius a is located on the boundary of a uniform magnetic field of induction B (Fig. 3.80). At the moment $t = 0$ the loop is set into rotation with a constant angular acceleration β about an axis O coinciding with a line of vector **B** on the boundary. Find the emf induced in the loop as a function of time t. Draw the approximate plot of this function. The arrow in the figure shows the emf direction taken to be positive.

3.293. A long straight wire carrying a current I and a П-shaped conductor with sliding connector are located in the same plane as shown in Fig. 3.81. The connector of length l and resistance R slides to the right with a constant velocity v. Find the current induced in

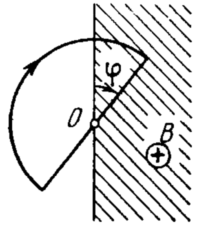


Fig. 3.80.

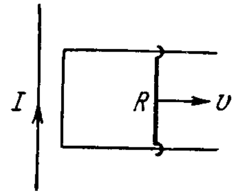


Fig. 3.81.

the loop as a function of separation r between the connector and the straight wire. The resistance of the П-shaped conductor and the self-inductance of the loop are assumed to be negligible.

3.294. A square frame with side a and a long straight wire carrying a current I are located in the same plane as shown in Fig. 3.82. The frame translates to the right with a constant velocity v. Find the emf induced in the frame as a function of distance x.

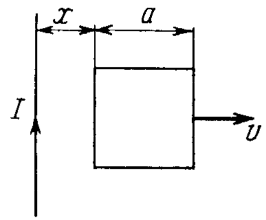


Fig. 3.82.

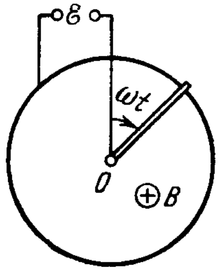


Fig. 3.83.

3.295. A metal rod of mass m can rotate about a horizontal axis O, sliding along a circular conductor of radius a (Fig. 3.83). The arrangement is located in a uniform magnetic field of induction B directed perpendicular to the ring plane. The axis and the ring are connected to an emf source to form a circuit of resistance R. Neglecting the friction, circuit inductance, and ring resistance, find the law according to which the source emf must vary to make the rod rotate with a constant angular velocity ω.

3.296. A copper connector of mass m slides down two smooth copper bars, set at an angle α to the horizontal, due to gravity (Fig. 3.84). At the top the bars are interconnected through a resistance R. The separation between the bars is equal to l. The system is located in a uniform magnetic field of induction B, perpendicular to the plane in which the connector slides. The resistances of the bars, the connector and the sliding contacts, as well as the self-inductance of the loop, are assumed to be negligible. Find the steady-state velocity of the connector.

3.297. The system differs from the one examined in the foregoing problem (Fig. 3.84) by a capacitor of capacitance C replacing the resistance R. Find the acceleration of the connector.

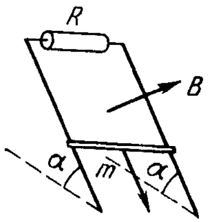


Fig. 3.84.

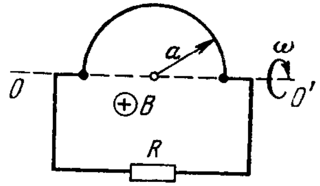


Fig. 3.85.

3.298. A wire shaped as a semi-circle of radius a rotates about an axis OO' with an angular velocity ω in a uniform magnetic field of induction B (Fig. 3.85). The rotation axis is perpendicular to the field direction. The total resistance of the circuit is equal to R. Neglecting the magnetic field of the induced current, find the mean amount of thermal power being generated in the loop during a rotation period.

3.299. A small coil is introduced between the poles of an electromagnet so that its axis coincides with the magnetic field direction. The cross-sectional area of the coil is equal to $S = 3.0$ mm^2, the number of turns is $N = 60$. When the coil turns through 180° about its diameter, a ballistic galvanometer connected to the coil indicates a charge $q = 4.5$ μC flowing through it. Find the magnetic induction magnitude between the poles provided the total resistance of the electric circuit equals $R = 40$ Ω.

3.300. A square wire frame with side a and a straight conductor carrying a constant current I are located in the same plane (Fig. 3.86).

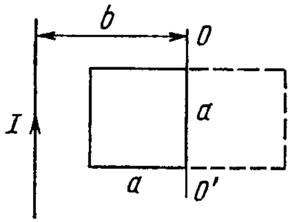


Fig. 3.86.

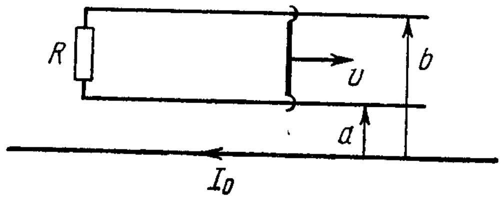


Fig. 3.87.

The inductance and the resistance of the frame are equal to L and R respectively. The frame was turned through 180° about the axis OO' separated from the current-carrying conductor by a distance b. Find the electric charge having flown through the frame.

3.301. A long straight wire carries a current I_0. At distances a and b from it there are two other wires, parallel to the former one, which are interconnected by a resistance R (Fig. 3.87). A connector slides without friction along the wires with a constant velocity v. Assuming the resistances of the wires, the connector, the sliding contacts, and the self-inductance of the freme to be negligible, find:

(a) the magnitude and the direction of the current induced in the connector;

(b) the force required to maintain the connector's velocity constant.

3.302. A conducting rod AB of mass m slides without friction over two long conducting rails separated by a distance l (Fig. 3.88). At the left end the rails are interconnected by a resistance R. The system is located in a uniform magnetic field perpendicular to the plane of the loop. At the moment $t = 0$ the rod AB starts moving to the right with an initial velocity v_0. Neglecting the resistances of the rails and the rod AB, as well as the self-inductance, find:

(a) the distance covered by the rod until it comes to a standstill;

(b) the amount of heat generated in the resistance R during this process.

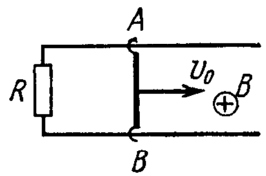


Fig. 3.88.

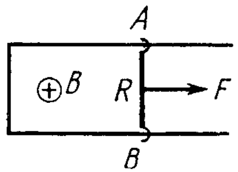


Fig. 3.89.

3.303. A connector AB can slide without friction along a П-shaped conductor located in a horizontal plane (Fig. 3.89). The connector has a length l, mass m, and resistance R. The whole system is located in a uniform magnetic field of induction B directed vertically. At the moment $t = 0$ a constant horizontal force F starts acting on the connector shifting it translationwise to the right. Find how the velocity of the connector varies with time t. The inductance of the loop and the resistance of the П-shaped conductor are assumed to be negligible.

3.304. Fig. 3.90 illustrates plane figures made of thin conductors which are located in a uniform magnetic field directed away from a

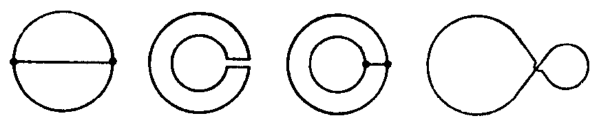


Fig. 3.90.

reader beyond the plane of the drawing. The magnetic induction starts diminishing. Find how the currents induced in these loops are directed.

3.305. A plane loop shown in Fig. 3.91 is shaped as two squares with sides $a = 20$ cm and $b = 10$ cm and is introduced into a uniform magnetic field at right angles to the loop's plane. The magnetic induction varies with time as $B = B_0 \sin \omega t$, where $B_0 = 10$ mT and $\omega = 100\ \text{s}^{-1}$. Find the amplitude of the current induced in the loop if its resistance per unit length is equal to $\rho = 50$ mΩ/m. The inductance of the loop is to be neglected.

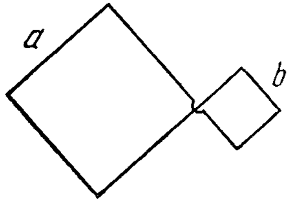


Fig. 3.91.

3.306. A plane spiral with a great number N of turns wound tightly to one another is located in a uniform magnetic field perpendicular to the spiral's plane. The outside radius of the spiral's turns is equal to a. The magnetic induction varies with time as $B = B_0 \sin \omega t$, where B_0 and ω are constants. Find the amplitude of emf induced in the spiral.

3.307. A Π-shaped conductor is located in a uniform magnetic field perpendicular to the plane of the conductor and varying with time at the rate $\dot{B} = 0.10$ T/s. A conducting connector starts moving with an acceleration $w = 10$ cm/s² along the parallel bars of the conductor. The length of the connector is equal to $l = 20$ cm. Find the emf induced in the loop $t = 2.0$ s after the beginning of the motion, if at the moment $t = 0$ the loop area and the magnetic induction are equal to zero. The inductance of the loop is to be neglected.

3.308. In a long straight solenoid with cross-sectional radius a and number of turns per unit length n a current varies with a constant velocity $\dot{I}$ A/s. Find the magnitude of the eddy current field strength as a function of the distance r from the solenoid axis. Draw the approximate plot of this function.

3.309. A long straight solenoid of cross-sectional diameter $d = 5$ cm and with $n = 20$ turns per one cm of its length has a round turn of copper wire of cross-sectional area $S = 1.0$ mm² tightly put on its winding. Find the current flowing in the turn if the current in the solenoid winding is increased with a constant velocity $\dot{I} = 100$ A/s. The inductance of the turn is to be neglected.

3.310. A long solenoid of cross-sectional radius a has a thin insulated wire ring tightly put on its winding; one half of the ring has the resistance η times that of the other half. The magnetic induction produced by the solenoid varies with time as $B = bt$, where b is a constant. Find the magnitude of the electric field strength in the ring.

3.311. A thin non-conducting ring of mass m carrying a charge q can freely rotate about its axis. At the initial moment the ring was at rest and no magnetic field was present. Then a practically uniform magnetic field was switched on, which was perpendicular to the plane of the ring and increased with time according to a certain law $\mathbf{B}(t)$. Find the angular velocity $\boldsymbol{\omega}$ of the ring as a function of the induction $\mathbf{B}(t)$.

3.312. A thin wire ring of radius a and resistance r is located inside a long solenoid so that their axes coincide. The length of the solenoid is equal to l, its cross-sectional radius, to b. At a certain moment the solenoid was connected to a source of a constant voltage V. The total resistance of the circuit is equal to R. Assuming the inductance of the ring to be negligible, find the maximum value of the radial force acting per unit length of the ring.

3.313. A magnetic flux through a stationary loop with a resistance R varies during the time interval τ as $\Phi = at(\tau - t)$. Find the amount of heat generated in the loop during that time. The inductance of the loop is to be neglected.

3.314. In the middle of a long solenoid there is a coaxial ring of square cross-section, made of conducting material with resistivity ρ. The thickness of the ring is equal to h, its inside and outside radii are equal to a and b respectively. Find the current induced in the ring if the magnetic induction produced by the solenoid varies with time as $B = \beta t$, where β is a constant. The inductance of the ring is to be neglected.

3.315. How many metres of a thin wire are required to manufacture a solenoid of length $l_0 = 100$ cm and inductance $L = 1.0$ mH if the solenoid's cross-sectional diameter is considerably less than its length?

3.316. Find the inductance of a solenoid of length l whose winding is made of copper wire of mass m. The winding resistance is equal to R. The solenoid diameter is considerably less than its length.

3.317. A coil of inductance $L = 300$ mH and resistance $R = 140$ mΩ is connected to a constant voltage source. How soon will the coil current reach $\eta = 50\%$ of the steady-state value?

3.318. Calculate the time constant τ of a straight solenoid of length $l = 1.0$ m having a single-layer winding of copper wire whose total mass is equal to $m = 1.0$ kg. The cross-sectional diameter of the solenoid is assumed to be considerably less than its length.

Note. The time constant τ is the ratio L/R, where L is inductance and R is active resistance.

3.319. Find the inductance of a unit length of a cable consisting of two thin-walled coaxial metallic cylinders if the radius of the outside cylinder is $\eta = 3.6$ times that of the inside one. The permeability of a medium between the cylinders is assumed to be equal to unity.

3.320. Calculate the inductance of a doughnut solenoid whose inside radius is equal to b and cross-section has the form of a square with side a. The solenoid winding consists of N turns. The space inside the solenoid is filled up with uniform paramagnetic having permeability μ.

3.321. Calculate the inductance of a unit length of a double tape line (Fig. 3.92) if the tapes are separated by a distance h which is considerably less than their width b, namely, $b/h = 50$.

3.322. Find the inductance of a unit length of a double line if the radius of each wire is η times less than the distance between the axes of the wires. The field inside the wires is to be neglected, the permeability is assumed to be equal to unity throughout, and $\eta \gg 1$.

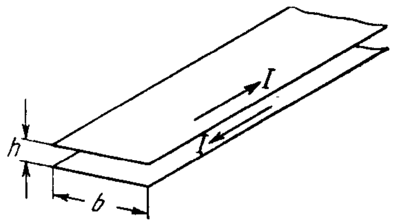


Fig. 3.92.

3.323. A superconducting round ring of radius a and inductance L was located in a uniform magnetic field of induction B. The ring plane was parallel to the vector **B**, and the current in the ring was equal to zero. Then the ring was turned through 90° so that its plane became perpendicular to the field. Find:

(a) the current induced in the ring after the turn;
(b) the work performed during the turn.

3.324. A current $I_0 = 1.9$ A flows in a long closed solenoid. The wire it is wound of is in a superconducting state. Find the current flowing in the solenoid when the length of the solenoid is increased by $\eta = 5\%$.

3.325. A ring of radius $a = 50$ mm made of thin wire of radius $b = 1.0$ mm was located in a uniform magnetic field with induction $B = 0.50$ mT so that the ring plane was perpendicular to the vector **B**. Then the ring was cooled down to a superconducting state, and the magnetic field was switched off. Find the ring current after that. Note that the inductance of a thin ring along which the surface current flows is equal to $L = \mu_0 a \left(\ln \frac{8a}{b} - 2\right)$.

3.326. A closed circuit consists of a source of constant emf $\mathscr{E}$ and a choke coil of inductance L connected in series. The active resistance of the whole circuit is equal to R. At the moment $t = 0$ the choke coil inductance was decreased abruptly η times. Find the current in the circuit as a function of time t.

Instruction. During a stepwise change of inductance the total magnetic flux (flux linkage) remains constant.

3.327. Find the time dependence of the current flowing through the inductance L of the circuit shown in Fig. 3.93 after the switch Sw is shorted at the moment $t = 0$.

3.328. In the circuit shown in Fig. 3.94 an emf $\mathscr{E}$, a resistance R, and coil inductances L_1 and L_2 are known. The internal resistance of the source and the coil resistances are negligible. Find the steady-state currents in the coils after the switch Sw was shorted.

3.329. Calculate the mutual inductance of a long straight wire and a rectangular frame with sides a and b. The frame and the wire lie

in the same plane, with the side b being closest to the wire, separated by a distance l from it and oriented parallel to it.

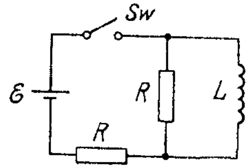


Fig. 3.93.

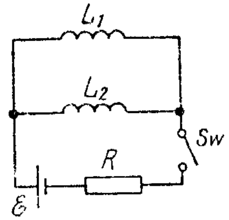


Fig. 3.94.

3.330. Determine the mutual inductance of a doughnut coil and an infinite straight wire passing along its axis. The coil has a rectangular cross-section, its inside radius is equal to a and the outside one, to b. The length of the doughnut's cross-sectional side parallel to the wire is equal to h. The coil has N turns. The system is located in a uniform magnetic with permeability μ.

3.331. Two thin concentric wires shaped as circles with radii a and b lie in the same plane. Allowing for $a \ll b$, find:

(a) their mutual inductance;
(b) the magnetic flux through the surface enclosed by the outside wire, when the inside wire carries a current I.

3.332. A small cylindrical magnet M (Fig. 3.95) is placed in the centre of a thin coil of radius a consisting of N turns. The coil is connected to a ballistic galvanometer. The active resistance of the whole circuit is equal to R. Find the magnetic moment of the magnet if its removal from the coil results in a charge q flowing through the galvanometer.

3.333. Find the approximate formula expressing the mutual inductance of two thin coaxial loops of the same radius a if their centres are separated by a distance l, with $l \gg a$.

Fig. 3.95.

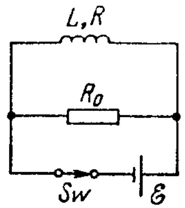


Fig. 3.96.

3.334. There are two stationary loops with mutual inductance L_{12}. The current in one of the loops starts to be varied as $I_1 = \alpha t$, where α is a constant, t is time. Find the time dependence $I_2(t)$ of the current in the other loop whose inductance is L_2 and resistance R.

3.335. A coil of inductance $L = 2.0$ μH and resistance $R = 1.0$ Ω is connected to a source of constant emf $\mathscr{E} = 3.0$ V (Fig. 3.96). A

resistance $R_0 = 2.0\ \Omega$ is connected in parallel with the coil. Find the amount of heat generated in the coil after the switch Sw is disconnected. The internal resistance of the source is negligible.

3.336. An iron tore supports $N = 500$ turns. Find the magnetic field energy if a current $I = 2.0$ A produces a magnetic flux across the tore's cross-section equal to $\Phi = 1.0$ mWb.

3.337. An iron core shaped as a doughnut with round cross-section of radius $a = 3.0$ cm carries a winding of $N = 1000$ turns through which a current $I = 1.0$ A flows. The mean radius of the doughnut is $b = 32$ cm. Using the plot in Fig. 3.76, find the magnetic energy stored up in the core. A field strength H is supposed to be the same throughout the cross-section and equal to its magnitude in the centre of the cross-section.

3.338. A thin ring made of a magnetic has a mean diameter $d = 30$ cm and supports a winding of $N = 800$ turns. The cross-sectional area of the ring is equal to $S = 5.0\ \text{cm}^2$. The ring has a cross-cut of width $b = 2.0$ mm. When the winding carries a certain current, the permeability of the magnetic equals $\mu = 1400$. Neglecting the dissipation of magnetic flux at the gap edges, find:

(a) the ratio of magnetic energies in the gap and in the magnetic;

(b) the inductance of the system; do it in two ways: using the flux and using the energy of the field.

3.339. A long cylinder of radius a carrying a uniform surface charge rotates about its axis with an angular velocity ω. Find the magnetic field energy per unit length of the cylinder if the linear charge density equals λ and $\mu = 1$.

3.340. At what magnitude of the electric field strength in vacuum the volume energy density of this field is the same as that of the magnetic field with induction $B = 1.0$ T (also in vacuum).

3.341. A thin uniformly charged ring of radius $a = 10$ cm rotates about its axis with an angular velocity $\omega = 100$ rad/s. Find the ratio of volume energy densities of magnetic and electric fields on the axis of the ring at a point removed from its centre by a distance $l = a$.

3.342. Using the expression for volume density of magnetic energy, demonstrate that the amount of work contributed to magnetization of a unit volume of para- or diamagnetic, is equal to $A = -\mathbf{JB}/2$.

3.343. Two identical coils, each of inductance L, are interconnected (a) in series, (b) in parallel. Assuming the mutual inductance of the coils to be negligible, find the inductance of the system in both cases.

3.344. Two solenoids of equal length and almost equal cross-sectional area are fully inserted into one another. Find their mutual inductance if their inductances are equal to L_1 and L_2.

3.345. Demonstrate that the magnetic energy of interaction of two current-carrying loops located in vacuum can be represented as $W_{ia} = (1/\mu_0) \int \mathbf{B}_1\mathbf{B}_2\, dV$, where $\mathbf{B}_1$ and $\mathbf{B}_2$ are the magnetic inductions

within a volume element dV, produced individually by the currents of the first and the second loop respectively.

3.346. Find the interaction energy of two loops carrying currents I_1 and I_2 if both loops are shaped as circles of radii a and b, with $a \ll b$. The loops' centres are located at the same point and their planes form an angle θ between them.

3.347. The space between two concentric metallic spheres is filled up with a uniform poorly conducting medium of resistivity ρ and permittivity ε. At the moment $t = 0$ the inside sphere obtains a certain charge. Find:

(a) the relation between the vectors of displacement current density and conduction current density at an arbitrary point of the medium at the same moment of time;

(b) the displacement current across an arbitrary closed surface wholly located in the medium and enclosing the internal sphere, if at the given moment of time the charge of that sphere is equal to q.

3.348. A parallel-plate capacitor is formed by two discs with a uniform poorly conducting medium between them. The capacitor was initially charged and then disconnected from a voltage source. Neglecting the edge effects, show that there is no magnetic field between capacitor plates.

3.349. A parallel-plate air condenser whose each plate has an area $S = 100\ \text{cm}^2$ is connected in series to an ac circuit. Find the electric field strength amplitude in the capacitor if the sinusoidal current amplitude in lead wires is equal to $I_m = 1.0$ mA and the current frequency equals $\omega = 1.6 \cdot 10^7\ \text{s}^{-1}$.

3.350. The space between the electrodes of a parallel-plate capacitor is filled with a uniform poorly conducting medium of conductivity σ and permittivity ε. The capacitor plates shaped as round discs are separated by a distance d. Neglecting the edge effects, find the magnetic field strength between the plates at a distance r from their axis if an ac voltage $V = V_m \cos \omega t$ is applied to the capacitor.

3.351. A long straight solenoid has n turns per unit length. An alternating current $I = I_m \sin \omega t$ flows through it. Find the displacement current density as a function of the distance r from the solenoid axis. The cross-sectional radius of the solenoid equals R.

3.352. A point charge q moves with a non-relativistic velocity $\mathbf{v} = \text{const}$. Find the displacement current density $\mathbf{j}_d$ at a point located at a distance r from the charge on a straight line

(a) coinciding with the charge path;

(b) perpendicular to the path and passing through the charge.

3.353. A thin wire ring of radius a carrying a charge q approaches the observation point P so that its centre moves rectilinearly with a constant velocity v. The plane of the ring remains perpendicular to the motion direction. At what distance x_m from the point P will the ring be located at the moment when the displacement current density at the point P becomes maximum? What is the magnitude of this maximum density?

3.354. A point charge q moves with a non-relativistic velocity $\mathbf{v} = \text{const}$. Applying the theorem for the circulation of the vector $\mathbf{H}$ around the dotted circle shown in Fig. 3.97, find $\mathbf{H}$ at the point A as a function of a radius vector $\mathbf{r}$ and velocity $\mathbf{v}$ of the charge.

3.355. Using Maxwell's equations, show that

(a) a time-dependent magnetic field cannot exist without an electric field;

(b) a uniform electric field cannot exist in the presence of a time-dependent magnetic field;

(c) inside an empty cavity a uniform electric (or magnetic) field can be time-dependent.

3.356. Demonstrate that the law of electric charge conservation, i.e. $\nabla \cdot \mathbf{j} = -\partial\rho/\partial t$, follows from Maxwell's equations.

3.357. Demonstrate that Maxwell's equations $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ and $\nabla \cdot \mathbf{B} = 0$ are compatible, i.e. the first one does not contradict the second one.

3.358. In a certain region of the inertial reference frame there is magnetic field with induction B rotating with angular velocity $\boldsymbol{\omega}$. Find $\nabla \times \mathbf{E}$ in this region as a function of vectors $\boldsymbol{\omega}$ and $\mathbf{B}$.

3.359. In the inertial reference frame K there is a uniform magnetic field with induction $\mathbf{B}$. Find the electric field strength in the frame K' which moves relative to the frame K with a non-relativistic velocity $\mathbf{v}$, with $\mathbf{v} \perp \mathbf{B}$. To solve this problem, consider the forces acting on an imaginary charge in both reference frames at the moment when the velocity of the charge in the frame K' is equal to zero.

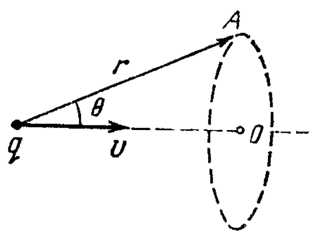


Fig. 3.97.

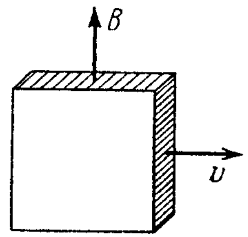


Fig. 3.98.

3.360. A large plate of non-ferromagnetic material moves with a constant velocity $v = 90$ cm/s in a uniform magnetic field with induction $B = 50$ mT as shown in Fig. 3.98. Find the surface density of electric charges appearing on the plate as a result of its motion.

3.361. A long solid aluminum cylinder of radius $a = 5.0$ cm rotates about its axis in a uniform magnetic field with induction $B = 10$ mT. The angular velocity of rotation equals $\omega = 45$ rad/s, with $\boldsymbol{\omega} \uparrow\uparrow \mathbf{B}$. Neglecting the magnetic field of appearing charges, find their space and surface densities.

3.362. A non-relativistic point charge q moves with a constant velocity $\mathbf{v}$. Using the field transformation formulas, find the magnetic induction $\mathbf{B}$ produced by this charge at the point whose position relative to the charge is determined by the radius vector $\mathbf{r}$.

3.363. Using Eqs. (3.6h), demonstrate that if in the inertial reference frame K there is only electric or only magnetic field, in any other inertial frame K' both electric and magnetic fields will coexist simultaneously, with $\mathbf{E}' \perp \mathbf{B}'$.

3.364. In an inertial reference frame K there is only magnetic field with induction $\mathbf{B} = b\,(y\mathbf{i} - x\mathbf{j})/(x^2 + y^2)$, where b is a constant, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes. Find the electric field strength $\mathbf{E}'$ in the frame K' moving relative to the frame K with a constant non-relativistic velocity $\mathbf{v} = v\mathbf{k}$; $\mathbf{k}$ is the unit vector of the z axis. The z' axis is assumed to coincide with the z axis. What is the shape of the field $\mathbf{E}'$?

3.365. In an inertial reference frame K there is only electric field of strength $\mathbf{E} = a\,(x\mathbf{i} + y\mathbf{j})/(x^2 + y^2)$, where a is a constant, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes. Find the magnetic induction $\mathbf{B}'$ in the frame K' moving relative to the frame K with a constant non-relativistic velocity $\mathbf{v} = v\mathbf{k}$; $\mathbf{k}$ is the unit vector of the z axis. The z' axis is assumed to coincide with the z axis. What is the shape of the magnetic induction $\mathbf{B}'$?

3.366. Demonstrate that the transformation formulas (3.6h) follow from the formulas (3.6i) at $v_0 \ll c$.

3.367. In an inertial reference frame K there is only a uniform electric field $E = 8$ kV/m in strength. Find the modulus and direction

(a) of the vector $\mathbf{E}'$, (b) of the vector $\mathbf{B}'$ in the inertial reference frame K' moving with a constant velocity $\mathbf{v}$ relative to the frame K at an angle $\alpha = 45°$ to the vector $\mathbf{E}$. The velocity of the frame K' is equal to a $\beta = 0.60$ fraction of the velocity of light.

3.368. Solve a problem differing from the foregoing one by a magnetic field with induction $B = 0.8$ T replacing the electric field.

3.369. Electromagnetic field has two invariant quantities. Using the transformation formulas (3.6i), demonstrate that these quantities are

(a) $\mathbf{EB}$; (b) $E^2 - c^2B^2$.

3.370. In an inertial reference frame K there are two uniform mutually perpendicular fields: an electric field of strength $E = 40$ kV/m and a magnetic field induction $B = 0.20$ mT. Find the electric strength E' (or the magnetic induction B') in the reference frame K' where only one field, electric or magnetic, is observed.

Instruction. Make use of the field invariants cited in the foregoing problem.

3.371. A point charge q moves uniformly and rectilinearly with a relativistic velocity equal to a β fraction of the velocity of light ($\beta = v/c$). Find the electric field strength $\mathbf{E}$ produced by the charge at the point whose radius vector relative to the charge is equal to $\mathbf{r}$ and forms an angle θ with its velocity vector.

3.7. MOTION OF CHARGED PARTICLES IN ELECTRIC AND MAGNETIC FIELDS

- Lorentz force:

$$\mathbf{F} = q\mathbf{E} + q\,[\mathbf{vB}]. \tag{3.7a}$$

- Motion equation of a relativistic particle:

$$\frac{d}{dt}\frac{m_0\mathbf{v}}{\sqrt{1-(v/c)^2}} = \mathbf{F}. \tag{3.7b}$$

- Period of revolution of a charged particle in a uniform magnetic field:

$$T = \frac{2\pi m}{qB}, \tag{3.7c}$$

where m is the relativistic mass of the particle, $m = m_0/\sqrt{1-(v/c)^2}$.

- Betatron condition, that is the condition for an electron to move along a circular orbit in a betatron:

$$B_0 = \frac{1}{2}\langle B\rangle, \tag{3.7d}$$

where B_0 is the magnetic induction at an orbit's point, $\langle B\rangle$ is the mean value of the induction inside the orbit.

3.372. At the moment $t = 0$ an electron leaves one plate of a parallel-plate capacitor with a negligible velocity. An accelerating voltage, varying as $V = at$, where $a = 100$ V/s, is applied between the plates. The separation between the plates is $l = 5.0$ cm. What is the velocity of the electron at the moment it reaches the opposite plate?

3.373. A proton accelerated by a potential difference V gets into the uniform electric field of a parallel-plate capacitor whose plates extend over a length l in the motion direction. The field strength varies with time as $E = at$, where a is a constant. Assuming the proton to be non-relativistic, find the angle between the motion directions of the proton before and after its flight through the capacitor; the proton gets in the field at the moment $t = 0$. The edge effects are to be neglected.

3.374. A particle with specific charge q/m moves rectilinearly due to an electric field $E = E_0 - ax$, where a is a positive constant, x is the distance from the point where the particle was initially at rest. Find:

(a) the distance covered by the particle till the moment it came to a standstill;

(b) the acceleration of the particle at that moment.

3.375. An electron starts moving in a uniform electric field of strength $E = 10$ kV/cm. How soon after the start will the kinetic energy of the electron become equal to its rest energy?

3.376. Determine the acceleration of a relativistic electron moving along a uniform electric field of strength E at the moment when its kinetic energy becomes equal to T.

3.377. At the moment $t = 0$ a relativistic proton flies with a velocity $\mathbf{v}_0$ into the region where there is a uniform transverse electric field of strength $\mathbf{E}$, with $\mathbf{v}_0 \perp \mathbf{E}$. Find the time dependence of

(a) the angle θ between the proton's velocity vector $\mathbf{v}$ and the initial direction of its motion;

(b) the projection v_x of the vector $\mathbf{v}$ on the initial direction of motion.

3.378. A proton accelerated by a potential difference $V = 500$ kV flies through a uniform transverse magnetic field with induction $B = 0.51$ T. The field occupies a region of space $d = 10$ cm in thickness (Fig. 3.99). Find the angle α through which the proton deviates from the initial direction of its motion.

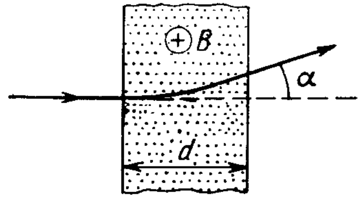


Fig. 3.99.

3.379. A charged particle moves along a circle of radius $r = 100$ mm in a uniform magnetic field with induction $B = 10.0$ mT. Find its velocity and period of revolution if that particle is

(a) a non-relativistic proton;

(b) a relativistic electron.

3.380. A relativistic particle with charge q and rest mass m_0 moves along a circle of radius r in a uniform magnetic field of induction B. Find:

(a) the modulus of the particle's momentum vector;

(b) the kinetic energy of the particle;

(c) the acceleration of the particle.

3.381. Up to what values of kinetic energy does the period of revolution of an electron and a proton in a uniform magnetic field exceed that at non-relativistic velocities by $\eta = 1.0\%$?

3.382. An electron accelerated by a potential difference $V = 1.0$ kV moves in a uniform magnetic field at an angle $\alpha = 30°$ to the vector $\mathbf{B}$ whose modulus is $B = 29$ mT. Find the pitch of the helical trajectory of the electron.

3.383. A slightly divergent beam of non-relativistic charged particles accelerated by a potential difference V propagates from a point A along the axis of a straight solenoid. The beam is brought into focus at a distance l from the point A at two successive values of magnetic induction B_1 and B_2. Find the specific charge q/m of the particles.

3.384. A non-relativistic electron originates at a point A lying on the axis of a straight solenoid and moves with velocity v at an angle α to the axis. The magnetic induction of the field is equal to B. Find the distance r from the axis to the point on the screen into which the electron strikes. The screen is oriented at right angles to the axis and is located at a distance l from the point A.

3.385. From the surface of a round wire of radius a carrying a direct current I an electron escapes with a velocity v_0 perpendicular to the surface. Find what will be the maximum distance of the electron from the axis of the wire before it turns back due to the action of the magnetic field generated by the current.

3.386. A non-relativistic charged particle flies through the electric field of a cylindrical capacitor and gets into a uniform transverse magnetic field with induction B (Fig. 3.100). In the capacitor the particle moves along the arc of a circle, in the magnetic field, along a semi-circle of radius r. The potential difference applied to the capacitor is equal to V, the radii of the electrodes are equal to a and b, with $a < b$. Find the velocity of the particle and its specific charge q/m.

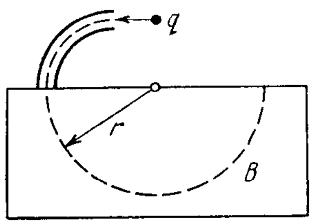


Fig. 3.100.

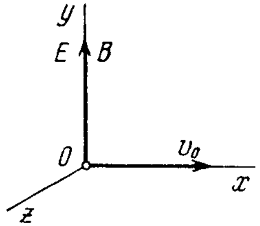


Fig. 3.101.

3.387. Uniform electric and magnetic fields with strength E and induction B respectively are directed along the y axis (Fig. 3.101). A particle with specific charge q/m leaves the origin O in the direction of the x axis with an initial non-relativistic velocity v_0. Find:

(a) the coordinate y_n of the particle when it crosses the y axis for the nth time;

(b) the angle α between the particle's velocity vector and the y axis at that moment.

3.388. A narrow beam of identical ions with specific charge q/m, possessing different velocities, enters the region of space, where there are uniform parallel electric and magnetic fields with strength E and induction B, at the point O (see Fig. 3.101). The beam direction coincides with the x axis at the point O. A plane screen oriented at right angles to the x axis is located at a distance l from the point O. Find the equation of the trace that the ions leave on the screen. Demonstrate that at $z \ll l$ it is the equation of a parabola.

3.389. A non-relativistic proton beam passes without deviation through the region of space where there are uniform transverse mutually perpendicular electric and magnetic fields with $E = 120$ kV/m and $B = 50$ mT. Then the beam strikes a grounded target. Find the force with which the beam acts on the target if the beam current is equal to $I = 0.80$ mA.

3.390. Non-relativistic protons move rectilinearly in the region of space where there are uniform mutually perpendicular electric and magnetic fields with $E = 4.0$ kV/m and $B = 50$ mT. The trajectory of the protons lies in the plane xz (Fig. 3.102) and forms an angle $\varphi = 30°$ with the x axis. Find the pitch of the helical trajectory along which the protons will move after the electric field is switched off.

3.391. A beam of non-relativistic charged particles moves without deviation through the region of space A (Fig. 3.103) where there are transverse mutually perpendicular electric and magnetic fields with

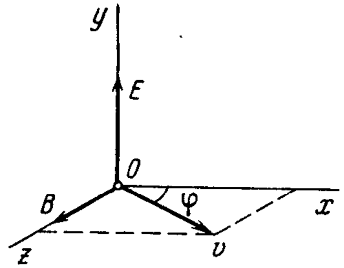


Fig. 3.102.

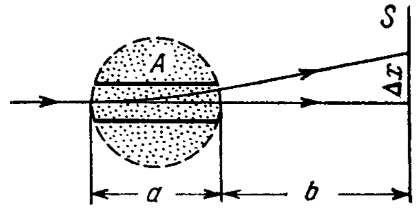


Fig. 3.103.

strength E and induction B. When the magnetic field is switched off, the trace of the beam on the screen S shifts by Δx. Knowing the distances a and b, find the specific charge q/m of the particles.

3.392. A particle with specific charge q/m moves in the region of space where there are uniform mutually perpendicular electric and magnetic fields with strength **E** and induction **B** (Fig. 3.104). At the moment $t = 0$ the particle was located at the point O and had zero velocity. For the non-relativistic case find:

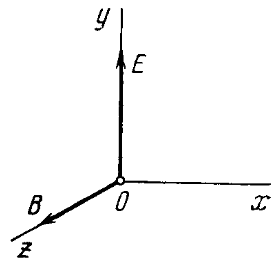


Fig. 3.104.

(a) the law of motion $x(t)$ and $y(t)$ of the particle; the shape of the trajectory;

(b) the length of the segment of the trajectory between two nearest points at which the velocity of the particle turns into zero;

(c) the mean value of the particle's velocity vector projection on the x axis (the drift velocity).

3.393. A system consists of a long cylindrical anode of radius a and a coaxial cylindrical cathode of radius b ($b < a$). A filament located along the axis of the system carries a heating current I producing a magnetic field in the surrounding space. Find the least potential difference between the cathode and anode at which the thermal electrons leaving the cathode without initial velocity start reaching the anode.

3.394. Magnetron is a device consisting of a filament of radius a and a coaxial cylindrical anode of radius b which are located in a uniform magnetic field parallel to the filament. An accelerating potential difference V is applied between the filament and the anode. Find the value of magnetic induction at which the electrons leaving the filament with zero velocity reach the anode.

3.395. A charged particle with specific charge q/m starts moving in the region of space where there are uniform mutually perpendicular electric and magnetic fields. The magnetic field is constant and

has an induction B while the strength of the electric field varies with time as $E = E_m \cos \omega t$, where $\omega = qB/m$. For the non-relativistic case find the law of motion $x(t)$ and $y(t)$ of the particle if at the moment $t = 0$ it was located at the point O (see Fig. 3.104). What is the approximate shape of the trajectory of the particle?

3.396. The cyclotron's oscillator frequency is equal to $\nu = 10$ MHz. Find the effective accelerating voltage applied across the dees of that cyclotron if the distance between the neighbouring trajectories of protons is not less than $\Delta r = 1.0$ cm, with the trajectory radius being equal to $r = 0.5$ m.

3.397. Protons are accelerated in a cyclotron so that the maximum curvature radius of their trajectory is equal to $r = 50$ cm. Find:

(a) the kinetic energy of the protons when the acceleration is completed if the magnetic induction in the cyclotron is $B = 1.0$ T;

(b) the minimum frequency of the cyclotron's oscillator at which the kinetic energy of the protons amounts to $T = 20$ MeV by the end of acceleration.

3.398. Singly charged ions He^+ are accelerated in a cyclotron so that their maximum orbital radius is $r = 60$ cm. The frequency of a cyclotron's oscillator is equal to $\nu = 10.0$ MHz, the effective accelerating voltage across the dees is $V = 50$ kV. Neglecting the gap between the dees, find:

(a) the total time of acceleration of the ion;

(b) the approximate distance covered by the ion in the process of its acceleration.

3.399. Since the period of revolution of electrons in a uniform magnetic field rapidly increases with the growth of energy, a cyclotron is unsuitable for their acceleration. This drawback is rectified in a *microtron* (Fig. 3.105) in which a change ΔT in the period of revolution of an electron is made multiple with the period of accelerating field T_0. How many times has an electron to cross the accelerating gap of a microtron to acquire an energy $W = 4.6$ MeV if $\Delta T = T_0$, the magnetic induction is equal to $B = 107$ mT, and the frequency of accelerating field to $\nu = 3000$ MHz?

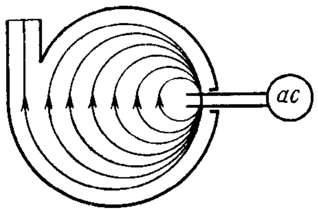


Fig. 3.105.

3.400. The ill effects associated with the variation of the period of revolution of the particle in a cyclotron due to the increase of its energy are eliminated by slow monitoring (modulating) the frequency of accelerating field. According to what law $\omega(t)$ should this frequency be monitored if the magnetic induction is equal to B and the particle acquires an energy ΔW per revolution? The charge of the particle is q and its mass is m.

3.401. A particle with specific charge q/m is located inside a round solenoid at a distance r from its axis. With the current switched into the winding, the magnetic induction of the field generated by the solenoid amounts to B. Find the velocity of the particle and the curvature radius of its trajectory, assuming that during the increase of current flowing in the solenoid the particle shifts by a negligible distance.

3.402. In a betatron the magnetic flux across an equilibrium orbit of radius $r = 25$ cm grows during the acceleration time at practically constant rate $\dot{\Phi} = 5.0$ Wb/s. In the process, the electrons acquire an energy $W = 25$ MeV. Find the number of revolutions made by the electron during the acceleration time and the corresponding distance covered by it.

3.403. Demonstrate that electrons move in a betatron along a round orbit of constant radius provided the magnetic induction on the orbit is equal to half the mean value of that inside the orbit (the betatron condition).

3.404. Using the betatron condition, find the radius of a round orbit of an electron if the magnetic induction is known as a function of distance r from the axis of the field. Examine this problem for the specific case $B = B_0 - ar^2$, where B_0 and a are positive constants.

3.405. Using the betatron condition, demonstrate that the strength of the eddy-current field has the extremum magnitude on an equilibrium orbit.

3.406. In a betatron the magnetic induction on an equilibrium orbit with radius $r = 20$ cm varies during a time interval $\Delta t = 1.0$ ms at practically constant rate from zero to $B = 0.40$ T. Find the energy acquired by the electron per revolution.

3.407. The magnetic induction in a betatron on an equilibrium orbit of radius r varies during the acceleration time at practically constant rate from zero to B. Assuming the initial velocity of the electron to be equal to zero, find:

(a) the energy acquired by the electron during the acceleration time;

(b) the corresponding distance covered by the electron if the acceleration time is equal to Δt.

PART FOUR

OSCILLATIONS AND WAVES

4.1. MECHANICAL OSCILLATIONS

- Harmonic motion equation and its solution:

$$\ddot{x}+\omega_0^2 x=0, \qquad x=a\cos(\omega_0 t+\alpha), \tag{4.1a}$$

where ω_0 is the natural oscillation frequency.

- Damped oscillation equation and its solution:

$$\ddot{x}+2\beta\dot{x}+\omega_0^2 x=0, \qquad x=a_0 e^{-\beta t}\cos(\omega t+\alpha), \tag{4.1b}$$

where β is the damping coefficient, ω is the frequency of damped oscillations:

$$\omega=\sqrt{\omega_0^2-\beta^2}. \tag{4.1c}$$

- Logarithmic damping decrement λ and quality factor Q:

$$\lambda=\beta T, \quad Q=\pi/\lambda, \tag{4.1d}$$

where $T=2\pi/\omega$.

- Forced oscillation equation and its steady-state solution:

$$\ddot{x}+2\beta\dot{x}+\omega_0^2 x=f_0\cos\omega t, \qquad x=a\cos(\omega t-\varphi), \tag{4.1e}$$

where

$$a=\frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}} \qquad \tan\varphi=\frac{2\beta\omega}{\omega_0^2-\omega^2}. \tag{4.1f}$$

- Maximum shift amplitude occurs at

$$\omega_{res}=\sqrt{\omega_0^2-2\beta^2}. \tag{4.1g}$$

4.1. A point oscillates along the x axis according to the law $x = a\cos(\omega t - \pi/4)$. Draw the approximate plots
(a) of displacement x, velocity projection v_x, and acceleration projection w_x as functions of time t;
(b) velocity projection v_x and acceleration projection w_x as functions of the coordinate x.

4.2. A point moves along the x axis according to the law $x = a\sin^2(\omega t - \pi/4)$. Find:
(a) the amplitude and period of oscillations; draw the plot $x(t)$;
(b) the velocity projection v_x as a function of the coordinate x; draw the plot $v_x(x)$.

4.3. A particle performs harmonic oscillations along the x axis about the equilibrium position $x = 0$. The oscillation frequency is $\omega = 4.00\ \text{s}^{-1}$. At a certain moment of time the particle has a coordinate $x_0 = 25.0$ cm and its velocity is equal to $v_{x0} = 100$ cm/s. Find the coordinate x and the velocity v_x of the particle $t = 2.40$ s after that moment.

4.4. Find the angular frequency and the amplitude of harmonic oscillations of a particle if at distances x_1 and x_2 from the equilibrium position its velocity equals v_1 and v_2 respectively.

4.5. A point performs harmonic oscillations along a straight line with a period $T = 0.60$ s and an amplitude $a = 10.0$ cm. Find the mean velocity of the point averaged over the time interval during which it travels a distance $a/2$, starting from
(a) the extreme position;
(b) the equilibrium position.

4.6. At the moment $t = 0$ a point starts oscillating along the x axis according to the law $x = a\sin\omega t$. Find:
(a) the mean value of its velocity vector projection $\langle v_x\rangle$;
(b) the modulus of the mean velocity vector $|\langle \mathbf{v}\rangle|$;
(c) the mean value of the velocity modulus $\langle v\rangle$ averaged over 3/8 of the period after the start.

4.7. A particle moves along the x axis according to the law $x = a\cos\omega t$. Find the distance that the particle covers during the time interval from $t = 0$ to t.

4.8. At the moment $t = 0$ a particle starts moving along the x axis so that its velocity projection varies as $v_x = 35\cos\pi t$ cm/s, where t is expressed in seconds. Find the distance that this particle covers during $t = 2.80$ s after the start.

4.9. A particle performs harmonic oscillations along the x axis according to the law $x = a\cos\omega t$. Assuming the probability P of the particle to fall within an interval from $-a$ to $+a$ to be equal to unity, find how the probability density dP/dx depends on x. Here dP denotes the probability of the particle falling within an interval from x to $x + dx$. Plot dP/dx as a function of x.

4.10. Using graphical means, find an amplitude a of oscillations resulting from the superposition of the following oscillations of the same direction:
(a) $x_1 = 3.0\cos(\omega t + \pi/3)$, $x_2 = 8.0\sin(\omega t + \pi/6)$;
(b) $x_1 = 3.0\cos\omega t$, $x_2 = 5.0\cos(\omega t + \pi/4)$, $x_3 = 6.0\sin\omega t$.

4.11. A point participates simultaneously in two harmonic oscillations of the same direction: $x_1 = a\cos\omega t$ and $x_2 = a\cos 2\omega t$. Find the maximum velocity of the point.

4.12. The superposition of two harmonic oscillations of the same direction results in the oscillation of a point according to the law $x = a\cos 2.1t\cos 50.0t$, where t is expressed in seconds. Find the angular frequencies of the constituent oscillations and the period with which they beat.

4.13. A point A oscillates according to a certain harmonic law in the reference frame K' which in its turn performs harmonic oscillations relative to the reference frame K. Both oscillations occur along the same direction. When the K' frame oscillates at the frequency 20 or 24 Hz, the beat frequency of the point A in the K frame turns

out to be equal to ν. At what frequency of oscillation of the frame K' will the beat frequency of the point A become equal to 2ν?

4.14. A point moves in the plane xy according to the law $x = a \sin \omega t$, $y = b \cos \omega t$, where a, b, and ω are positive constants. Find:

(a) the trajectory equation $y(x)$ of the point and the direction of its motion along this trajectory;

(b) the acceleration $\mathbf{w}$ of the point as a function of its radius vector $\mathbf{r}$ relative to the origin of coordinates.

4.15. Find the trajectory equation $y(x)$ of a point if it moves according to the following laws:

(a) $x = a \sin \omega t$, $y = a \sin 2\omega t$;

(b) $x = a \sin \omega t$, $y = a \cos 2\omega t$.

Plot these trajectories.

4.16. A particle of mass m is located in a unidimensional potential field where the potential energy of the particle depends on the coordinate x as $U(x) = U_0 (1 - \cos ax)$; U_0 and a are constants. Find the period of small oscillations that the particle performs about the equilibrium position.

4.17. Solve the foregoing problem if the potential energy has the form $U(x) = a/x^2 - b/x$, where a and b are positive constants.

4.18. Find the period of small oscillations in a vertical plane performed by a ball of mass $m = 40$ g fixed at the middle of a horizontally stretched string $l = 1.0$ m in length. The tension of the string is assumed to be constant and equal to $F = 10$ N.

4.19. Determine the period of small oscillations of a mathematical pendulum, that is a ball suspended by a thread $l = 20$ cm in length, if it is located in a liquid whose density is $\eta = 3.0$ times less than that of the ball. The resistance of the liquid is to be neglected.

4.20. A ball is suspended by a thread of length l at the point O on the wall, forming a small angle α with the vertical (Fig. 4.1). Then

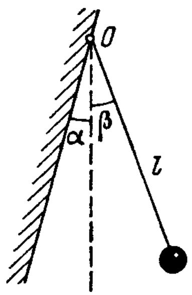


Fig. 4.1.

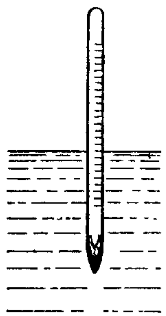


Fig. 4.2.

the thread with the ball was deviated through a small angle β ($\beta > \alpha$) and set free. Assuming the collision of the ball against the wall to be perfectly elastic, find the oscillation period of such a pendulum.

4.21. A pendulum clock is mounted in an elevator car which starts going up with a constant acceleration w, with $w < g$. At a height h the acceleration of the car reverses, its magnitude remaining constant. How soon after the start of the motion will the clock show the right time again?

4.22. Calculate the period of small oscillations of a hydrometer (Fig. 4.2) which was slightly pushed down in the vertical direction. The mass of the hydrometer is $m = 50$ g, the radius of its tube is $r = 3.2$ mm, the density of the liquid is $\rho = 1.00$ g/cm^3. The resistance of the liquid is assumed to be negligible.

4.23. A non-deformed spring whose ends are fixed has a stiffness $\varkappa = 13$ N/m. A small body of mass $m = 25$ g is attached at the point removed from one of the ends by $\eta = 1/3$ of the spring's length. Neglecting the mass of the spring, find the period of small longitudinal oscillations of the body. The force of gravity is assumed to be absent.

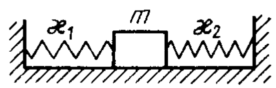


Fig. 4.3.

4.24. Determine the period of small longitudinal oscillations of a body with mass m in the system shown in Fig. 4.3. The stiffness values of the springs are $\varkappa_1$ and $\varkappa_2$. The friction and the masses of the springs are negligible.

4.25. Find the period of small vertical oscillations of a body with mass m in the system illustrated in Fig. 4.4. The stiffness values of the springs are $\varkappa_1$ and $\varkappa_2$, their masses are negligible.

4.26. A small body of mass m is fixed to the middle of a stretched string of length $2l$. In the equilibrium position the string tension is equal to T_0. Find the angular frequency of small oscillations of the body in the transverse direction. The mass of the string is negligible, the gravitational field is absent.

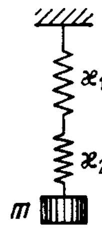


Fig. 4.4.

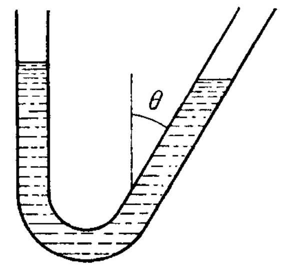


Fig. 4.5.

4.27. Determine the period of oscillations of mercury of mass $m = 200$ g poured into a bent tube (Fig. 4.5) whose right arm forms an angle $\theta = 30°$ with the vertical. The cross-sectional area of the tube is $S = 0.50$ cm^2. The viscosity of mercury is to be neglected.

4.28. A uniform rod is placed on two spinning wheels as shown in Fig. 4.6. The axes of the wheels are separated by a distance $l = 20$ cm, the coefficient of friction between the rod and the wheels is $k = 0.18$. Demonstrate that in this case the rod performs harmonic oscillations. Find the period of these oscillations.

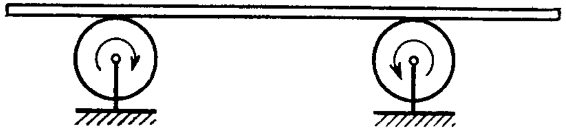


Fig. 4.6.

4.29. Imagine a shaft going all the way through the Earth from pole to pole along its rotation axis. Assuming the Earth to be a homogeneous ball and neglecting the air drag, find:

(a) the equation of motion of a body falling down into the shaft;

(b) how long does it take the body to reach the other end of the shaft;

(c) the velocity of the body at the Earth's centre.

4.30. Find the period of small oscillations of a mathematical pendulum of length l if its point of suspension O moves relative to the Earth's surface in an arbitrary direction with a constant acceleration **w** (Fig. 4.7). Calculate that period if $l = 21$ cm, $w = g/2$, and the angle between the vectors **w** and **g** equals $\beta = 120°$.

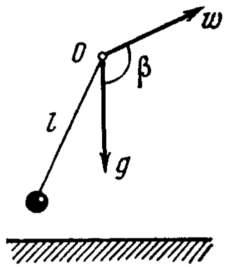


Fig. 4.7.

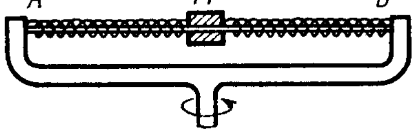


Fig. 4.8.

4.31. In the arrangement shown in Fig. 4.8 the sleeve M of mass $m = 0.20$ kg is fixed between two identical springs whose combined stiffness is equal to $\varkappa = 20$ N/m. The sleeve can slide without friction over a horizontal bar AB. The arrangement rotates with a constant angular velocity $\omega = 4.4$ rad/s about a vertical axis passing through the middle of the bar. Find the period of small oscillations of the sleeve. At what values of ω will there be no oscillations of the sleeve?

4.32. A plank with a bar placed on it performs horizontal harmonic oscillations with amplitude $a = 10$ cm. Find the coefficient of friction between the bar and the plank if the former starts sliding along the plank when the amplitude of oscillation of the plank becomes less than $T = 1.0$ s.

4.33. Find the time dependence of the angle of deviation of a mathematical pendulum 80 cm in length if at the initial moment the pendulum

(a) was deviated through the angle 3.0° and then set free without push;

(b) was in the equilibrium position and its lower end was imparted the horizontal velocity 0.22 m/s;

(c) was deviated through the angle 3.0° and its lower end was imparted the velocity 0.22 m/s directed toward the equilibrium position.

4.34. A body A of mass $m_1 = 1.00$ kg and a body B of mass $m_2 = 4.10$ kg are interconnected by a spring as shown in Fig. 4.9. The body A performs free vertical harmonic oscillations with amplitude $a = 1.6$ cm and frequency $\omega = 25\ \text{s}^{-1}$. Neglecting the mass of the spring, find the maximum and minimum values of force that this system exerts on the bearing surface.

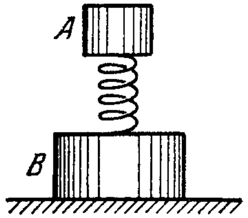


Fig. 4.9.

4.35. A plank with a body of mass m placed on it starts moving straight up according to the law $y = a(1 - \cos \omega t)$, where y is the displacement from the initial position, $\omega = 11\ \text{s}^{-1}$. Find:

(a) the time dependence of the force that the body exerts on the plank if $a = 4.0$ cm; plot this dependence;

(b) the minimum amplitude of oscillation of the plank at which the body starts falling behind the plank;

(c) the amplitude of oscillation of the plank at which the body springs up to a height $h = 50$ cm relative to the initial position (at the moment $t = 0$).

4.36. A body of mass m was suspended by a non-stretched spring, and then set free without push. The stiffness of the spring is $\varkappa$. Neglecting the mass of the spring, find:

(a) the law of motion $y(t)$, where y is the displacement of the body from the equilibrium position;

(b) the maximum and minimum tensions of the spring in the process of motion.

4.37. A particle of mass m moves due to the force $\mathbf{F} = -\alpha m \mathbf{r}$, where α is a positive constant, $\mathbf{r}$ is the radius vector of the particle relative to the origin of coordinates. Find the trajectory of its motion if at the initial moment $\mathbf{r} = r_0\mathbf{i}$ and the velocity $\mathbf{v} = v_0\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes.

4.38. A body of mass m is suspended from a spring fixed to the ceiling of an elevator car. The stiffness of the spring is $\varkappa$. At the moment $t = 0$ the car starts going up with an acceleration w. Neglecting the mass of the spring, find the law of motion $y(t)$ of the body relative to the elevator car if $y(0) = 0$ and $\dot{y}(0) = 0$. Consider the following two cases:

(a) $w = \text{const}$;

(b) $w = \alpha t$, where α is a constant.

4.39. A body of mass $m = 0.50$ kg is suspended from a rubber cord with elasticity coefficient $k = 50$ N/m. Find the maximum distance over which the body can be pulled down for the body's oscillations to remain harmonic. What is the energy of oscillation in this case?

4.40. A body of mass m fell from a height h onto the pan of a spring balance (Fig. 4.10). The masses of the pan and the spring are negligible, the stiffness of the latter is $\varkappa$. Having stuck to the pan, the body starts performing harmonic oscillations in the vertical direction. Find the amplitude and the energy of these oscillations.

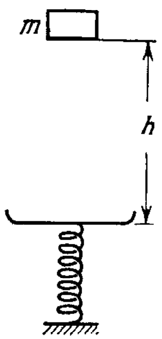


Fig. 4.10.

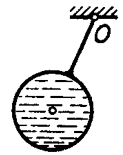


Fig. 4.11.

4.41. Solve the foregoing problem for the case of the pan having a mass M. Find the oscillation amplitude in this case.

4.42. A particle of mass m moves in the plane xy due to the force varying with velocity as $\mathbf{F} = a(\dot{y}\mathbf{i} - \dot{x}\mathbf{j})$, where a is a positive constant, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of the x and y axes. At the initial moment $t = 0$ the particle was located at the point $x = y = 0$ and possessed a velocity $\mathbf{v}_0$ directed along the unit vector $\mathbf{j}$. Find the law of motion $x(t)$, $y(t)$ of the particle, and also the equation of its trajectory.

4.43. A pendulum is constructed as a light thin-walled sphere of radius R filled up with water and suspended at the point O from a light rigid rod (Fig. 4.11). The distance between the point O and the centre of the sphere is equal to l. How many times will the small oscillations of such a pendulum change after the water freezes? The viscosity of water and the change of its volume on freezing are to be neglected.

4.44. Find the frequency of small oscillations of a thin uniform vertical rod of mass m and length l hinged at the point O (Fig. 4.12). The combined stiffness of the springs is equal to $\varkappa$. The mass of the springs is negligible.

4.45. A uniform rod of mass $m = 1.5$ kg suspended by two identical threads $l = 90$ cm in length (Fig. 4.13) was turned through a small angle about the vertical axis passing through its middle point C. The threads deviated in the process through an angle $\alpha = 5.0°$. Then the rod was released to start performing small oscillations. Find:

(a) the oscillation period;

(b) the rod's oscillation energy.

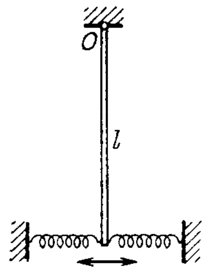


Fig. 4.12.

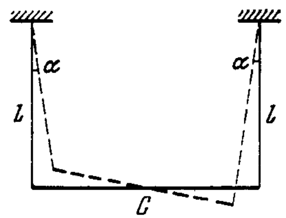


Fig. 4.13.

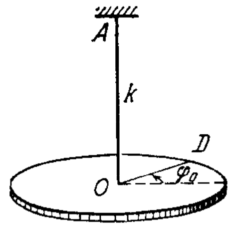


Fig. 4.14.

4.46. An arrangement illustrated in Fig. 4.14 consists of a horizontal uniform disc D of mass m and radius R and a thin rod AO whose torsional coefficient is equal to k. Find the amplitude and the energy of small torsional oscillations if at the initial moment the disc was deviated through an angle φ_0 from the equilibrium position and then imparted an angular velocity $\dot{\varphi}_0$.

4.47. A uniform rod of mass m and length l performs small oscillations about the horizontal axis passing through its upper end. Find the mean kinetic energy of the rod averaged over one oscillation period if at the initial moment it was deflected from the vertical by an angle θ_0 and then imparted an angular velocity $\dot{\theta}_0$.

4.48. A physical pendulum is positioned so that its centre of gravity is above the suspension point. From that position the pendulum started moving toward the stable equilibrium and passed it with an angular velocity ω. Neglecting the friction find the period of small oscillations of the pendulum.

4.49. A physical pendulum performs small oscillations about the horizontal axis with frequency $\omega_1 = 15.0\ \text{s}^{-1}$. When a small body of mass $m = 50$ g is fixed to the pendulum at a distance $l = 20$ cm below the axis, the oscillation frequency becomes equal to $\omega_2 = 10.0\ \text{s}^{-1}$. Find the moment of inertia of the pendulum relative to the oscillation axis.

4.50. Two physical pendulums perform small oscillations about the same horizontal axis with frequencies ω_1 and ω_2. Their moments of inertia relative to the given axis are equal to I_1 and I_2 respectively. In a state of stable equilibrium the pendulums were fastened rigidly together. What will be the frequency of small oscillations of the compound pendulum?

4.51. A uniform rod of length l performs small oscillations about the horizontal axis OO' perpendicular to the rod and passing through

one of its points. Find the distance between the centre of inertia of the rod and the axis OO' at which the oscillation period is the shortest. What is it equal to?

4.52. A thin uniform plate shaped as an equilateral triangle with a height h performs small oscillations about the horizontal axis coinciding with one of its sides. Find the oscillation period and the reduced length of the given pendulum.

4.53. A smooth horizontal disc rotates about the vertical axis O (Fig. 4.15) with a constant angular velocity ω. A thin uniform rod AB of length l performs small oscillations about the vertical axis A fixed to the disc at a distance a from the axis of the disc. Find the frequency ω_0 of these oscillations.

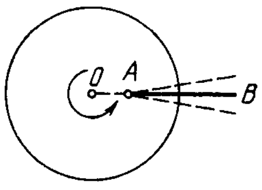


Fig. 4.15.

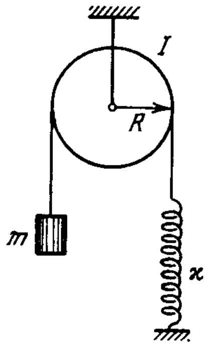


Fig. 4.16.

4.54. Find the frequency of small oscillations of the arrangement illustrated in Fig. 4.16. The radius of the pulley is R, its moment of inertia relative to the rotation axis is I, the mass of the body is m, and the spring stiffness is $\varkappa$. The mass of the thread and the spring is negligible, the thread does not slide over the pulley, there is no friction in the axis of the pulley.

4.55. A uniform cylindrical pulley of mass M and radius R can freely rotate about the horizontal axis O (Fig. 4.17). The free end of

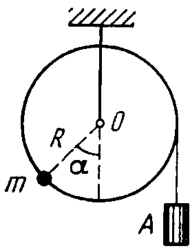


Fig. 4.17.

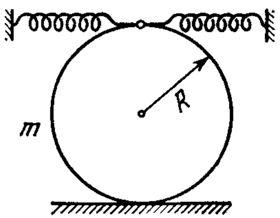


Fig. 4.18.

a thread tightly wound on the pulley carries a deadweight A. At a certain angle α it counterbalances a point mass m fixed at the rim

of the pulley. Find the frequency of small oscillations of the arrangement.

4.56. A solid uniform cylinder of radius r rolls without sliding along the inside surface of a cylinder of radius R, performing small oscillations. Find their period.

4.57. A solid uniform cylinder of mass m performs small oscillations due to the action of two springs whose combined stiffness is equal to $\varkappa$ (Fig. 4.18). Find the period of these oscillations in the absence of sliding.

4.58. Two cubes with masses m_1 and m_2 were interconnected by a weightless spring of stiffness $\varkappa$ and placed on a smooth horizontal surface. Then the cubes were drawn closer to each other and released simultaneously. Find the natural oscillation frequency of the system.

4.59. Two balls with masses $m_1 = 1.0$ kg and $m_2 = 2.0$ kg are slipped on a thin smooth horizontal rod (Fig. 4.19). The balls are

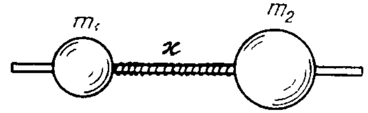


Fig. 4.19.

interconnected by a light spring of stiffness $\varkappa = 24$ N/m. The left-hand ball is imparted the initial velocity $v_1 = 12$ cm/s. Find:

(a) the oscillation frequency of the system in the process of motion;

(b) the energy and the amplitude of oscillations.

4.60. Find the period of small torsional oscillations of a system consisting of two discs slipped on a thin rod with torsional coefficient k. The moments of inertia of the discs relative to the rod's axis are equal to I_1 and I_2.

4.61. A mock-up of a CO_2 molecule consists of three balls interconnected by identical light springs and placed along a straight line in the state of equilibrium. Such a system can freely perform oscillations of two types, as shown by the arrows in Fig. 4.20. Knowing the masses of the atoms, find the ratio of frequencies of these oscillations.

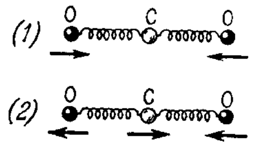


Fig. 4.20.

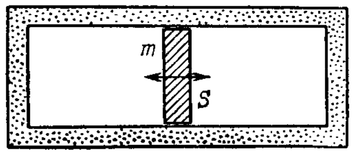


Fig. 4.21.

4.62. In a cylinder filled up with ideal gas and closed from both ends there is a piston of mass m and cross-sectional area S (Fig. 4.21).

In equilibrium the piston divides the cylinder into two equal parts, each with volume V_0. The gas pressure is p_0. The piston was slighlty displaced from the equilibrium position and released. Find its oscillation frequency, assuming the processes in the gas to be adiabatic and the friction negligible.

4.63. A small ball of mass $m = 21$ g suspended by an insulating thread at a height $h = 12$ cm from a large horizontal conducting plane performs small oscillations (Fig. 4.22). After a charge q had been imparted to the ball, the oscillation period changed $\eta = 2.0$ times. Find q.

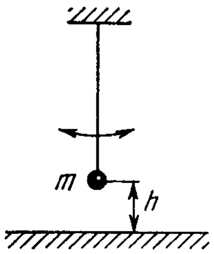


Fig. 4.22.

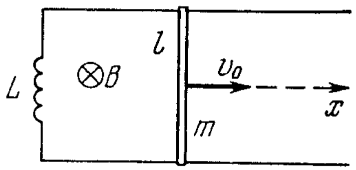


Fig. 4.23.

4.64. A small magnetic needle performs small oscillations about an axis perpendicular to the magnetic induction vector. On changing the magnetic induction the needle's oscillation period decreased $\eta = 5.0$ times. How much and in what way was the magnetic induction changed? The oscillation damping is assumed to be negligible.

4.65. A loop (Fig. 4.23) is formed by two parallel conductors connected by a solenoid with inductance L and a conducting rod of mass m which can freely (without friction) slide over the conductors. The conductors are located in a horizontal plane in a uniform vertical magnetic field with induction B. The distance between the conductors is equal to l. At the moment $t = 0$ the rod is imparted an initial velocity v_0 directed to the right. Find the law of its motion $x(t)$ if the electric resistance of the loop is negligible.

4.66. A coil of inductance L connects the upper ends of two vertical copper bars separated by a distance l. A horizontal conducting connector of mass m starts falling with zero initial velocity along the bars without losing contact with them. The whole system is located in a uniform magnetic field with induction B perpendicular to the plane of the bars. Find the law of motion $x(t)$ of the connector.

4.67. A point performs damped oscillations according to the law $x = a_0 e^{-\beta t} \sin \omega t$. Find:

(a) the oscillation amplitude and the velocity of the point at the moment $t = 0$;

(b) the moments of time at which the point reaches the extreme positions.

4.68. A body performs torsional oscillations according to the law $\varphi = \varphi_0 e^{-\beta t} \cos \omega t$. Find:

(a) the angular velocity $\dot{\varphi}$ and the angular acceleration $\ddot{\varphi}$ of the body at the moment $t = 0$;

(b) the moments of time at which the angular velocity becomes maximum.

4.69. A point performs damped oscillations with frequency ω and damping coefficient β according to the law (4.1b). Find the initial amplitude a_0 and the initial phase α if at the moment $t = 0$ the displacement of the point and its velocity projection are equal to

(a) $x(0) = 0$ and $v_x(0) = \dot{x}_0$;

(b) $x(0) = x_0$ and $v_x(0) = 0$.

4.70. A point performs damped oscillations with frequency $\omega = 25\ \text{s}^{-1}$. Find the damping coefficient β if at the initial moment the velocity of the point is equal to zero and its displacement from the equilibrium position is $\eta = 1.020$ times less than the amplitude at that moment.

4.71. A point performs damped oscillations with frequency ω and damping coefficient β. Find the velocity amplitude of the point as a function of time t if at the moment $t = 0$

(a) its displacement amplitude is equal to a_0;

(b) the displacement of the point $x(0) = 0$ and its velocity projection $v_x(0) = \dot{x}_0$.

4.72. There are two damped oscillations with the following periods T and damping coefficients β: $T_1 = 0.10$ ms, $\beta_1 = 100\ \text{s}^{-1}$ and $T_2 = 10$ ms, $\beta_2 = 10\ \text{s}^{-1}$. Which of them decays faster?

4.73. A mathematical pendulum oscillates in a medium for which the logarithmic damping decrement is equal to $\lambda_0 = 1.50$. What will be the logarithmic damping decrement if the resistance of the medium increases $n = 2.00$ times? How many times has the resistance of the medium to be increased for the oscillations to become impossible?

4.74. A deadweight suspended from a weightless spring extends it by $\Delta x = 9.8$ cm. What will be the oscillation period of the deadweight when it is pushed slightly in the vertical direction? The logarithmic damping decrement is equal to $\lambda = 3.1$.

4.75. Find the quality factor of the oscillator whose displacement amplitude decreases $\eta = 2.0$ times every $n = 110$ oscillations.

4.76. A particle was displaced from the equilibrium position by a distance $l = 1.0$ cm and then left alone. What is the distance that the particle covers in the process of oscillations till the complete stop, if the logarithmic damping decrement is equal to $\lambda = 0.020$?

4.77. Find the quality factor of a mathematical pendulum $l = 50$ cm long if during the time interval $\tau = 5.2$ min its total mechanical energy decreases $\eta = 4.0 \cdot 10^4$ times.

4.78. A uniform disc of radius $R = 13$ cm can rotate about a horizontal axis perpendicular to its plane and passing through the edge of the disc. Find the period of small oscillations of that disc if the logarithmic damping decrement is equal to $\lambda = 1.00$.

12 9451

4.79. A thin uniform disc of mass m and radius R suspended by an elastic thread in the horizontal plane performs torsional oscillations in a liquid. The moment of elastic forces emerging in the thread is equal to $N = \alpha\varphi$, where α is a constant and φ is the angle of rotation from the equilibrium position. The resistance force acting on a unit area of the disc is equal to $F_1 = \eta v$, where η is a constant and v is the velocity of the given element of the disc relative to the liquid. Find the frequency of small oscillation.

4.80. A disc A of radius R suspended by an elastic thread between two stationary planes (Fig. 4.24) performs torsional oscillations about its axis OO'. The moment of inertia of the disc relative to that axis is equal to I, the clearance between the disc and each of the planes is equal to h, with $h \ll R$. Find the viscosity of the gas surrounding the disc A if the oscillation period of the disc equals T and the logarithmic damping decrement, λ.

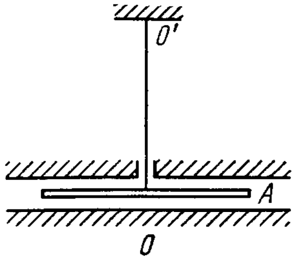


Fig. 4.24.

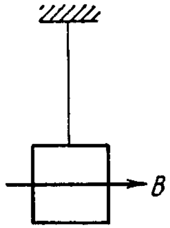


Fig. 4.25.

4.81. A conductor in the shape of a square frame with side a suspended by an elastic thread is located in a uniform horizontal magnetic field with induction B. In equilibrium the plane of the frame is parallel to the vector **B** (Fig. 4.25). Having been displaced from the equilibrium position, the frame performs small oscillations about a vertical axis passing through its centre. The moment of inertia of the frame relative to that axis is equal to I, its electric resistance is R. Neglecting the inductance of the frame, find the time interval after which the amplitude of the frame's deviation angle decreases e-fold.

4.82. A bar of mass $m = 0.50$ kg lying on a horizontal plane with a friction coefficient $k = 0.10$ is attached to the wall by means of a horizontal non-deformed spring. The stiffness of the spring is equal to $\varkappa = 2.45$ N/cm, its mass is negligible. The bar was displaced so that the spring was stretched by $x_0 = 3.0$ cm, and then released. Find:

(a) the period of oscillation of the bar;

(b) the total number of oscillations that the bar performs until it stops completely.

4.83. A ball of mass m can perform undamped harmonic oscillations about the point $x = 0$ with natural frequency ω_0. At the moment $t = 0$, when the ball was in equilibrium, a force $F_x = F_0 \cos \omega t$ coinciding with the x axis was applied to it. Find the law of forced oscillation $x(t)$ for that ball.

4.84. A particle of mass m can perform undamped harmonic oscillations due to an electric force with coefficient k. When the particle was in equilibrium, a permanent force F was applied to it for τ seconds. Find the oscillation amplitude that the particle acquired after the action of the force ceased. Draw the approximate plot $x(t)$ of oscillations. Investigate possible cases.

4.85. A ball of mass m when suspended by a spring stretches the latter by Δl. Due to external vertical force varying according to a harmonic law with amplitude F_0 the ball performs forced oscillations. The logarithmic damping decrement is equal to λ. Neglecting the mass of the spring, find the angular frequency of the external force at which the displacement amplitude of the ball is maximum. What is the magnitude of that amplitude?

4.86. The forced harmonic oscillations have equal displacement amplitudes at frequencies $\omega_1 = 400$ s^{-1} and $\omega_2 = 600$ s^{-1}. Find the resonance frequency at which the displacement amplitude is maximum.

4.87. The velocity amplitude of a particle is equal to half the maximum value at the frequencies ω_1 and ω_2 of external harmonic force. Find:

(a) the frequency corresponding to the velocity resonance;

(b) the damping coefficient β and the damped oscillation frequency ω of the particle.

4.88. A certain resonance curve describes a mechanical oscillating system with logarithmic damping decrement $\lambda = 1.60$. For this curve find the ratio of the maximum displacement amplitude to the displacement amplitude at a very low frequency.

4.89. Due to the external vertical force $F_x = F_0 \cos \omega t$ a body suspended by a spring performs forced steady-state oscillations according to the law $x = a \cos(\omega t - \varphi)$. Find the work performed by the force F during one oscillation period.

4.90. A ball of mass $m = 50$ g is suspended by a weightless spring with stiffness $\varkappa = 20.0$ N/m. Due to external vertical harmonic force with frequency $\omega = 25.0$ s^{-1} the ball performs steady-state oscillations with amplitude $a = 1.3$ cm. In this case the displacement of the ball lags in phase behind the external force by $\varphi = \frac{3}{4}\pi$. Find:

(a) the quality factor of the given oscillator;

(b) the work performed by the external force during one oscillation period.

4.91. A ball of mass m suspended by a weightless spring can perform vertical oscillations with damping coefficient β. The natural oscillation frequency is equal to ω_0. Due to the external vertical force varying as $F = F_0 \cos \omega t$ the ball performs steady-state harmonic oscillations. Find:

(a) the mean power $\langle P \rangle$, developed by the force F, averaged over one oscillation period;

(b) the frequency ω of the force F at which $\langle P\rangle$ is maximum; what is $\langle P\rangle_{max}$ equal to?

4.92. An external harmonic force F whose frequency can be varied, with amplitude maintained constant, acts in a vertical direction on a ball suspended by a weightless spring. The damping coefficient is η times less than the natural oscillation frequency ω_0 of the ball. How much, in per cent, does the mean power $\langle P\rangle$ developed by the force F at the frequency of displacement resonance differ from the maximum mean power $\langle P\rangle_{max}$? Averaging is performed over one oscillation period.

4.93. A uniform horizontal disc fixed at its centre to an elastic vertical rod performs forced torsional oscillations due to the moment of forces $N_z = N_m \cos \omega t$. The oscillations obey the law $\varphi = \varphi_m \cos (\omega t - \alpha)$. Find:

(a) the work performed by friction forces acting on the disc during one oscillation period;

(b) the quality factor of the given oscillator if the moment of inertia of the disc relative to the axis is equal to I.

4.2. ELECTRIC OSCILLATIONS

• Damped oscillation in a circuit

$$q = q_m e^{-\beta t} \cos (\omega t + \alpha),$$

where

$$\omega = \sqrt{\omega_0^2 - \beta^2}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}, \qquad \beta = \frac{R}{2L}. \tag{4.2a}$$

• Logarithmic damping decrement λ and quality factor Q of a circuit are defined by Eqs. (4.1d). When damping is low:

$$\lambda = \pi R \sqrt{\frac{C}{L}}, \qquad Q = \frac{1}{R}\sqrt{\frac{L}{C}}. \tag{4.2b}$$

• Steady-state forced oscillation in a circuit with a voltage $V = V_m \cos \omega t$ connected in series:

$$I = I_m \cos (\omega t - \varphi), \tag{4.2c}$$

where

$$I_m = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}, \qquad \tan \varphi = \frac{\omega L - \frac{1}{\omega C}}{R}. \tag{4.2d}$$

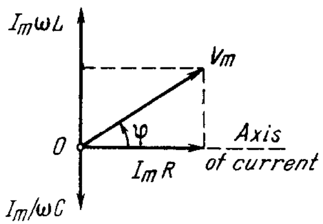


Fig. 4.26.

The corresponding vector diagram for voltages is shown in Fig. 4.26.

• Power generated in an ac circuit:

$$P = VI \cos \varphi, \tag{4.2e}$$

where V and I are the effective values of voltage and current:

$$V = V_m/\sqrt{2}, \quad I = I_m/\sqrt{2}. \tag{4.2f}$$

4.94. Due to a certain cause the free electrons in a plane copper plate shifted over a small distance x at right angles to its surface. As a result, a surface charge and a corresponding restoring force emerged, giving rise to so-called plasma oscillations. Find the angular frequency of these oscillations if the free electron concentration in copper is $n = 0.85 \cdot 10^{29}$ m^{-1}.

4.95. An oscillating circuit consisting of a capacitor with capacitance C and a coil of inductance L maintains free undamped oscillations with voltage amplitude across the capacitor equal to V_m. For an arbitrary moment of time find the relation between the current I in the circuit and the voltage V across the capacitor. Solve this problem using Ohm's law and then the energy conservation law.

4.96. An oscillating circuit consists of a capacitor with capacitance C, a coil of inductance L with negligible resistance, and a switch. With the switch disconnected, the capacitor was charged to a voltage V_m and then at the moment $t = 0$ the switch was closed. Find:

(a) the current $I(t)$ in the circuit as a function of time;

(b) the emf of self-inductance in the coil at the moments when the electric energy of the capacitor is equal to that of the current in the coil.

4.97. In an oscillating circuit consisting of a parallel-plate capacitor and an inductance coil with negligible active resistance the oscillations with energy W are sustained. The capacitor plates were slowly drawn apart to increase the oscillation frequency η-fold. What work was done in the process?

4.98. In an oscillating circuit shown in Fig. 4.27 the coil inductance is equal to $L = 2.5$ mH and the capacitor have capacitances $C_1 = 2.0$ µF and $C_2 = 3.0$ µF. The capacitors were charged to a voltage $V = 180$ V, and then the switch Sw was closed. Find:

(a) the natural oscillation frequency;

(b) the peak value of the current flowing through the coil.

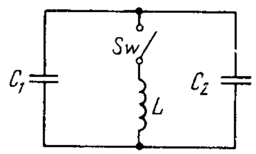


Fig. 4.27.

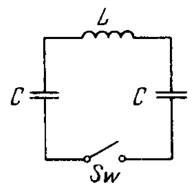


Fig. 4.28.

4.99. An electric circuit shown in Fig. 4.28 has a negligibly small active resistance. The left-hand capacitor was charged to a voltage V_0 and then at the moment $t = 0$ the switch Sw was closed. Find the time dependence of the voltages in left and right capacitors.

4.100. An oscillating circuit consists of an inductance coil L and a capacitor with capacitance C. The resistance of the coil and the lead

wires is negligible. The coil is placed in a permanent magnetic field so that the total flux passing through all the turns of the coil is equal to Φ. At the moment $t = 0$ the magnetic field was switched off. Assuming the switching off time to be negligible compared to the natural oscillation period of the circuit, find the circuit current as a function of time t.

4.101. The free damped oscillations are maintained in a circuit, such that the voltage across the capacitor varies as $V = V_m e^{-\beta t} \cos \omega t$. Find the moments of time when the modulus of the voltage across the capacitor reaches

(a) peak values;

(b) maximum (extremum) values.

4.102. A certain oscillating circuit consists of a capacitor with capacitance C, a coil with inductance L and active resistance R, and a switch. When the switch was disconnected, the capacitor was charged; then the switch was closed and oscillations set in. Find the ratio of the voltage across the capacitor to its peak value at the moment immediately after closing the switch.

4.103. A circuit with capacitance C and inductance L generates free damped oscillations with current varying with time as $I = I_m e^{-\beta t} \sin \omega t$. Find the voltage across the capacitor as a function of time, and in particular, at the moment $t = 0$.

4.104. An oscillating circuit consists of a capacitor with capacitance $C = 4.0$ μF and a coil with inductance $L = 2.0$ mH and active resistance $R = 10$ Ω. Find the ratio of the energy of the coil's magnetic field to that of the capacitor's electric field at the moment when the current has the maximum value.

4.105. An oscillating circuit consists of two coils connected in series whose inductances are L_1 and L_2, active resistances are R_1 and R_2, and mutual inductance is negligible. These coils are to be replaced by one, keeping the frequency and the quality factor of the circuit constant. Find the inductance and the active resistance of such a coil.

4.106. How soon does the current amplitude in an oscillating circuit with quality factor $Q = 5000$ decrease $\eta = 2.0$ times if the oscillation frequency is $\nu = 2.2$ MHz?

4.107. An oscillating circuit consists of capacitance $C = 10$ μF, inductance $L = 25$ mH, and active resistance R 1.0 Ω. How many oscillation periods does it take for the current amplitude to decrease e-fold?

4.108. How much (in per cent) does the free oscillation frequency ω of a circuit with quality factor $Q = 5.0$ differ from the natural oscillation frequency ω_0 of that circuit?

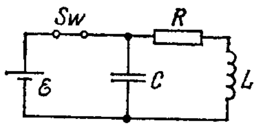


Fig. 4.29.

4.109. In a circuit shown in Fig. 4.29 the battery emf is equal to $\mathscr{E} = 2.0$ V, its internal resistance is $r = 9.0$ Ω, the capacitance of the capacitor is $C = 10$ μF, the coil inductance is $L = 100$ mH, and the resistance

is $R = 1.0$ Ω. At a certain moment the switch Sw was disconnected. Find the energy of oscillations in the circuit

(a) immediately after the switch was disconnected;

(b) $t = 0.30$ s after the switch was disconnected.

4.110. Damped oscillations are induced in a circuit whose quality factor is $Q = 50$ and natural oscillation frequency is $\nu_0 = 5.5$ kHz. How soon will the energy stored in the circuit decrease $\eta = 2.0$ times?

4.111. An oscillating circuit incorporates a leaking capacitor. Its capacitance is equal to C and active resistance to R. The coil inductance is L. The resistance of the coil and the wires is negligible. Find:

(a) the damped oscillation frequency of such a circuit;

(b) its quality factor.

4.112. Find the quality factor of a circuit with capacitance $C = 2.0$ μF and inductance $L = 5.0$ mH if the maintenance of undamped oscillations in the circuit with the voltage amplitude across the capacitor being equal to $V_m = 1.0$ V requires a power $\langle P \rangle = 0.10$ mW. The damping of oscillations is sufficiently low.

4.113. What mean power should be fed to an oscillating circuit with active resistance $R = 0.45$ Ω to maintain undamped harmonic oscillations with current amplitude $I_m = 30$ mA?

4.114. An oscillating circuit consists of a capacitor with capacitance $C = 1.2$ nF and a coil with inductance $L = 6.0$ μH and active resistance $R = 0.50$ Ω. What mean power should be fed to the circuit to maintain undamped harmonic oscillations with voltage amplitude across the capacitor being equal to $V_m = 10$ V?

4.115. Find the damped oscillation frequency of the circuit shown in Fig. 4.30. The capacitance C, inductance L, and active resistance R are supposed to be known. Find how must C, L, and R be interrelated to make oscillations possible.

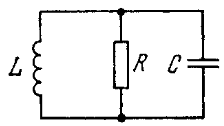


Fig. 4.30.

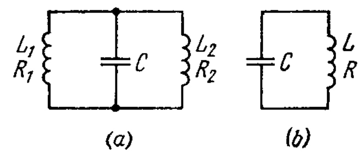


Fig. 4.31.

4.116. There are two oscillating circuits (Fig. 4.31) with capacitors of equal capacitances. How must inductances and active resistances of the coils be interrelated for the frequencies and damping of free oscillations in both circuits to be equal? The mutual inductance of coils in the left circuit is negligible.

4.117. A circuit consists of a capacitor with capacitance C and a coil of inductance L connected in series, as well as a switch and a resistance equal to the critical value for this circuit. With the switch

disconnected, the capacitor was charged to a voltage V_0, and at the moment $t = 0$ the switch was closed. Find the current I in the circuit as a function of time t. What is I_{max} equal to?

4.118. A coil with active resistance R and inductance L was connected at the moment $t = 0$ to a source of voltage $V = V_m \cos \omega t$. Find the current in the coil as a function of time t.

4.119. A circuit consisting of a capacitor with capacitance C and a resistance R connected in series was connected at the moment $t = 0$ to a source of ac voltage $V = V_m \cos \omega t$. Find the current in the circuit as a function of time t.

4.120. A long one-layer solenoid tightly wound of wire with resistivity ρ has n turns per unit length. The thickness of the wire insulation is negligible. The cross-sectional radius of the solenoid is equal to a. Find the phase difference between current and alternating voltage fed to the solenoid with frequency ν.

4.121. A circuit consisting of a capacitor and an active resistance $R = 110\ \Omega$ connected in series is fed an alternating voltage with amplitude $V_m = 110$ V. In this case the amplitude of steady-state current is equal to $I_m = 0.50$ A. Find the phase difference between the current and the voltage fed.

4.122. Fig. 4.32 illustrates the simplest ripple filter. A voltage $V = V_0 (1 + \cos \omega t)$ is fed to the left input. Find:

(a) the output voltage $V'(t)$;

(b) the magnitude of the product RC at which the output amplitude of alternating voltage component is $\eta = 7.0$ times less than the direct voltage component, if $\omega = 314\ \text{s}^{-1}$.

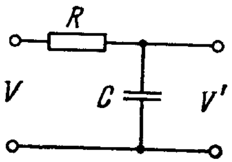
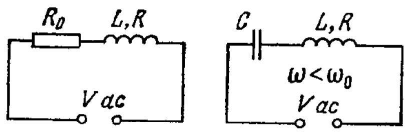


Fig. 4.32.



(a) (b)

Fig. 4.33.

4.123. Draw the approximate voltage vector diagrams in the electric circuits shown in Fig. 4.33 *a*, *b*. The external voltage V is assumed to be alternating harmonically with frequency ω.

4.124. A series circuit consisting of a capacitor with capacitance $C = 22\ \mu$F and a coil with active resistance $R = 20\ \Omega$ and inductance $L = 0.35$ H is connected to a source of alternating voltage with amplitude $V_m = 180$ V and frequency $\omega = 314\ \text{s}^{-1}$. Find:

(a) the current amplitude in the circuit;

(b) the phase difference between the current and the external voltage;

(c) the amplitudes of voltage across the capacitor and the coil.

4.125. A series circuit consisting of a capacitor with capacitance C, a resistance R, and a coil with inductance L and negligible active resistance is connected to an oscillator whose frequency can be varied without changing the voltage amplitude. Find the frequency at which the voltage amplitude is maximum

(a) across the capacitor;

(b) across the coil.

4.126. An alternating voltage with frequency $\omega = 314\ \text{s}^{-1}$ and amplitude $V_m = 180$ V is fed to a series circuit consisting of a capacitor and a coil with active resistance $R = 40\ \Omega$ and inductance $L = 0.36$ H. At what value of the capacitor's capacitance will the voltage amplitude across the coil be maximum? What is this amplitude equal to? What is the corresponding voltage amplitude across the condenser?

4.127. A capacitor with capacitance C whose interelectrode space is filled up with poorly conducting medium with active resistance R is connected to a source of alternating voltage $V = V_m \cos \omega t$. Find the time dependence of the steady-state current flowing in lead wires. The resistance of the wires is to be neglected.

4.128. An oscillating circuit consists of a capacitor of capacitance C and a solenoid with inductance L_1. The solenoid is inductively connected with a short-circuited coil having an inductance L_2 and a negligible active resistance. Their mutual inductance coefficient is equal to L_{12}. Find the natural frequency of the given oscillating circuit.

4.129. Find the quality factor of an oscillating circuit connected in series to a source of alternating emf if at resonance the voltage across the capacitor is n times that of the source.

4.130. An oscillating circuit consisting of a coil and a capacitor connected in series is fed an alternating emf, with coil inductance being chosen to provide the maximum current in the circuit. Find the quality factor of the system, provided an n-fold increase of inductance results in an η-fold decrease of the current in the circuit.

4.131. A series circuit consisting of a capacitor and a coil with active resistance is connected to a source of harmonic voltage whose frequency can be varied, keeping the voltage amplitude constant. At frequencies ω_1 and ω_2 the current amplitudes are n times less than the resonance amplitude. Find:

(a) the resonance frequency;

(b) the quality factor of the circuit.

4.132. Demonstrate that at low damping the quality factor Q of a circuit maintaining forced oscillations is approximately equal to $\omega_0/\Delta\omega$, where ω_0 is the natural oscillation frequency, $\Delta\omega$ is the width of the resonance curve $I(\omega)$ at the "height" which is $\sqrt{2}$ times less than the resonance current amplitude.

4.133. A circuit consisting of a capacitor and a coil connected in series is fed two alternating voltages of equal amplitudes but different frequencies. The frequency of one voltage is equal to the natural oscillation frequency (ω_0) of the circuit, the frequency of the other voltage is η times higher. Find the ratio of the current amplitudes

(I_0/I) generated by the two voltages if the quality factor of the system is equal to Q. Calculate this ratio for $Q = 10$ and 100, if $\eta = 1.10$.

4.134. It takes t_0 hours for a direct current I_0 to charge a storage battery. How long will it take to charge such a battery from the mains using a half-wave rectifier, if the effective current value is also equal to I_0?

4.135. Find the effective value of current if its mean value is I_0 and its time dependence is

(a) shown in Fig. 4.34;

(b) $I \sim |\sin \omega t|$.

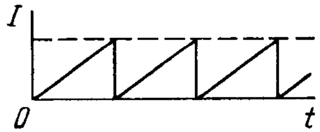


Fig. 4.34.

4.136. A solenoid with inductance $L = 7$ mH and active resistance $R = 44\ \Omega$ is first connected to a source of direct voltage V_0 and then to a source of sinusoidal voltage with effective value $V = V_0$. At what frequency of the oscillator will the power consumed by the solenoid be $\eta = 5.0$ times less than in the former case?

4.137. A coil with inductive resistance $X_L = 30\ \Omega$ and impedance $Z = 50\ \Omega$ is connected to the mains with effective voltage value $V = 100$ V. Find the phase difference between the current and the voltage, as well as the heat power generated in the coil.

4.138. A coil with inductance $L = 0.70$ H and active resistance $r = 20\ \Omega$ is connected in series with an inductance-free resistance R. An alternating voltage with effective value $V = 220$ V and frequency $\omega = 314\ \mathrm{s}^{-1}$ is applied across the terminals of this circuit. At what value of the resistance R will the maximum heat power be generated in the circuit? What is it equal to?

4.139. A circuit consisting of a capacitor and a coil in series is connected to the mains. Varying the capacitance of the capacitor, the heat power generated in the coil was increased $n = 1.7$ times. How much (in per cent) was the value of $\cos \varphi$ changed in the process?

4.140. A source of sinusoidal emf with constant voltage is connected in series with an oscillating circuit with quality factor $Q = 100$. At a certain frequency of the external voltage the heat power generated in the circuit reaches the maximum value. How much (in per cent) should this frequency be shifted to decrease the power generated $n = 2.0$ times?

4.141. A series circuit consisting of an inductance-free resistance $R = 0.16$ kΩ and a coil with active resistance is connected to the mains with effective voltage $V = 220$ V. Find the heat power generated in the coil if the effective voltage values across the resistance R and the coil are equal to $V_1 = 80$ V and $V_2 = 180$ V respectively.

4.142. A coil and an inductance-free resistance $R = 25\ \Omega$ are connected in parallel to the ac mains. Find the heat power generated in the coil provided a current $I = 0.90$ A is drawn from the mains. The coil and the resistance R carry currents $I_1 = 0.50$ A and $I_2 = 0.60$ A respectively.

4.143. An alternating current of frequency $\omega = 314\ \mathrm{s}^{-1}$ is fed to a circuit consisting of a capacitor of capacitance $C = 73\ \mu$F and an active resistance $R = 100\ \Omega$ connected in parallel. Find the impedance of the circuit.

4.144. Draw the approximate vector diagrams of currents in the circuits shown in Fig. 4.35. The voltage applied across the points A and B is assumed to be sinusoidal; the parameters of each circuit are so chosen that the total current I_0 lags in phase behind the external voltage by an angle φ.

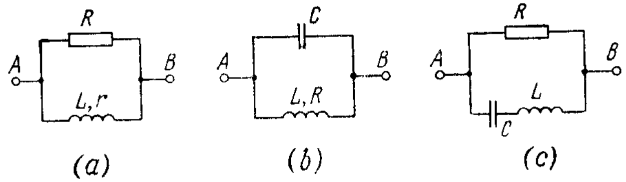


Fig. 4.35.

4.145. A capacitor with capacitance $C = 1.0\ \mu$F and a coil with active resistance $R = 0.10\ \Omega$ and inductance $L = 1.0$ mH are connected in parallel to a source of sinusoidal voltage $V = 31$ V. Find:

(a) the frequency ω at which the resonance sets in;

(b) the effective value of the fed current in resonance, as well as the corresponding currents flowing through the coil and through the capacitor.

4.146. A capacitor with capacitance C and a coil with active resistance R and inductance L are connected in parallel to a source of sinusoidal voltage of frequency ω. Find the phase difference between the current fed to the circuit and the source voltage.

4.147. A circuit consists of a capacitor with capacitance C and a coil with active resistance R and inductance L connected in parallel. Find the impedance of the circuit at frequency ω of alternating voltage.

4.148. A ring of thin wire with active resistance R and inductance L rotates with constant angular velocity ω in the external uniform magnetic field perpendicular to the rotation axis. In the process, the flux of magnetic induction of external field across the ring varies with time as $\Phi = \Phi_0 \cos \omega t$. Demonstrate that

(a) the inductive current in the ring varies with time as $I = I_m \sin(\omega t - \varphi)$, where $I_m = \omega\Phi_0/\sqrt{R^2 + \omega^2 L^2}$ with $\tan \varphi = \omega L/R$;

(b) the mean mechanical power developed by external forces to maintain rotation is defined by the formula $P = {}^1/_2\,\omega^2\Phi_0^2 R/(R^2 + \omega^2 L^2)$.

4.149. A wooden core (Fig. 4.36) supports two coils: coil *1* with inductance L_1 and short-circuited coil *2* with active resistance R and inductance L_2. The mutual inductance of the coils depends on

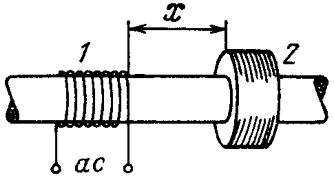


Fig. 4.36.

the distance x between them according to the law $L_{12}(x)$. Find the mean (averaged over time) value of the interaction force between the coils when coil *1* carries an alternating current $I_1 = I_0 \cos \omega t$.

4.3. ELASTIC WAVES. ACOUSTICS

- Equations of plane and spherical waves:

$$\xi = a \cos(\omega t - kx), \quad \xi = \frac{a_0}{r} \cos(\omega t - kr). \tag{4.3a}$$

In the case of a homogeneous absorbing medium the factors $e^{-\gamma x}$ and $e^{-\gamma r}$ respectively appear in the formulas, where γ is the wave damping coefficient.

- Wave equation:

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \xi}{\partial t^2}. \tag{4.3b}$$

- Phase velocity of longitudinal waves in an elastic medium ($v_{\parallel}$) and transverse waves in a string ($v_{\perp}$):

$$v_{\parallel} = \sqrt{E/\rho}, \quad v_{\perp} = \sqrt{T/\rho_1}, \tag{4.3c}$$

where E is Young's modulus, ρ is the density of the medium, T is the tension of the string, ρ_1 is its linear density.

- Volume density of energy of an elastic wave:

$$w = \rho a^2 \omega^2 \sin^2(\omega t - kx), \quad \langle w \rangle = {}^1/_2 \rho a^2 \omega^2. \tag{4.3d}$$

- Energy flow density, or the Umov vector for a travelling wave:

$$\mathbf{j} = w\mathbf{v}, \quad \langle \mathbf{j} \rangle = {}^1/_2 \rho a^2 \omega^2 \mathbf{v}. \tag{4.3e}$$

- Standing wave equation:

$$\xi = a \cos kx \cdot \cos \omega t. \tag{4.3f}$$

- Acoustical Doppler effect:

$$\nu = \nu_0 \frac{v + v_{ob}}{v - v_s}. \tag{4.3g}$$

- Loudness level (in bels):

$$L = \log(I/I_0). \tag{4.3h}$$

- Relation between the intensity I of a sound wave and the pressure oscillation amplitude $(\Delta p)_m$:

$$I = (\Delta p)_m^2 / 2\rho v. \tag{4.3i}$$

4.150. How long will it take sound waves to travel the distance l between the points A and B if the air temperature between them varies linearly from T_1 to T_2? The velocity of sound propagation in air is equal to $v = \alpha\sqrt{T}$, where α is a constant.

4.151. A plane harmonic wave with frequency ω propagates at a velocity v in a direction forming angles α, β, γ with the x, y, z axes. Find the phase difference between the oscillations at the points of medium with coordinates x_1, y_1, z_1 and x_2, y_2, z_2.

4.152. A plane wave of frequency ω propagates so that a certain phase of oscillation moves along the x, y, z axes with velocities v_1, v_2, v_3 respectively. Find the wave vector $\mathbf{k}$, assuming the unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ of the coordinate axes to be assigned.

4.153. A plane elastic wave $\xi = a \cos(\omega t - kx)$ propagates in a medium K. Find the equation of this wave in a reference frame K' moving in the positive direction of the x axis with a constant velocity V relative to the medium K. Investigate the expression obtained.

4.154. Demonstrate that any differentiable function $f(t + \alpha x)$, where α is a constant, provides a solution of wave equation. What is the physical meaning of the constant α?

4.155. The equation of a travelling plane sound wave has the form $\xi = 60 \cos(1800t - 5.3x)$, where ξ is expressed in micrometres, t in seconds, and x in metres. Find:

(a) the ratio of the displacement amplitude, with which the particles of medium oscillate, to the wavelength;

(b) the velocity oscillation amplitude of particles of the medium and its ratio to the wave propagation velocity;

(c) the oscillation amplitude of relative deformation of the medium and its relation to the velocity oscillation amplitude of particles of the medium.

4.156. A plane wave $\xi = a \cos(\omega t - kx)$ propagates in a homogeneous elastic medium. For the moment $t = 0$ draw

(a) the plots of ξ, $\partial\xi/\partial t$, and $\partial\xi/\partial x$ vs x;

(b) the velocity direction of the particles of the medium at the points where $\xi = 0$, for the cases of longitudinal and transverse waves;

(c) the approximate plot of density distribution $\rho(x)$ of the medium for the case of longitudinal waves.

4.157. A plane elastic wave $\xi = ae^{-\gamma x} \cos(\omega t - kx)$, where a, γ, ω, and k are constants, propagates in a homogeneous medium. Find the phase difference between the oscillations at the points where the particles' displacement amplitudes differ by $\eta = 1.0\%$, if $\gamma = 0.42\ \text{m}^{-1}$ and the wavelength is $\lambda = 50$ cm.

4.158. Find the radius vector defining the position of a point source of spherical waves if that source is known to be located on the straight line between the points with radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ at which the oscillation amplitudes of particles of the medium are equal to a_1 and a_2. The damping of the wave is negligible, the medium is homogeneous.

4.159. A point isotropic source generates sound oscillations with frequency $\nu = 1.45$ kHz. At a distance $r_0 = 5.0$ m from the source the displacement amplitude of particles of the medium is equal to $a_0 = 50$ μm, and at the point A located at a distance $r = 10.0$ m from the source the displacement amplitude is $\eta = 3.0$ times less than a_0. Find:

(a) the damping coefficient γ of the wave;

(b) the velocity oscillation amplitude of particles of the medium at the point A.

4.160. Two plane waves propagate in a homogeneous elastic medium, one along the x axis and the other along the y axis: $\xi_1 = a \cos(\omega t - kx)$, $\xi_2 = a \cos(\omega t - ky)$. Find the wave motion pattern of particles in the plane xy if both waves

(a) are transverse and their oscillation directions coincide;

(b) are longitudinal.

4.161. A plane undamped harmonic wave propagates in a medium. Find the mean space density of the total oscillation energy $\langle w \rangle$, if at any point of the medium the space density of energy becomes equal to w_0 one-sixth of an oscillation period after passing the displacement maximum.

4.162. A point isotropic sound source is located on the perpendicular to the plane of a ring drawn through the centre O of the ring. The distance between the point O and the source is $l = 1.00$ m, the radius of the ring is $R = 0.50$ m. Find the mean energy flow across the area enclosed by the ring if at the point O the intensity of sound is equal to $I_0 = 30$ μW/m². The damping of the waves is negligible.

4.163. A point isotropic source with sonic power $P = 0.10$ W is located at the centre of a round hollow cylinder with radius $R = 1.0$ m and height $h = 2.0$ m. Assuming the sound to be completely absorbed by the walls of the cylinder, find the mean energy flow reaching the lateral surface of the cylinder.

4.164. The equation of a plane standing wave in a homogeneous elastic medium has the form $\xi = a \cos kx \cdot \cos \omega t$. Plot:

(a) ξ and $\partial\xi/\partial x$ as functions of x at the moments $t = 0$ and $t = T/2$, where T is the oscillation period;

(b) the distribution of density $\rho(x)$ of the medium at the moments $t = 0$ and $t = T/2$ in the case of longitudinal oscillations;

(c) the velocity distribution of particles of the medium at the moment $t = T/4$; indicate the directions of velocities at the antinodes, both for longitudinal and transverse oscillations.

4.165. A longitudinal standing wave $\xi = a \cos kx \cdot \cos \omega t$ is maintained in a homogeneous medium of density ρ. Find the expressions for the space density of

(a) potential energy $w_p(x, t)$;

(b) kinetic energy $w_k(x, t)$.

Plot the space density distribution of the total energy w in the space between the displacement nodes at the moments $t = 0$ and $t = T/4$, where T is the oscillation period.

4.166. A string 120 cm in length sustains a standing wave, with the points of the string at which the displacement amplitude is equal to 3.5 mm being separated by 15.0 cm. Find the maximum displacement amplitude. To which overtone do these oscillations correspond?

4.167. Find the ratio of the fundamental tone frequencies of two identical strings after one of them was stretched by $\eta_1 = 2.0\%$ and the other, by $\eta_2 = 4.0\%$. The tension is assumed to be proportional to the elongation.

4.168. Determine in what way and how many times will the fundamental tone frequency of a stretched wire change if its length is shortened by 35% and the tension increased by 70%.

4.169. To determine the sound propagation velocity in air by acoustic resonance technique one can use a pipe with a piston and a sonic membrane closing one of its ends. Find the velocity of sound if the distance between the adjacent positions of the piston at which resonance is observed at a frequency $\nu = 2000$ Hz is equal to $l = 8.5$ cm.

4.170. Find the number of possible natural oscillations of air column in a pipe whose frequencies lie below $\nu_0 = 1250$ Hz. The length of the pipe is $l = 85$ cm. The velocity of sound is $v = 340$ m/s. Consider the two cases:

(a) the pipe is closed from one end;

(b) the pipe is opened from both ends.

The open ends of the pipe are assumed to be the antinodes of displacement.

4.171. A copper rod of length $l = 50$ cm is clamped at its midpoint. Find the number of natural longitudinal oscillations of the rod in the frequency range from 20 to 50 kHz. What are those frequencies equal to?

4.172. A string of mass m is fixed at both ends. The fundamental tone oscillations are excited with circular frequency ω and maximum displacement amplitude a_{max}. Find:

(a) the maximum kinetic energy of the string;

(b) the mean kinetic energy of the string averaged over one oscillation period.

4.173. A standing wave $\xi = a \sin kx \cdot \cos \omega t$ is maintained in a homogeneous rod with cross-sectional area S and density ρ. Find the total mechanical energy confined between the sections corresponding to the adjacent displacement nodes.

4.174. A source of sonic oscillations with frequency $\nu_0 = 1000$ Hz moves at right angles to the wall with a velocity $u = 0.17$ m/s. Two stationary receivers R_1 and R_2 are located on a straight line, coinciding with the trajectory of the source, in the following succession: R_1-source-R_2-wall. Which receiver registers the beatings and what is the beat frequency? The velocity of sound is equal to $v = 340$ m/s.

4.175. A stationary observer receives sonic oscillations from two tuning forks one of which approaches, and the other recedes with

the same velocity. As this takes place, the observer hears the beatings with frequency $\nu = 2.0$ Hz. Find the velocity of each tuning fork if their oscillation frequency is $\nu_0 = 680$ Hz and the velocity of sound in air is $v = 340$ m/s.

4.176. A receiver and a source of sonic oscillations of frequency $\nu_0 = 2000$ Hz are located on the x axis. The source swings harmonically along that axis with a circular frequency ω and an amplitude $a = 50$ cm. At what value of ω will the frequency bandwidth registered by the stationary receiver be equal to $\Delta\nu = 200$ Hz? The velocity of sound is equal to $v = 340$ m/s.

4.177. A source of sonic oscillations with frequency $\nu_0 = 1700$ Hz and a receiver are located at the same point. At the moment $t = 0$ the source starts receding from the receiver with constant acceleration $w = 10.0$ m/s^2. Assuming the velocity of sound to be equal to $v = 340$ m/s, find the oscillation frequency registered by the stationary receiver $t = 10.0$ s after the start of motion.

4.178. A source of sound with natural frequency $\nu_0 = 1.8$ kHz moves uniformly along a straight line separated from a stationary observer by a distance $l = 250$ m. The velocity of the source is equal to $\eta = 0.80$ fraction of the velocity of sound. Find:

(a) the frequency of sound received by the observer at the moment when the source gets closest to him;

(b) the distance between the source and the observer at the moment when the observer receives a frequency $\nu = \nu_0$.

4.179. A stationary source sends forth monochromatic sound. A wall approaches it with velocity $u = 33$ cm/s. The propagation velocity of sound in the medium is $v = 330$ m/s. In what way and how much, in per cent, does the wavelength of sound change on reflection from the wall?

4.180. A source of sonic oscillations with frequency $\nu_0 = 1700$ Hz and a receiver are located on the same normal to a wall. Both the source and the receiver are stationary, and the wall recedes from the source with velocity $u = 6.0$ cm/s. Find the beat frequency registered by the receiver. The velocity of sound is equal to $v = 340$ m/s.

4.181. Find the damping coefficient γ of a sound wave if at distances $r_1 = 10$ m and $r_2 = 20$ m from a point isotropic source of sound the sound wave intensity values differ by a factor $\eta = 4.5$.

4.182. A plane sound wave propagates along the x axis. The damping coefficient of the wave is $\gamma = 0.0230$ m^{-1}. At the point $x = 0$ the loudness level is $L = 60$ dB. Find:

(a) the loudness level at a point with coordinate $x = 50$ m;

(b) the coordinate x of the point at which the sound is not heard any more.

4.183. At a distance $r_0 = 20.0$ m from a point isotropic source of sound the loudness level $L_0 = 30.0$ dB. Neglecting the damping of the sound wave, find:

(a) the loudness level at a distance $r = 10.0$ m from the source;

(b) the distance from the source at which the sound is not heard.

4.184. An observer A located at a distance $r_A = 5.0$ m from a ringing tuning fork notes the sound to fade away $\tau = 19$ s later than an observer B who is located at a distance $r_B = 50$ m from the tuning fork. Find the damping coefficient β of oscillations of the tuning fork. The sound velocity $v = 340$ m/s.

4.185. A plane longitudinal harmonic wave propagates in a medium with density ρ. The velocity of the wave propagation is v. Assuming that the density variations of the medium, induced by the propagating wave, $\Delta\rho \ll \rho$, demonstrate that

(a) the pressure increment in the medium $\Delta p = -\rho v^2 (\partial\xi/\partial x)$, where $\partial\xi/\partial x$ is the relative deformation;

(b) the wave intensity is defined by Eq. (4.3i).

4.186. A ball of radius $R = 50$ cm is located in the way of propagation of a plane sound wave. The sonic wavelength is $\lambda = 20$ cm, the frequency is $\nu = 1700$ Hz, the pressure oscillation amplitude in air is $(\Delta p)_m = 3.5$ Pa. Find the mean energy flow, averaged over an oscillation period, reaching the surface of the ball.

4.187. A point A is located at a distance $r = 1.5$ m from a point isotropic source of sound of frequency $\nu = 600$ Hz. The sonic power of the source is $P = 0.80$ W. Neglecting the damping of the waves and assuming the velocity of sound in air to be equal to $v = 340$ m/s, find at the point A:

(a) the pressure oscillation amplitude $(\Delta p)_m$ and its ratio to the air pressure;

(b) the oscillation amplitude of particles of the medium; compare it with the wavelength of sound.

4.188. At a distance $r = 100$ m from a point isotropic source of sound of frequency 200 Hz the loudness level is equal to $L = 50$ dB. The audibility threshold at this frequency corresponds to the sound intensity $I_0 = 0.10$ nW/m^2. The damping coefficient of the sound wave is $\gamma = 5.0 \cdot 10^{-4}$ m^{-1}. Find the sonic power of the source.

4.4. ELECTROMAGNETIC WAVES. RADIATION

- Phase velocity of an electromagnetic wave:

$$v = c/\sqrt{\varepsilon\mu}, \quad \text{where } c = 1/\sqrt{\varepsilon_0\mu_0}. \tag{4.4a}$$

- In a travelling electromagnetic wave:

$$E\sqrt{\varepsilon\varepsilon_0} = H\sqrt{\mu\mu_0}. \tag{4.4b}$$

- Space density of the energy of an electromagnetic field:

$$w = \frac{\mathbf{ED}}{2} + \frac{\mathbf{BH}}{2}. \tag{4.4c}$$

- Flow density of electromagnetic energy, the Poynting vector:

$$\mathbf{S} = [\mathbf{EH}]. \tag{4.4d}$$

- Energy flow density of electric dipole radiation in a far field zone:

$$S \sim \frac{1}{r^2}\sin^2\theta, \tag{4.4e}$$

where r is the distance from the dipole, θ is the angle between the radius vector $\mathbf{r}$ and the axis of the dipole.

• Radiation power of an electric dipole with moment $\mathbf{p}(t)$ and of a charge q, moving with acceleration $\mathbf{w}$:

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2\ddot{\mathbf{p}}^2}{3c^3}, \quad P = \frac{1}{4\pi\varepsilon_0}\frac{2q^2\mathbf{w}^2}{3c^3}. \tag{4.4f}$$

4.189. An electromagnetic wave of frequency $\nu = 3.0$ MHz passes from vacuum into a non-magnetic medium with permittivity $\varepsilon = 4.0$. Find the increment of its wavelength.

4.190. A plane electromagnetic wave falls at right angles to the surface of a plane-parallel plate of thickness l. The plate is made of non-magnetic substance whose permittivity decreases exponentially from a value ε_1 at the front surface down to a value ε_2 at the rear one. How long does it take a given wave phase to travel across this plate?

4.191. A plane electromagnetic wave of frequency $\nu = 10$ MHz propagates in a poorly conducting medium with conductivity $\sigma = 10$ mS/m and permittivity $\varepsilon = 9$. Find the ratio of amplitudes of conduction and displacement current densities.

4.192. A plane electromagnetic wave $\mathbf{E} = \mathbf{E}_m \cos(\omega t - \mathbf{kr})$ propagates in vacuum. Assuming the vectors $\mathbf{E}_m$ and $\mathbf{k}$ to be known, find the vector $\mathbf{H}$ as a function of time t at the point with radius vector $\mathbf{r} = 0$.

4.193. A plane electromagnetic wave $\mathbf{E} = \mathbf{E}_m \cos(\omega t - \mathbf{kr})$, where $\mathbf{E}_m = E_m\mathbf{e}_y$, $\mathbf{k} = k\mathbf{e}_x$, $\mathbf{e}_x$, $\mathbf{e}_y$ are the unit vectors of the x, y axes, propagates in vacuum. Find the vector $\mathbf{H}$ at the point with radius vector $\mathbf{r} = x\mathbf{e}_x$ at the moment (a) $t = 0$, (b) $t = t_0$. Consider the case when $E_m = 160$ V/m, $k = 0.51$ m^{-1}, $x = 7.7$ m, and $t_0 = 33$ ns.

4.194. A plane electromagnetic wave $\mathbf{E} = \mathbf{E}_m \cos(\omega t - kx)$ propagating in vacuum induces the emf $\mathscr{E}_{ind}$ in a square frame with side l. The orientation of the frame is shown in Fig. 4.37. Find the amplitude value ε_{ind}, if $E_m = 0.50$ mV/m, the frequency $\nu = 5.0$ MHz and $l = 50$ cm.

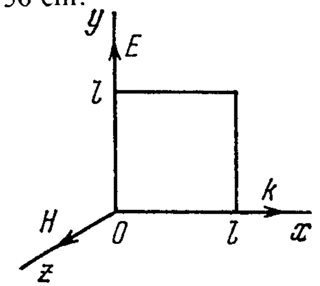


Fig. 4.37.

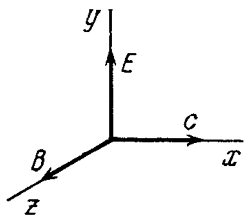


Fig. 4.38.

4.195. Proceeding from Maxwell's equations show that in the case of a plane electromagnetic wave (Fig. 4.38) propagating in vacuum the following relations hold:

$$\frac{\partial E}{\partial t} = -c^2\frac{\partial B}{\partial x}, \quad \frac{\partial B}{\partial t} = -\frac{\partial E}{\partial x}.$$

4.196. Find the mean Poynting vector $\langle\mathbf{S}\rangle$ of a plane electromagnetic wave $\mathbf{E} = \mathbf{E}_m \cos(\omega t - \mathbf{kr})$ if the wave propagates in vacuum.

4.197. A plane harmonic electromagnetic wave with plane polarization propagates in vacuum. The electric component of the wave has a strength amplitude $E_m = 50$ mV/m, the frequency is $\nu = 100$ MHz. Find:

(a) the efficient value of the displacement current density;

(b) the mean energy flow density averaged over an oscillation period.

4.198. A ball of radius $R = 50$ cm is located in a non-magnetic medium with permittivity $\varepsilon = 4.0$. In that medium a plane electromagnetic wave propagates, the strength amplitude of whose electric component is equal to $E_m = 200$ V/m. What amount of energy reaches the ball during a time interval $t = 1.0$ min?

4.199. A standing electromagnetic wave with electric component $\mathbf{E} = \mathbf{E}_m \cos kx \cdot \cos\omega t$ is sustained along the x axis in vacuum. Find the magnetic component of the wave $\mathbf{B}(x, t)$. Draw the approximate distribution pattern of the wave's electric and magnetic components ($\mathbf{E}$ and $\mathbf{B}$) at the moments $t = 0$ and $t = T/4$, where T is the oscillation period.

4.200. A standing electromagnetic wave $\mathbf{E} = \mathbf{E}_m \cos kx \cdot \cos\omega t$ is sustained along the x axis in vacuum. Find the projection of the Poynting vector on the x axis $S_x(x, t)$ and the mean value of that projection averaged over an oscillation period.

4.201. A parallel-plate air capacitor whose electrodes are shaped as discs of radius $R = 6.0$ cm is connected to a source of an alternating sinusoidal voltage with frequency $\omega = 1000$ s^{-1}. Find the ratio of peak values of magnetic and electric energies within the capacitor.

4.202. An alternating sinusoidal current of frequency $\omega = 1000$ s^{-1} flows in the winding of a straight solenoid whose cross-sectional radius is equal to $R = 6.0$ cm. Find the ratio of peak values of electric and magnetic energies within the solenoid.

4.203. A parallel-plate capacity whose electrodes are shaped as round discs is charged slowly. Demonstrate that the flux of the Poynting vector across the capacitor's lateral surface is equal to the increment of the capacitor's energy per unit time. The dissipation of field at the edge is to be neglected in calculations.

4.204. A current I flows along a straight conductor with round cross-section. Find the flux of the Poynting vector across the lateral surface of the conductor's segment with resistance R.

4.205. Non-relativistic protons accelerated by a potential difference U form a round beam with current I. Find the magnitude and

direction of the Poynting vector outside the beam at a distance r from its axis.

4.206. A current flowing in the winding of a long straight solenoid is increased at a sufficiently slow rate. Demonstrate that the rate at which the energy of the magnetic field in the solenoid increases is equal to the flux of the Poynting vector across the lateral surface of the solenoid.

4.207. Fig. 4.39 illustrates a segment of a double line carrying direct current whose direction is indicated by the arrows. Taking into account that the potential $\varphi_2 > \varphi_1$, and making use of the Poynting vector, establish on which side (left or right) the source of the current is located.

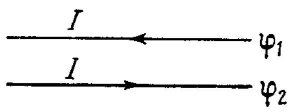


Fig. 4.39.

4.208. The energy is transferred from a source of constant voltage V to a consumer by means of a long straight coaxial cable with negligible active resistance. The consumed current is I. Find the energy flux across the cross-section of the cable. The conductive sheath is supposed to be thin.

4.209. A source of ac voltage $V = V_0 \cos \omega t$ delivers energy to a consumer by means of a long straight coaxial cable with negligible active resistance. The current in the circuit varies as $I = I_0 \cos (\omega t - \varphi)$. Find the time-averaged energy flux through the cross-section of the cable. The sheath is thin.

4.210. Demonstrate that at the boundary between two media the normal components of the Poynting vector are continuous, i.e. $S_{1n} = S_{2n}$.

4.211. Demonstrate that a closed system of charged non-relativistic particles with identical specific charges emits no dipole radiation.

4.212. Find the mean radiation power of an electron performing harmonic oscillations with amplitude $a = 0.10$ nm and frequency $\omega = 6.5 \cdot 10^{14}$ s^{-1}

4.213. Find the radiation power developed by a non-relativistic particle with charge e and mass m, moving along a circular orbit of radius R in the field of a stationary point charge q.

4.214. A particle with charge e and mass m flies with non-relativistic velocity v at a distance b past a stationary particle with charge q. Neglecting the bending of the trajectory of the moving particle, find the energy lost by this particle due to radiation during the total flight time.

4.215. A non-relativistic proton enters a half-space along the normal to the transverse uniform magnetic field whose induction

equals $B = 1.0$ T. Find the ratio of the energy lost by the proton due to radiation during its motion in the field to its initial kinetic energy.

4.216. A non-relativistic charged particle moves in a transverse uniform magnetic field with induction B. Find the time dependence of the particle's kinetic energy diminishing due to radiation. How soon will its kinetic energy decrease e-fold? Calculate this time interval for the case (a) of an electron, (b) of a proton.

4.217. A charged particle moves along the y axis according to the law $y = a \cos \omega t$, and the point of observation P is located on the x axis at a distance l from the particle ($l \gg a$). Find the ratio of electromagnetic radiation flow densities S_1/S_2 at the point P at the moments when the coordinate of the particle $y_1 = 0$ and $y_2 = a$. Calculate that ratio if $\omega = 3.3 \cdot 10^6$ s^{-1} and $l = 190$ m.

4.218. A charged particle moves uniformly with velocity v along a circle of radius R in the plane xy (Fig. 4.40). An observer is located

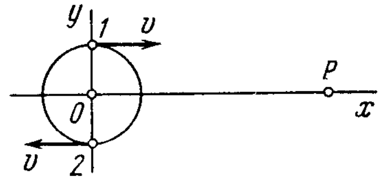


Fig. 4.40.

on the x axis at a point P which is removed from the centre of the circle by a distance much exceeding R. Find:

(a) the relationship between the observed values of the y projection of the particle's acceleration and the y coordinate of the particle;

(b) the ratio of electromagnetic radiation flow densities S_1/S_2 at the point P at the moments of time when the particle moves, from the standpoint of the observer P, toward him and away from him, as shown in the figure.

4.219. An electromagnetic wave emitted by an elementary dipole propagates in vacuum so that in the far field zone the mean value of the energy flow density is equal to S_0 at the point removed from the dipole by a distance r along the perpendicular drawn to the dipole's axis. Find the mean radiation power of the dipole.

4.220. The mean power radiated by an elementary dipole is equal to P_0. Find the mean space density of energy of the electromagnetic field in vacuum in the far field zone at the point removed from the dipole by a distance r along the perpendicular drawn to the dipole's axis.

4.221. An electric dipole whose modulus is constant and whose moment is equal to p rotates with constant angular velocity ω about the axis drawn at right angles to the axis of the dipole and passing through its midpoint. Find the power radiated by such a dipole.

4.222. A free electron is located in the field of a plane electromagnetic wave. Neglecting the magnetic component of the wave disturbing its motion, find the ratio of the mean energy radiated by the oscillating electron per unit time to the mean value of the energy flow density of the incident wave.

4.223. A plane electromagnetic wave with frequency ω falls upon an elastically bonded electron whose natural frequency equals ω_0. Neglecting the damping of oscillations, find the ratio of the mean energy dissipated by the electron per unit time to the mean value of the energy flow density of the incident wave.

4.224. Assuming a particle to have the form of a ball and to absorb all incident light, find the radius of a particle for which its gravitational attraction to the Sun is counterbalanced by the force that light exerts on it. The power of light radiated by the Sun equals $P = 4 \cdot 10^{26}$ W, and the density of the particle is $\rho = 1.0$ g/cm^3.

PART FIVE

OPTICS

5.1. PHOTOMETRY AND GEOMETRICAL OPTICS

• Spectral response of an eye $V(\lambda)$ is shown in Fig. **5.1**.

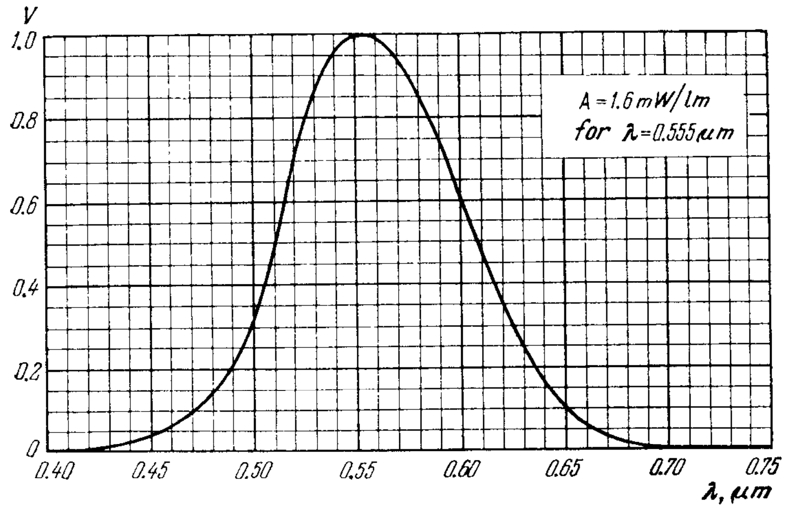


Fig. 5.1.

• Luminous intensity I and illuminance E:

$$I = \frac{d\Phi}{d\Omega}, \quad E = \frac{d\Phi_{inc}}{dS}. \tag{5.1a}$$

• Illuminance produced by a point isotropic source:

$$E = \frac{I \cos\alpha}{r^2}, \tag{5.1b}$$

where α is the angle between the normal to the surface and the direction to the source.

• Luminosity M and luminance L:

$$M = \frac{d\Phi_{emit}}{dS}, \quad L = \frac{d\Phi}{d\Omega\, \Delta S \cos\theta}. \tag{5.1c}$$

• For a Lambert source $L = $ const and luminosity

$$M = \pi L. \tag{5.1d}$$

• Relation between **refractive angle** θ of a prism and least deviation angle α:

$$\sin\frac{\alpha+\theta}{2}=n\sin\frac{\theta}{2}, \tag{5.1e}$$

where n is the refractive index of the prism.

• Equation of spherical mirror:

$$\frac{1}{s'}+\frac{1}{s}=\frac{2}{R}, \tag{5.1f}$$

where R is the curvature radius of the mirror.

• Equations for aligned optical system (Fig. 5.2):

$$\frac{n'}{s'}-\frac{n}{s}=\Phi,\quad \frac{f'}{s'}+\frac{f}{s}=1,\quad xx'=ff'. \tag{5.1g}$$

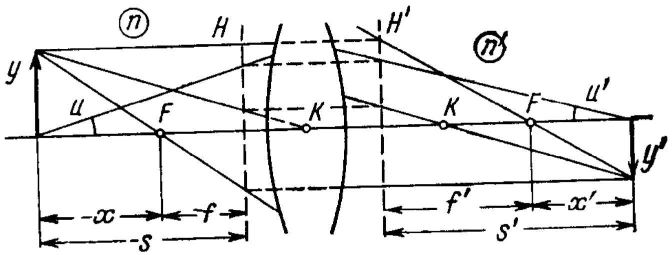


Fig. 5.2.

• Relations between focal lengths and optical power:

$$f'=\frac{n'}{\Phi},\quad f=-\frac{n}{\Phi},\quad \frac{f'}{f}=-\frac{n'}{n}. \tag{5.1h}$$

• Optical power of a spherical refractive surface:

$$\Phi=\frac{n'-n}{R}. \tag{5.1i}$$

• Optical power of a thin lens in a medium with refractive index n_0:

$$\Phi=(n-n_0)\left(\frac{1}{R_1}-\frac{1}{R_2}\right), \tag{5.1j}$$

where n is the refractive index of the lens.

• Optical power of a thick lens:

$$\Phi=\Phi_1+\Phi_2-\frac{d}{n}\Phi_1\Phi_2, \tag{5.1k}$$

where d is the thickness of the lens. This equation is also valid for a system of two thin lenses separated by a medium with refractive index n.

• Principal planes H and H' are removed from the crest points O and O' of surfaces of a thick lens (Fig. 5.3) by the following distances:

$$X=\frac{d}{n}\frac{\Phi_2}{\Phi},\quad X'=-\frac{d}{n}\frac{\Phi_1}{\Phi}. \tag{5.1l}$$

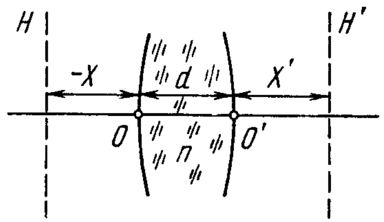


Fig. 5.3.

• Lagrange-Helmholtz invariant:

$$nyu=\text{const}. \tag{5.1m}$$

• Magnifying power of an optical device:

$$\Gamma=\frac{\tan\psi'}{\tan\psi}, \tag{5.1n}$$

where ψ' and ψ are the angles subtended at the eye by an image formed by the optical device and by the corresponding object at a distance for convenient viewing (in the case of a microscope or magnifying glass that distance is equal to l_0 = 25 cm).

5.1. Making use of the spectral response curve for an eye (see Fig. 5.1), find:

(a) the energy flux corresponding to the luminous flux of 1.0 lm at the wavelengths 0.51 and 0.64 μm;

(b) the luminous flux corresponding to the wavelength interval from 0.58 to 0.63 μm if the respective energy flux, equal to Φ_e = = 4.5 mW, is uniformly distributed over all wavelengths of the interval. The function $V(\lambda)$ is assumed to be linear in the given spectral interval.

5.2. A point isotropic source emits a luminous flux Φ = 10 lm with wavelength λ = 0.59 μm. Find the peak strength values of electric and magnetic fields in the luminous flux at a distance r = = 1.0 m from the source. Make use of the curve illustrated in Fig. 5.1.

5.3. Find the mean illuminance of the irradiated part of an opaque sphere receiving

(a) a parallel luminous flux resulting in illuminance E_0 at the point of normal incidence;

(b) light from a point isotropic source located at a distance l = = 100 cm from the centre of the sphere; the radius of the sphere is R = 60 cm and the luminous intensity is I = 36 cd.

5.4. Determine the luminosity of a surface whose luminance depends on direction as $L = L_0 \cos \theta$, where θ is the angle between the radiation direction and the normal to the surface.

5.5. A certain luminous surface obeys Lambert's law. Its luminance is equal to L. Find:

(a) the luminous flux emitted by an element ΔS of this surface into a cone whose axis is normal to the given element and whose aperture angle is equal to θ;

(b) the luminosity of such a source.

5.6. An illuminant shaped as a plane horizontal disc $S = 100$ cm^2 in area is suspended over the centre of a round table of radius $R = 1.0$ m. Its luminance does not depend on direction and is equal to $L = 1.6 \cdot 10^4$ cd/m^2. At what height over the table should the illuminant be suspended to provide maximum illuminance at the circumference of the table? How great will that illuminance be? The illuminant is assumed to be a point source.

5.7. A point source is suspended at a height $h = 1.0$ m over the centre of a round table of radius $R = 1.0$ m. The luminous intensity I of the source depends on direction so that illuminance at all points of the table is the same. Find the function $I(\theta)$, where θ is the angle between the radiation direction and the vertical, as well as the luminous flux reaching the table if $I(0) = I_0 = 100$ cd.

5.8. A vertical shaft of light from a projector forms a light spot $S = 100$ cm^2 in area on the ceiling of a round room of radius $R = 2.0$ m. The illuminance of the spot is equal to $E = 1000$ lx. The reflection coefficient of the ceiling is equal to $\rho = 0.80$. Find the maximum illuminance of the wall produced by the light reflected from the ceiling. The reflection is assumed to obey Lambert's law.

5.9. A luminous dome shaped as a hemisphere rests on a horizontal plane. Its luminosity is uniform. Determine the illuminance at the centre of that plane if its luminance equals L and is independent of direction.

5.10. A Lambert source has the form of an infinite plane. Its luminance is equal to L. Find the illuminance of an area element oriented parallel to the given source.

5.11. An illuminant shaped as a plane horizontal disc of radius $R = 25$ cm is suspended over a table at a height $h = 75$ cm. The illuminance of the table below the centre of the illuminant is equal to $E_0 = 70$ lx. Assuming the source to obey Lambert's law, find its luminosity.

5.12. A small lamp having the form of a uniformly luminous sphere of radius $R = 6.0$ cm is suspended at a height $h = 3.0$ m above the floor. The luminance of the lamp is equal to $L = 2.0 \cdot 10^4$ cd/m^2 and is independent of direction. Find the illuminance of the floor directly below the lamp.

5.13. Write the law of reflection of a light beam from a mirror in vector form, using the directing unit vectors $\mathbf{e}$ and $\mathbf{e}'$ of the incident and reflected beams and the unit vector $\mathbf{n}$ of the outside normal to the mirror surface.

5.14. Demonstrate that a light beam reflected from three mutually perpendicular plane mirrors in succession reverses its direction.

5.15. At what value of the angle of incident θ_1 is a shaft of light reflected from the surface of water perpendicular to the refracted shaft?

5.16. Two optical media have a plane boundary between them. Suppose θ_{1cr} is the critical angle of incidence of a beam and θ_1 is the angle of incidence at which the refracted beam is perpendicular to the reflected one (the beam is assumed to come from an optically denser medium). Find the relative refractive index of these media if $\sin \theta_{1cr} / \sin \theta_1 = \eta = 1.28$.

5.17. A light beam falls upon a plane-parallel glass plate $d = 6.0$ cm in thickness. The angle of incidence is $\theta = 60°$. Find the value of deflection of the beam which passed through that plate.

5.18. A man standing on the edge of a swimming pool looks at a stone lying on the bottom. The depth of the swimming pool is equal to h. At what distance from the surface of water is the image of the stone formed if the line of vision makes an angle θ with the normal to the surface?

5.19. Demonstrate that in a prism with small refracting angle θ the shaft of light deviates through the angle $\alpha \simeq (n - 1)\,\theta$ regardless of the angle of incidence, provided that the latter is also small.

5.20. A shaft of light passes through a prism with refracting angle θ and refractive index n. Let α be the diffraction angle of the shaft. Demonstrate that if the shaft of light passes through the prism symmetrically,

(a) the angle α is the least;

(b) the relationship between the angles α and θ is defined by Eq. (5.1e).

5.21. The least deflection angle of a certain glass prism is equal to its refracting angle. Find the latter.

5.22. Find the minimum and maximum deflection angles for a light ray passing through a glass prism with refracting angle $\theta = 60°$.

5.23. A trihedral prism with refracting angle 60° provides the least deflection angle 37° in air. Find the least deflection angle of that prism in water.

5.24. A light ray composed of two monochromatic components passes through a trihedral prism with refracting angle $\theta = 60°$. Find the angle $\Delta\alpha$ between the components of the ray after its passage through the prism if their respective indices of refraction are equal to 1.515 and 1.520. The prism is oriented to provide the least deflection angle.

5.25. Using Fermat's principle derive the laws of deflection and refraction of light on the plane interface between two media.

5.26. By means of plotting find:

(a) the path of a light ray after reflection from a concave and convex spherical mirrors (see Fig. 5.4, where F is the focal point, OO' is the optical axis);

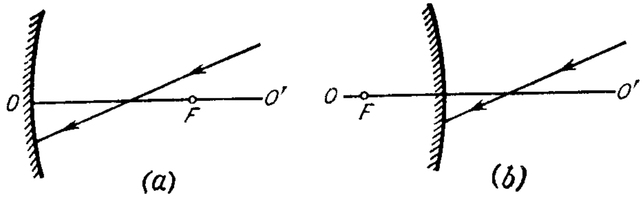


Fig. 5.4.

(b) the positions of the mirror and its focal point in the cases illustrated in Fig. 5.5, where P and P' are the conjugate points.

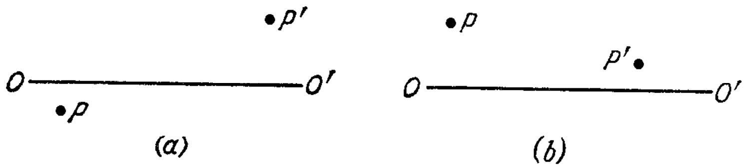


Fig. 5.5.

5.27. Determine the focal length of a concave mirror if:

(a) with the distance between an object and its image being equal to $l = 15$ cm, the transverse magnification $\beta = -2.0$;

(b) in a certain position of the object the transverse magnification is $\beta_1 = -0.50$ and in another position displaced with respect to the former by a distance $l = 5.0$ cm the transverse magnification $\beta_2 = -0.25$.

5.28. A point source with luminous intensity $I_0 = 100$ cd is positioned at a distance $s = 20.0$ cm from the crest of a concave mirror with focal length $f = 25.0$ cm. Find the luminous intensity of the reflected ray if the reflection coefficient of the mirror is $\rho = 0.80$.

5.29. Proceeding from Fermat's principle derive the refraction formula for paraxial rays on a spherical boundary surface of radius R between media with refractive indices n and n'.

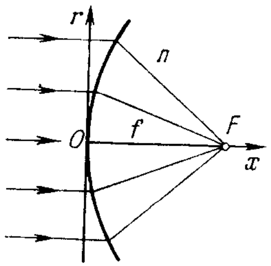


Fig. 5.6.

5.30. A parallel beam of light falls from vacuum on a surface enclosing a medium with refractive index n (Fig. 5.6). Find the shape of that surface, $x(r)$, if the beam is brought into focus at the point F at a distance f from the crest O. What is the maximum radius of a beam that can still be focussed?

5.31. A point source is located at a distance of 20 cm from the front surface of a symmetrical glass biconvex lens. The lens is 5.0 cm thick and the curvature radius of its surfaces is 5.0 cm. How far beyond the rear surface of this lens is the image of the source formed?

5.32. An object is placed in front of convex surface of a glass plano-convex lens of thickness $d = 9.0$ cm. The image of that object is formed on the plane surface of the lens serving as a screen. Find:

(a) the transverse magnification if the curvature radius of the lens's convex surface is $R = 2.5$ cm;

(b) the image illuminance if the luminance of the object is $L = 7700$ cd/m^2 and the entrance aperture diameter of the lens is $D = 5.0$ mm; losses of light are negligible.

5.33. Find the optical power and the focal lengths

(a) of a thin glass lens in liquid with refractive index $n_0 = 1.7$ if its optical power in air is $\Phi_0 = -5.0$ D;

(b) of a thin symmetrical biconvex glass lens, with air on one side and water on the other side, if the optical power of that lens in air is $\Phi_0 = +10$ D.

5.34. By means of plotting find:

(a) the path of a ray of light beyond thin converging and diverging lenses (Fig. 5.7, where OO' is the optical axis, F and F' are the front and rear focal points);

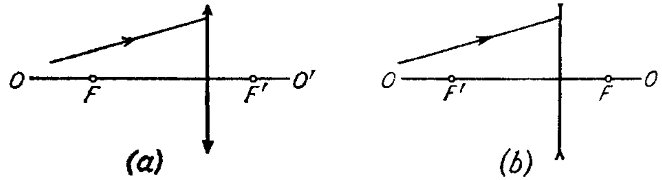


Fig. 5.7.

(b) the position of a thin lens and its focal points if the position of the optical axis OO' and the positions of the cojugate points P, P' (see Fig. 5.5) are known; the media on both sides of the lenses are identical;

(c) the path of ray *2* beyond the converging and diverging lenses (Fig. 5.8) if the path of ray *1* and the positions of the lens and of its optical axis OO' are all known; the media on both sides of the lenses are identical.

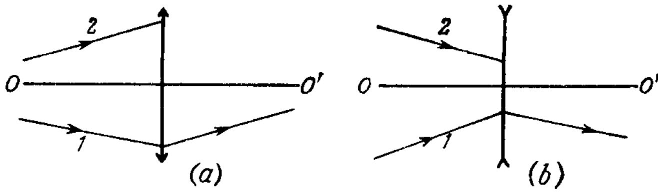


Fig. 5.8.

5.35. A thin converging lens with focal length $f = 25$ cm projects the image of an object on a screen removed from the lens by a dis-

tance $l=5.0$ m. Then the screen was drawn closer to the lens by a distance $\Delta l = 18$ cm. By what distance should the object be shifted for its image to become sharp again?

5.36. A source of light is located at a distance $l = 90$ cm from a screen. A thin converging lens provides the sharp image of the source when placed between the source of light and the screen at two positions. Determine the focal length of the lens if

(a) the distance between the two positions of the lens is $\Delta l = 30$ cm;

(b) the transverse dimensions of the image at one position of the lens are $\eta = 4.0$ greater than those at the other position.

5.37. A thin converging lens is placed between an object and a screen whose positions are fixed. There are two positions of the lens at which the sharp image of the object is formed on the screen. Find the transverse dimension of the object if at one position of the lens the image dimension equals $h' = 2.0$ mm and at the other, $h'' = 4.5$ mm.

5.38. A thin converging lens with aperture ratio $D : f = 1 : 3.5$ (D is the lens diameter, f is its focal length) provides the image of a sufficiently distant object on a photographic plate. The object luminance is $L = 260$ cd/m^2. The losses of light in the lens amount to $\alpha = 0.10$. Find the illuminance of the image.

5.39. How does the luminance of a real image depend on diameter D of a thin converging lens if that image is observed

(a) directly;

(b) on a white screen backscattering according to Lambert's law?

5.40. There are two thin symmetrical lenses: one is converging, with refractive index $n_1 = 1.70$, and the other is diverging with refractive index $n_2 = 1.51$. Both lenses have the same curvature radius of their surfaces equal to $R = 10$ cm. The lenses were put close together and submerged into water. What is the focal length of this system in water?

5.41. Determine the focal length of a concave spherical mirror which is manufactured in the form of a thin symmetric biconvex glass lens one of whose surfaces is silvered. The curvature radius of the lens surface is $R = 40$ cm.

5.42. Figure 5.9 illustrates an aligned system consisting of three thin lenses. The system is located in air. Determine:

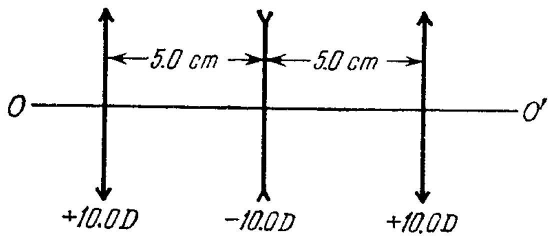


Fig. 5.9.

(a) the position of the point of convergence of a parallel ray incoming from the left after passing through the system;

(b) the distance between the first lens and a point lying on the axis to the left of the system, at which that point and its image are located symmetrically with respect to the lens system.

5.43. A Galilean telescope of 10-fold magnification has the length of 45 cm when adjusted to infinity. Determine:

(a) the focal lengths of the telescope's objective and ocular;

(b) by what distance the ocular should be displaced to adjust the telescope to the distance of 50 m.

5.44. Find the magnification of a Keplerian telescope adjusted to infinity if the mounting of the objective has a diameter D and the image of that mounting formed by the telescope's ocular has a diameter d.

5.45. On passing through a telescope a flux of light increases its intensity $\eta = 4.0 \cdot 10^4$ times. Find the angular dimension of a distant object if its image formed by that telescope has an angular dimension $\psi' = 2.0°$.

5.46. A Keplerian telescope with magnification $\Gamma = 15$ was submerged into water which filled up the inside of the telescope. To make the system work as a telescope again within the former dimensions, the objective was replaced. What has the magnification of the telescope become equal to? The refractive index of the glass of which the ocular is made is equal to $n = 1.50$.

5.47. At what magnification Γ of a telescope with a diameter of the objective $D = 6.0$ cm is the illuminance of the image of an object on the retina not less than without the telescope? The pupil diameter is assumed to be equal to $d_0 = 3.0$ mm. The losses of light in the telescope are negligible.

5.48. The optical powers of the objective and the ocular of a microscope are equal to 100 and 20 D respectively. The microscope magnification is equal to 50. What will the magnification of the microscope be when the distance between the objective and the ocular is increased by 2.0 cm?

5.49. A microscope has a numerical aperture $\sin \alpha = 0.12$, where α is the aperture angle subtended by the entrance pupil of the microscope. Assuming the diameter of an eye's pupil to be equal to $d_0 = 4.0$ mm, determine the microscope magnification at which

(a) the diameter of the beam of light coming from the microscope is equal to the diameter of the eye's pupil;

(b) the illuminance of the image on the retina is independent of magnification (consider the case when the beam of light passing through the system "microscope-eye" is bounded by the mounting of the objective).

5.50. Find the positions of the principal planes, the focal and nodal points of a thin biconvex symmetric glass lens with curvature radius of its surfaces equal to $R = 7.50$ cm. There is air on one side of the lens and water on the other.

5.51. By means of plotting find the positions of focal points and principal planes of aligned optical systems illustrated in Fig. 5.10:

(a) a telephoto lens, that is a combination of a converging and a diverging thin lenses ($f_1 = 1.5\ a$, $f_2 = -1.5\ a$);

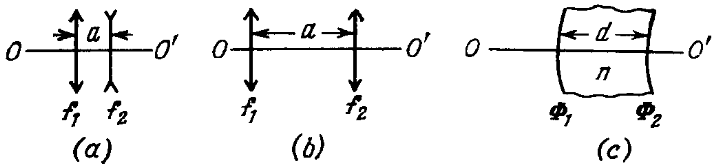


Fig. 5.10.

(b) a system of two thin converging lenses ($f_1 = 1.5\ a$, $f_2 = 0.5\ a$);

(c) a thick convex-concave lens ($d = 4$ cm, $n = 1.5$, $\Phi_1 = +50$ D, $\Phi_2 = -50$ D).

5.52. An optical system is located in air. Let OO' be its optical axis, F and F' are the front and rear focal points, H and H' are the front and rear principal planes, P and P' are the conjugate points. By means of plotting find:

(a) the positions F' and H' (Fig. 5.11a);

(b) the position of the point S' conjugate to the point S (Fig. 5.11b);

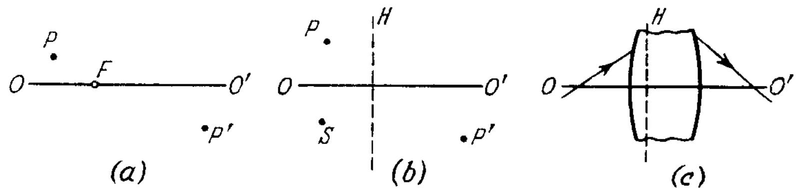


Fig. 5.11.

(c) the positions F, F', and H' (Fig. 5.11c, where the path of the ray of light is shown before and after passing through the system).

5.53. Suppose F and F' are the front and rear focal points of an optical system, and H and H' are its front and rear principal points. By means of plotting find the position of the image S' of the point S for the following relative positions of the points S, F, F', H, and H':

(a) $FSHH'F'$; (b) $HSF'FH'$; (c) $H'SF'FH$; (d) $F'H'SHF$.

5.54. A telephoto lens consists of two thin lenses, the front converging lens and the rear diverging lens with optical powers $\Phi_1 = +10$ D and $\Phi_2 = -10$ D. Find:

(a) the focal length and the positions of principal axes of that system if the lenses are separated by a distance $d = 4.0$ cm;

(b) the distance d between the lenses at which the ratio of a focal length f of the system to a distance l between the converging lens and the rear principal focal point is the highest. What is this ratio equal to?

5.55. Calculate the positions of the principal planes and focal points of a thick convex-concave glass lens if the curvature radius of the convex surface is equal to $R_1 = 10.0$ cm and of the concave surface to $R_2 = 5.0$ cm and the lens thickness is $d = 3.0$ cm.

5.56. An aligned optical system consists of two thin lenses with focal lengths f_1 and f_2, the distance between the lenses being equal to d. The given system has to be replaced by one thin lens which, at any position of an object, would provide the same transverse magnification as the system. What must the focal length of this lens be equal to and in what position must it be placed with respect to the two-lens system?

5.57. A system consists of a thin symmetrical converging glass lens with the curvature radius of its surfaces $R = 38$ cm and a plane mirror oriented at right angles to the optical axis of the lens. The distance between the lens and the mirror is $l = 12$ cm. What is the optical power of this system when the space between the lens and the mirror is filled up with water?

5.58. At what thickness will a thick convex-concave glass lens in air

(a) serve as a telescope provided the curvature radius of its convex surface is $\Delta R = 1.5$ cm greater than that of its concave surface?

(b) have the optical power equal to -1.0 D if the curvature radii of its convex and concave surfaces are equal to 10.0 and 7.5 cm respectively?

5.59. Find the positions of the principal planes, the focal length and the sign of the optical power of a thick convex-concave glass lens

(a) whose thickness is equal to d and curvature radii of the surfaces are the same and equal to R;

(b) whose refractive surfaces are concentric and have the curvature radii R_1 and R_2 ($R_2 > R_1$).

5.60. A telescope system consists of two glass balls with radii $R_1 = 5.0$ cm and $R_2 = 1.0$ cm. What are the distance between the centres of the balls and the magnification of the system if the bigger ball serves as an objective?

5.61. Two identical thick symmetrical biconvex lenses are put close together. The thickness of each lens equals the curvature radius of its surfaces, i.e. $d = R = 3.0$ cm. Find the optical power of this system in air.

5.62. A ray of light propagating in an isotropic medium with refractive index n varying gradually from point to point has a curvature radius ρ determined by the formula

$$\frac{1}{\rho} = \frac{\partial}{\partial N}(\ln n),$$

where the derivative is taken with respect to the principal normal to the ray. Derive this formula, assuming that in such a medium the law of refraction $n \sin\theta = \text{const}$ holds. Here θ is the angle between the ray and the direction of the vector ∇n at a given point.

5.63. Find the curvature radius of a ray of light propagating in a horizontal direction close to the Earth's surface where the gradient of the refractive index in air is equal to approximately $3 \cdot 10^{-8}\ m^{-1}$. At what value of that gradient would the ray of light propagate all the way round the Earth?

5.2. INTERFERENCE OF LIGHT

- Width of a fringe:

$$\Delta x = \frac{l}{d}\lambda, \tag{5.2a}$$

where l is the distance from the sources to the screen, d is the distance between the sources.

- Temporal and spatial coherences. Coherence length and coherence radius:

$$l_{coh} \approx \frac{\lambda^2}{\Delta\lambda}, \quad \rho_{coh} \approx \frac{\lambda}{\psi}, \tag{5.2b}$$

where ψ is the angular dimension of the source.

- Condition for interference maxima in the case of light reflected from a thin plate of thickness b:

$$2b\sqrt{n^2 - \sin^2\theta_1} = (k + 1/2)\lambda, \tag{5.2c}$$

where k is an integer.

- Newton's rings produced on reflection of light from the surfaces of an air interlayer formed between a lens of radius R and a glass plate with which the convex surface of the lens is in contact. The radii of the rings:

$$r = \sqrt{\lambda R k/2}, \tag{5.2d}$$

with the rings being bright if $k = 1, 3, 5, \ldots$, and dark if $k = 2, 4, 6, \ldots$ The value $k = 0$ corresponds to the middle of the central dark spot.

5.64. Demonstrate that when two harmonic oscillations are added, the time-averaged energy of the resultant oscillation is equal to the sum of the energies of the constituent oscillations, if both of them

(a) have the same direction and are incoherent, and all the values of the phase difference between the oscillations are equally probable;

(b) are mutually perpendicular, have the same frequency and an arbitrary phase difference.

5.65. By means of plotting find the amplitude of the oscillation resulting from the addition of the following three oscillations of the same direction:

$$\xi_1 = a\cos\omega t, \quad \xi_2 = 2a\sin\omega t, \quad \xi_3 = 1.5a\cos(\omega t + \pi/3).$$

5.66. A certain oscillation results from the addition of coherent oscillations of the same direction $\xi_k = a\cos[\omega t + (k-1)\varphi]$, where k is the number of the oscillation ($k = 1, 2, \ldots, N$), φ is the phase difference between the kth and $(k-1)$th oscillations. Find the amplitude of the resultant oscillation.

5.67. A system illustrated in Fig. 5.12 consists of two coherent point sources *1* and *2* located in a certain plane so that their dipole moments are oriented at right angles to that plane. The sources are separated by a distance d, the radiation wavelength is equal to λ. Taking into account that the oscillations of source *2* lag in phase behind the oscillations of source *1* by φ ($\varphi < \pi$), find:

(a) the angles θ at which the radiation intensity is maximum;

(b) the conditions under which the radiation intensity in the direction $\theta = \pi$ is maximum and in the opposite direction, minimum.

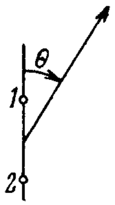


Fig. 5.12.

5.68. A stationary radiating system consists of a linear chain of parallel oscillators separated by a distance d, with the oscillation phase varying linearly along the chain. Find the time dependence of the phase difference $\Delta\varphi$ between the neighbouring oscillators at which the principal radiation maximum of the system will be "scanning" the surroundings with the constant angular velocity ω.

5.69. In Lloyd's mirror experiment (Fig. 5.13) a light wave emitted directly by the source S (narrow slit) interferes with the wave reflected from a mirror M. As a result, an interference fringe pattern is

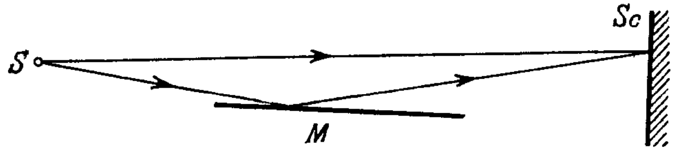


Fig. 5.13.

formed on the screen Sc. The source and the mirror are separated by a distance $l = 100$ cm. At a certain position of the source the fringe width on the screen was equal to $\Delta x = 0.25$ mm, and after the source was moved away from the mirror plane by $\Delta h = 0.60$ mm, the fringe width decreased $\eta = 1.5$ times. Find the wavelength of light.

5.70. Two coherent plane light waves propagating with a divergence angle $\psi \ll 1$ fall almost normally on a screen. The amplitudes of the waves are equal. Demonstrate that the distance between the neighbouring maxima on the screen is equal to $\Delta x = \lambda/\psi$, where λ is the wavelength.

5.71. Figure 5.14 illustrates the interference experiment with Fresnel mirrors. The angle between the mirrors is $\alpha = 12'$, the distances from the mirrors' intersection line to the narrow slit S and the screen Sc are equal to $r = 10.0$ cm and $b = 130$ cm respectively. The wavelength of light is $\lambda = 0.55\ \mu m$. Find:

(a) the width of a fringe on the screen and the number of possible maxima;

(b) the shift of the interference pattern on the screen when the slit is displaced by $\delta l = 1.0$ mm along the arc of radius r with centre at the point O;

(c) at what maximum width δ_{max} of the slit the interference fringes on the screen are still observed sufficiently sharp.

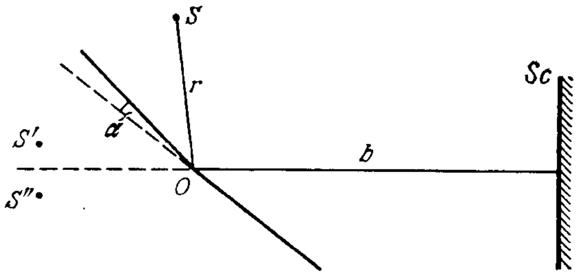


Fig. 5.14.

5.72. A plane light wave falls on Fresnel mirrors with an angle $\alpha = 2.0'$ between them. Determine the wavelength of light if the width of the fringe on the screen $\Delta x = 0.55$ mm.

5.73. A lens of diameter 5.0 cm and focal length $f = 25.0$ cm was cut along the diameter into two identical halves. In the process, the layer of the lens $a = 1.00$ mm in thickness was lost. Then the halves were put together to form a composite lens. In this focal plane a narrow slit was placed, emitting monochromatic light with wavelength $\lambda = 0.60$ μm. Behind the lens a screen was located at a distance $b = 50$ cm from it. Find:

(a) the width of a fringe on the screen and the number of possible maxima;

(b) the maximum width of the slit δ_{max} at which the fringes on the screen will be still observed sufficiently sharp.

5.74. The distances from a Fresnel biprism to a narrow slit and a screen are equal to $a = 25$ cm and $b = 100$ cm respectively. The refracting angle of the glass biprism is equal to $\theta = 20'$. Find the wavelength of light if the width of the fringe on the screen is $\Delta x = 0.55$ mm.

5.75. A plane light wave with wavelength $\lambda = 0.70$ μm falls normally on the base of a biprism made of glass ($n = 1.520$) with refracting angle $\theta = 5.0°$. Behind the biprism (Fig. 5.15) there is a plane-parallel plate, with the space between them filled up with benzene ($n' = 1.500$). Find the width of a fringe on the screen Sc placed behind this system.

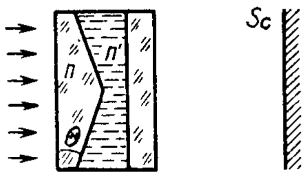


Fig. 5.15.

5.76. A plane monochromatic light wave falls normally on a diaphragm with two narrow slits separated by a distance $d = 2.5$ mm. A fringe pattern is formed on a screen placed at a distance $l = 100$ cm behind the diaphragm. By what distance and in which direction will these fringes be displaced when one of the slits is covered by a glass plate of thickness $h = 10$ μm?

5.77. Figure 5.16 illustrates an interferometer used in measurements of refractive indices of transparent substances. Here S is

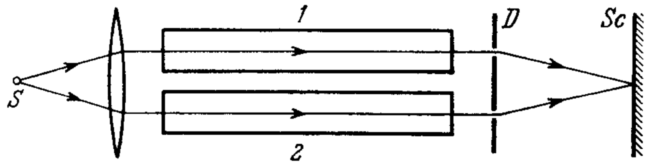


Fig. 5.16.

a narrow slit illuminated by monochromatic light with wavelength $\lambda = 589$ nm, 1 and 2 are identical tubes with air of length $l = 10.0$ cm each, D is a diaphragm with two slits. After the air in tube 1 was replaced with ammonia gas, the interference pattern on the screen Sc was displaced upward by $N = 17$ fringes. The refractive index of air is equal to $n = 1.000277$. Determine the refractive index of ammonia gas.

5.78. An electromagnetic wave falls normally on the boundary between two isotropic dielectrics with refractive indices n_1 and n_2. Making use of the continuity condition for the tangential components, **E** and **H** across the boundary, demonstrate that at the interface the electric field vector **E**

(a) of the transmitted wave experiences no phase jump;

(b) of the reflected wave is subjected to the phase jump equal to π if it is reflected from a medium of higher optical density.

5.79. A parallel beam of white light falls on a thin film whose refractive index is equal to $n = 1.33$. The angle of indices is $\theta_1 = 52°$. What must the film thickness be equal to for the reflected light to be coloured yellow ($\lambda = 0.60$ μm) most intensively?

5.80. Find the minimum thickness of a film with refractive index 1.33 at which light with wavelength 0.64 μm experiences maximum reflection while light with wavelength 0.40 μm is not reflected at all. The incidence angle of light is equal to 30°.

5.81. To decrease light losses due to reflection from the glass surface the latter is coated with a thin layer of substance whose refractive index $n' = \sqrt{n}$, where n is the refractive index of the glass. In this case the amplitudes of electromagnetic oscillations reflected from both coated surfaces are equal. At what thickness of that coating is the glass reflectivity in the direction of the normal equal to zero for light with wavelength λ?

5.82. Diffused monochromatic light with wavelength $\lambda = 0.60$ μm falls on a thin film with refractive index $n = 1.5$. Determine the

film thickness if the angular separation of neighbouring maxima observed in reflected light at the angles close to $\theta = 45°$ to the normal is equal to $\delta\theta = 3.0°$.

5.83. Monochromatic light passes through an orifice in a screen *Sc* (Fig. 5.17) and being reflected from a thin transparent plate *P* produces fringes of equal inclination on the screen. The thickness of the plate is equal to d, the distance between the plate and the screen is l, the radii of the ith and kth dark rings are r_i and r_k. Find the wavelength of light taking into account that $r_{i,k} \ll l$.

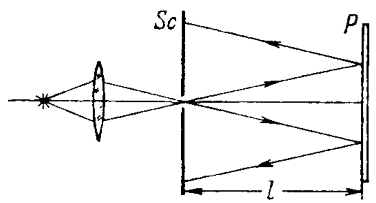


Fig. 5.17.

5.84. A plane monochromatic light wave with wavelength λ falls on the surface of a glass wedge whose faces form an angle $\alpha \ll 1$. The plane of incidence is perpendicular to the edge, the angle of incidence is θ_1. Find the distance between the neighbouring fringe maxima on the screen placed at right angles to reflected light.

5.85. Light with wavelength $\lambda = 0.55$ μm from a distant point source falls normally on the surface of a glass wedge. A fringe pattern whose neighbouring maxima on the surface of the wedge are separated by a distance $\Delta x = 0.21$ mm is observed in reflected light. Find:

(a) the angle between the wedge faces;

(b) the degree of light monochromatism ($\Delta\lambda/\lambda$) if the fringes disappear at a distance $l \simeq 1.5$ cm from the wedge's edge.

5.86. The convex surface of a plano-convex glass lens comes into contact with a glass plate. The curvature radius of the lens's convex surface is R, the wavelength of light is equal to λ. Find the width Δr of a Newton ring as a function of its radius r in the region where $\Delta r \ll r$.

5.87. The convex surface of a plano-convex glass lens with curvature radius $R = 40$ cm comes into contact with a glass plate. A certain ring observed in reflected light has a radius $r = 2.5$ mm. Watching the given ring, the lens was gradually removed from the plate by a distance $\Delta h = 5.0$ μm. What has the radius of that ring become equal to?

5.88. At the crest of a spherical surface of a plano-convex lens there is a ground-off plane spot of radius $r_0 = 3.0$ mm through which the lens comes into contact with a glass plate. The curvature radius of the lens's convex surface is equal to $R = 150$ cm. Find the radius of the sixth bright ring when observed in reflected light with wavelength $\lambda = 655$ nm.

5.89. A plano-convex glass lens with curvature radius of spherical surface $R = 12.5$ cm is *pressed* against a glass plate. The diameters of the tenth and fifteenth dark Newton's rings in reflected light are equal to $d_1 = 1.00$ mm and $d_2 = 1.50$ mm. Find the wavelength of light.

5.90. Two plano-convex thin glass lenses are brought into contact with their spherical surfaces. Find the optical power of such a system if in reflected light with wavelength $\lambda = 0.60$ μm the diameter of the fifth bright ring is $d = 1.50$ mm.

5.91. Two thin symmetric glass lenses, one biconvex and the other biconcave, are brought into contact to make a system with optical power $\Phi = 0.50$ D. Newton's rings are observed in reflected light with wavelength $\lambda = 0.61$ μm. Determine:

(a) the radius of the tenth dark ring;

(b) how the radius of that ring will change when the space between the lenses is filled up with water.

5.92. The spherical surface of a plano-convex lens comes into contact with a glass plate. The space between the lens and the plate is filled up with carbon dioxide. The refractive indices of the lens, carbon dioxide, and the plate are equal to $n_1 = 1.50$, $n_2 = 1.63$, and $n_3 = 1.70$ respectively. The curvature radius of the spherical surface of the lens is equal to $R = 100$ cm. Determine the radius of the fifth dark Newton's ring in reflected light with wavelength $\lambda = 0.50$ μm.

5.93. In a two-beam interferometer the orange mercury line composed of two wavelengths $\lambda_1 = 576.97$ nm and $\lambda_2 = 579.03$ nm is employed. What is the least order of interference at which the sharpness of the fringe pattern is the worst?

5.94. In Michelson's interferometer the yellow sodium line composed of two wavelengths $\lambda_1 = 589.0$ nm and $\lambda_2 = 589.6$ nm was used. In the process of translational displacement of one of the mirrors the interference pattern vanished periodically (why?). Find the displacement of the mirror between two successive appearances of the sharpest pattern.

5.95. When a Fabry-Perot étalon is illuminated by monochromatic light with wavelength λ an interference pattern, the system of con-

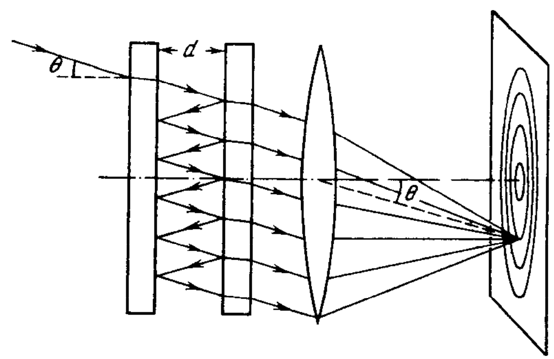


Fig. 5.18.

centric rings, appears in the focal plane of a lens (Fig. 5.18). The thickness of the étalon is equal to d. Determine how

(a) the position of rings;
(b) the angular width of fringes
depends on the order of interference.

5.96. For the Fabry-Perot étalon of thickness $d = 2.5$ cm find:
(a) the highest order of interference of light with wavelength $\lambda = 0.50$ μm;
(b) the dispersion region $\Delta\lambda$, i.e. the spectral interval of wavelengths, within which there is still no overlap with other orders of interference if the observation is carried out approximately at wavelength $\lambda = 0.50$ μm.

5.3. DIFFRACTION OF LIGHT

• Radius of the periphery of the kth Fresnel zone:

$$r_k = \sqrt{k\lambda \frac{ab}{a+b}}, \quad k = 1, 2, 3, \ldots, \tag{5.3a}$$

• Cornu's spiral (Fig. 5.19). The numbers along that spiral correspond to the values of parameter v. In the case of a plane wave $v = x\sqrt{2/b\lambda}$, where x and b are the distances defining the position of the element dS of a wavefront relative to the observation point P as shown in the upper left corner of the figure.

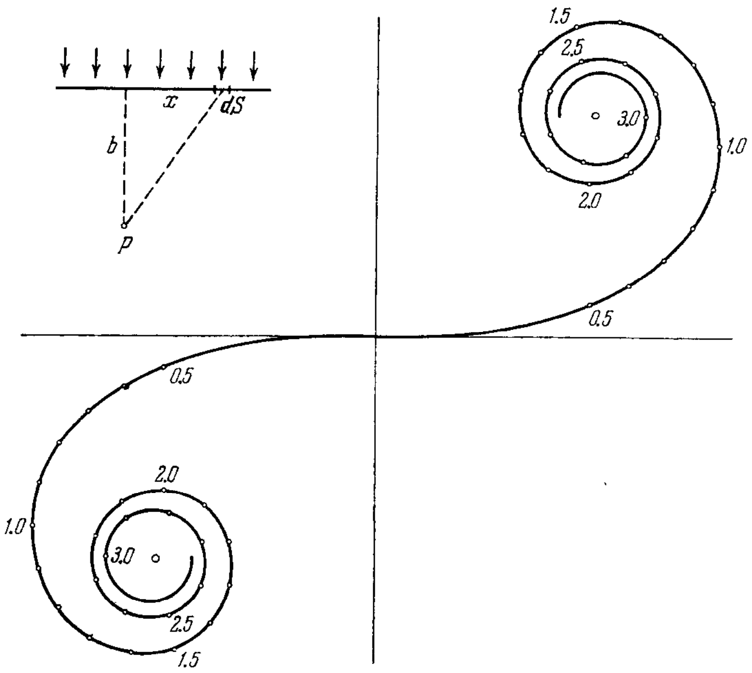


Fig. 5.19.

• Fraunhofer diffraction produced by light falling normally from a slit. Condition of intensity minima:

$$b \sin\theta = \pm k\lambda, \quad k = 1, 2, 3, \ldots, \tag{5.3b}$$

where b is the width of the slit, θ is the diffraction angle.

• Diffraction grating, with light falling normally. The main Fraunhofer maxima appear under the condition

$$d \sin\theta = \pm k\lambda, \quad k = 0, 1, 2, \ldots, \tag{5.3c}$$

the condition of additional minima:

$$d \sin\theta = \pm \frac{k'}{N}\lambda, \tag{5.3d}$$

where $k' = 1, 2, \ldots$, except for $0, N, 2N, \ldots$.

• Angular dispersion of a diffraction grating:

$$D = \frac{\delta\theta}{\delta\lambda} = \frac{k}{d\cos\theta}. \tag{5.3e}$$

• Resolving power of a diffraction grating:

$$R = \frac{\lambda}{\delta\lambda} = kN, \tag{5.3f}$$

where N is the number of lines of the grating.

• Resolving power of an objective

$$R = \frac{1}{\delta\psi} = \frac{D}{1.22\lambda}, \tag{5.3g}$$

where $\delta\psi$ is the least angular separation resolved by the objective, D is the diameter of the objective.

• Bragg's equation. The condition of diffraction maxima:

$$2d \sin\alpha = \pm k\lambda, \tag{5.3h}$$

where d is the interplanar distance, α is the glancing angle, $k = 1, 2, 3, \ldots$.

5.97. A plane light wave falls normally on a diaphragm with round aperture opening the first N Fresnel zones for a point P on a screen located at a distance b from the diaphragm. The wavelength of light is equal to λ. Find the intensity of light I_0 in front of the diaphragm if the distribution of intensity of light $I(r)$ on the screen is known. Here r is the distance from the point P.

5.98. A point source of light with wavelength $\lambda = 0.50$ μm is located at a distance $a = 100$ cm in front of a diaphragm with round aperture of radius $r = 1.0$ mm. Find the distance b between the diaphragm and the observation point for which the number of Fresnel zones in the aperture equals $k = 3$.

5.99. A diaphragm with round aperture, whose radius r can be varied during the experiment, is placed between a point source of light and a screen. The distances from the diaphragm to the source and the screen are equal to $a = 100$ cm and $b = 125$ cm. Determine the wavelength of light if the intensity maximum at the centre of the diffraction pattern of the screen is observed at $r_1 = 1.00$ mm and the next maximum at $r_2 = 1.29$ mm.

5.100. A plane monochromatic light wave with intensity I_0 falls normally on an opaque screen with a round aperture. What is the intensity of light I behind the screen at the point for which the aperture

(a) is equal to the first Fresnel zone; to the internal half of the first zone;

(b) was made equal to the first Fresnel zone and then half of it was closed (along the diameter)?

5.101. A plane monochromatic light wave with intensity I_0 falls normally on an opaque disc closing the first Fresnel zone for the observation point P. What did the intensity of light I at the point P become equal to after

(a) half of the disc (along the diameter) was removed;

(b) half of the external half of the first Fresnel zone was removed (along the diameter)?

5.102. A plane monochromatic light wave with intensity I_0 falls normally on the surfaces of the opaque screens shown in Fig. 5.20. Find the intensity of light I at a point P

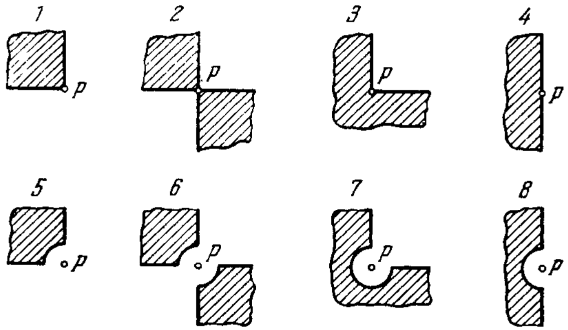


Fig. 5.20.

(a) located behind the corner points of screens *1-3* and behind the edge of half-plane *4*;

(b) for which the rounded-off edge of screens *5-8* coincides with the boundary of the first Fresnel zone.

Derive the general formula describing the results obtained for screens *1-4*; the same, for screens *5-8*.

5.103. A plane light wave with wavelength $\lambda = 0.60$ μm falls normally on a sufficiently large glass plate having a round recess on the opposite side (Fig. 5.21). For the observation point P that recess corresponds to the first one and a half Fresnel zones. Find the depth h of the recess at which the intensity of light at the point P is

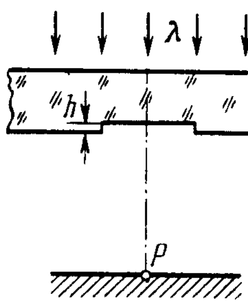


Fig. 5.21.

(a) maximum;

(b) minimum;

(c) equal to the intensity of incident light.

5.104. A plane light wave with wavelength λ and intensity I_0 falls normally on a large glass plate whose opposite side serves as an opaque screen with a round aperture equal to the first Fresnel zone for the observation point P. In the middle of the aperture there is a round recess equal to half the Fresnel zone. What must the depth h of that recess be for the intensity of light at the point P to be the highest? What is this intensity equal to?

5.105. A plane light wave with wavelength $\lambda = 0.57$ μm falls normally on a surface of a glass ($n = 1.60$) disc which shuts one and a half Fresnel zones for the observation point P. What must the minimum thickness of that disc be for the intensity of light at the point P to be the highest? Take into account the interference of light on its passing through the disc.

5.106. A plane light wave with wavelength $\lambda = 0.54$ μm goes through a thin converging lens with focal length $f = 50$ cm and an aperture stop fixed immediately after the lens, and reaches a screen placed at a distance $b = 75$ cm from the aperture stop. At what aperture radii has the centre of the diffraction pattern on the screen the maximum illuminance?

5.107. A plane monochromatic light wave falls normally on a round aperture. At a distance $b = 9.0$ m from it there is a screen showing a certain diffraction pattern. The aperture diameter was decreased $\eta = 3.0$ times. Find the new distance b' at which the screen should be positioned to obtain the diffraction pattern similar to the previous one but diminished η times.

5.108. An opaque ball of diameter $D = 40$ mm is placed between a source of light with wavelength $\lambda = 0.55$ μm and a photographic plate. The distance between the source and the ball is equal to $a = 12$ m and that between the ball and the photographic plate is equal to $b = 18$ m. Find:

(a) the image dimension y' on the plate if the transverse dimension of the source is $y = 6.0$ mm;

(b) the minimum height of irregularities, covering the surface of the ball at random, at which the ball obstructs light.

Note. As calculations and experience show, that happens when the height of irregularities is comparable with the width of the Fresnel zone along which the edge of an opaque screen passes.

5.109. A point source of monochromatic light is positioned in front of a zone plate at a distance $a = 1.5$ m from it. The image of the source is formed at a distance $b = 1.0$ m from the plate. Find the focal length of the zone plate.

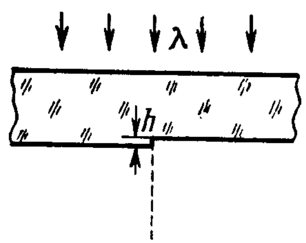


Fig. 5.22.

5.110. A plane light wave with wavelength $\lambda = 0.60$ μm and intensity I_0 falls normally on a large glass plate whose side view is shown in

Fig. 5.22. At what height h of the ledge will the intensity of light at points located directly below be

(a) minimum;

(b) twice as low as I_0 (the losses due to reflection are to be neglected).

5.111. A plane monochromatic light wave falls normally on an opaque half-plane. A screen is located at a distance $b = 100$ cm behind the half-plane. Making use of the Cornu spiral (Fig. 5.19), find:

(a) the ratio of intensities of the first maximum and the neighbouring minimum;

(b) the wavelength of light if the first two maxima are separated by a distance $\Delta x = 0.63$ mm.

5.112. A plane light wave with wavelength 0.60 μm falls normally on a long opaque strip 0.70 mm wide. Behind it a screen is placed at a distance 100 cm. Using Fig. 5.19, find the ratio of intensities of light in the middle of the diffraction pattern and at the edge of the geometrical shadow.

5.113. A plane monochromatic light wave falls normally on a long rectangular slit behind which a screen is positioned at a distance $b = 60$ cm. First the width of the slit was adjusted so that in the middle of the diffraction pattern the lowest minimum was observed. After widening the slit by $\Delta h = 0.70$ mm, the next minimum was obtained in the centre of the pattern. Find the wavelength of light.

5.114. A plane light wave with wavelength $\lambda = 0.65$ μm falls normally on a large glass plate whose opposite side has a long rectangular recess 0.60 mm wide. Using Fig. 5.19, find the depth h of the recess at which the diffraction pattern on the screen 77 cm away from the plate has the maximum illuminance at its centre.

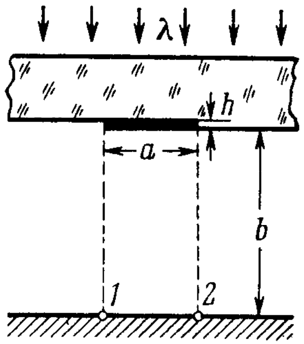


Fig. 5.23.

5.115. A plane light wave with wavelength $\lambda = 0.65$ μm falls normally on a large glass plate whose opposite side has a ledge and an opaque strip of width $a = 0.30$ mm (Fig. 5.23). A screen is placed at a distance $b = 110$ cm from the plate. The height h of the ledge is such that the intensity of light at point *2* of the screen is the highest possible. Making use of Fig. 5.19, find the ratio of intensities at points *1* and *2*.

5.116. A plane monochromatic light wave of intensity I_0 falls normally on an opaque screen with a long slit having a semicircular

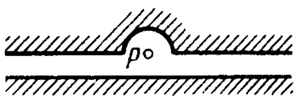


Fig. 5.24.

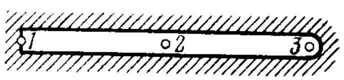


Fig. 5.25.

cut on one side (Fig. 5.24). The edge of the cut coincides with the boundary line of the first Fresnel zone for the observation point P. The width of the slit measures 0.90 of the radius of the cut. Using Fig. 5.19, find the intensity of light at the point P.

5.117. A plane monochromatic light wave falls normally on an opaque screen with a long slit whose shape is shown in Fig. 5.25. Making use of Fig. 5.19, find the ratio of intensities of light at points *1*, *2*, and *3* located behind the screen at equal distances from it. For point *3* the rounded-off edge of the slit coincides with the boundary line of the first Fresnel zone.

5.118. A plane monochromatic light wave falls normally on an opaque screen shaped as a long strip with a round hole in the middle. For the observation point P the hole corresponds to half the Fresnel zone, with the hole diameter being $\eta = 1.07$ times less than the width of the strip. Using Fig. 5.19, find the intensity of light at the point P provided that the intensity of the incident light is equal to I_0.

5.119. Light with wavelength λ falls normally on a long rectangular slit of width b. Find the angular distribution of the intensity of light in the case of Fraunhofer diffraction, as well as the angular position of minima.

5.120. Making use of the result obtained in the foregoing problem, find the conditions defining the angular position of maxima of the first, the second, and the third order.

5.121. Light with wavelength $\lambda = 0.50$ μm falls on a slit of width $b = 10$ μm at an angle $\theta_0 = 30°$ to its normal. Find the angular position of the first minima located on both sides of the central Fraunhofer maximum.

5.122. A plane light wave with wavelength $\lambda = 0.60$ μm falls normally on the face of a glass wedge with refracting angle $\Theta = 15°$. The opposite face of the wedge is opaque and has a slit of width $b = 10$ μm parallel to the edge. Find:

(a) the angle $\Delta\theta$ between the direction to the Fraunhofer maximum of zeroth order and that of incident light;

(b) the angular width of the Fraunhofer maximum of the zeroth order.

5.123. A monochromatic beam falls on a reflection grating with period $d = 1.0$ mm at a glancing angle $\alpha_0 = 1.0°$. When it is diffracted at a glancing angle $\alpha = 3.0°$ a Fraunhofer maximum of second order occurs. Find the wavelength of light.

5.124. Draw the approximate diffraction pattern originating in the case of the Fraunhofer diffraction from a grating consisting of three identical slits if the ratio of the grating period to the slit width is equal to

(a) two;

(b) three.

5.125. With light falling normally on a diffraction grating, the angle of diffraction of second order is equal to 45° for a wavelength

$\lambda_1 = 0.65$ μm. Find the angle of diffraction of third order for a wave length $\lambda_2 = 0.50$ μm.

5.126. Light with wavelength 535 nm falls normally on a diffraction grating. Find its period if the diffraction angle 35° corresponds to one of the Fraunhofer maxima and the highest order of spectrum is equal to five.

5.127. Find the wavelength of monochromatic light falling normally on a diffraction grating with period $d = 2.2$ μm if the angle between the directions to the Fraunhofer maxima of the first and the second order is equal to $\Delta\theta = 15°$.

5.128. Light with wavelength 530 nm falls on a transparent diffraction grating with period 1.50 μm. Find the angle, relative to the grating normal, at which the Fraunhofer maximum of highest order is observed provided the light falls on the grating

(a) at right angles;

(b) at the angle 60° to the normal.

5.129. Light with wavelength $\lambda = 0.60$ μm falls normally on a diffraction grating inscribed on a plane surface of a plano-convex cylindrical glass lens with curvature radius $R = 20$ cm. The period of the grating is equal to $d = 6.0$ μm. Find the distance between the principal maxima of first order located symmetrically in the focal plane of that lens.

5.130. A plane light wave with wavelength $\lambda = 0.50$ μm falls normally on the face of a glass wedge with an angle $\Theta = 30°$. On the opposite face of the wedge a transparent diffraction grating with period $d = 2.00$ μm is inscribed, whose lines are parallel to the wedge's edge. Find the angles that the direction of incident light forms with the directions to the principal Fraunhofer maxima of the zero and the first order. What is the highest order of the spectrum? At what angle to the direction of incident light is it observed?

5.131. A plane light wave with wavelength λ falls normally on a phase diffraction grating whose side view is shown in Fig. 5.26. The grating is cut on a glass plate with refractive index n. Find the depth h of the lines at which the intensity of the central Fraunhofer maximum is equal to zero. What is in this case the diffraction angle corresponding to the first maximum?

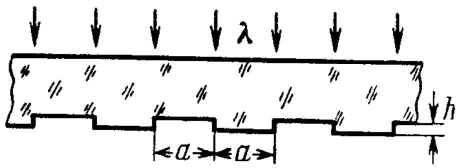


Fig. 5.26.

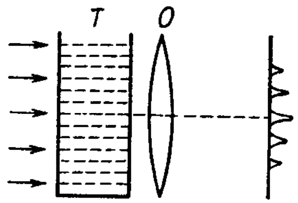


Fig. 5.27.

5.132. Figure 5.27 illustrates an arrangement employed in observations of diffraction of light by ultrasound. A plane light wave with wavelength $\lambda = 0.55$ μm passes through the water-filled tank T

in which a standing ultrasonic wave is sustained at a frequency $\nu = 4.7$ MHz. As a result of diffraction of light by the optically inhomogeneous periodic structure a diffraction spectrum can be observed in the focal plane of the objective O with focal length $f = 35$ cm. The separation between neighbouring maxima is $\Delta x = 0.60$ mm. Find the propagation velocity of ultrasonic oscillations in water.

5.133. To measure the angular distance ψ between the components of a double star by Michelson's method, in front of a telescope's lens a diaphragm was placed, which had two narrow parallel slits separated by an adjustable distance d. While diminishing d, the first smearing of the pattern was observed in the focal plane of the objective at $d = 95$ cm. Find ψ, assuming the wavelength of light to be equal to $\lambda = 0.55$ μm.

5.134. A transparent diffraction grating has a period $d = 1.50$ μm. Find the angular dispersion D (in angular minutes per nanometres) corresponding to the maximum of highest order for a spectral line of wavelength $\lambda = 530$ nm of light falling on the grating

(a) at right angles;

(b) at the angle $\theta_0 = 45°$ to the normal.

5.135. Light with wavelength λ falls on a diffraction grating at right angles. Find the angular dispersion of the grating as a function of diffraction angle θ.

5.136. Light with wavelength $\lambda = 589.0$ nm falls normally on a diffraction grating with period $d = 2.5$ μm, comprising $N = 10\,000$ lines. Find the angular width of the diffraction maximum of second order.

5.137. Demonstrate that when light falls on a diffraction grating at right angles, the maximum resolving power of the grating cannot exceed the value l/λ, where l is the width of the grating and λ is the wavelength of light.

5.138. Using a diffraction grating as an example, demonstrate that the frequency difference of two maxima resolved according to Rayleigh's criterion is equal to the reciprocal of the difference of propagation times of the extreme interfering oscillations, i.e. $\delta\nu = 1/\delta t$.

5.139. Light composed of two spectral lines with wavelengths 600.000 and 600.050 nm falls normally on a diffraction grating 10.0 mm wide. At a certain diffraction angle θ these lines are close to being resolved (according to Rayleigh's criterion). Find θ.

5.140. Light falls normally on a transparent diffraction grating of width $l = 6.5$ cm with 200 lines per millimetre. The spectrum under investigation includes a spectral line with $\lambda = 670.8$ nm consisting of two components differing by $\delta\lambda = 0.015$ nm. Find:

(a) in what order of the spectrum these components will be resolved;

(b) the least difference of wavelengths that can be resolved by this grating in a wavelength region $\lambda \approx 670$ nm.

5.141. With light falling normally on a transparent diffraction grating 10 mm wide, it was found that the components of the yellow line of sodium (589.0 and 589.6 nm) are resolved beginning with the fifth order of the spectrum. Evaluate:

(a) the period of this grating;

(b) what must be the width of the grating with the same period for a doublet $\lambda = 460.0$ nm whose components differ by 0.13 nm to be resolved in the third order of the spectrum.

5.142. A transparent diffraction grating of a quartz spectrograph is 25 mm wide and has 250 lines per millimetre. The focal length of an objective in whose focal plane a photographic plate is located is equal to 80 cm. Light falls on the grating at right angles. The spectrum under investigation includes a doublet with components of wavelengths 310.154 and 310.184 nm. Determine:

(a) the distances on the photographic plate between the components of this doublet in the spectra of the first and the second order;

(b) whether these components will be resolved in these orders of the spectrum.

5.143. The ultimate resolving power $\lambda/\delta\lambda$ of the spectrograph's trihedral prism is determined by diffraction of light at the prism edges (as in the case of a slit). When the prism is oriented to the least deviation angle in accordance with Rayleigh's criterion,

$$\lambda/\delta\lambda = b \mid dn/d\lambda \mid,$$

where b is the width of the prism's base (Fig. 5.28), and $dn/d\lambda$ is the dispersion of its material. Derive this formula.

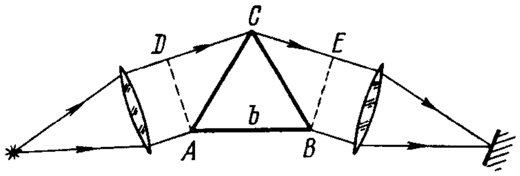


Fig. 5.28.

5.144. A spectrograph's trihedral prism is manufactured from glass whose refractive index varies with wavelength as $n = A + B/\lambda^2$, where A and B are constants, with B being equal to 0.010 μm^2. Making use of the formula from the foregoing problem, find:

(a) how the resolving power of the prism depends on λ; calculate the value of $\lambda/\delta\lambda$ in the vicinity of $\lambda_1 = 434$ nm and $\lambda_2 = 656$ nm if the width of the prism's base is $b = 5.0$ cm;

(b) the width of the prism's base capable of resolving the yellow doublet of sodium (589.0 and 589.6 nm).

5.145. How wide is the base of a trihedral prism which has the same resolving power as a diffraction grating with 10 000 lines in the second order of the spectrum if $|dn/d\lambda| = 0.10\ \mu m^{-1}$?

5.146. There is a telescope whose objective has a diameter $D = 5.0$ cm. Find the resolving power of the objective and the minimum separation between two points at a distance $l = 3.0$ km from the telescope, which it can resolve (assume $\lambda = 0.55$ μm).

5.147. Calculate the minimum separation between two points on the Moon which can be resolved by a reflecting telescope with mirror diameter 5 m. The wavelength of light is assumed to be equal to $\lambda = 0.55$ μm.

5.148. Determine the minimum multiplication of a telescope with diameter of objective $D = 5.0$ cm with which the resolving power of the objective is totally employed if the diameter of the eye's pupil is $d_0 = 4.0$ mm.

5.149. There is a microscope whose objective's numerical aperture is $\sin\alpha = 0.24$, where α is the half-angle subtended by the objective's rim. Find the minimum separation resolved by this microscope when an object is illuminated by light with wavelength $\lambda = 0.55$ μm.

5.150. Find the minimum magnification of a microscope, whose objective's numerical aperture is $\sin\alpha = 0.24$, at which the resolving power of the objective is totally employed if the diameter of the eye's pupil is $d_0 = 4.0$ mm.

5.151. A beam of X-rays with wavelength λ falls at a glancing angle 60.0° on a linear chain of scattering centres with period a. Find the angles of incidence corresponding to all diffraction maxima if $\lambda = 2a/5$.

5.152. A beam of X-rays with wavelength $\lambda = 40$ pm falls normally on a plane rectangular array of scattering centres and produces a system of diffraction maxima (Fig. 5.29) on a plane screen removed from the array by a distance $l = 10$ cm. Find the array periods a and b along the x and y axes if the distances between symmetrically located maxima of second order are equal to $\Delta x = 60$ mm (along the x axis) and $\Delta y = 40$ mm (along the y axis).

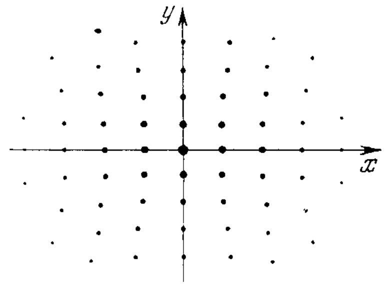


Fig. 5.29.

5.153. A beam of X-rays impinges on a three-dimensional rectangular array whose periods are a, b, and c. The direction of the incident beam coincides with the direction along which the array period is equal to a. Find the directions to the diffraction maxima and the wavelengths at which these maxima will be observed.

5.154. A narrow beam of X-rays impinges on the natural facet of a NaCl single crystal, whose density is $\rho = 2.16$ g/cm^3 at a glancing angle $\alpha = 60.0°$. The mirror reflection from this facet produces a maximum of second order. Find the wavelength of radiation.

5.155. A beam of X-rays with wavelength $\lambda = 174$ pm falls on the surface of a single crystal rotating about its axis which is paral-

lel to its surface and perpendicular to the direction of the incident beam. In this case the directions to the maxima of second and third order from the system of planes parallel to the surface of the single crystal form an angle $\alpha = 60°$ between them. Find the corresponding interplanar distance.

5.156. On transmitting a beam of X-rays with wavelength $\lambda = 17.8$ pm through a polycrystalline specimen a system of diffraction rings is produced on a screen located at a distance $l = 15$ cm from the specimen. Determine the radius of the bright ring corresponding to second order of reflection from the system of planes with interplanar distance $d = 155$ pm.

5.4. POLARIZATION OF LIGHT

- Degree of polarization of light:

$$P = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}. \tag{5.4a}$$

- Malus's law:

$$I = I_0 \cos^2 \varphi. \tag{5.4b}$$

- Brewster's law:

$$\tan \theta_B = n_2/n_1. \tag{5.4c}$$

- Fresnel equations for intensity of light reflected at the boundary between two dielectrics:

$$I'_\perp = I_\perp \frac{\sin^2(\theta_1 - \theta_2)}{\sin^2(\theta_1 + \theta_2)}, \quad I'_\parallel = I_\parallel \frac{\tan^2(\theta_1 - \theta_2)}{\tan^2(\theta_1 + \theta_2)}, \tag{5.4d}$$

where $I_\perp$ and $I_\parallel$ are the intensities of incident light whose electric vector oscillations are respectively perpendicular and parallel to the plane of incidence.

- A crystalline plate between two polarizers P and P'. If the angle between the plane of polarizer P and the optical axis OO' of the plate is equal to 45°, the intensity I' of light which passes through the polarizer P' turns out to be either maximum or minimum under the following conditions:

Polarizers P and P'	$\delta = 2\pi k$	$\delta = (2k+1)\pi$
parallel	$I'_\parallel = \max$	$I'_\parallel = \min$
crossed	$I'_\perp = \min$	$I'_\perp = \max$

(5.4e)

Here $\delta = 2\pi(n_0 - n_e)d/\lambda$ is the phase difference between the ordinary and extraordinary rays, $k = 0, 1, 2, \ldots$

- Natural and magnetic rotation of the plane of polarization:

$$\varphi_{nat} = \alpha l, \quad \varphi_{magn} = VlH, \tag{5.4f}$$

where α is the rotation constant, V is Verdet's constant.

5.157. A plane monochromatic wave of natural light with intensity I_0 falls normally on a screen composed of two touching Polaroid half-planes. The principal direction of one Polaroid is parallel, and of the other perpendicular, to the boundary between them. What kind of diffraction pattern is formed behind the screen? What is the intensity of light behind the screen at the points of the plane perpendicular to the screen and passing through the boundary between the Polaroids?

5.158. A plane monochromatic wave of natural light with intensity I_0 falls normally on an opaque screen with round hole corresponding to the first Fresnel zone for the observation point P. Find the intensity of light at the point P after the hole was covered with two identical Polaroids whose principal directions are mutually perpendicular and the boundary between them passes

(a) along the diameter of the hole;

(b) along the circumference of the circle limiting the first half of the Fresnel zone.

5.159. A beam of plane-polarized light falls on a polarizer which rotates about the axis of the ray with angular velocity $\omega = 21$ rad/s. Find the energy of light passing through the polarizer per one revolution if the flux of energy of the incident ray is equal to $\Phi_0 = 4.0$ mW.

5.160. A beam of natural light falls on a system of $N = 6$ Nicol prisms whose transmission planes are turned each through an angle $\varphi = 30°$ with respect to that of the foregoing prism. What fraction of luminous flux passes through this system?

5.161. Natural light falls on a system of three identical in-line Polaroids, the principal direction of the middle Polaroid forming an angle $\varphi = 60°$ with those of two other Polaroids. The maximum transmission coefficient of each Polaroid is equal to $\tau = 0.81$ when plane-polarized light falls on them. How many times will the intensity of the light decrease after its passing through the system?

5.162. The degree of polarization of partially polarized light is $P = 0.25$. Find the ratio of intensities of the polarized component of this light and the natural component.

5.163. A Nicol prism is placed in the way of partially polarized beam of light. When the prism is turned from the position of maximum transmission through an angle $\varphi = 60°$, the intensity of transmitted light decreased by a factor of $\eta = 3.0$. Find the degree of polarization of incident light.

5.164. Two identical imperfect polarizers are placed in the way of a natural beam of light. When the polarizers' planes are parallel, the system transmits $\eta = 10.0$ times more light than in the case of crossed planes. Find the degree of polarization of light produced

(a) by each polarizer separately;

(b) by the whole system when the planes of the polarizers are parallel.

5.165. Two parallel plane-polarized beams of light of equal intensity whose oscillation planes N_1 and N_2 form a small angle φ between

them (Fig. 5.30) fall on a Nicol prism. To equalize the intensities of the beams emerging behind the prism, its principal direction N must be aligned along the bisecting line A or B. Find the value of the angle φ at which the rotation of the Nicol prism through a small angle $\delta\varphi \ll \varphi$ from the position A results in the fractional change of intensities of the beams $\Delta I/I$ by the value $\eta = 100$ times exceeding that resulting due to rotation through the same angle from the position B.

5.166. Resorting to the Fresnel equations, demonstrate that light reflected from the surface of dielectric will be totally polarized if the angle of incidence θ_1 satisfies the condition $\tan\theta_1 = n$, where n is the refractive index of the dielectric. What is in this case the angle between the reflected and refracted rays?

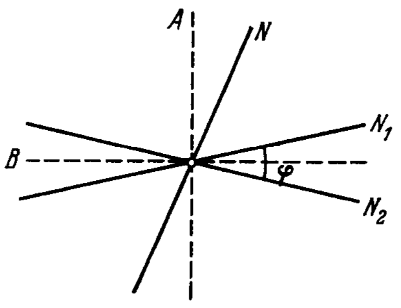


Fig. 5.30.

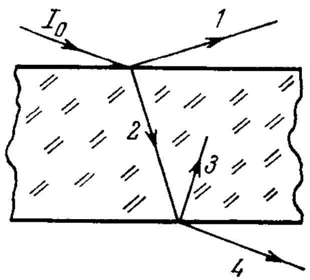


Fig. 5.31.

5.167. Natural light falls at the Brewster angle on the surface of glass. Using the Fresnel equations, find
(a) the reflection coefficient;
(b) the degree of polarization of refracted light.

5.168. A plane beam of natural light with intensity I_0 falls on the surface of water at the Brewster angle. A fraction $\rho = 0.039$ of luminous flux is reflected. Find the intensity of the refracted beam.

5.169. A beam of plane-polarized light falls on the surface of water at the Brewster angle. The polarization plane of the electric vector of the electromagnetic wave makes an angle $\varphi = 45°$ with the incidence plane. Find the reflection coefficient.

5.170. A narrow beam of natural light falls on the surface of a thick transparent plane-parallel plate at the Brewster angle. As a result, a fraction $\rho = 0.080$ of luminous flux is reflected from its top surface. Find the degree of polarization of beams *1-4* (Fig. 5.31)

5.171. A narrow beam of light of intensity I_0 falls on a plane-parallel glass plate (Fig. 5.31) at the Brewster angle. Using the Fresnel equations, find:
(a) the intensity of the transmitted beam I_4 if the oscillation plane of the incident plane-polarized light is perpendicular to the incidence plane;

(b) the degree of polarization of the transmitted light if the light falling on the plate is natural.

5.172. A narrow beam of natural light falls on a set of N thick plane-parallel glass plates at the Brewster angle. Find:
(a) the degree P of polarization of the transmitted beam;
(b) what P is equal to when $N = 1$, 2, 5, and 10.

5.173. Using the Fresnel equations, find:
(a) the reflection coefficient of natural light falling normally on the surface of glass;
(b) the relative loss of luminous flux due to reflections of a paraxial ray of natural light passing through an aligned optical system comprising five glass lenses (secondary reflections of light are to be neglected).

5.174. A light wave falls normally on the surface of glass coated with a layer of transparent substance. Neglecting secondary reflections, demonstrate that the amplitudes of light waves reflected from the two surfaces of such a layer will be equal under the condition $n' = \sqrt{n}$, where n' and n are the refractive indices of the layer and the glass respectively.

5.175. A beam of natural light falls on the surface of glass at an angle of 45°. Using the Fresnel equations, find the degree of polarization of
(a) reflected light;
(b) refracted light.

5.176. Using Huygens's principle, construct the wavefronts and the propagation directions of the ordinary and extraordinary rays in a positive uniaxial crystal whose optical axis
(a) is perpendicular to the incidence plane and parallel to the surface of the crystal;
(b) lies in the incidence plane and is parallel to the surface of the crystal;
(c) lies in the incidence plane at an angle of 45° to the surface of the crystal, and light falls at right angles to the optical axis.

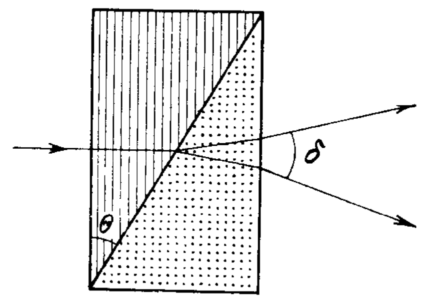


Fig. 5.32.

5.177. A narrow beam of natural light with wavelength $\lambda = 589$ nm falls normally on the surface of a Wollaston polarizing prism made of Iceland spar as shown in Fig. 5.32. The optical axes of the two parts of the prism are mutually perpendicular. Find the angle δ between the directions of the beams behind the prism if the angle θ is equal to 30°.

5.178. What kind of polarization has a plane electromagnetic wave if the projections of the vector $\mathbf{E}$ on the x and y axes are perpendicular to the propagation direction and are defined by the following equations:

(a) $E_x = E\cos(\omega t - kz)$, $E_y = E\sin(\omega t - kz)$;
(b) $E_x = E\cos(\omega t - kz)$, $E_y = E\cos(\omega t - kz + \pi/4)$;
(c) $E_x = E\cos(\omega t - kz)$, $E_y = E\cos(\omega t - kz + \pi)$?

5.179. One has to manufacture a quartz plate cut parallel to its optical axis and not exceeding 0.50 mm in thickness. Find the maximum thickness of the plate allowing plane-polarized light with wavelength $\lambda = 589$ nm

(a) to experience only rotation of polarization plane;
(b) to acquire circular polarization

after passing through that plate.

5.180. A quartz plate cut parallel to the optical axis is placed between two crossed Nicol prisms. The angle between the principal directions of the Nicol prisms and the plate is equal to 45°. The thickness of the plate is $d = 0.50$ mm. At what wavelengths in the interval from 0.50 to 0.60 μm is the intensity of light which passed through that system independent of rotation of the rear prism? The difference of refractive indices for ordinary and extraordinary rays in that wavelength interval is assumed to be $\Delta n = 0.0090$.

5.181. White natural light falls on a system of two crossed Nicol prisms having between them a quartz plate 1.50 mm thick, cut parallel to the optical axis. The axis of the plate forms an angle of 45° with the principal directions of the Nicol prisms. The light transmitted through that system was split into the spectrum. How many dark fringes will be observed in the wavelength interval from 0.55 to 0.66 μm? The difference of refractive indices for ordinary and extraordinary rays in that wavelength interval is assumed to be equal to 0.0090.

5.182. A crystalline plate cut parallel to its optical axis is 0.25 mm thick and serves as a quarter-wave plate for a wavelength $\lambda = 530$ nm. At what other wavelengths of visible spectrum will it also serve as a quarter-wave plate? The difference of refractive indices for extraordinary and ordinary rays is assumed to be constant and equal to $n_e - n_o = 0.0090$ at all wavelengths of the visible spectrum.

5.183. A quartz plate cut parallel to its optical axis is placed between two crossed Nicol prisms so that their principle directions form an angle of 45° with the optical axis of the plate. What is the minimum thickness of that plate transmitting light of wavelength $\lambda_1 = 643$ nm with maximum intensity while greatly reducing the intensity of transmitting light of wavelength $\lambda_2 = 564$ nm? The difference of refractive indices for extraordinary and ordinary rays is assumed to be equal to $n_e - n_o = 0.0090$ for both wavelengths.

5.184. A quartz wedge with refracting angle $\Theta = 3.5°$ is inserted between two crossed Polaroids. The optical axis of the wedge is parallel to its edge and forms an angle of 45° with the principal directions of the Polaroids. On transmission of light with wavelength $\lambda = 550$ nm through this system, an interference fringe pattern is formed. The width of each fringe is $\Delta x = 1.0$ mm. Find the difference of refractive indices of quartz for ordinary and extraordinary rays at the wavelength indicated above.

5.185. Natural monochromatic light of intensity I_0 falls on a system of two Polaroids between which a crystalline plate is inserted, cut parallel to its optical axis. The plate introduces a phase difference δ between the ordinary and extraordinary rays. Demonstrate that the intensity of light transmitted through that system is equal to

$$I = \frac{1}{2} I_0 [\cos^2(\varphi - \varphi') - \sin 2\varphi \cdot \sin 2\varphi' \; \sin^2(\delta/2)],$$

where φ and φ' are the angles between the optical axis of the crystal and the principal directions of the Polaroids. In particular, consider the cases of crossed and parallel Polaroids.

5.186. Monochromatic light with circular polarization falls normally on a crystalline plate cut parallel to the optical axis. Behind the plate there is a Nicol prism whose principal direction forms an angle φ with the optical axis of the plate. Demonstrate that the intensity of light transmitted through that system is equal to

$$I = I_0 (1 + \sin 2\varphi \cdot \sin \delta),$$

where δ is the phase difference between the ordinary and extraordinary rays which is introduced by the plate.

5.187. Explain how, using a Polaroid and a quarter-wave plate made of positive uniaxial crystal ($n_e > n_o$), to distinguish

(a) light with left-hand circular polarization from that with right-hand polarization;
(b) natural light from light with circular polarization and from the composition of natural light and that with circular polarization.

5.188. Light with wavelength λ falls on a system of crossed polarizer P and analyzer A between which a Babinet compensator C is inserted (Fig. 5.33). The compensator consists of two quartz wedges with the optical axis of one of them being parallel to the edge, and of the other, perpendicular to it. The principal directions of the polarizer and the analyser form an angle of 45° with the optical axes of the compensator. The refracting angle of the wedges is equal to Θ ($\Theta \ll 1$) and the difference of refractive indices of quartz is $n_e - n_o$. The insertion of investigated birefringent sample S, with the optical axis oriented as shown in the figure, results in displacement of the fringes upward by δx mm. Find:

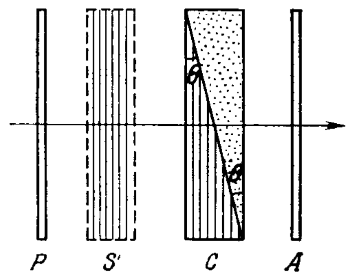


Fig. 5.33.

(a) the width of the fringe Δx;
(b) the magnitude and the sign of the optical path difference of ordinary and extraordinary rays, which appears due to the sample S.

5.189. Using the tables of the Appendix, calculate the difference of refractive indices of quartz for light of wavelength $\lambda = 589.5$ nm with right-hand and left-hand circular polarizations.

5.190. Plane-polarized light of wavelength 0.59 μm falls on a trihedral quartz prism P (Fig. 5.34) with refracting angle $\Theta = 30°$. Inside the prism light propagates along the optical axis whose direction is shown by hatching. Behind the Polaroid Pol an interference pattern of bright and dark fringes of width $\Delta x = 15.0$ mm is observed. Find the specific rotation constant of quartz and the distribution of intensity of light behind the Polaroid.

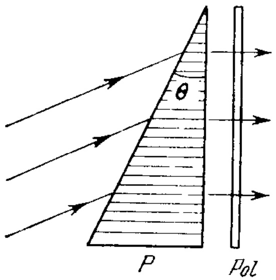


Fig. 5.34.

5.191. Natural monochromatic light falls on a system of two crossed Nicol prisms between which a quartz plate cut at right angles to its optical axis is inserted. Find the minimum thickness of the plate at which this system will transmit $\eta = 0.30$ of luminous flux if the specific rotation constant of quartz is equal to $\alpha = 17$ ang.deg/mm.

5.192. Light passes through a system of two crossed Nicol prisms between which a quartz plate cut at right angles to its optical axis is placed. Determine the minimum thickness of the plate which allows light of wavelength 436 nm to be completely cut off by the system and transmits half the light of wavelength 497 nm. The specific rotation constant of quartz for these wavelengths is equal to 41.5 and 31.1 angular degrees per mm respectively.

5.193. Plane-polarized light of wavelength 589 nm propagates along the axis of a cylindrical glass vessel filled with slightly turbid sugar solution of concentration 500 g/l. Viewing from the side, one can see a system of helical fringes, with 50 cm between neighbouring dark fringes along the axis. Explain the emergence of the fringes and determine the specific rotation constant of the solution.

5.194. A Kerr cell is positioned between two crossed Nicol prisms so that the direction of electric field **E** in the capacitor forms an angle of 45° with the principal directions of the prisms. The capacitor has the length $l = 10$ cm and is filled up with nitrobenzene. Light of wavelength $\lambda = 0.50$ μm passes through the system. Taking into account that in this case the Kerr constant is equal to $B = 2.2 \cdot 10^{-10}$ cm/V^2, find:

(a) the minimum strength of electric field E in the capacitor at which the intensity of light that passes through this system is independent of rotation of the rear prism;

(b) how many times per second light will be interrupted when a sinusoidal voltage of frequency $\nu = 10$ MHz and strength amplitude $E_m = 50$ kV/cm is applied to the capacitor.

Note. The Kerr constant is the coefficient B in the equation $n_e - n_o = B\lambda E^2$.

5.195. Monochromatic plane-polarized light with angular frequency ω passes through a certain substance along a uniform magnetic field H. Find the difference of refractive indices for right-hand and left-hand components of light beam with circular polarization if the Verdet constant is equal to V.

5.196. A certain substance is placed in a longitudinal magnetic field of a solenoid located between two Polaroids. The length of the tube with substance is equal to $l = 30$ cm. Find the Verdet constant if at a field strength $H = 56.5$ kA/m the angle of rotation of polarization plane is equal to $\varphi_1 = +5°10'$ for one direction of the field and to $\varphi_2 = -3°20'$, for the opposite direction.

5.197. A narrow beam of plane-polarized light passes through dextrorotatory positive compound placed into a longitudinal magnetic field as shown in Fig. 5.35. Find the angle through which the

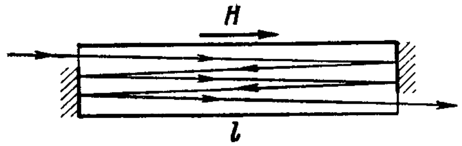


Fig. 5.35.

polarization plane of the transmitted beam will turn if the length of the tube with the compound is equal to l, the specific rotation constant of the compound is equal to α, the Verdet constant is V, and the magnetic field strength is H.

5.198. A tube of length $l = 26$ cm is filled with benzene and placed in a longitudinal magnetic field of a solenoid positioned between two Polaroids. The angle between the principle directions of the Polaroids is equal to 45°. Find the minimum strength of the magnetic field at which light of the wavelength 589 nm propagates through that system only in one direction (optical valve). What happens if the direction of the given magnetic field is changed to the opposite one?

5.199. Experience shows that a body irradiated with light with circular polarization acquires a torque. This happens because such a light possesses an angular momentum whose flow density in vacuum is equal to $M = I/\omega$, where I is the intensity of light, ω is the angular oscillation frequency. Suppose light with circular polarization and wavelength $\lambda = 0.70$ μm falls normally on a uniform black disc of mass $m = 10$ mg which can freely rotate about its axis. How soon will its angular velocity become equal to $\omega_0 = 1.0$ rad/s provided $I = 10$ W/cm^2?

5.5. DISPERSION AND ABSORPTION OF LIGHT

- Permittivity of substance according to elementary theory of dispersion:

$$\varepsilon = 1 + \sum_k \frac{n_k e^2/m\varepsilon_0}{\omega_{0k}^2 - \omega^2}, \tag{5.5a}$$

where n_k is the concentration of electrons of natural frequency ω_{0k}.

- Relation between refractive index and permittivity of substance:

$$n = \sqrt{\varepsilon}. \tag{5.5b}$$

- Phase velocity v and group velocity u:

$$v = \omega/k, \quad u = d\omega/dk. \tag{5.5c}$$

- Rayleigh's formula:

$$u = v - \lambda \frac{dv}{d\lambda}. \tag{5.5d}$$

- Attenuation of a narrow beam of electromagnetic radiation:

$$I = I_0 e^{-\mu d}, \tag{5.5e}$$

where $\mu = \varkappa + \varkappa'$, μ, $\varkappa$, $\varkappa'$ are the coefficients of linear attenuation, absorption, and scattering.

5.200. A free electron is located in the field of a monochromatic light wave. The intensity of light is $I = 150$ W/m², its frequency is $\omega = 3.4 \cdot 10^{15}$ s⁻¹. Find:

(a) the electron's oscillation amplitude and its velocity amplitude;

(b) the ratio F_m/F_e, where F_m and F_e are the amplitudes of forces with which the magnetic and electric components of the light wave field act on the electron; demonstrate that that ratio is equal to $\frac{1}{2} v/c$, where v is the electron's velocity amplitude and c is the velocity of light.

Instruction. The action of the magnetic field component can be disregarded in the equation of motion of the electron since the calculations show it to be negligible.

5.201. An electromagnetic wave of frequency ω propagates in dilute plasma. The free electron concentration in plasma is equal to n_0. Neglecting the interaction of the wave and plasma ions, find:

(a) the frequency dependence of plasma permittivity;

(b) how the phase velocity of the electromagnetic wave depends on its wavelength λ in plasma.

5.202. Find the free electron concentration in ionosphere if its refractive index is equal to $n = 0.90$ for radiowaves of frequency $\nu = 100$ MHz.

5.203. Assuming electrons of substance to be free when subjected to hard X-rays, determine by what magnitude the refractive index of graphite differs from unity in the case of X-rays whose wavelength in vacuum is equal to $\lambda = 50$ pm.

5.204. An electron experiences a quasi-elastic force kx and a "friction force" $\gamma\dot{x}$ in the field of electromagnetic radiation. The E-component of the field varies as $E = E_0 \cos \omega t$. Neglecting the action of the magnetic component of the field, find:

(a) the motion equation of the electron;

(b) the mean power absorbed by the electron; the frequency at which that power is maximum and the expression for the maximum mean power.

5.205. In some cases permittivity of substance turns out to be a complex or a negative quantity, and refractive index, respectively, a complex ($n' = n + i\varkappa$) or an imaginary ($n' = i\varkappa$) quantity. Write the equation of a plane wave for both of these cases and find out the physical meaning of such refractive indices.

5.206. A sounding of dilute plasma by radiowaves of various frequencies reveals that radiowaves with wavelengths exceeding $\lambda_0 = 0.75$ m experience total internal reflection. Find the free electron concentration in that plasma.

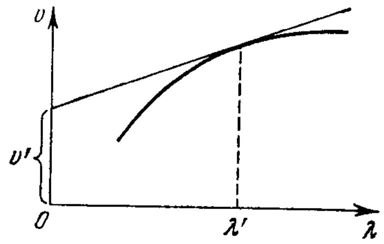


Fig. 5.36.

5.207. Using the definition of the group velocity u, derive Rayleigh's formula (5.5d). Demonstrate that in the vicinity of $\lambda = \lambda'$ the velocity u is equal to the segment v' cut by the tangent of the curve $v(\lambda)$ at the point λ' (Fig. 5.36).

5.208. Find the relation between the group velocity u and phase velocity v for the following dispersion laws:

(a) $v \propto 1/\sqrt{\lambda}$;

(b) $v \propto k$;

(c) $v \propto 1/\omega^2$.

Here λ, k, and ω are the wavelength, wave number, and angular frequency.

5.209. In a certain medium the relationship between the group and phase velocities of an electromagnetic wave has the form $uv = c^2$, where c is the velocity of light in vacuum. Find the dependence of permittivity of that medium on wave frequency, $\varepsilon(\omega)$.

5.210. The refractive index of carbon dioxide at the wavelengths 509, 534, and 589 nm is equal to 1.647, 1.640, and 1.630 respectively. Calculate the phase and group velocities of light in the vicinity of $\lambda = 534$ nm.

5.211. A train of plane light waves propagates in the medium where the phase velocity v is a linear function of wavelength: $v = a + b\lambda$, where a and b are some positive constants. Demonstrate that in such a medium the shape of an arbitrary train of light waves is restored after the time interval $\tau = 1/b$.

5.212. A beam of natural light of intensity I_0 falls on a system of two crossed Nicol prisms between which a tube filled with certain

solution is placed in a longitudinal magnetic field of strength H. The length of the tube is l, the coefficient of linear absorption of solution is $\varkappa$, and the Verdet constant is V. Find the intensity of light transmitted through that system.

5.213. A plane monochromatic light wave of intensity I_0 falls normally on a plane-parallel plate both of whose surfaces have a reflection coefficient ρ. Taking into account multiple reflections, find the intensity of the transmitted light if

(a) the plate is perfectly transparent, i.e. the absorption is absent;

(b) the coefficient of linear absorption is equal to $\varkappa$, and the plate thickness is d.

5.214. Two plates, one of thickness $d_1 = 3.8$ mm and the other of thickness $d_2 = 9.0$ mm, are manufactured from a certain substance. When placed alternately in the way of monochromatic light, the first transmits $\tau_1 = 0.84$ fraction of luminous flux and the second, $\tau_2 = 0.70$. Find the coefficient of linear absorption of that substance. Light falls at right angles to the plates. The secondary reflections are to be neglected.

5.215. A beam of monochromatic light passes through a pile of $N = 5$ identical plane-parallel glass plates each of thickness $l = 0.50$ cm. The coefficient of reflection at each surface of the plates is $\rho = 0.050$. The ratio of the intensity of light transmitted through the pile of plates to the intensity of incident light is $\tau = 0.55$. Neglecting the secondary reflections of light, find the absorption coefficient of the given glass.

5.216. A beam of monochromatic light falls normally on the surface of a plane-parallel plate of thickness l. The absorption coefficient of the substance the plate is made of varies linearly along the normal to its surface from $\varkappa_1$ to $\varkappa_2$. The coefficient of reflection at each surface of the plate is equal to ρ. Neglecting the secondary reflections, find the transmission coefficient of such a plate.

5.217. A beam of light of intensity I_0 falls normally on a transparent plane-parallel plate of thickness l. The beam contains all the wavelengths in the interval from λ_1 to λ_2 of equal spectral intensity. Find the intensity of the transmitted beam if in this wavelength interval the absorption coefficient is a linear function of λ, with extreme values $\varkappa_1$ and $\varkappa_2$. The coefficient of reflection at each surface is equal to ρ. The secondary reflections are to be neglected.

5.218. A light filter is a plate of thickness d whose absorption coefficient depends on wavelength λ as

$$\varkappa(\lambda) = \alpha(1 - \lambda/\lambda_0)^2 \text{ cm}^{-1},$$

where α and λ_0 are constants. Find the passband $\Delta\lambda$ of this light filter, that is the band at whose edges the attenuation of light is η times that at the wavelength λ_0. The coefficient of reflection from the surfaces of the light filter is assumed to be the same at all wavelengths.

5.219. A point source of monochromatic light emitting a luminous flux Φ is positioned at the centre of a spherical layer of substance. The inside radius of the layer is a, the outside one is b. The coefficient of linear absorption of the substance is equal to $\varkappa$, the reflection coefficient of the surfaces is equal to ρ. Neglecting the secondary reflections, find the intensity of light that passes through that layer.

5.220. How many times will the intensity of a narrow X-ray beam of wavelength 20 pm decrease after passing through a lead plate of thickness $d = 1.0$ mm if the mass absorption coefficient for the given radiation wavelength is equal to $\mu/\rho = 3.6$ cm²/g?

5.221. A narrow beam of X-ray radiation of wavelength 62 pm penetrates an aluminium screen 2.6 cm thick. How thick must a lead screen be to attenuate the beam just as much? The mass absorption coefficients of aluminium and lead for this radiation are equal to 3.48 and 72.0 cm²/g respectively.

5.222. Find the thickness of aluminium layer which reduces by half the intensity of a narrow monochromatic X-ray beam if the corresponding mass absorption coefficient is $\mu/\rho = 0.32$ cm²/g.

5.223. How many 50%-absorption layers are there in the plate reducing the intensity of a narrow X-ray beam $\eta = 50$ times?

5.6. OPTICS OF MOVING SOURCES

- Doppler effect for $\ll c$:

$$\frac{\Delta\omega}{\omega} = \frac{v}{c}\cos\theta \tag{5.6a}$$

where v is the velocity of a source, θ is the angle between the source's motion direction and the observation line.

- Doppler effect in the general case:

$$\omega = \omega_0 \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta}, \tag{5.6b}$$

where $\beta = v/c$.

- If $\theta = 0$, the Doppler effect is called radial, and if $\theta = \pi/2$, transverse.
- Vavilov-Cherenkov effect:

$$\cos\theta = \frac{c}{nv} \tag{5.6c}$$

where θ is the angle between the radiation propagation direction and the velocity vector v of a particle.

5.224. In the Fizeau experiment on measurement of the velocity of light the distance between the gear wheel and the mirror is $l = 7.0$ km, the number of teeth is $z = 720$. Two successive disappearances of light are observed at the following rotation velocities of the wheel: $n_1 = 283$ rps and $n_2 = 313$ rps. Find the velocity of light.

5.225. A source of light moves with velocity v relative to a receiver. Demonstrate that for $v \ll c$ the fractional variation of frequency of light is defined by Eq. (5.6a).

5.226. One of the spectral lines emitted by excited He^+ ions has a wavelength $\lambda = 410$ nm. Find the Doppler shift $\Delta\lambda$ of that line when observed at an angle $\theta = 30°$ to the beam of moving ions possessing kinetic energy $T = 10$ MeV.

5.227. When a spectral line of wavelength $\lambda = 0.59$ μm is observed in the directions to the opposite edges of the solar disc along its equator, there is a difference in wavelengths equal to $\delta\lambda = 8.0$ pm. Find the period of the Sun's revolution about its own axis.

5.228. The Doppler effect has made it possible to discover the double stars which are so distant that their resolution by means of a telescope is impossible. The spectral lines of such stars periodically become doublets indicating that the radiation does come from two stars revolving about their centre of mass. Assuming the masses of the two stars to be equal, find the distance between them and their masses if the maximum splitting of the spectral lines is equal to $(\Delta\lambda/\lambda)_m = 1.2 \cdot 10^{-4}$ and occurs every $\tau = 30$ days.

5.229. A plane electromagnetic wave of frequency ω_0 falls normally on the surface of a mirror approaching with a relativistic velocity V. Making use of the Doppler formula, find the frequency of the reflected wave. Simplify the obtained expression for the case $V \ll c$.

5.230. A radar operates at a wavelength $\lambda = 50.0$ cm. Find the velocity of an approaching aircraft if the beat frequency between the transmitted signal and the signal reflected from the aircraft is equal to $\Delta\nu = 1.00$ kHz at the radar location.

5.231. Taking into account that the wave phase $\omega t - kx$ is an invariant, i.e. it retains its value on transition from one inertial frame to another, determine how the frequency ω and the wave number k entering the expression for the wave phase are transformed. Examine the unidimensional case.

5.232. How fast does a certain nebula recede if the hydrogen line $\lambda = 434$ nm in its spectrum is displaced by 130 nm toward longer wavelengths?

5.233. How fast should a car move for the driver to perceive a red traffic light ($\lambda \approx 0.70$ μm) as a green one ($\lambda' \approx 0.55$ μm)?

5.234. An observer moves with velocity $v_1 = \frac{1}{2}c$ along a straight line. In front of him a source of monochromatic light moves with velocity $v_2 = \frac{3}{4}c$ in the same direction and along the same straight line. The proper frequency of light is equal to ω_0. Find the frequency of light registered by the observer.

5.235. One of the spectral lines of atomic hydrogen has the wavelength $\lambda = 656.3$ nm. Find the Doppler shift $\Delta\lambda$ of that line when observed at right angles to the beam of hydrogen atoms with kinetic energy $T = 1.0$ MeV (the transverse Doppler effect).

5.236. A source emitting electromagnetic signals with proper frequency $\omega_0 = 3.0 \cdot 10^{10}$ s^{-1} moves at a constant velocity $v = 0.80\,c$ along a straight line separated from a stationary observer P by a distance l (Fig. 5.37). Find the frequency of the signals perceived by the observer at the moment when

(a) the source is at the point O;

(b) the observer sees it at the point O.

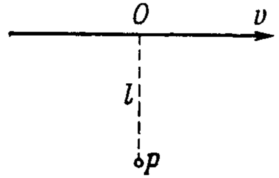


Fig. 5.37.

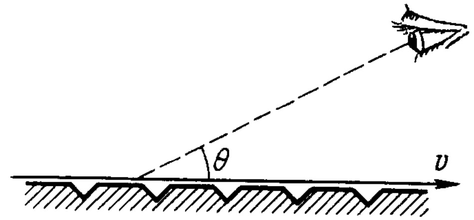


Fig. 5.38.

5.237. A narrow beam of electrons passes immediately over the surface of a metallic mirror with a diffraction grating with period $d = 2.0$ μm inscribed on it. The electrons move with velocity v, comparable to c, at right angles to the lines of the grating. The trajectory of the electrons can be seen in the form of a strip, whose colouring depends on the observation angle θ (Fig. 5.38). Interpret this phenomenon. Find the wavelength of the radiation observed at an angle $\theta = 45°$.

5.238. A gas consists of atoms of mass m being in thermodynamic equilibrium at temperature T. Suppose ω_0 is the natural frequency of light emitted by the atoms.

(a) Demonstrate that the spectral distribution of the emitted light is defined by the formula

$$I_\omega = I_0 e^{-a(1-\omega/\omega_0)^2},$$

(I_0 is the spectral intensity corresponding to the frequency ω_0, $a = mc^2/2kT$).

(b) Find the relative width $\Delta\omega/\omega_0$ of a given spectral line, i.e. the width of the line between the frequencies at which $I_\omega = I_0/2$.

5.239. A plane electromagnetic wave propagates in a medium moving with constant velocity $V \ll c$ relative to an inertial frame K. Find the velocity of that wave in the frame K if the refractive index of the medium is equal to n and the propagation direction of the wave coincides with that of the medium.

5.240. Aberration of light is the apparent displacement of stars attributable to the effect of the orbital motion of the Earth. The direction to a star in the ecliptic plane varies periodically, and the star performs apparent oscillations within an angle $\delta\theta = 41''$. Find the orbital velocity of the Earth.

5.241. Demonstrate that the angle θ between the propagation direction of light and the x axis transforms on transition from the reference frame K to K' according to the formula

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta},$$

where $\beta = V/c$ and V is the velocity of the frame K' with respect to the frame K. The x and x' axes of the reference frames coincide.

5.242. Find the aperture angle of a cone in which all the stars located in the semi-sphere for an observer on the Earth will be visible if one moves relative to the Earth with relativistic velocity V differing by 1.0% from the velocity of light. Make use of the formula of the foregoing problem.

5.243. Find the conditions under which a charged particle moving uniformly through a medium with refractive index n emits light (the Vavilov-Cherenkov effect). Find also the direction of that radiation.

Instruction. Consider the interference of oscillations induced by the particle at various moments of time.

5.244. Find the lowest values of the kinetic energy of an electron and a proton causing the emergence of Cherenkov's radiation in a medium with refractive index $n = 1.60$. For what particles is this minimum value of kinetic energy equal to $T_{min} = 29.6$ MeV?

5.245. Find the kinetic energy of electrons emitting light in a medium with refractive index $n = 1.50$ at an angle $\theta = 30°$ to their propagation direction.

5.7. THERMAL RADIATION. QUANTUM NATURE OF LIGHT

• Radiosity

$$M_e = \frac{c}{4}u, \tag{5.7a}$$

where u is the space density of thermal radiation energy.

• Wien's formula and Wien's displacement law:

$$u_\omega = \omega^3 F(\omega/T), \quad T\lambda_m = b, \tag{5.7b}$$

where λ_m is the wavelength corresponding to the maximum of the function u_λ.

• Stefan-Boltzmann law:

$$M_e = \sigma T^4, \tag{5.7c}$$

where σ is the Stefan-Boltzmann constant.

• Planck's formula:

$$u_\omega = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{e^{\hbar\omega/kT} - 1}. \tag{5.7d}$$

• Einstein's photoelectric equation:

$$\hbar\omega = A + \frac{mv_{max}^2}{2}. \tag{5.7e}$$

• Compton effect:

$$\Delta\lambda = 2\pi\lambda\!\!\!^{-}_C (1 - \cos\theta), \tag{5.7f}$$

where $\lambda\!\!\!^{-}_C = \hbar/mc$ is Compton's wavelength.

5.246. Using Wien's formula, demonstrate that

(a) the most probable radiation frequency $\omega_{pr} \propto T$;

(b) the maximum spectral density of thermal radiation $(u_\omega)_{max} \propto T^3$;

(c) the radiosity $M_e \propto T^4$.

5.247. The temperature of one of the two heated black bodies is $T_1 = 2500$ K. Find the temperature of the other body if the wavelength corresponding to its maximum emissive capacity exceeds by $\Delta\lambda = 0.50$ μm the wavelength corresponding to the maximum emissive capacity of the first black body.

5.248. The radiosity of a black body is $M_e = 3.0$ W/cm². Find the wavelength corresponding to the maximum emissive capacity of that body.

5.249. The spectral composition of solar radiation is much the same as that of a black body whose maximum emission corresponds to the wavelength 0.48 μm. Find the mass lost by the Sun every second due to radiation. Evaluate the time interval during which the mass of the Sun diminishes by 1 per cent.

5.250. Find the temperature of totally ionized hydrogen plasma of density $\rho = 0.10$ g/cm³ at which the thermal radiation pressure is equal to the gas kinetic pressure of the particles of plasma. Take into account that the thermal radiation pressure $p = u/3$, where u is the space density of radiation energy, and at high temperatures all substances obey the equation of state of an ideal gas.

5.251. A copper ball of diameter $d = 1.2$ cm was placed in an evacuated vessel whose walls are kept at the absolute zero temperature. The initial temperature of the ball is $T_0 = 300$ K. Assuming the surface of the ball to be absolutely black, find how soon its temperature decreases $\eta = 2.0$ times.

5.252. There are two cavities (Fig. 5.39) with small holes of equal diameters $d = 1.0$ cm and perfectly reflecting outer surfaces. The

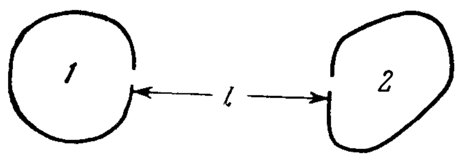


Fig. 5.39.

distance between the holes is $l = 10$ cm. A constant temperature $T_1 = 1700$ K is maintained in cavity *1*. Calculate the steady-state temperature inside cavity *2*.

Instruction. Take into account that a black body radiation obeys the cosine emission law.

5.253. A cavity of volume $V = 1.0$ l is filled with thermal radiation at a temperature $T = 1000$ K. Find:

(a) the heat capacity C_V; (b) the entropy S of that radiation.

5.254. Assuming the spectral distribution of thermal radiation energy to obey Wien's formula $u(\omega, T) = A\omega^3 \exp(-a\omega/T)$, where $a = 7.64$ ps·K, find for a temperature $T = 2000$ K the most probable

(a) radiation frequency; (b) radiation wavelength.

5.255. Using Planck's formula, derive the approximate expressions for the space spectral density u_ω of radiation

(a) in the range where $\hbar\omega \ll kT$ (Rayleigh-Jeans formula);

(b) in the range where $\hbar\omega \gg kT$ (Wien's formula).

5.256. Transform Planck's formula for space spectral density u_ω of radiation from the variable ω to the variables ν (linear frequency) and λ (wavelength).

5.257. Using Planck's formula, find the power radiated by a unit area of a black body within a narrow wavelength interval $\Delta\lambda = 1.0$ nm close to the maximum of spectral radiation density at a temperature $T = 3000$ K of the body.

5.258. Fig. 5.40 shows the plot of the function $y(x)$ representing a fraction of the total power of thermal radiation falling within

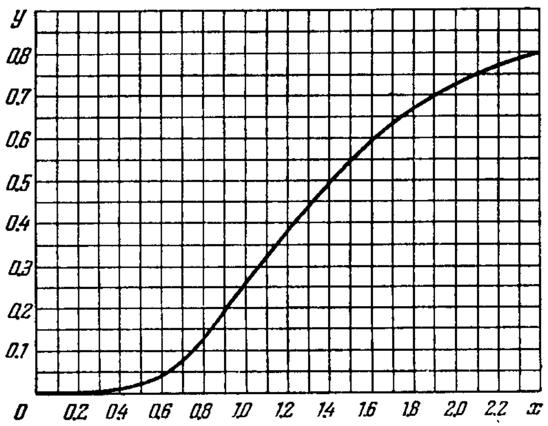


Fig. 5.40.

the spectral interval from 0 to x. Here $x = \lambda/\lambda_m$ (λ_m is the wavelength corresponding to the maximum of spectral radiation density).

Using this plot, find:

(a) the wavelength which divides the radiation spectrum into two equal (in terms of energy) parts at the temperature 3700 K;

(b) the fraction of the total radiation power falling within the visible range of the spectrum (0.40-0.76 μm) at the temperature 5000 K;

(c) how many times the power radiated at wavelengths exceeding 0.76 μm will increase if the temperature rises from 3000 to 5000 K.

5.259. Making use of Planck's formula, derive the expressions determining the number of photons per 1 cm³ of a cavity at a temperature T in the spectral intervals $(\omega, \omega + d\omega)$ and $(\lambda, \lambda + d\lambda)$.

5.260. An isotropic point source emits light with wavelength $\lambda = 589$ nm. The radiation power of the source is $P = 10$ W. Find:

(a) the mean density of the flow of photons at a distance $r = 2.0$ m from the source;

(b) the distance between the source and the point at which the mean concentration of photons is equal to $n = 100$ cm^{-3}.

5.261. From the standpoint of the corpuscular theory demonstrate that the momentum transferred by a beam of parallel light rays per unit time does not depend on its spectral composition but depends only on the energy flux Φ_e.

5.262. A laser emits a light pulse of duration $\tau = 0.13$ ms and energy $E = 10$ J. Find the mean pressure exerted by such a light pulse when it is focussed into a spot of diameter $d = 10$ μm on a surface perpendicular to the beam and possessing a reflection coefficient $\rho = 0.50$.

5.263. A short light pulse of energy $E = 7.5$ J falls in the form of a narrow and almost parallel beam on a mirror plate whose reflection coefficient is $\rho = 0.60$. The angle of incidence is 30°. In terms of the corpuscular theory find the momentum transferred to the plate.

5.264. A plane light wave of intensity $I = 0.20$ W/cm² falls on a plane mirror surface with reflection coefficient $\rho = 0.8$. The angle of incidence is 45°. In terms of the corpuscular theory find the magnitude of the normal pressure exerted by light on that surface.

5.265. A plane light wave of intensity $I = 0.70$ W/cm² illuminates a sphere with ideal mirror surface. The radius of the sphere is $R = 5.0$ cm. From the standpoint of the corpuscular theory find the force that light exerts on the sphere.

5.266. An isotropic point source of radiation power P is located on the axis of an ideal mirror plate. The distance between the source and the plate exceeds the radius of the plate η-fold. In terms of the corpuscular theory find the force that light exerts on the plate.

5.267. In a reference frame K a photon of frequency ω falls normally on a mirror approaching it with relativistic velocity V. Find the momentum imparted to the mirror during the reflection of the photon

(a) in the reference frame fixed to the mirror;

(b) in the frame K.

5.268. A small ideal mirror of mass $m = 10$ mg is suspended by a weightless thread of length $l = 10$ cm. Find the angle through which the thread will be deflected when a short laser pulse with energy $E = 13$ J is shot in the horizontal direction at right angles to the mirror. Where does the mirror get its kinetic energy?

5.269. A photon of frequency ω_0 is emitted from the surface of a star whose mass is M and radius R. Find the gravitational shift

of frequency $\Delta\omega/\omega_0$ of the photon at a very great distance from the star.

5.270. A voltage applied to an X-ray tube being increased $\eta = 1.5$ times, the short-wave limit of an X-ray continuous spectrum shifts by $\Delta\lambda = 26$ pm. Find the initial voltage applied to the tube.

5.271. A narrow X-ray beam falls on a NaCl single crystal. The least angle of incidence at which the mirror reflection from the system of crystallographic planes is still observed is equal to $\alpha = 4.1°$. The interplanar distance is $d = 0.28$ nm. How high is the voltage applied to the X-ray tube?

5.272. Find the wavelength of the short-wave limit of an X-ray continuous spectrum if electrons approach the anticathode of the tube with velocity $v = 0.85\ c$, where c is the velocity of light.

5.273. Find the photoelectric threshold for zinc and the maximum velocity of photoelectrons liberated from its surface by electromagnetic radiation with wavelength 250 nm.

5.274. Illuminating the surface of a certain metal alternately with light of wavelengths $\lambda_1 = 0.35$ μm and $\lambda_2 = 0.54$ μm, it was found that the corresponding maximum velocities of photoelectrons differ by a factor $\eta = 2.0$. Find the work function of that metal.

5.275. Up to what maximum potential will a copper ball, remote from all other bodies, be charged when irradiated by electromagnetic radiation of wavelength $\lambda = 140$ nm?

5.276. Find the maximum kinetic energy of photoelectrons liberated from the surface of lithium by electromagnetic radiation whose electric component varies with time as $E = a\,(1 + \cos\omega t)\cos\omega_0 t$, where a is a constant, $\omega = 6.0\cdot10^{14}\ \text{s}^{-1}$ and $\omega_0 = 3.60\cdot10^{15}\ \text{s}^{-1}$.

5.277. Electromagnetic radiation of wavelength $\lambda = 0.30$ μm falls on a photocell operating in the saturation mode. The corresponding spectral sensitivity of the photocell is $J = 4.8$ mA/W. Find the yield of photoelectrons, i.e. the number of photoelectrons produced by each incident photon.

5.278. There is a vacuum photocell whose one electrode is made of cesium and the other of copper. Find the maximum velocity of photoelectrons approaching the copper electrode when the cesium electrode is subjected to electromagnetic radiation of wavelength 0.22 μm and the electrodes are shorted outside the cell.

5.279. A photoelectric current emerging in the circuit of a vacuum photocell when its zinc electrode is subjected to electromagnetic radiation of wavelength 262 nm is cancelled if an external decelerating voltage 1.5 V is applied. Find the magnitude and polarity of the outer contact potential difference of the given photocell.

5.280. Compose the expression for a quantity whose dimension is length, using velocity of light c, mass of a particle m, and Planck's constant $\hbar$. What is that quantity?

5.281. Using the conservation laws, demonstrate that a free electron cannot absorb a photon completely.

5.282. Explain the following features of Compton scattering of light by matter:

(a) the increase in wavelength $\Delta\lambda$ is independent of the nature of the scattering substance;

(b) the intensity of the displaced component of scattered light grows with the increasing angle of scattering and with the diminishing atomic number of the substance;

(c) the presence of a non-displaced component in the scattered radiation.

5.283. A narrow monochromatic X-ray beam falls on a scattering substance. The wavelengths of radiation scattered at angles $\theta_1 = 60°$ and $\theta_2 = 120°$ differ by a factor $\eta = 2.0$. Assuming the free electrons to be responsible for the scattering, find the incident radiation wavelength.

5.284. A photon with energy $\hbar\omega = 1.00$ MeV is scattered by a stationary free electron. Find the kinetic energy of a Compton electron if the photon's wavelength changed by $\eta = 25\%$ due to scattering.

5.285. A photon of wavelength $\lambda = 6.0$ pm is scattered at right angles by a stationary free electron. Find:

(a) the frequency of the scattered photon;

(b) the kinetic energy of the Compton electron.

5.286. A photon with energy $\hbar\omega = 250$ keV is scattered at an angle $\theta = 120°$ by a stationary free electron. Find the energy of the scattered photon.

5.287. A photon with momentum $p = 1.02$ MeV/c, where c is the velocity of light, is scattered by a stationary free electron, changing in the process its momentum to the value $p' = 0.255$ MeV/c. At what angle is the photon scattered?

5.288. A photon is scattered at an angle $\theta = 120°$ by a stationary free electron. As a result, the electron acquires a kinetic energy $T = 0.45$ MeV. Find the energy that the photon had prior to scattering.

5.289. Find the wavelength of X-ray radiation if the maximum kinetic energy of Compton electrons is $T_{max} = 0.19$ MeV.

5.290. A photon with energy $\hbar\omega = 0.15$ MeV is scattered by a stationary free electron changing its wavelength by $\Delta\lambda = 3.0$ pm. Find the angle at which the Compton electron moves.

5.291. A photon with energy exceeding $\eta = 2.0$ times the rest energy of an electron experienced a head-on collision with a stationary free electron. Find the curvature radius of the trajectory of the Compton electron in a magnetic field $B = 0.12\ T$. The Compton electron is assumed to move at right angles to the direction of the field.

5.292. Having collided with a relativistic electron, a photon is deflected through an angle $\theta = 60°$ while the electron stops. Find the Compton displacement of the wavelength of the scattered photon.

PART SIX

ATOMIC AND NUCLEAR PHYSICS*

6.1. SCATTERING OF PARTICLES. RUTHERFORD-BOHR ATOM

• Angle θ at which a charged particle is deflected by the Coulomb field of a stationary atomic nucleus is defined by the formula:

$$\tan\frac{\theta}{2}=\frac{q_1q_2}{2bT}, \qquad (6.1a)$$

where q_1 and q_2 are the charges of the particle and the nucleus, b is the aiming parameter, T is the kinetic energy of a striking particle.

• Rutherford formula. The relative number of particles scattered into an elementary solid angle $d\Omega$ at an angle θ to their initial propagation direction:

$$\frac{dN}{N}=n\left(\frac{q_1q_2}{4T}\right)^2\frac{d\Omega}{\sin^4(\theta/2)}, \qquad (6.1b)$$

where n is the number of nuclei of the foil per unit area of its surface, $d\Omega=\sin\theta\, d\theta\, d\varphi$.

• Generalized Balmer formula (Fig. 6.1):

$$\omega=RZ^2\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right), \quad R=\frac{me^4}{2\hbar^3}, \qquad (6.1c)$$

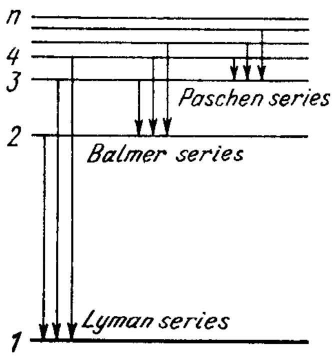


Fig. 6.1.

where ω is the transition frequency (in s^{-1}) between energy levels with quantum numbers n_1 and n_2, R is the Rydberg constant, Z is the serial number of a hydrogen-like ion.

6.1. Employing Thomson's model, calculate the radius of a hydrogen atom and the wavelength of emitted light if the ionization energy of the atom is known to be equal to $E = 13.6$ eV.

6.2. An alpha particle with kinetic energy 0.27 MeV is deflected through an angle of 60° by a golden foil. Find the corresponding value of the aiming parameter.

6.3. To what minimum distance will an alpha particle with kinetic energy $T = 0.40$ MeV approach in the case of a head-on collision to

(a) a stationary Pb nucleus;

(b) a stationary free Li^7 nucleus?

6.4. An alpha particle with kinetic energy $T = 0.50$ MeV is deflected through an angle of $\theta = 90°$ by the Coulomb field of a stationary Hg nucleus. Find:

(a) the least curvature radius of its trajectory;

(b) the minimum approach distance between the particle and the nucleus.

6.5. A proton with kinetic energy T and aiming parameter b was deflected by the Coulomb field of a stationary Au nucleus. Find the momentum imparted to the given nucleus as a result of scattering.

6.6. A proton with kinetic energy $T = 10$ MeV flies past a stationary free electron at a distance $b = 10$ pm. Find the energy acquired by the electron, assuming the proton's trajectory to be rectilinear and the electron to be practically motionless as the proton flies by.

6.7. A particle with kinetic energy T is deflected by a spherical potential well of radius R and depth U_0, i.e. by the field in which the potential energy of the particle takes the form

$$U=\begin{cases} 0 & \text{for } r>R, \\ -U_0 & \text{for } r<R, \end{cases}$$

where r is the distance from the centre of the well. Find the relationship between the aiming parameter b of the particle and the angle θ through which it deflects from the initial motion direction.

6.8. A stationary ball of radius R is irradiated by a parallel stream of particles whose radius is r. Assuming the collision of a particle and the ball to be elastic, find:

(a) the deflection angle θ of a particle as a function of its aiming parameter b;

(b) the fraction of particles which after a collision with the ball are scattered into the angular interval between θ and $\theta + d\theta$;

(c) the probability of a particle to be deflected, after a collision with the ball, into the front hemisphere $\left(\theta<\frac{\pi}{2}\right)$.

6.9. A narrow beam of alpha particles with kinetic energy 1.0 MeV falls normally on a platinum foil 1.0 μm thick. The scattered particles are observed at an angle of 60° to the incident beam direction by means of a counter with a circular inlet area 1.0 cm² located at the distance 10 cm from the scattering section of the foil. What fraction of scattered alpha particles reaches the counter inlet?

6.10. A narrow beam of alpha particles with kinetic energy $T = 0.50$ MeV and intensity $I = 5.0\cdot10^5$ particles per second falls normally on a golden foil. Find the thickness of the foil if at a distance $r = 15$ cm from a scattering section of that foil the flux density of scattered particles at the angle $\theta = 60°$ to the incident beam is equal to $J = 40$ particles/(cm²·s).

6.11. A narrow beam of alpha particles falls normally on a silver foil behind which a counter is set to register the scattered particles. On substitution of platinum foil of the same mass thickness for the silver foil, the number of alpha particles registered per unit time increased $\eta = 1.52$ times. Find the atomic number of platinum,

* All the formulas in this Part are given in the Gaussian system of units.

assuming the atomic number of silver and the atomic masses of both platinum and silver to be known.

6.12. A narrow beam of alpha particles with kinetic energy $T = 0.50$ MeV falls normally on a golden foil whose mass thickness is $\rho d = 1.5$ mg/cm^2. The beam intensity is $I_0 = 5.0 \cdot 10^5$ particles per second. Find the number of alpha particles scattered by the foil during a time interval $\tau = 30$ min into the angular interval:

(a) 59-61°; (b) over $\theta_0 = 60°$.

6.13. A narrow beam of protons with velocity $v = 6 \cdot 10^6$ m/s falls normally on a silver foil of thickness $d = 1.0$ μm. Find the probability of the protons to be scattered into the rear hemisphere ($\theta > 90°$).

6.14. A narrow beam of alpha particles with kinetic energy $T = 600$ keV falls normally on a golden foil incorporating $n = 1.1 \cdot 10^{19}$ nuclei/cm^2. Find the fraction of alpha particles scattered through the angles $\theta < \theta_0 = 20°$.

6.15. A narrow beam of protons with kinetic energy $T = 1.4$ MeV falls normally on a brass foil whose mass thickness $\rho d = 1.5$ mg/cm^2. The weight ratio of copper and zinc in the foil is equal to 7 : 3 respectively. Find the fraction of the protons scattered through the angles exceeding $\theta_0 = 30°$.

6.16. Find the effective cross section of a uranium nucleus corresponding to the scattering of alpha particles with kinetic energy $T = 1.5$ MeV through the angles exceeding $\theta_0 = 60°$.

6.17. The effective cross section of a gold nucleus corresponding to the scattering of monoenergetic alpha particles within the angular interval from 90° to 180° is equal to $\Delta\sigma = 0.50$ kb. Find:

(a) the energy of alpha particles;

(b) the differential cross section of scattering $d\sigma/d\Omega$ (kb/sr) corresponding to the angle $\theta = 60°$.

6.18. In accordance with classical electrodynamics an electron moving with acceleration **w** loses its energy due to radiation as

$$\frac{dE}{dt} = -\frac{2e^2}{3c^3} \mathbf{w}^2,$$

where e is the electron charge, c is the velocity of light. Estimate the time during which the energy of an electron performing almost harmonic oscillations with frequency $\omega = 5 \cdot 10^{15}$ s^{-1} will decrease $\eta = 10$ times.

6.19. Making use of the formula of the foregoing problem, estimate the time during which an electron moving in a hydrogen atom along a circular orbit of radius $r = 50$ pm would have fallen onto the nucleus. For the sake of simplicity assume the vector **w** to be permanently directed toward the centre of the atom.

6.20. Demonstrate that the frequency ω of a photon emerging when an electron jumps between neighbouring circular orbits of a hydrogen-like ion satisfies the inequality $\omega_n > \omega > \omega_{n+1}$, where ω_n and ω_{n+1} are the frequencies of revolution of that electron around the nucleus along the circular orbits. Make sure that as $n \to \infty$ the frequency of the photon $\omega \to \omega_n$.

6.21. A particle of mass m moves along a circular orbit in a centrosymmetrical potential field $U(r) = kr^2/2$. Using the Bohr quantization condition, find the permissible orbital radii and energy levels of that particle.

6.22. Calculate for a hydrogen atom and a He^+ ion:

(a) the radius of the first Bohr orbit and the velocity of an electron moving along it;

(b) the kinetic energy and the binding energy of an electron in the ground state;

(c) the ionization potential, the first excitation potential and the wavelength of the resonance line ($n' = 2 \to n = 1$).

6.23. Calculate the angular frequency of an electron occupying the second Bohr orbit of He^+ ion.

6.24. For hydrogen-like systems find the magnetic moment μ_n corresponding to the motion of an electron along the n-th orbit and the ratio of the magnetic and mechanical moments μ_n/M_n. Calculate the magnetic moment of an electron occupying the first Bohr orbit.

6.25. Calculate the magnetic field induction at the centre of a hydrogen atom caused by an electron moving along the first Bohr orbit.

6.26. Calculate and draw on the wavelength scale the spectral intervals in which the Lyman, Balmer, and Paschen series for atomic hydrogen are confined. Show the visible portion of the spectrum.

6.27. To what series does the spectral line of atomic hydrogen belong if its wave number is equal to the difference between the wave numbers of the following two lines of the Balmer series: 486.1 and 410.2 nm? What is the wavelength of that line?

6.28. For the case of atomic hydrogen find:

(a) the wavelengths of the first three lines of the Balmer series;

(b) the minimum resolving power $\lambda/\delta\lambda$ of a spectral instrument capable of resolving the first 20 lines of the Balmer series.

6.29. Radiation of atomic hydrogen falls normally on a diffraction grating of width $l = 6.6$ mm. The 50th line of the Balmer series in the observed spectrum is close to resolution at a diffraction angle θ (in accordance with Rayleigh's criterion). Find that angle.

6.30. What element has a hydrogen-like spectrum whose lines have wavelengths four times shorter than those of atomic hydrogen?

6.31. How many spectral lines are emitted by atomic hydrogen excited to the n-th energy level?

6.32. What lines of atomic hydrogen absorption spectrum fall within the wavelength range from 94.5 to 130.0 nm?

6.33. Find the quantum number n corresponding to the excited state of He^+ ion if on transition to the ground state that ion emits two photons in succession with wavelengths 108.5 and 30.4 nm.

6.34. Calculate the Rydberg constant R if He^+ ions are known to have the wavelength difference between the first (of the longest wavelength) lines of the Balmer and Lyman series equal to $\Delta\lambda = 133.7$ nm.

6.35. What hydrogen-like ion has the wavelength difference between the first lines of the Balmer and Lyman series equal to 59.3 nm?

6.36. Find the wavelength of the first line of the He^+ ion spectral series whose interval between the extreme lines is $\Delta\omega = 5.18 \cdot 10^{15}$ s^{-1}.

6.37. Find the binding energy of an electron in the ground state of hydrogen-like ions in whose spectrum the third line of the Balmer series is equal to 108.5 nm.

6.38. The binding energy of an electron in the ground state of He atom is equal to $E_0 = 24.6$ eV. Find the energy required to remove both electrons from the atom.

6.39. Find the velocity of photoelectrons liberated by electromagnetic radiation of wavelength $\lambda = 18.0$ nm from stationary He^+ ions in the ground state.

6.40. At what minimum kinetic energy must a hydrogen atom move for its inelastic head-on collision with another, stationary, hydrogen atom to make one of them capable of emitting a photon? Both atoms are supposed to be in the ground state prior to the collision.

6.41. A stationary hydrogen atom emits a photon corresponding to the first line of the Lyman series. What velocity does the atom acquire?

6.42. From the conditions of the foregoing problem find how much (in per cent) the energy of the emitted photon differs from the energy of the corresponding transition in a hydrogen atom.

6.43. A stationary He^+ ion emitted a photon corresponding to the first line of the Lyman series. That photon liberated a photoelectron from a stationary hydrogen atom in the ground state. Find the velocity of the photoelectron.

6.44. Find the velocity of the excited hydrogen atoms if the first line of the Lyman series is displaced by $\Delta\lambda = 0.20$ nm when their radiation is observed at an angle $\theta = 45°$ to their motion direction.

6.45. According to the Bohr-Sommerfeld postulate the periodic motion of a particle in a potential field must satisfy the following quantization rule:

$$\oint p\, dq = 2\pi\hbar n,$$

where q and p are generalized coordinate and momentum of the particle, n are integers. Making use of this rule, find the permitted values of energy for a particle of mass m moving

(a) in a unidimensional rectangular potential well of width l with infinitely high walls;

(b) along a circle of radius r;

(c) in a unidimensional potential field $U = \alpha x^2/2$, where α is a positive constant;

(d) along a round orbit in a central field, where the potential energy of the particle is equal to $U = -\alpha/r$ (α is a positive constant).

6.46. Taking into account the motion of the nucleus of a hydrogen atom, find the expressions for the electron's binding energy in the ground state and for the Rydberg constant. How much (in per cent) do the binding energy and the Rydberg constant, obtained without taking into account the motion of the nucleus, differ from the more accurate corresponding values of these quantities?

6.47. For atoms of light and heavy hydrogen (H and D) find the difference

(a) between the binding energies of their electrons in the ground state;

(b) between the wavelengths of first lines of the Lyman series.

6.48. Calculate the separation between the particles of a system in the ground state, the corresponding binding energy, and the wavelength of the first line of the Lyman series, if such a system is

(a) a mesonic hydrogen atom whose nucleus is a proton (in a mesonic atom an electron is replaced by a meson whose charge is the same and mass is 207 that of an electron);

(b) a positronium consisting of an electron and a positron revolving around their common centre of masses.

6.2. WAVE PROPERTIES OF PARTICLES. SCHRÖDINGER EQUATION

- The de Broglie wavelength of a particle with momentum p:

$$\lambda = \frac{2\pi\hbar}{p}. \tag{6.2a}$$

- Uncertainty principle:

$$\Delta x \cdot \Delta p_x \geqslant \hbar. \tag{6.2b}$$

- Schrödinger time-dependent and time-independent equations:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + U\Psi,$$
$$\nabla^2 \psi + \frac{2m}{\hbar^2}(E - U)\psi = 0, \tag{6.2c}$$

where Ψ is the total wave function, ψ is its coordinate part, ∇^2 is the Laplace operator, E and U are the total and potential energies of the particle. In spherical coordinates:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{6.2d}$$

• Coefficient of transparency of a potential barrier $V(x)$:

$$D \approx \exp\left[-\frac{2}{\hbar}\int_{x_1}^{x_2}\sqrt{2m(V-E)}\,dx\right], \qquad (6.2e)$$

where x_1 and x_2 are the coordinates of the points between which $V > E$.

6.49. Calculate the de Broglie wavelengths of an electron, proton, and uranium atom, all having the same kinetic energy 100 eV.

6.50. What amount of energy should be added to an electron to reduce its de Broglie wavelength from 100 to 50 pm?

6.51. A neutron with kinetic energy $T = 25$ eV strikes a stationary deuteron (heavy hydrogen nucleus). Find the de Broglie wavelengths of both particles in the frame of their centre of inertia.

6.52. Two identical non-relativistic particles move at right angles to each other, possessing de Broglie wavelengths λ_1 and λ_2. Find the de Broglie wavelength of each particle in the frame of their centre of inertia.

6.53. Find the de Broglie wavelength of hydrogen molecules, which corresponds to their most probable velocity at room temperature.

6.54. Calculate the most probable de Broglie wavelength of hydrogen molecules being in thermodynamic equilibrium at room temperature.

6.55. Derive the expression for a de Broglie wavelength λ of a relativistic particle moving with kinetic energy T. At what values of T does the error in determining λ using the non-relativistic formula not exceed 1% for an electron and a proton?

6.56. At what value of kinetic energy is the de Broglie wavelength of an electron equal to its Compton wavelength?

6.57. Find the de Broglie wavelength of relativistic electrons reaching the anticathode of an X-ray tube if the short wavelength limit of the continuous X-ray spectrum is equal to $\lambda_{sh} = 10.0$ pm?

6.58. A parallel stream of monoenergetic electrons falls normally on a diaphragm with narrow square slit of width $b = 1.0$ μm. Find the velocity of the electrons if the width of the central diffraction maximum formed on a screen located at a distance $l = 50$ cm from the slit is equal to $\Delta x = 0.36$ mm.

6.59. A parallel stream of electrons accelerated by a potential difference $V = 25$ V falls normally on a diaphragm with two narrow slits separated by a distance $d = 50$ μm. Calculate the distance between neighbouring maxima of the diffraction pattern on a screen located at a distance $l = 100$ cm from the slits.

6.60. A narrow stream of monoenergetic electrons falls at an angle of incidence $\theta = 30°$ on the natural facet of an aluminium single crystal. The distance between the neighbouring crystal planes parallel to that facet is equal to $d = 0.20$ nm. The maximum mirror reflection is observed at a certain accelerating voltage V_0. Find V_0 if the next maximum mirror reflection is known to be observed when the accelerating voltage is increased $\eta = 2.25$ times.

6.61. A narrow beam of monoenergetic electrons falls normally on the surface of a Ni single crystal. The reflection maximum of fourth order is observed in the direction forming an angle $\theta = 55°$ with the normal to the surface at the energy of the electrons equal to $T = 180$ eV. Calculate the corresponding value of the interplanar distance.

6.62. A narrow stream of electrons with kinetic energy $T = 10$ keV passes through a polycrystalline aluminium foil, forming a system of diffraction fringes on a screen. Calculate the interplanar distance corresponding to the reflection of third order from a certain system of crystal planes if it is responsible for a diffraction ring of diameter $D = 3.20$ cm. The distance between the foil and the screen is $l = 10.0$ cm.

6.63. A stream of electrons accelerated by a potential difference V falls on the surface of a metal whose inner potential is $V_i = 15$ V. Find:

(a) the refractive index of the metal for the electrons accelerated by a potential difference $V = 150$ V;

(b) the values of the ratio V/V_i at which the refractive index differs from unity by not more than $\eta = 1.0\%$.

6.64. A particle of mass m is located in a unidimensional square potential well with infinitely high walls. The width of the well is equal to l. Find the permitted values of energy of the particle taking into account that only those states of the particle's motion are realized for which the whole number of de Broglie half-waves are fitted within the given well.

6.65. Describe the Bohr quantum conditions in terms of the wave theory: demonstrate that an electron in a hydrogen atom can move only along those round orbits which accommodate a whole number of de Broglie waves.

6.66. Estimate the minimum errors in determining the velocity of an electron, a proton, and a ball of mass of 1 mg if the coordinates of the particles and of the centre of the ball are known with uncertainly 1 μm.

6.67. Employing the uncertainty principle, evaluate the indeterminancy of the velocity of an electron in a hydrogen atom if the size of the atom is assumed to be $l = 0.10$ nm. Compare the obtained magnitude with the velocity of an electron in the first Bohr orbit of the given atom.

6.68. Show that for the particle whose coordinate uncertainty is $\Delta x = \lambda/2\pi$, where λ is its de Broglie wavelength, the velocity uncertainty is of the same order of magnitude as the particle's velocity itself.

6.69. A free electron was initially confined within a region with linear dimensions $l = 0.10$ nm. Using the uncertainty principle, evaluate the time over which the width of the corresponding train of waves becomes $\eta = 10$ times as large.

6.70. Employing the uncertainty principle, estimate the minimum kinetic energy of an electron confined within a region whose size is $l = 0.20$ nm.

6.71. An electron with kinetic energy $T \approx 4$ eV is confined within a region whose linear dimension is $l = 1$ μm. Using the uncertainty principle, evaluate the relative uncertainty of its velocity.

6.72. An electron is located in a unidimensional square potential well with infinitely high walls. The width of the well is l. From the uncertainty principle estimate the force with which the electron possessing the minimum permitted energy acts on the walls of the well.

6.73. A particle of mass m moves in a unidimensional potential field $U = kx^2/2$ (harmonic oscillator). Using the uncertainty principle, evaluate the minimum permitted energy of the particle in that field.

6.74. Making use of the uncertainty principle, evaluate the minimum permitted energy of an electron in a hydrogen atom and its corresponding apparent distance from the nucleus.

6.75. A parallel stream of hydrogen atoms with velocity $v = 600$ m/s falls normally on a diaphragm with a narrow slit behind which a screen is placed at a distance $l = 1.0$ m. Using the uncertainty principle, evaluate the width of the slit δ at which the width of its image on the screen is minimum.

6.76. Find a particular solution of the time-dependent Schrödinger equation for a freely moving particle of mass m.

6.77. A particle in the ground state is located in a unidimensional square potential well of length l with absolutely impenetrable walls ($0 < x < l$). Find the probability of the particle staying within a region $\frac{1}{3}l \leqslant x \leqslant \frac{2}{3}l$.

6.78. A particle is located in a unidimensional square potential well with infinitely high walls. The width of the well is l. Find the normalized wave functions of the stationary states of the particle, taking the midpoint of the well for the origin of the x coordinate.

6.79. Demonstrate that the wave functions of the stationary states of a particle confined in a unidimensional potential well with infinitely high walls are orthogonal, i.e. they satisfy the condition $\int\limits_0^l \psi_n \psi_{n'}\, dx = 0$ if $n' \neq n$. Here l is the width of the well, n are integers.

6.80. An electron is located in a unidimensional square potential well with infinitely high walls. The width of the well equal to l is such that the energy levels are very dense. Find the density of energy levels dN/dE, i.e. their number per unit energy interval, as a function of E. Calculate dN/dE for $E = 1.0$ eV if $l = 1.0$ cm.

6.81. A particle of mass m is located in a two-dimensional square potential well with absolutely impenetrable walls. Find:

(a) the particle's permitted energy values if the sides of the well are l_1 and l_2;

(b) the energy values of the particle at the first four levels if the well has the shape of a square with side l.

6.82. A particle is located in a two-dimensional square potential well with absolutely impenetrable walls ($0 < x < a$, $0 < y < b$). Find the probability of the particle with the lowest energy to be located within a region $0 < x < a/3$.

6.83. A particle of mass m is located in a three-dimensional cubic potential well with absolutely impenetrable walls. The side of the cube is equal to a. Find:

(a) the proper values of energy of the particle;

(b) the energy difference between the third and fourth levels;

(c) the energy of the sixth level and the number of states (the degree of degeneracy) corresponding to that level.

6.84. Using the Schrödinger equation, demonstrate that at the point where the potential energy $U(x)$ of a particle has a finite discontinuity, the wave function remains smooth, i.e. its first derivative with respect to the coordinate is continuous.

6.85. A particle of mass m is located in a unidimensional potential field $U(x)$ whose shape is shown in Fig. 6.2, where $U(0) = \infty$. Find:

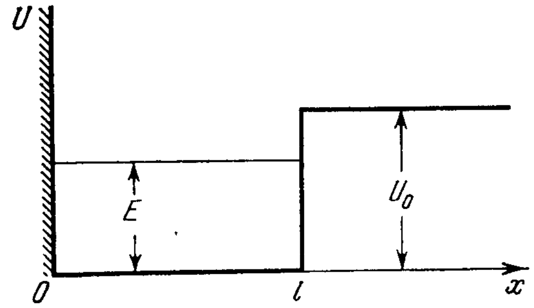


Fig. 6.2.

(a) the equation defining the possible values of energy of the particle in the region $E < U_0$; reduce that equation to the form

$$\sin kl = \pm kl \sqrt{\hbar^2/2ml^2U_0},$$

where $k = \sqrt{2mE}/\hbar$. Solving this equation by graphical means, demonstrate that the possible values of energy of the particle form a discontinuous spectrum;

(b) the minimum value of the quantity l^2U_0 at which the first energy level appears in the region $E < U_0$. At what minimum value of l^2U_0 does the nth level appear?

6.86. Making use of the solution of the foregoing problem, determine the probability of the particle with energy $E = U_0/2$ to be located in the region $x > l$, if $l^2U_0 = \left(\frac{3}{4}\pi\right)^2 \frac{\hbar^2}{m}$.

6.87. Find the possible values of energy of a particle of mass m located in a spherically symmetrical potential well $U(r) = 0$ for $r < r_0$ and $U(r) = \infty$ for $r = r_0$, in the case when the motion of the particle is described by a wave function $\psi(r)$ depending only on r.

Instruction. When solving the Schrödinger equation, make the substitution $\psi(r) = \chi(r)/r$.

6.88. From the conditions of the foregoing problem find:

(a) normalized eigenfunctions of the particle in the states for which $\psi(r)$ depends only on r;

(b) the most probable value r_{pr} for the ground state of the particle and the probability of the particle to be in the region $r < r_{pr}$.

6.89. A particle of mass m is located in a spherically symmetrical potential well $U(r) = 0$ for $r < r_0$ and $U(r) = U_0$ for $r > r_0$.

(a) By means of the substitution $\psi(r) = \chi(r)/r$ find the equation defining the proper values of energy E of the particle for $E < U_0$, when its motion is described by a wave function $\psi(r)$ depending only on r. Reduce that equation to the form

$$\sin kr_0 = \pm kr_0 \sqrt{\hbar^2/2mr_0^2U_0}, \quad \text{where } k = \sqrt{2mE}/\hbar.$$

(b) Calculate the value of the quantity $r_0^2U_0$ at which the first level appears.

6.90. The wave function of a particle of mass m in a unidimensional potential field $U(x) = kx^2/2$ has in the ground state the form $\psi(x) = Ae^{-\alpha x^2}$, where A is a normalization factor and α is a positive constant. Making use of the Schrödinger equation, find the constant α and the energy E of the particle in this state.

6.91. Find the energy of an electron of a hydrogen atom in a stationary state for which the wave function takes the form $\psi(r) = A(1 + ar)e^{-\alpha r}$, where A, a, and α are constants.

6.92. The wave function of an electron of a hydrogen atom in the ground state takes the form $\psi(r) = Ae^{-r/r_1}$, where A is a certain constant, r_1 is the first Bohr radius. Find:

(a) the most probable distance between the electron and the nucleus;

(b) the mean value of modulus of the Coulomb force acting on the electron;

(c) the mean value of the potential energy of the electron in the field of the nucleus.

6.93. Find the mean electrostatic potential produced by an electron in the centre of a hydrogen atom if the electron is in the ground state for which the wave function is $\psi(r) = Ae^{-r/r_1}$, where A is a certain constant, r_1 is the first Bohr radius.

6.94. Particles of mass m and energy E move from the left to the potential barrier shown in Fig. 6.3. Find:

(a) the reflection coefficient R of the barrier for $E > U_0$;

(b) the effective penetration depth of the particles into the region $x > 0$ for $E < U_0$, i.e. the distance from the barrier boundary to the point at which the probability of finding a particle decreases e-fold.

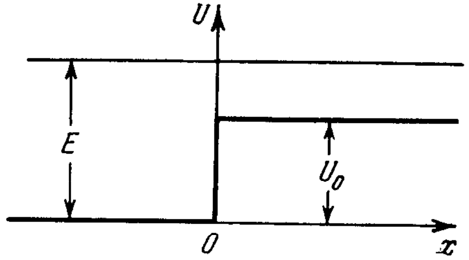


Fig. 6.3.

6.95. Employing Eq. (6.2e), find the probability D of an electron with energy E tunnelling through a potential barrier of width l and height U_0 provided the barrier is shaped as shown:

(a) in Fig. 6.4;

(b) in Fig. 6.5.

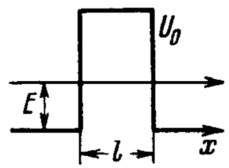


Fig. 6.4.

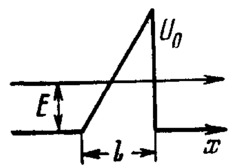


Fig. 6.5.

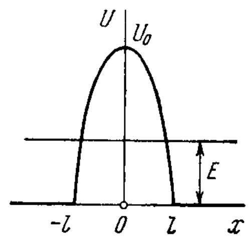


Fig. 6.6.

6.96. Using Eq. (6.2e), find the probability D of a particle of mass m and energy E tunnelling through the potential barrier shown in Fig. 6.6, where $U(x) = U_0(1 - x^2/l^2)$.

6.3. PROPERTIES OF ATOMS. SPECTRA

• Spectral labelling of terms: ${}^{\varkappa}(L)_J$, where $\varkappa = 2S + 1$ is the multiplicity, L, S, J are quantum numbers,

$$L = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots$$
$$(L):\ S,\ P,\ D,\ F,\ G,\ H,\ I,\ \ldots$$

• Terms of alkali metal atoms:

$$T=\frac{R}{(n+\alpha)^2}, \tag{6.3a}$$

where R is the Rydberg constant, α is the Rydberg correction.
Fig. 6.7 illustrates the diagram of a lithium atom terms.

• Angular momenta of an atom:

$$M_L = \hbar\sqrt{L(L+1)}, \tag{6.3b}$$

with similar expressions for M_S and M_J.

• Hund rules:

(1) For a certain electronic configuration, the terms of the largest S value are the lowest in energy, and among the terms of S_{max} that of the largest L usually lies lowest;

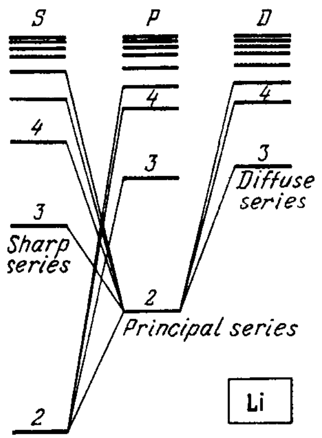


Fig. 6.7.

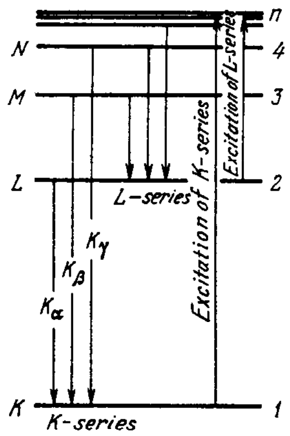


Fig. 6.8.

(2) for the basic (normal) term $J = |L - S|$ if the subshell is less than half-filled, and $J = L + S$ in the remaining cases.

• Boltzmann's formula:

$$\frac{N_2}{N_1}=\frac{g_2}{g_1}e^{-(E_2-E_1)/kT}, \tag{6.3c}$$

where g_1 and g_2 are the statistical weights (degeneracies) of the corresponding levels.

• Probabilities of atomic transitions per unit time between level *1* and a higher level *2* for the cases of spontaneous radiation, induced radiation, and absorption:

$$P_{21}^{sp}=A_{21},\quad P_{21}^{ind}=B_{21}u_\omega,\quad P_{12}^{abs}=B_{12}u_\omega, \tag{6.3d}$$

where A_{21}, B_{21}, B_{12} are Einstein coefficients, u_ω is the spectral density of radiation corresponding to frequency ω of transition between the given levels.

• Relation between Einstein coefficients:

$$g_1B_{12}=g_2B_{21},\quad B_{21}=\frac{\pi^2c^3}{\hbar\omega^3}A_{21}. \tag{6.3e}$$

• Diagram showing formation of X-ray spectra (Fig. 6.8).

• Moseley's law for K_α lines:

$$\omega_{K_\alpha}=\frac{3}{4}R(Z-\sigma)^2, \tag{6.3f}$$

where σ is the correction constant which is equal to unity for light elements.

• Magnetic moment of an atom and Landé g factor:

$$\mu=g\sqrt{J(J+1)}\,\mu_B,\qquad g=1+\frac{J(J+1)+S(S+1)-L(L+1)}{2J(J+1)}. \tag{6.3g}$$

• Zeeman splitting of spectral lines in a weak magnetic field:

$$\Delta\omega = (m_1g_1 - m_2g_2)\,\mu_B B/\hbar. \tag{6.3h}$$

• With radiation directed along the magnetic field, the Zeeman components caused by the transition $m_1 = m_2$ are absent.

6.97. The binding energy of a valence electron in a Li atom in the states $2S$ and $2P$ is equal to 5.39 and 3.54 eV respectively. Find the Rydberg corrections for S and P terms of the atom.

6.98. Find the Rydberg correction for the $3P$ term of a Na atom whose first excitation potential is 2.10 V and whose valence electron in the normal $3S$ state has the binding energy 5.14 eV.

6.99. Find the binding energy of a valence electron in the ground state of a Li atom if the wavelength of the first line of the sharp series is known to be equal to $\lambda_1 = 813$ nm and the short-wave cut-off wavelength of that series to $\lambda_2 = 350$ nm.

6.100. Determine the wavelengths of spectral lines appearing on transition of excited Li atoms from the state $3S$ down to the ground state $2S$. The Rydberg corrections for the S and P terms are -0.41 and -0.04.

6.101. The wavelengths of the yellow doublet components of the resonance Na line caused by the transition $3P \rightarrow 3S$ are equal to 589.00 and 589.56 nm. Find the splitting of the $3P$ term in eV units.

6.102. The first line of the sharp series of atomic cesium is a doublet with wavelengths 1358.8 and 1469.5 nm. Find the frequency intervals (in rad/s units) between the components of the sequent lines of that series.

6.103. Write the spectral designations of the terms of the hydrogen atom whose electron is in the state with principal quantum number $n = 3$.

6.104. How many and which values of the quantum number J can an atom possess in the state with quantum numbers S and L equal respectively to

(a) 2 and 3; (b) 3 and 3; (c) 5/2 and 2?

6.105. Find the possible values of total angular momenta of atoms in the states ^4P and ^5D.

6.106. Find the greatest possible total angular momentum and the corresponding spectral designation of the term

(a) of a Na atom whose valence electron possesses the principal quantum number $n = 4$;

(b) of an atom with electronic configuration $1s^22p3d$.

6.107. It is known that in F and D states the number of possible values of the quantum number J is the same and equal to five. Find the spin angular momentum in these states.

6.108. An atom is in the state whose multiplicity is three and the total angular momentum is $\hbar\sqrt{20}$. What can the corresponding quantum number L be equal to?

6.109. Find the possible multiplicities $\varkappa$ of the terms of the types

(a) ${}^{\varkappa}D_2$; (b) ${}^{\varkappa}P_{3/2}$; (c) ${}^{\varkappa}F_1$.

6.110. A certain atom has three electrons (s, p, and d), in addition to filled shells, and is in a state with the greatest possible total mechanical moment for a given configuration. In the corresponding vector model of the atom find the angle between the spin momentum and the total angular momentum of the given atom.

6.111. An atom possessing the total angular momentum $\hbar\sqrt{6}$ is in the state with spin quantum number $S = 1$. In the corresponding vector model the angle between the spin momentum and the total angular momentum is $\theta = 73.2°$. Write the spectral symbol for the term of that state.

6.112. Write the spectral symbols for the terms of a two-electron system consisting of one p electron and one d electron.

6.113. A system comprises an atom in ${}^2P_{3/2}$ state and a d electron. Find the possible spectral terms of that system.

6.114. Find out which of the following transitions are forbidden by the selection rules: ${}^2D_{3/2} \rightarrow {}^2P_{1/2}$, ${}^3P_1 \rightarrow {}^2S_{1/2}$, ${}^3F_3 \rightarrow {}^3P_2$, ${}^4F_{7/2} \rightarrow {}^4D_{5/2}$.

6.115. Determine the overall degeneracy of a $3D$ state of a Li atom. What is the physical meaning of that value?

6.116. Find the degeneracy of the states ${}^2P$, ${}^3D$, and ${}^4F$ possessing the greatest possible values of the total angular momentum.

6.117. Write the spectral designation of the term whose degeneracy is equal to seven and the quantum numbers L and S are interrelated as $L = 3S$.

6.118. What element has the atom whose K, L, and M shells and $4s$ subshell are filled completely and $4p$ subshell is half-filled?

6.119. Using the Hund rules, find the basic term of the atom whose partially filled subshell contains

(a) three p electrons; (b) four p electrons.

6.120. Using the Hund rules, find the total angular momentum of the atom in the ground state whose partially filled subshell contains

(a) three d electrons; (b) seven d electrons.

6.121. Making use of the Hund rules, find the number of electrons in the only partially filled subshell of the atom whose basic term is

(a) ${}^3F_2$; (b) ${}^2P_{3/2}$; (c) ${}^6S_{5/2}$.

6.122. Using the Hund rules, write the spectral symbol of the basic term of the atom whose only partially filled subshell

(a) is filled by 1/3, and $S = 1$;

(b) is filled by 70%, and $S = 3/2$.

6.123. The only partially filled subshell of a certain atom contains three electrons, the basic term of the atom having $L = 3$. Using the Hund rules, write the spectral symbol of the ground state of the given atom.

6.124. Using the Hund rules, find the magnetic moment of the ground state of the atom whose open subshell is half-filled with five electrons.

6.125. What fraction of hydrogen atoms is in the state with the principal quantum number $n = 2$ at a temperature $T = 3000$ K?

6.126. Find the ratio of the number of atoms of gaseous sodium in the state $3P$ to that in the ground state $3S$ at a temperature $T = 2400$ K. The spectral line corresponding to the transition $3P \rightarrow 3S$ is known to have the wavelength $\lambda = 589$ nm.

6.127. Calculate the mean lifetime of excited atoms if it is known that the intensity of the spectral line appearing due to transition to the ground state diminishes by a factor $\eta = 25$ over a distance $l = 2.5$ mm along the stream of atoms whose velocity is $v = 600$ m/s.

6.128. Rarefied Hg gas whose atoms are practically all in the ground state was lighted by a mercury lamp emitting a resonance line of wavelength $\lambda = 253.65$ nm. As a result, the radiation power of Hg gas at that wavelength turned out to be $P = 35$ mW. Find the number of atoms in the state of resonance excitation whose mean lifetime is $\tau = 0.15$ µs.

6.129. Atomic lithium of concentration $n = 3.6\cdot10^{16}$ cm^{-3} is at a temperature $T = 1500$ K. In this case the power emitted at the resonant line's wavelength $\lambda = 671$ nm ($2P \rightarrow 2S$) per unit volume of gas is equal to $P = 0.30$ W/cm^3. Find the mean lifetime of Li atoms in the resonance excitation state.

6.130. Atomic hydrogen is in thermodynamic equilibrium with its radiation. Find:

(a) the ratio of probabilities of induced and spontaneous radiations of the atoms from the level $2P$ at a temperature $T = 3000$ K;

(b) the temperature at which these probabilities become equal.

6.131. A beam of light of frequency ω, equal to the resonant frequency of transition of atoms of gas, passes through that gas heated to temperature T. In this case $\hbar\omega \gg kT$. Taking into account induced radiation, demonstrate that the absorption coefficient of the gas $\varkappa$ varies as $\varkappa = \varkappa_0 (1 - e^{-\hbar\omega/kT})$, where $\varkappa_0$ is the absorption coefficient for $T \rightarrow 0$.

6.132. The wavelength of a resonant mercury line is $\lambda = 253.65$ nm. The mean lifetime of mercury atoms in the state of resonance excitation is $\tau = 0.15$ µs. Evaluate the ratio of the Doppler line broadening to the natural linewidth at a gas temperature $T = 300$ K.

6.133. Find the wavelength of the K_α line in copper ($Z = 29$) if the wavelength of the K_α line in iron ($Z = 26$) is known to be equal to 193 pm.

6.134. Proceeding from Moseley's law find:

(a) the wavelength of the K_α line in aluminium and cobalt:

(b) the difference in binding energies of K and L electrons in vanadium.

6.135. How many elements are there in a row between those whose wavelengths of K_α lines are equal to 250 and 179 pm?

6.136. Find the voltage applied to an X-ray tube with nickel anticathode if the wavelength difference between the K_α line and the short-wave cut-off of the continuous X-ray spectrum is equal to 84 pm.

6.137. At a certain voltage applied to an X-ray tube with aluminium anticathode the short-wave cut-off wavelength of the continuous X-ray spectrum is equal to 0.50 nm. Will the K series of the characteristic spectrum whose excitation potential is equal to 1.56 kV be also observed in this case?

6.138. When the voltage applied to an X-ray tube increased from $V_1 = 10$ kV to $V_2 = 20$ kV, the wavelength interval between the K_α line and the short-wave cut-off of the continuous X-ray spectrum increases by a factor $n = 3.0$. Find the atomic number of the element of which the tube's anticathode is made.

6.139. What metal has in its absorption spectrum the difference between the frequencies of X-ray K and L absorption edges equal to $\Delta\omega = 6.85 \cdot 10^{18}$ s^{-1}?

6.140. Calculate the binding energy of a K electron in vanadium whose L absorption edge has the wavelength $\lambda_L = 2.4$ nm.

6.141. Find the binding energy of an L electron in titanium if the wavelength difference between the first line of the K series and its short-wave cut-off is $\Delta\lambda = 26$ pm.

6.142. Find the kinetic energy and the velocity of the photoelectrons liberated by K_α radiation of zinc from the K shell of iron whose K band absorption edge wavelength is $\lambda_K = 174$ pm.

6.143. Calculate the Landé g factor for atoms

(a) in S states; (b) in singlet states.

6.144. Calculate the Landé g factor for the following terms:

(a) $^6F_{1/2}$; (b) $^4D_{1/2}$; (c) $^5F_2$; (d) $^5P_1$; (e) $^3P_0$.

6.145. Calculate the magnetic moment of an atom (in Bohr magnetons)

(a) in ^1F state;

(b) in $^2D_{3/2}$ state;

(c) in the state in which $S = 1$, $L = 2$, and Landé factor $g = 4/3$.

6.146. Determine the spin angular momentum of an atom in the state D_2 if the maximum value of the magnetic moment projection in that state is equal to four Bohr magnetons.

6.147. An atom in the state with quantum numbers $L = 2$, $S = 1$ is located in a weak magnetic field. Find its magnetic moment if the least possible angle between the angular momentum and the field direction is known to be equal to 30°.

6.148. A valence electron in a sodium atom is in the state with principal quantum number $n = 3$, with the total angular momentum being the greatest possible. What is its magnetic moment in that state?

6.149. An excited atom has the electronic configuration $1s^2 2s^2 2p 3d$ being in the state with the greatest possible total angular momentum. Find the magnetic moment of the atom in that state.

6.150. Find the total angular momentum of an atom in the state with $S = 3/2$ and $L = 2$ if its magnetic moment is known to be equal to zero.

6.151. A certain atom is in the state in which $S = 2$, the total angular momentum $M = \sqrt{2}\hbar$, and the magnetic moment is equal to zero. Write the spectral symbol of the corresponding term.

6.152. An atom in the state $^2P_{3/2}$ is located in the external magnetic field of induction $B = 1.0$ kG. In terms of the vector model find the angular precession velocity of the total angular momentum of that atom.

6.153. An atom in the state $^2P_{1/2}$ is located on the axis of a loop of radius $r = 5$ cm carrying a current $I = 10$ A. The distance between the atom and the centre of the loop is equal to the radius of the latter. How great may be the maximum force that the magnetic field of that current exerts on the atom?

6.154. A hydrogen atom in the normal state is located at a distance $r = 2.5$ cm from a long straight conductor carrying a current $I = 10$ A. Find the force acting on the atom.

6.155. A narrow stream of vanadium atoms in the ground state $^4F_{3/2}$ is passed through a transverse strongly inhomogeneous magnetic field of length $l_1 = 5.0$ cm as in the Stern-Gerlach experiment. The beam splitting is observed on a screen located at a distance $l_2 = 15$ cm from the magnet. The kinetic energy of the atoms is $T = 22$ MeV. At what value of the gradient of the magnetic field induction B is the distance between the extreme components of the split beam on the screen equal to $\delta = 2.0$ mm?

6.156. Into what number of sublevels are the following terms split in a weak magnetic field:

(a) $^3P_0$; (b) $^2F_{5/2}$; (c) $^4D_{1/2}$?

6.157. An atom is located in a magnetic field of induction $B = 2.50$ kG. Find the value of the total splitting of the following terms (expressed in eV units):

(a) ^1D; (b) $^3F_4$.

6.158. What kind of Zeeman effect, normal or anomalous, is observed in a weak magnetic field in the case of spectral lines caused by the following transitions:

(a) $^1P \to {}^1S$; (b) $^2D_{5/2} \to {}^2P_{3/2}$; (c) $^3D_1 \to {}^3P_0$; (d) $^5I_5 \to {}^5H_4$?

6.159. Determine the spectral symbol of an atomic singlet term if the total splitting of that term in a weak magnetic field of induction $B = 3.0$ kG amounts to $\Delta E = 104$ µeV.

6.160. It is known that a spectral line $\lambda = 612$ nm of an atom is caused by a transition between singlet terms. Calculate the interval $\Delta\lambda$ between the extreme components of that line in the magnetic field with induction $B = 10.0$ kG.

6.161. Find the minimum magnitude of the magnetic field induction B at which a spectral instrument with resolving power $\lambda/\delta\lambda = 1.0 \cdot 10^5$ is capable of resolving the components of the spectral line $\lambda = 536$ nm caused by a transition between singlet terms. The observation line is at right angles to the magnetic field direction.

6.162. A spectral line caused by the transition $^3D_1 \rightarrow {}^3P_0$ experiences the Zeeman splitting in a weak magnetic field. When observed at right angles to the magnetic field direction, the interval between the neighbouring components of the split line is $\Delta\omega = 1.32 \cdot 10^{10}$ s^{-1}. Find the magnetic field induction B at the point where the source is located.

6.163. The wavelengths of the Na yellow doublet ($^2P \rightarrow {}^2S$) are equal to 589.59 and 589.00 nm. Find:

(a) the ratio of the intervals between neighbouring sublevels of the Zeeman splitting of the terms $^2P_{3/2}$ and $^2P_{1/2}$ in a weak magnetic field;

(b) the magnetic field induction B at which the interval between neighbouring sublevels of the Zeeman splitting of the term $^2P_{3/2}$ is $\eta = 50$ times smaller than the natural splitting of the term ^2P.

6.164. Draw a diagram of permitted transitions between the terms $^2P_{3/2}$ and $^2S_{1/2}$ in a weak magnetic field. Find the displacements (in rad/s units) of Zeeman components of that line in a magnetic field $B = 4.5$ kG.

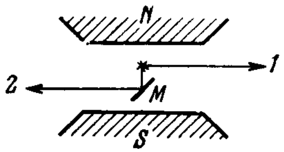


Fig. 6.9.

6.165. The same spectral line undergoing anomalous Zeeman splitting is observed in direction *1* and, after reflection from the mirror M (Fig. 6.9), in direction *2*. How many Zeeman components are observed in both directions if the spectral line is caused by the transition

(a) $^2P_{3/2} \rightarrow {}^2S_{1/2}$; (b) $^3P_2 \rightarrow {}^3S_1$?

6.166. Calculate the total splitting $\Delta\omega$ of the spectral line $^3D_3 \rightarrow {}^3P_2$ in a weak magnetic field with induction $B = 3.4$ kG.

6.4. MOLECULES AND CRYSTALS

• Rotational energy of a diatomic molecule:

$$E_J = \frac{\hbar^2}{2I} J(J+1), \tag{6.4a}$$

where I is the molecule's moment of inertia.

• Vibrational energy of a diatomic molecule:

$$E_v = \hbar\omega \left(v + \frac{1}{2}\right), \tag{6.4b}$$

where ω is the natural frequency of oscillations of the molecule.

• Mean energy of a quantum harmonic oscillator at a temperature T:

$$\langle E \rangle = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}. \tag{6.4c}$$

• Debye formula for molar vibrational energy of a crystal:

$$U = 9R\Theta \left[\frac{1}{8} + \left(\frac{T}{\Theta}\right)^4 \int\limits_0^{\Theta/T} \frac{x^3\,dx}{e^x - 1}\right], \tag{6.4d}$$

where Θ is the Debye temperature,

$$\Theta = \hbar\omega_{max}/k. \tag{6.4e}$$

• Molar vibrational heat capacity of a crystal for $T \ll \Theta$:

$$C = \frac{12}{5}\pi^4 R \left(\frac{T}{\Theta}\right)^3. \tag{6.4f}$$

• Distribution of free electrons in metal in the vicinity of the absolute zero:

$$dn = \frac{\sqrt{2}\, m^{3/2}}{\pi^2\hbar^3} \sqrt{E}\, dE, \tag{6.4g}$$

where dn is the concentration of electrons whose energy falls within the interval E, $E + dE$. The energy E is counted off the bottom of the conduction band.

• Fermi level at $T = 0$:

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}, \tag{6.4h}$$

where n is the concentration of free electrons in metal.

6.167. Determine the angular rotation velocity of an S_2 molecule promoted to the first excited rotational level if the distance between its nuclei is $d = 189$ pm.

6.168. For an HCl molecule find the rotational quantum numbers of two neighbouring levels whose energies differ by 7.86 meV. The nuclei of the molecule are separated by the distance of 127.5 pm.

6.169. Find the angular momentum of an oxygen molecule whose rotational energy is $E = 2.16$ meV and the distance between the nuclei is $d = 121$ pm.

6.170. Show that the frequency intervals between the neighbouring spectral lines of a true rotational spectrum of a diatomic molecule are equal. Find the moment of inertia and the distance between the nuclei of a CH molecule if the intervals between the neighbouring lines of the true rotational spectrum of these molecules are equal to $\Delta\omega = 5.47 \cdot 10^{12}$ s^{-1}.

6.171. For an HF molecule find the number of rotational levels located between the zeroth and first excited vibrational levels assuming rotational states to be independent of vibrational ones. The natural vibration frequency of this molecule is equal to $7.79 \cdot 10^{14}$ rad/s, and the distance between the nuclei is 91.7 pm.

6.172. Evaluate how many lines there are in a true rotational spectrum of CO molecules whose natural vibration frequency is $\omega = 4.09 \cdot 10^{14}\ s^{-1}$ and moment of inertia $I = 1.44 \cdot 10^{-39}\ g \cdot cm^2$.

6.173. Find the number of rotational levels per unit energy interval, dN/dE, for a diatomic molecule as a function of rotational energy E. Calculate that magnitude for an iodine molecule in the state with rotational quantum number $J = 10$. The distance between the nuclei of that molecule is equal to 267 pm.

6.174. Find the ratio of energies required to excite a diatomic molecule to the first vibrational and to the first rotational level. Calculate that ratio for the following molecules:

Molecule	ω, $10^{14}\ s^{-1}$	d, pm
(a) H_2	8.3	74
(b) HI	4.35	160
(c) I_2	0 40	267

Here ω is the natural vibration frequency of a molecule, d is the distance between nuclei.

6.175. The natural vibration frequency of a hydrogen molecule is equal to $8.25 \cdot 10^{14}\ s^{-1}$, the distance between the nuclei is 74 pm. Find the ratio of the number of these molecules at the first excited vibrational level ($v = 1$) to the number of molecules at the first excited rotational level ($J = 1$) at a temperature $T = 875$ K. It should be remembered that the degeneracy of rotational levels is equal to $2J + 1$.

6.176. Derive Eq. (6.4c), making use of the Boltzmann distribution. From Eq. (6.4c) obtain the expression for molar vibration heat capacity $C_{V\ vib}$ of diatomic gas. Calculate $C_{V\ vib}$ for Cl_2 gas at the temperature 300 K. The natural vibration frequency of these molecules is equal to $1.064 \cdot 10^{14}\ s^{-1}$.

6.177. In the middle of the rotation-vibration band of emission spectrum of HCl molecule, where the "zeroth" line is forbidden by the selection rules, the interval between neighbouring lines is $\Delta\omega = 0.79 \cdot 10^{13}\ s^{-1}$. Calculate the distance between the nuclei of an HCl molecule.

6.178. Calculate the wavelengths of the red and violet satellites, closest to the fixed line, in the vibration spectrum of Raman scattering by F_2 molecules if the incident light wavelength is equal to $\lambda_0 = 404.7$ nm and the natural vibration frequency of the molecule is $\omega = 2.15 \cdot 10^{14}\ s^{-1}$.

6.179. Find the natural vibration frequency and the quasielastic force coefficient of an S_2 molecule if the wavelengths of the red and violet satellites, closest to the fixed line, in the vibration spectrum of Raman scattering are equal to 346.6 and 330.0 nm.

6.180. Find the ratio of intensities of the violet and red satellites, closest to the fixed line, in the vibration spectrum of Raman scattering by Cl_2 molecules at a temperature $T = 300$ K if the natural vibration frequency of these molecules is $\omega = 1.06 \cdot 10^{14}\ s^{-1}$. By what factor will this ratio change if the temperature is doubled?

6.181. Consider the possible vibration modes in the following linear molecules:

(a) CO_2 (O—C—O); (b) C_2H_2 (H—C—C—H).

6.182. Find the number of natural transverse vibrations of a string of length l in the frequency interval from ω to $\omega + d\omega$ if the propagation velocity of vibrations is equal to v. All vibrations are supposed to occur in one plane.

6.183. There is a square membrane of area S. Find the number of natural vibrations perpendicular to its plane in the frequency interval from ω to $\omega + d\omega$ if the propagation velocity of vibrations is equal to v.

6.184. Find the number of natural transverse vibrations of a right-angled parallelepiped of volume V in the frequency interval from ω to $\omega + d\omega$ if the propagation velocity of vibrations is equal to v.

6.185. Assuming the propagation velocities of longitudinal and transverse vibrations to be the same and equal to v, find the Debye temperature

(a) for a unidimensional crystal, i.e. a chain of identical atoms, incorporating n_0 atoms per unit length;

(b) for a two-dimensional crystal, i.e. a plane square grid consisting of identical atoms, containing n_0 atoms per unit area;

(c) for a simple cubic lattice consisting of identical atoms, containing n_0 atoms per unit volume.

6.186. Calculate the Debye temperature for iron in which the propagation velocities of longitudinal and transverse vibrations are equal to 5.85 and 3.23 km/s respectively.

6.187. Evaluate the propagation velocity of acoustic vibrations in aluminium whose Debye temperature is $\Theta = 396$ K.

6.188. Derive the formula expressing molar heat capacity of a unidimensional crystal, a chain of identical atoms, as a function of temperature T if the Debye temperature of the chain is equal to Θ. Simplify the obtained expression for the case $T \gg \Theta$.

6.189. In a chain of identical atoms the vibration frequency ω depends on wave number k as $\omega = \omega_{max} \sin(ka/2)$, where ω_{max} is the maximum vibration frequency, $k = 2\pi/\lambda$ is the wave number corresponding to frequency ω, a is the distance between neighbouring atoms. Making use of this dispersion relation, find the dependence of the number of longitudinal vibrations per unit frequency interval on ω, i.e. $dN/d\omega$, if the length of the chain is l. Having obtained $dN/d\omega$, find the total number N of possible longitudinal vibrations of the chain.

6.190. Calculate the zero-point energy per one gram of copper whose Debye temperature is $\Theta = 330$ K.

6.191. Fig. 6.10 shows heat capacity of a crystal vs temperature in terms of the Debye theory. Here C_{cl} is classical heat capacity, Θ is the Debye temperature. Using this plot, find:

(a) the Debye temperature for silver if at a temperature $T = 65$ K its molar heat capacity is equal to 15 J/(mol·K);

(b) the molar heat capacity of aluminium at $T = 80$ K if at $T = 250$ K it is equal to 22.4 J/(mol·K);

(c) the maximum vibration frequency for copper whose heat capacity at $T = 125$ K differs from the classical value by 25%.

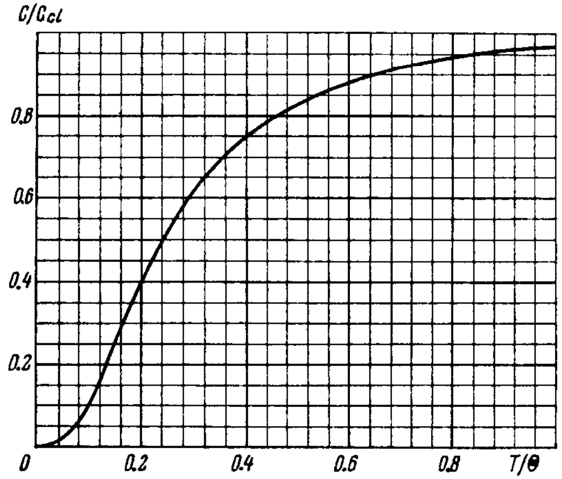


Fig. 6.10.

6.192. Demonstrate that molar heat capacity of a crystal at a temperature $T \ll \Theta$, where Θ is the Debye temperature, is defined by Eq. (6.4f).

6.193. Can one consider the temperatures 20 and 30 K as low for a crystal whose heat capacities at these temperatures are equal to 0.226 and 0.760 J/(mol·K)?

6.194. Calculate the mean zero-point energy per one oscillator of a crystal in terms of the Debye theory if the Debye temperature of the crystal is equal to Θ.

6.195. Draw the vibration energy of a crystal as a function of frequency (neglecting the zero-point vibrations). Consider two cases: $T = \Theta/2$ and $T = \Theta/4$, where Θ is the Debye temperature.

6.196. Evaluate the maximum values of energy and momentum of a phonon (acoustic quantum) in copper whose Debye temperature is equal to 330 K.

6.197. Employing Eq. (6.4g), find at $T = 0$:

(a) the maximum kinetic energy of free electrons in a metal if their concentration is equal to n;

(b) the mean kinetic energy of free electrons if their maximum kinetic energy T_{max} is known.

6.198. What fraction (in per cent) of free electrons in a metal at $T = 0$ has a kinetic energy exceeding half the maximum energy?

6.199. Find the number of free electrons per one sodium atom at $T = 0$ if the Fermi level is equal to $E_F = 3.07$ eV and the density of sodium is 0.97 g/cm³.

6.200. Up to what temperature has one to heat classical electronic gas to make the mean energy of its electrons equal to that of free electrons in copper at $T = 0$? Only one free electron is supposed to correspond to each copper atom.

6.201. Calculate the interval (in eV units) between neighbouring levels of free electrons in a metal at $T = 0$ near the Fermi level, if the concentration of free electrons is $n = 2.0 \cdot 10^{22}$ cm^{-3} and the volume of the metal is $V = 1.0$ cm³.

6.202. Making use of Eq. (6.4g), find at $T = 0$:

(a) the velocity distribution of free electrons;

(b) the ratio of the mean velocity of free electrons to their maximum velocity.

6.203. On the basis of Eq. (6.4g) find the number of free electrons in a metal at $T = 0$ as a function of de Broglie wavelengths.

6.204. Calculate the electronic gas pressure in metallic sodium, at $T = 0$, in which the concentration of free electrons is $n = 2.5 \cdot 10^{22}$ cm^{-3}. Use the equation for the pressure of ideal gas.

6.205. The increase in temperature of a cathode in electronic tube by $\Delta T = 1.0$ K from the value $T = 2000$ K results in the increase of saturation current by $\eta = 1.4\%$. Find the work function of electron for the material of the cathode.

6.206. Find the refractive index of metallic sodium for electrons with kinetic energy $T = 135$ eV. Only one free electron is assumed to correspond to each sodium atom.

6.207. Find the minimum energy of electron-hole pair formation in an impurity-free semiconductor whose electric conductance increases $\eta = 5.0$ times when the temperature increases from $T_1 = 300$ K to $T_2 = 400$ K.

6.208. At very low temperatures the photoelectric threshold short wavelength in an impurity-free germanium is equal to $\lambda_{th} = 1.7$ µm. Find the temperature coefficient of resistance of this germanium sample at room temperature.

6.209. Fig. 6.11 illustrates logarithmic electric conductance as a function of reciprocal temperature (T in kK units) for some

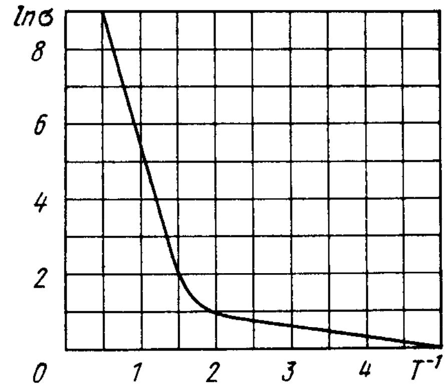


Fig. 6.11.

n-type semiconductor. Using this plot, find the width of the forbidden band of the semiconductor and the activation energy of donor levels.

6.210. The resistivity of an impurity-free semiconductor at room temperature is $\rho = 50\ \Omega\cdot$cm. It becomes equal to $\rho_1 = 40\ \Omega\cdot$cm when the semiconductor is illuminated with light, and $t = 8$ ms after switching off the light source the resistivity becomes equal to $\rho_2 = 45\ \Omega\cdot$cm. Find the mean lifetime of conduction electrons and holes.

6.211. In Hall effect measurements a plate of width $h = 10$ mm and length $l = 50$ mm made of p-type semiconductor was placed in a magnetic field with induction $B = 5.0$ kG. A potential difference $V = 10$ V was applied across the edges of the plate. In this case the Hall field is $V_H = 50$ mV and resistivity $\rho = 2.5\ \Omega\cdot$cm. Find the concentration of holes and hole mobility.

6.212. In Hall effect measurements in a magnetic field with induction $B = 5.0$ kG the transverse electric field strength in an impurity-free germanium turned out to be $\eta = 10$ times less than the longitudinal electric field strength. Find the difference in the mobilities of conduction electrons and holes in the given semiconductor.

6.213. The Hall effect turned out to be not observable in a semiconductor whose conduction electron mobility was $\eta = 2.0$ times that of the hole mobility. Find the ratio of hole and conduction electron concentrations in that semiconductor.

6.5. RADIOACTIVITY

• Fundamental law of radioactive decay:

$$N = N_0 e^{-\lambda t}. \tag{6.5a}$$

• Relation between the decay constant λ, the mean lifetime τ, and the half-life T:

$$\lambda = \frac{1}{\tau} = \frac{\ln 2}{T}. \tag{6.5b}$$

• Specific activity is the activity of a unit mass of a radioisotope.

6.214. Knowing the decay constant λ of a nucleus, find:
(a) the probability of decay of the nucleus during the time from 0 to t;
(b) the mean lifetime τ of the nucleus.

6.215. What fraction of the radioactive cobalt nuclei whose half-life is 71.3 days decays during a month?

6.216. How many beta-particles are emitted during one hour by 1.0 μg of Na^{24} radionuclide whose half-life is 15 hours?

6.217. To investigate the beta-decay of Mg^{23} radionuclide, a counter was activated at the moment $t = 0$. It registered N_1 beta-particles by a moment $t_1 = 2.0$ s, and by a moment $t_2 = 3t_1$ the number of registered beta-particles was 2.66 times greater. Find the mean lifetime of the given nuclei.

6.218. The activity of a certain preparation decreases 2.5 times after 7.0 days. Find its half-life.

6.219. At the initial moment the activity of a certain radionuclide totalled 650 particles per minute. What will be the activity of the preparation after half its half-life period?

6.220. Find the decay constant and the mean lifetime of Co^{55} radionuclide if its activity is known to decrease 4.0% per hour. The decay product is nonradioactive.

6.221. A U^{238} preparation of mass 1.0 g emits $1.24\cdot10^4$ alpha-particles per second. Find the half-life of this nuclide and the activity of the preparation.

6.222. Determine the age of ancient wooden items if it is known that the specific activity of C^{14} nuclide in them amounts to 3/5 of that in lately felled trees. The half-life of C^{14} nuclei is 5570 years.

6.223. In a uranium ore the ratio of U^{238} nuclei to Pb^{206} nuclei is $\eta = 2.8$. Evaluate the age of the ore, assuming all the lead Pb^{206} to be a final decay product of the uranium series. The half-life of U^{238} nuclei is $4.5\cdot10^9$ years.

6.224. Calculate the specific activities of Na^{24} and U^{235} nuclides whose half-lifes are 15 hours and $7.1\cdot10^8$ years respectively.

6.225. A small amount of solution containing Na^{24} radionuclide with activity $A = 2.0\cdot10^3$ disintegrations per second was injected in the bloodstream of a man. The activity of 1 cm³ of blood sample taken $t = 5.0$ hours later turned out to be $A' = 16$ disintegrations per minute per cm³. The half-life of the radionuclide is $T = 15$ hours. Find the volume of the man's blood.

6.226. The specific activity of a preparation consisting of radioactive Co^{58} and nonradioactive Co^{59} is equal to $2.2\cdot10^{12}$ dis/(s·g). The half-life of Co^{58} is 71.3 days. Find the ratio of the mass of radioactive cobalt in that preparation to the total mass of the preparation (in per cent).

6.227. A certain preparation includes two beta-active components with different half-lifes. The measurements resulted in the following dependence of the natural logarithm of preparation activity on time t expressed in hours:

t	0	1	2	3	5	7	10	14	20
$\ln A$	4.10	3.60	3.10	2.60	2.06	1.82	1.60	1.32	0.90

Find the half-lifes of both components and the ratio of radioactive nuclei of these components at the moment $t = 0$.

6.228. A P^{32} radionuclide with half-life $T = 14.3$ days is produced in a reactor at a constant rate $q = 2.7\cdot10^9$ nuclei per second. How soon after the beginning of production of that radionuclide will its activity be equal to $A = 1.0\cdot10^9$ dis/s?

6.229. A radionuclide A_1 with decay constant λ_1 transforms into a radionuclide A_2 with decay constant λ_2. Assuming that at the

initial moment the preparation contained only the radionuclide A_1, find:

(a) the equation describing accumulation of the radionuclide A_2 with time;

(b) the time interval after which the activity of radionuclide A_2 reaches the maximum value.

6.230. Solve the foregoing problem if $\lambda_1 = \lambda_2 = \lambda$.

6.231. A radionuclide A_1 goes through the transformation chain $A_1 \rightarrow A_2 \rightarrow A_3$ (stable) with respective decay constants λ_1 and λ_2. Assuming that at the initial moment the preparation contained only the radionuclide A_1 equal in quantity to N_{10} nuclei, find the equation describing accumulation of the stable isotope A_3.

6.232. A Bi^{210} radionuclide decays via the chain

$$Bi^{210} \xrightarrow[\lambda_1]{} Po^{210} \xrightarrow[\lambda_2]{} Pb^{206} \text{ (stable)},$$

where the decay constants are $\lambda_1 = 1.60 \cdot 10^{-6}$ s^{-1}, $\lambda_2 = 5.80 \cdot 10^{-8}$ s^{-1}. Calculate alpha- and beta-activities of the Bi^{210} preparation of mass 1.00 mg a month after its manufacture.

6.233. (a) What isotope is produced from the alpha-radioactive Ra^{226} as a result of five alpha-disintegrations and four β^--disintegrations?

(b) How many alpha- and β^--decays does U^{238} experience before turning finally into the stable Pb^{206} isotope?

6.234. A stationary Pb^{200} nucleus emits an alpha-particle with kinetic energy $T_\alpha = 5.77$ MeV. Find the recoil velocity of a daughter nucleus. What fraction of the total energy liberated in this decay is accounted for by the recoil energy of the daughter nucleus?

6.235. Find the amount of heat generated by 1.00 mg of a Po^{210} preparation during the mean lifetime period of these nuclei if the emitted alpha-particles are known to possess the kinetic energy 5.3 MeV and practically all daughter nuclei are formed directly in the ground state.

6.236. The alpha-decay of Po^{210} nuclei (in the ground state) is accompanied by emission of two groups of alpha-particles with kinetic energies 5.30 and 4.50 MeV. Following the emission of these particles the daughter nuclei are found in the ground and excited states. Find the energy of gamma-quanta emitted by the excited nuclei.

6.237. The mean path length of alpha-particles in air under standard conditions is defined by the formula $R = 0.98 \cdot 10^{-27} v_0^3$ cm, where v_0 (cm/s) is the initial velocity of an alpha-particle. Using this formula, find for an alpha-particle with initial kinetic energy 7.0 MeV:

(a) its mean path length;

(b) the average number of ion pairs formed by the given alpha-particle over the whole path R as well as over its first half, assuming the ion pair formation energy to be equal to 34 eV.

6.238. Find the energy Q liberated in β^-- and β^+-decays and in K-capture if the masses of the parent atom M_p, the daughter atom M_d and an electron m are known.

6.239. Taking the values of atomic masses from the tables, find the maximum kinetic energy of beta-particles emitted by Be^{10} nuclei and the corresponding kinetic energy of recoiling daughter nuclei formed directly in the ground state.

6.240. Evaluate the amount of heat produced during a day by a β^--active Na^{24} preparation of mass $m = 1.0$ mg. The beta-particles are assumed to possess an average kinetic energy equal to 1/3 of the highest possible energy of the given decay. The half-life of Na^{24} is $T = 15$ hours.

6.241. Taking the values of atomic masses from the tables, calculate the kinetic energies of a positron and a neutrino emitted by C^{11} nucleus for the case when the daughter nucleus does not recoil.

6.242. Find the kinetic energy of the recoil nucleus in the positronic decay of a N^{13} nucleus for the case when the energy of positrons is maximum.

6.243. From the tables of atomic masses determine the velocity of a nucleus appearing as a result of K-capture in a Be^7 atom provided the daughter nucleus turns out to be in the ground state.

6.244. Passing down to the ground state, excited Ag^{109} nuclei emit either gamma quanta with energy 87 keV or K conversion electrons whose binding energy is 26 keV. Find the velocity of these electrons.

6.245. A free stationary Ir^{191} nucleus with excitation energy $E = 129$ keV passes to the ground state, emitting a gamma quantum. Calculate the fractional change of gamma quanta energy due to recoil of the nucleus.

6.246. What must be the relative velocity of a source and an absorber consisting of free Ir^{191} nuclei to observe the maximum absorption of gamma quanta with energy $\varepsilon = 129$ keV?

6.247. A source of gamma quanta is placed at a height $h = 20$ m above an absorber. With what velocity should the source be displaced upward to counterbalance completely the gravitational variation of gamma quanta energy due to the Earth's gravity at the point where the absorber is located?

6.248. What is the minimum height to which a gamma quanta source containing excited Zn^{67} nuclei has to be raised for the gravitational displacement of the Mössbauer line to exceed the line width itself, when registered on the Earth's surface? The registered gamma quanta are known to have an energy $\varepsilon = 93$ keV and appear on transition of Zn^{67} nuclei to the ground state, and the mean lifetime of the excited state is $\tau = 14$ µs.

• Binding energy of a nucleus:

$$E_b = Zm_H + (A - Z)\, m_n - M, \qquad (6.6a)$$

where Z is the charge of the nucleus (in units of e), A is the mass number, m_H, m_n, and M are the masses of a hydrogen atom, a neutron, and an *atom* corresponding to the given nucleus.

In calculations the following formula is more convenient to use:

$$E_b = Z\Delta_H + (A - Z)\Delta_n - \Delta, \qquad (6.6b)$$

where Δ_H, Δ_n, and Δ are the mass surpluses of a hydrogen atom, a neutron, and an atom corresponding to the given nucleus.

• Energy diagram of a nuclear reaction

$$m + M \to M^* \to m' + M' + Q \qquad (6.6c)$$

is illustrated in Fig. 6.12, where $m + M$ and $m' + M'$ are the sums of rest masses of particles before and after the reaction, $\widetilde{T}$ and $\widetilde{T}'$ are the total kinetic energies of particles before and after the reaction (in the frame of the centre of inertia), E^* is the excitation energy of the transitional nucleus, Q is the energy of the reaction, E and E' are the binding energies of the particles m and m' in the transitional nucleus, *1, 2, 3* are the energy levels of the transitional nucleus.

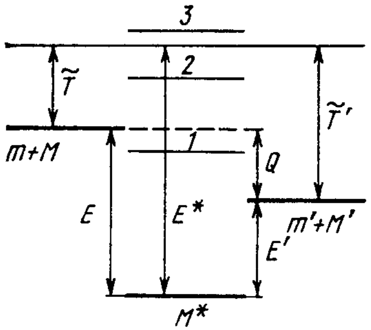


Fig. 6.12.

• Threshold (minimum) kinetic energy of an incoming particle at which an endoergic nuclear reaction

$$T_{th} = \frac{m + M}{M}\,|Q| \qquad (6.6d)$$

becomes possible; here m and M are the masses of the incoming particle and the target nucleus.

6.249. An alpha-particle with kinetic energy $T_\alpha = 7.0$ MeV is scattered elastically by an initially stationary Li^6 nucleus. Find the kinetic energy of the recoil nucleus if the angle of divergence of the two particles is $\Theta = 60°$.

6.250. A neutron collides elastically with an initially stationary deuteron. Find the fraction of the kinetic energy lost by the neutron

(a) in a head-on collision;

(b) in scattering at right angles.

6.251. Find the greatest possible angle through which a deuteron is scattered as a result of elastic collision with an initially stationary proton.

6.252. Assuming the radius of a nucleus to be equal to $R = 0.13\sqrt[3]{A}$ pm, where A is its mass number, evaluate the density of nuclei and the number of nucleons per unit volume of the nucleus.

6.253. Write missing symbols, denoted by x, in the following nuclear reactions:

(a) $B^{10}\,(x,\ \alpha)\,Be^8$;

(b) $O^{17}\,(d,\ n)\,x$;

(c) $Na^{23}\,(p,\ x)\,Ne^{20}$;

(d) $x\,(p,\ n)\,Ar^{37}$.

6.254. Demonstrate that the binding energy of a nucleus with mass number A and charge Z can be found from Eq. (6.6b).

6.255. Find the binding energy of a nucleus consisting of equal numbers of protons and neutrons and having the radius one and a half times smaller than that of Al^{27} nucleus.

6.256. Making use of the tables of atomic masses, find:

(a) the mean binding energy per one nucleon in O^{16} nucleus;

(b) the binding energy of a neutron and an alpha-particle in a B^{11} nucleus;

(c) the energy required for separation of an O^{16} nucleus into four identical particles.

6.257. Find the difference in binding energies of a neutron and a proton in a B^{11} nucleus. Explain why there is the difference.

6.258. Find the energy required for separation of a Ne^{20} nucleus into two alpha-particles and a C^{12} nucleus if it is known that the binding energies per one nucleon in Ne^{20}, He^4, and C^{12} nuclei are equal to 8.03, 7.07, and 7.68 MeV respectively.

6.259. Calculate in atomic mass units the mass of

(a) a Li^8 atom whose nucleus has the binding energy 41.3 MeV;

(b) a C^{10} nucleus whose binding energy per nucleon is equal to 6.04 MeV.

6.260. The nuclei involved in the nuclear reaction $A_1 + A_2 \to A_3 + A_4$ have the binding energies E_1, E_2, E_3, and E_4. Find the energy of this reaction.

6.261. Assuming that the splitting of a U^{235} nucleus liberates the energy of 200 MeV, find:

(a) the energy liberated in the fission of one kilogram of U^{235} isotope, and the mass of coal with calorific value of 30 kJ/g which is equivalent to that for one kg of U^{235};

(b) the mass of U^{235} isotope split during the explosion of the atomic bomb with 30 kt trotyl equivalent if the calorific value of trotyl is 4.1 kJ/g.

6.262. What amount of heat is liberated during the formation of one gram of He^4 from deuterium H^2? What mass of coal with calorific value of 30 kJ/g is thermally equivalent to the magnitude obtained?

6.263. Taking the values of atomic masses from the tables, calculate the energy per nucleon which is liberated in the nuclear reaction $Li^6 + H^2 \to 2He^4$. Compare the obtained magnitude with the energy per nucleon liberated in the fission of U^{235} nucleus.

6.264. Find the energy of the reaction $Li^7 + p \to 2He^4$ if the binding energies per nucleon in Li^7 and He^4 nuclei are known to be equal to 5.60 and 7.06 MeV respectively.

6.265. Find the energy of the reaction $N^{14}\,(\alpha,\ p)\,O^{17}$ if the kinetic energy of the incoming alpha-particle is $T_\alpha = 4.0$ MeV and the

proton outgoing at an angle $\theta = 60°$ to the motion direction of the alpha-particle has a kinetic energy $T_p = 2.09$ MeV.

6.266. Making use of the tables of atomic masses, determine the energies of the following reactions:

(a) $Li^7 (p, n) Be^7$;
(b) $Be^9 (n, \gamma) Be^{10}$;
(c) $Li^7 (\alpha, n) B^{10}$;
(d) $O^{16} (d, \alpha) N^{14}$.

6.267. Making use of the tables of atomic masses, find the velocity with which the products of the reaction $B^{10} (n, \alpha) Li^7$ come apart; the reaction proceeds via interaction of very slow neutrons with stationary boron nuclei.

6.268. Protons striking a stationary lithium target activate a reaction $Li^7 (p, n) Be^7$. At what value of the proton's kinetic energy can the resulting neutron be stationary?

6.269. An alpha particle with kinetic energy $T = 5.3$ MeV initiates a nuclear reaction $Be^9 (\alpha, n) C^{12}$ with energy yield $Q = +5.7$ MeV. Find the kinetic energy of the neutron outgoing at right angles to the motion direction of the alpha-particle.

6.270. Protons with kinetic energy $T = 1.0$ MeV striking a lithium target induce a nuclear reaction $p + Li^7 \rightarrow 2He^4$. Find the kinetic energy of each alpha-particle and the angle of their divergence provided their motion directions are symmetrical with respect to that of incoming protons.

6.271. A particle of mass m strikes a stationary nucleus of mass M and activates an endoergic reaction. Demonstrate that the threshold (minimal) kinetic energy required to initiate this reaction is defined by Eq. (6.6d).

6.272. What kinetic energy must a proton possess to split a deuteron H^2 whose binding energy is $E_b = 2.2$ MeV?

6.273. The irradiation of lithium and beryllium targets by a monoergic stream of protons reveals that the reaction $Li^7(p, n)Be^7 - 1.65$ MeV is initiated whereas the reaction $Be^9(p, n)B^9 - 1.85$ MeV does not take place. Find the possible values of kinetic energy of the protons.

6.274. To activate the reaction (n, α) with stationary B^{11} nuclei, neutrons must have the threshold kinetic energy $T_{th} = 4.0$ MeV. Find the energy of this reaction.

6.275. Calculate the threshold kinetic energies of protons required to activate the reactions (p, n) and (p, d) with Li^7 nuclei.

6.276. Using the tabular values of atomic masses, find the threshold kinetic energy of an alpha particle required to activate the nuclear reaction $Li^7 (\alpha, n) B^{10}$. What is the velocity of the B^{10} nucleus in this case?

6.277. A neutron with kinetic energy $T = 10$ MeV activates a nuclear reaction $C^{12} (n, \alpha) Be^9$ whose threshold is $T_{th} = 6.17$ MeV. Find the kinetic energy of the alpha-particles outgoing at right angles to the incoming neutrons' direction.

6.278. How much, in per cent, does the threshold energy of gamma quantum exceed the binding energy of a deuteron ($E_b = 2.2$ MeV) in the reaction $\gamma + H^2 \rightarrow n + p$?

6.279. A proton with kinetic energy $T = 1.5$ MeV is captured by a deuteron H^2. Find the excitation energy of the formed nucleus.

6.280. The yield of the nuclear reaction $C^{13}(d, n)N^{14}$ has maximum magnitudes at the following values of kinetic energy T_i of bombarding deuterons: 0.60, 0.90, 1.55, and 1.80 MeV. Making use of the table of atomic masses, find the corresponding energy levels of the transitional nucleus through which this reaction proceeds.

6.281. A narrow beam of thermal neutrons is attenuated $\eta = 360$ times after passing through a cadmium plate of thickness $d = 0.50$ mm. Determine the effective cross-section of interaction of these neutrons with cadmium nuclei.

6.282. Determine how many times the intensity of a narrow beam of thermal neutrons will decrease after passing through the heavy water layer of thickness $d = 5.0$ cm. The effective cross-sections of interaction of deuterium and oxygen nuclei with thermal neutrons are equal to $\sigma_1 = 7.0$ b and $\sigma_2 = 4.2$ b respectively.

6.283. A narrow beam of thermal neutrons passes through a plate of iron whose absorption and scattering effective cross-sections are equal to $\sigma_a = 2.5$ b and $\sigma_s = 11$ b respectively. Find the fraction of neutrons quitting the beam due to scattering if the thickness of the plate is $d = 0.50$ cm.

6.284. The yield of a nuclear reaction producing radionuclides may be described in two ways: either by the ratio w of the number of nuclear reactions to the number of bombarding particles, or by the quantity k, the ratio of the activity of the formed radionuclide to the number of bombarding particles. Find:

(a) the half-life of the formed radionuclide, assuming w and k to be known;

(b) the yield w of the reaction $Li^7(p, n)Be^7$ if after irradiation of a lithium target by a beam of protons (over $t = 2.0$ hours and with beam current $I = 10\ \mu A$) the activity of Be^7 became equal to $A = 1.35 \cdot 10^8$ dis/s and its half-life to $T = 53$ days.

6.285. Thermal neutrons fall normally on the surface of a thin gold foil consisting of stable Au^{197} nuclide. The neutron flux density is $J = 1.0 \cdot 10^{10}$ part./(s·cm²). The mass of the foil is $m = 10$ mg. The neutron capture produces beta-active Au^{198} nuclei with half-life $T = 2.7$ days. The effective capture cross-section is $\sigma = 98$ b. Find:

(a) the irradiation time after which the number of Au^{197} nuclei decreases by $\eta = 1.0\%$;

(b) the maximum number of Au^{198} nuclei that can be formed during protracted irradiation.

6.286. A thin foil of certain stable isotope is irradiated by thermal neutrons falling normally on its surface. Due to the capture of neutrons a radionuclide with decay constant λ appears. Find the law

describing accumulation of that radionuclide $N(t)$ per unit area of the foil's surface. The neutron flux density is J, the number of nuclei per unit area of the foil's surface is n, and the effective cross-section of formation of active nuclei is σ.

6.287. A gold foil of mass $m = 0.20$ g was irradiated during $t = 6.0$ hours by a thermal neutron flux falling normally on its surface. Following $\tau = 12$ hours after the completion of irradiation the activity of the foil became equal to $A = 1.9 \cdot 10^7$ dis/s. Find the neutron flux density if the effective cross-section of formation of a radioactive nucleus is $\sigma = 96$ b, and the half-life is equal to $T = 2.7$ days.

6.288. How many neutrons are there in the hundredth generation if the fission process starts with $N_0 = 1000$ neutrons and takes place in a medium with multiplication constant $k = 1.05$?

6.289. Find the number of neutrons generated per unit time in a uranium reactor whose thermal power is $P = 100$ MW if the average number of neutrons liberated in each nuclear splitting is $\nu = 2.5$. Each splitting is assumed to release an energy $E = 200$ MeV.

6.290. In a thermal reactor the mean lifetime of one generation of thermal neutrons is $\tau = 0.10$ s. Assuming the multiplication constant to be equal to $k = 1.010$, find:

(a) how many times the number of neutrons in the reactor, and consequently its power, will increase over $t = 1.0$ min;

(b) the period T of the reactor, i.e. the time period over which its power increases e-fold.

6.7. ELEMENTARY PARTICLES

• Total energy and momentum of a relativistic particle:

$$E = m_0c^2 + T, \quad pc = \sqrt{T(T + 2m_0c^2)}, \tag{6.7a}$$

where T is the kinetic energy of the particle.

• When examining collisions of particles it pays to use the invariant:

$$E^2 - p^2c^2 = m_0^2c^4, \tag{6.7b}$$

where E and p are the total energy and the total momentum of the system prior to collision, m_0 is the rest mass of the formed particle.

• Threshold (minimal) kinetic energy of a particle m striking a stationary particle M and activating the endoergic reaction $m + M \to m_1 + m_2 + \ldots$:

$$T_{th} = \frac{(m_1 + m_2 + \ldots)^2 - (m+M)^2}{2M}c^2, \tag{6.7c}$$

where $m, M, m_1, m_2, \ldots$ are the rest masses of the respective particles.

• Quantum numbers classifying elementary particles:

Q, electric charge,
L, lepton charge,
B, baryon charge,
T, isotopic spin, T_z, its projection,
S, strangeness, $S = 2\langle Q\rangle - B$,
Y, hypercharge, $Y = B + S$.

• Relation between quantum numbers of strongly interacting particles:

$$Q = T_z + \frac{Y}{2} = T_z + \frac{B+S}{2}. \tag{6.7d}$$

• Interactions of particles obey the laws of conservation of the Q, L and B charges. In strong interactions the laws of conservation of S (or Y), T, and its projection T_z are also valid.

6.291. Calculate the kinetic energies of protons whose momenta are 0.10, 1.0, and 10 GeV/c, where c is the velocity of light.

6.292. Find the mean path travelled by pions whose kinetic energy exceeds their rest energy $\eta = 1.2$ times. The mean lifetime of very slow pions is $\tau_0 = 25.5$ ns.

6.293. Negative pions with kinetic energy $T = 100$ MeV travel an average distance $l = 11$ m from their origin to decay. Find the proper lifetime of these pions.

6.294. There is a narrow beam of negative pions with kinetic energy T equal to the rest energy of these particles. Find the ratio of fluxes at the sections of the beam separated by a distance $l = 20$ m. The proper mean lifetime of these pions is $\tau_0 = 25.5$ ns.

6.295. A stationary positive pion disintegrated into a muon and a neutrino. Find the kinetic energy of the muon and the energy of the neutrino.

6.296. Find the kinetic energy of a neutron emerging as a result of the decay of a stationary Σ^- hyperon ($\Sigma^- \to n + \pi^-$).

6.297. A stationary positive muon disintegrated into a positron and two neutrinos. Find the greatest possible kinetic energy of the positron.

6.298. A stationary neutral particle disintegrated into a proton with kinetic energy $T = 5.3$ MeV and a negative pion. Find the mass of that particle. What is its name?

6.299. A negative pion with kinetic energy $T = 50$ MeV disintegrated during its flight into a muon and a neutrino. Find the energy of the neutrino outgoing at right angles to the pion's motion direction.

6.300. A Σ^+ hyperon with kinetic energy $T_\Sigma = 320$ MeV disintegrated during its flight into a neutral particle and a positive pion outgoing with kinetic energy $T_\pi = 42$ MeV at right angles to the hyperon's motion direction. Find the rest mass of the neutral particle (in MeV units).

6.301. A neutral pion disintegrated during its flight into two gamma quanta with equal energies. The angle of divergence of gamma quanta is $\Theta = 60°$. Find the kinetic energy of the pion and of each gamma quantum.

6.302. A relativistic particle with rest mass m collides with a stationary particle of mass M and activates a reaction leading to formation of new particles: $m + M \to m_1 + m_2 + \ldots$, where the rest masses of newly formed particles are written on the right-hand side. Making use of the invariance of the quantity $E^2 - p^2c^2$, dem-

onstrate that the threshold kinetic energy of the particle m required for this reaction is defined by Eq. (6.7c).

6.303. A positron with kinetic energy $T = 750$ keV strikes a stationary free electron. As a result of annihilation, two gamma quanta with equal energies appear. Find the angle of divergence between them.

6.304. Find the threshold energy of gamma quantum required to form

(a) an electron-positron pair in the field of a stationary electron;

(b) a pair of pions of opposite signs in the field of a stationary proton.

6.305. Protons with kinetic energy T strike a stationary hydrogen target. Find the threshold values of T for the following reactions:

(a) $p + p \to p + p + p + \tilde{p}$; (b) $p + p \to p + p + \pi^0$.

6.306. A hydrogen target is bombarded by pions. Calculate the threshold values of kinetic energies of these pions making possible the following reactions:

(a) $\pi^- + p \to K^+ + \Sigma^-$; (b) $\pi^0 + p \to K^+ + \Lambda^0$.

6.307. Find the strangeness S and the hypercharge Y of a neutral elementary particle whose isotopic spin projection is $T_z = +1/2$ and baryon charge $B = +1$. What particle is this?

6.308. Which of the following processes are forbidden by the law of conservation of lepton charge:

(1) $n \to p + e^- + \nu$; (4) $p + e^- \to n + \nu$;
(2) $\pi^+ \to \mu^+ + e^- + e^+$; (5) $\mu^+ \to e^+ + \nu + \tilde{\nu}$;
(3) $\pi^- \to \mu^- + \nu$; (6) $K^- \to \mu^- + \tilde{\nu}$?

6.309. Which of the following processes are forbidden by the law of conservation of strangeness:

(1) $\pi^- + p \to \Sigma^- + K^+$; (4) $n + p \to \Lambda^0 + \Sigma^+$;
(2) $\pi^- + p \to \Sigma^+ + K^-$; (5) $\pi^- + n \to \Xi^- + K^+ + K^-$;
(3) $\pi^- + p \to K^+ + K^- + n$; (6) $K^- + p \to \Omega^- + K^+ + K^0$?

6.310. Indicate the reasons why the following processes are forbidden:

(1) $\Sigma^- \to \Lambda^0 + \pi^-$; (4) $n + p \to \Sigma^+ + \Lambda^0$;
(2) $\pi^- + p \to K^+ + K^-$; (5) $\pi^- \to \mu^- + e^+ + e^-$;
(3) $K^- + n \to \Omega^- + K^+ + K^0$; (6) $\mu^- \to e^- + \nu_e + \tilde{\nu}_\mu$.

ANSWERS AND SOLUTIONS

1.1. $v = l/2\tau = 3.0$ km per hour.
1.2. $\langle v \rangle = 2v_0 (v_1 + v_2)/(2v_0 + v_1 + v_2)$.
1.3. $\Delta t = \tau \sqrt{1 - 4\langle v \rangle / w\tau} = 15$ s.
1.4. (a) 10 cm/s; (b) 25 cm/s; (c) $t_0 = 16$ s.
1.5. $(\mathbf{r}_1 - \mathbf{r}_2)/|\mathbf{r}_1 - \mathbf{r}_2| = (\mathbf{v}_2 - \mathbf{v}_1)/|\mathbf{v}_2 - \mathbf{v}_1|$.
1.6. $v' = \sqrt{v_0^2 + v^2 + 2v_0 v \cos\varphi} \approx 40$ km per hour, $\varphi' = 19°$.
1.7. $u = \dfrac{v_0}{(1 - v_0^2/v'^2)^{-1/2} - 1} = 3.0$ km per hour.
1.8. $\tau_A/\tau_B = \eta/\sqrt{\eta^2 - 1} = 1.8$.
1.9. $\theta = \arcsin (1/n) + \pi/2 = 120°$.
1.10. $l = v_0 t \sqrt{2(1 - \sin\theta)} = 22$ m.
1.11. $l = (v_1 + v_2)\sqrt{v_1 v_2}/g = 2.5$ m.
1.12. $t = 2a/3v$.
1.13. It is seen from Fig. 1*a* that the points A and B converge with velocity $v - u \cos\alpha$, where the angle α varies with time. The

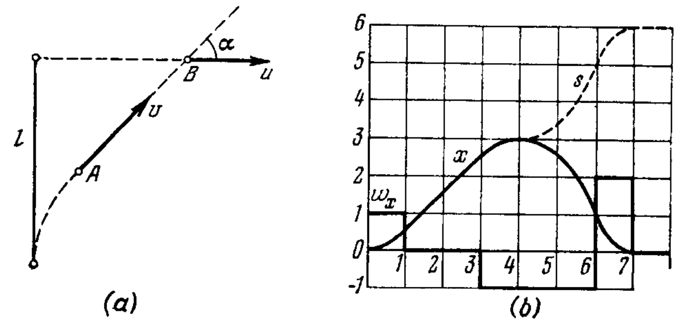


Fig. 1.

points merge provided the following two conditions are met:

$$\int_0^\tau (v - u\cos\alpha)\, dt = l, \qquad \int_0^\tau v\cos\alpha\, dt = u\tau,$$

where τ is the sought time. It follows from these two equations that

$$\tau = vl/(v^2 - u^2).$$

1.14. $x_1 - x_2 = l - w\tau\,(t + \tau/2) = 0.24$ km. Toward the train with velocity $V = 4.0$ m/s.
1.15. (a) 0.7 s; (b) 0.7 and 1.3 m respectively.

1.16. $t_m = \frac{v_1 l_1 + v_2 l_2}{v_1^2+v_2^2}$, $l_{min} = \frac{|l_1 v_2 - l_2 v_1|}{\sqrt{v_1^2+v_2^2}}$.

1.17. $CD = l/\sqrt{\eta^2-1}$.

1.18. See Fig. 1b.

1.19. (a) $\langle v\rangle = \pi R/\tau = 50$ cm/s; (b) $|\langle \mathbf{v}\rangle| = 2R/\tau = 32$ cm/s; (c) $|\langle \mathbf{w}\rangle| = 2\pi R/\tau^2 = 10$ cm/s^2.

1.20. (a) $\mathbf{v} = \mathbf{a}(1-2\alpha t)$, $\mathbf{w} = -2\alpha\mathbf{a} = \text{const}$; (b) $\Delta t = 1/\alpha$, $s = a/2\alpha$.

1.21. (a) $x = v_0 t(1 - t/2\tau)$, $x = 0.24$, 0 and -4.0 m; (b) 1.1, 9 and 11 s; (c) $s = \begin{cases}(1-t/2\tau)v_0 t & \text{for } t\leqslant\tau,\\ [1+(1-t/\tau)^2]v_0t/2 & \text{for } t\geqslant\tau.\end{cases}$ 24 and 34 cm respectively.

1.22. (a) $v = \alpha^2 t/2$, $w=\alpha^2/2$; (b) $\langle v\rangle = \alpha\sqrt{s}/2$.

1.23. (a) $s = (^2/_3a) v_0^{3/2}$; (b) $t = 2\sqrt{v_0}/a$.

1.24. (a) $y = -x^2 b/a^2$; (b) $\mathbf{v} = a\mathbf{i} - 2bt\mathbf{j}$, $\mathbf{w} = -2b\mathbf{j}$, $v = \sqrt{a^2+4b^2t^2}$, $w = 2b$; (c) $\tan\alpha = a/2bt$; (d) $\langle\mathbf{v}\rangle = a\mathbf{i} - bt\mathbf{j}$, $|\langle\mathbf{v}\rangle| = \sqrt{a^2+b^2t^2}$.

1.25. (a) $y = x - x^2\alpha/a$; (b) $v = a\sqrt{1+(1-2\alpha t)^2}$, $w = 2\alpha a = \text{const}$; (c) $t_0 = 1/\alpha$.

1.26. (a) $s = a\omega\tau$; (b) $\pi/2$.

1.27. $v_0 = \sqrt{(1+a^2)\,w/2b}$.

1.28. (a) $\mathbf{r} = \mathbf{v}_0 t + \mathbf{g}t^2/2$; (b) $\langle\mathbf{v}\rangle_t = \mathbf{v}_0 + \mathbf{g}t/2$, $\langle\mathbf{v}\rangle = \mathbf{v}_0 - \mathbf{g}\,(\mathbf{v}_0\mathbf{g})/g^2$.

1.29. (a) $\tau = 2\,(v_0/g)\sin\alpha$; (b) $h = (v_0^2/2g)\sin^2\alpha$, $l = (v_0^2/g)\sin 2\alpha$, $\alpha = 76°$; (c) $y = x\tan\alpha - (g/2v_0^2\cos^2\alpha)\,x^2$; (d) $R_1 = v_0^2/g\cos\alpha$, $R_2 = (v_0^2/g)\cos^2\alpha$.

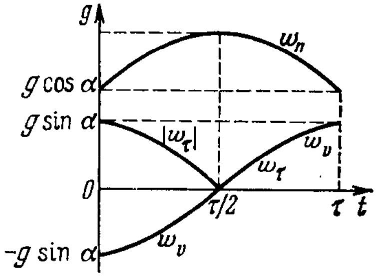


Fig. 2.

1.30. See Fig. 2.

1.31. $l = 8h\sin\alpha$.

1.32. 0.41 or 0.71 min later, depending on the initial angle.

1.33. $\Delta t = \frac{2v_0}{g}\,\frac{\sin(\theta_1-\theta_2)}{\cos\theta_1+\cos\theta_2} = 11$ s.

1.34. (a) $x = (a/2v_0)\,y^2$; (b) $w = av_0$, $w_\tau = a^2y/\sqrt{1+(ay/v_0)^2}$, $w_n = av_0/\sqrt{1+(ay/v_0)^2}$.

1.35. (a) $y = (b/2a)\,x^2$; (b) $R = v^2/w_n = v^2/\sqrt{w^2-w_\tau^2} = (a/b)\,[1+(xb/a)^2]^{3/2}$.

1.36. $v = \sqrt{2ax}$.

1.37. $w = a\sqrt{1+(4\pi n)^2} = 0.8$ m/s^2.

1.38. (a) $v = v_0/(1+v_0t/R) = v_0 e^{-s/R}$; (b) $w = \sqrt{2}\,v_0^2/Re^{2s/R} = \sqrt{2}\,v^2/R$.

1.39. $\tan\alpha = 2s/R$.

1.40. (a) $w_0 = a^2\omega^2/R = 2.6$ m/s^2, $w_a = a\omega^2 = 3.2$ m/s^2; (b) $w_{min} = a\omega^2\sqrt{1-(R/2a)^2} = 2.5$ m/s^2, $l_m = \pm a\sqrt{1-R^2/2a^2} = \pm 0.37$ m.

1.41. $R = a^3/2bs$, $w = a\sqrt{1+(4bs^2/a^3)^2}$.

1.42. (a) $w = 2av^2$, $R = {}^1/_2a$; (b) $w = bv^2/a^2$, $R = a^2/b$.

1.43. $v = 2R\omega = 0.40$ m/s, $w = 4R\omega^2 = 0.32$ m/s^2.

1.44. $w = (v/t)\sqrt{1+4a^2t^4} = 0.7$ m/s^2.

1.45. $\omega = 2\pi nv/l = 2.0\cdot10^3$ rad/s.

1.46. (a) $\langle\omega\rangle = 2a/3 = 4$ rad/s, $\langle\beta\rangle = \sqrt{3ab} = 6$ rad/s^2; (b) $\beta = 2\sqrt{3ab} = 12$ rad/s^2.

1.47. $t = \sqrt[3]{(4/a)\tan\alpha} = 7$ s.

1.48. $\langle\omega\rangle = \omega_0/3$.

1.49. (a) $\varphi = (1-e^{-at})\,\omega_0/a$; (b) $\omega = \omega_0 e^{-at}$.

1.50. $\omega_z = \pm\sqrt{2\beta_0\sin\varphi}$, see Fig. 3.

1.51. (a) $y = v^2/\beta x$ (hyperbola); (b) $y = \sqrt{2wx}/\omega$ (parabola).

1.52. (a) $w_A = v^2/R = 2.0$ m/s^2, the vector $\mathbf{w}_A$ is permanently directed to the centre of the wheel; (b) $s = 8R = 4.0$ m.

1.53. (a) $v_A = 2wt = 10.0$ cm/s, $v_B = \sqrt{2}\,wt = 7.1$ cm/s, $v_O = 0$; (b) $w_A = 2w\sqrt{1+(wt^2/2R)^2} = 5.6$ cm/s^2, $w_B = w\sqrt{1+(1-wt^2/R)^2} = 2.5$ cm/s^2, $w_O = w^2t^2/R = 2.5$ cm/s^2.

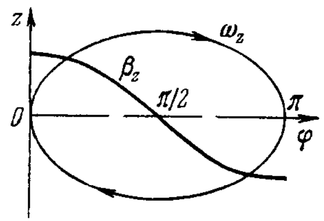


Fig. 3.

1.54. $R_A = 4r$, $R_B = 2\sqrt{2}\,r$.

1.55. $\omega = \sqrt{\omega_1^2+\omega_2^2} = 5$ rad/s, $\beta = \omega_1\omega_2 = 12$ rad/s^2.

1.56. (a) $\omega = at\sqrt{1+(bt/a)^2} = 8$ rad/s, $\beta = a\sqrt{1+(2bt/a)^2} = 1.3$ rad/s^2; (b) 17°.

1.57. (a) $\omega = v/R\cos\alpha = 2.3$ rad/s, 60°; (b) $\beta = (v/R)^2\tan\alpha = 2.3$ rad/s^2.

1.58. $\omega = \omega_0\sqrt{1+(\beta_0t/\omega_0)^2} = 0.6$ rad/s, $\beta = \beta_0\sqrt{1+\omega_0^2t^2} = 0.2$ rad/s^2.

1.59. $\Delta m = 2mw/(g+w)$.

1.60. $\mathbf{w} = \frac{m_0 - k(m_1+m_2)}{m_0+m_1+m_2}\,\mathbf{g}$, $T = \frac{(1+k)\,m_0}{m_0+m_1+m_2}\,m_2 g$.

1.61. (a) $F = \frac{(k_1-k_2)\,m_1m_2g\cos\alpha}{m_1+m_2}$; (b) $\tan\alpha_{min} = \frac{k_1m_1+k_2m_2}{m_1+m_2}$.

1.62. $k = [(\eta^2-1)/(\eta^2+1)]\tan\alpha = 0.16$.

1.63. (a) $m_2/m_1 > \sin\alpha + k\cos\alpha$; (b) $m_2/m_1 < \sin\alpha - k\cos\alpha$; (c) $\sin\alpha - k\cos\alpha < m_2/m_1 < \sin\alpha + k\cos\alpha$.

1.64. $\mathbf{w}_2 = \mathbf{g}\,(\eta - \sin\alpha - k\cos\alpha)/(\eta+1) = 0.05\,\mathbf{g}$.

1.65. When $t\leqslant t_0$, the accelerations $w_1 = w_2 = at/(m_1+m_2)$; when $t\geqslant t_0$ $w_1 = kgm_2/m_1$, $w_2 = (at - km_2g)/m_2$. Here $t_0 = kgm_2\,(m_1+m_2)/am$. See Fig. 4.

1.66. $\tan 2\alpha = -1/k$, $\alpha = 49°$; $t_{min} = 1.0$ s.

1.67. $\tan\beta = k$; $T_{min} = mg\,(\sin\alpha + k\cos\alpha)/\sqrt{1+k^2}$.

1.68. (a) $v = \frac{mg^2\cos\alpha}{2a\sin^2\alpha}$; (b) $s = \frac{m^2g^3\cos\alpha}{6a^2\sin^3\alpha}$.

1.69. $v = \sqrt{(2g/3a)\sin\alpha}$.

1.70. $\tau = \sqrt{2l/(3w+kg)}$.

1.71. (a) $\mathbf{w}_1 = \dfrac{(m_1-m_2)\,\mathbf{g}+2m_2\mathbf{w}_0}{m_1+m_2}$, $\quad \mathbf{w}'_1 = \dfrac{m_1-m_2}{m_1+m_2}(\mathbf{g}-\mathbf{w}_0)$;
(b) $\mathbf{F} = \dfrac{4m_1m_2}{m_1+m_2}(\mathbf{g}-\mathbf{w}_0)$.

1.72. $\mathbf{w} = 2\mathbf{g}\,(2\eta-\sin\alpha)/(4\eta+1)$.

1.73. $\mathbf{w}_1 = \dfrac{4m_1m_2+m_0\,(m_1-m_2)}{4m_1m_2+m_0\,(m_1+m_2)}\,\mathbf{g}$.

1.74. $F_{fr} = 2lmM/(M-m)\,t^2$.

1.75. $t = \sqrt{2l\,(4+\eta)/3g\,(2-\eta)} = 1.4$ s.

1.76. $H = 6h\eta\,/(\eta+4) = 0.6$ m.

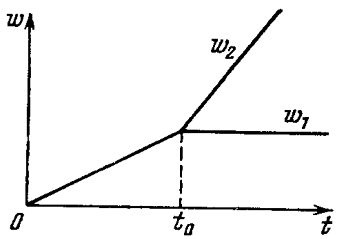


Fig. 4.

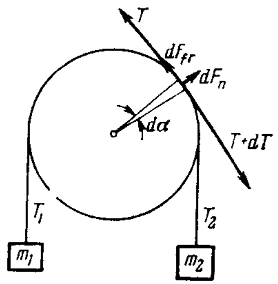


Fig. 5.

1.77. $w_A = g/(1+\eta\cot^2\alpha)$, $w_B = g/(\tan\alpha+\eta\cot\alpha)$.

1.78. $w = g\sqrt{2}/(2+k+M/m)$.

1.79. $w_{min} = g\,(1-k)/(1+k)$.

1.80. $w_{max} = g\,(1+k\cot\alpha)/(\cot\alpha-k)$.

1.81. $w = g\sin\alpha\cos\alpha/(\sin^2\alpha+m_1/m_2)$.

1.82. $w = \dfrac{mg\sin\alpha}{M+2m\,(1-\cos\alpha)}$.

1.83. (a) $|\langle\mathbf{F}\rangle| = 2\sqrt{2}\,mv^2/\pi R$; (b) $|\langle\mathbf{F}\rangle| = mw_\tau$.

1.84. 2.1, 0.7 and 1.5 kN.

1.85. (a) $w = g\sqrt{1+3\cos^2\theta}$, $\quad T = 3mg\cos\theta$;
(b) $T = mg\sqrt{3}$; (c) $\cos\theta = 1/\sqrt{3}$, $\theta = 54.7°$.

1.86. $\approx 53°$.

1.87. $\theta = \arccos(2/3) \approx 48°$, $\quad v = \sqrt{2gR/3}$.

1.88. $\varepsilon = 1/(\varkappa/m\omega^2-1)$. Is independent of the rotation direction.

1.89. $r = R/2$, $\quad v_{max} = {}^1/_2\sqrt{kgR}$.

1.90. $s = {}^1/_2R\sqrt{(kg/w_\tau)^2-1} = 60$ m.

1.91. $v \leqslant \alpha\sqrt{kg/a}$.

1.92. $T = (\cot\theta+\omega^2R/g)\,mg/2\pi$.

1.93. (a) Let us examine a small element of the thread in contact with the pulley (Fig. 5). Since the element is weightless, $dT = dF_{fr} = k\,dF_n$ and $dF_n = T\,d\alpha$. Hence, $dT/T = k\,d\alpha$. Integrating this equation, we obtain $k = (\ln\eta_0)/\pi$; (b) $w = g\,(\eta-\eta_0)/(\eta+\eta_0)$.

1.94. $F = (mv_0^2/R)\cos^2\alpha$.

1.95. $\mathbf{F} = -m\omega^2\mathbf{r}$, where $\mathbf{r}$ is the radius vector of the particle relative to the origin of coordinates; $F = m\omega^2\sqrt{x^2+y^2}$.

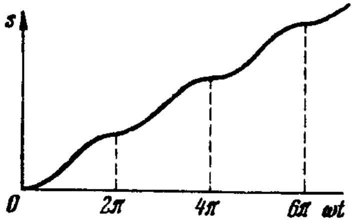


Fig. 6.

1.96. (a) $\Delta\mathbf{p} = m\mathbf{g}t$; (b) $|\Delta\mathbf{p}| = -2m(\mathbf{v}_0\mathbf{g})/g$.

1.97. (a) $\mathbf{p} = \mathbf{a}\tau^3/6$; (b) $s = a\tau^4/12m$.

1.98. $s = (\omega t-\sin\omega t)\,F_0/m\omega^2$, see Fig. 6.

1.99. $t = \pi/\omega$; $\quad s = 2F_0/m\omega^2$; $\quad v_{max} = F_0/m\omega$.

1.100. (a) $v = v_0e^{-tr/m}$, $\quad t\to\infty$; $\quad$ (b) $v = v_0 - sr/m$, $\quad s_{total} = \dfrac{mv_0}{r}$; (c) $\langle v\rangle = v_0\dfrac{\eta-1}{\eta\ln\eta}$.

1.101. $t = \dfrac{h\,(v_0-v)}{v_0v\ln(v_0/v)}$.

1.102. $s = \dfrac{2}{a}\tan\alpha$, $\quad v_{max} = \sqrt{\dfrac{g}{a}\sin\alpha\tan\alpha}$.

Instruction. To reduce the equation to the form which is convenient to integrate, the acceleration must be represented as dv/dt and then a change of variables made according to the formula $dt = dx/v$.

1.103. $s = {}^1/_6\,a\,(t-t_0)^3/m$, where $t_0 = kmg/a$ is the moment of time at which the motion starts. At $t \leqslant t_0$ the distance is $s = 0$.

1.104. $v' = v_0/\sqrt{1+kv_0^2/mg}$.

1.105. (a) $v = (2F/m\omega)\,|\sin(\omega t/2)|$; (b) $\Delta s = 8F/m\omega^2$, $\langle v\rangle = 4F/\pi m\omega$.

1.106. $v = v_0/(1+\cos\varphi)$. **Instruction.** Here $w_\tau = -w_x$, and therefore $v = -v_x + \text{const}$. From the initial condition it follows that const $= v_0$. Besides, $v_x = v\cos\varphi$.

1.107. $w = [1-\cos(l/R)]\,Rg/l$.

1.108. (a) $v = \sqrt{2gR/3}$; (b) $\cos\theta_0 = \dfrac{2+\eta\sqrt{5+9\eta^2}}{3\,(1+\eta^2)}$, where $\eta = w_0/g$, $\theta_0 \approx 17°$.

1.109. For $n < 1$, including negative values.

1.110. When $\omega^2R > g$, there are two steady equilibrium positions: $\theta_1 = 0$ and $\theta_2 = \arccos(g/\omega^2R)$. When $\omega^2R < g$, there is only one equilibrium position: $\theta_1 = 0$. As long as there is only one lower equilibrium position, it is steady. Whenever the second equilibrium position appears (which is permanently steady) the lower one becomes unsteady.

1.111. $h \approx (\omega s^2/v)\sin\varphi = 7$ cm, where ω is the angular velocity of the Earth's rotation.

1.112. $F = m\sqrt{g^2+\omega^4r^2+(2v'\omega)^2} = 8$ N.

1.113. $F_{cor} = 2m\omega^2r\sqrt{1+(v_0/\omega r)^2} = 2.8$ N.

1.114. (a) $w' = \omega^2 R$; (b) $F_{in} = m\omega^2 r\sqrt{(2R/r)^2 - 1}$.

1.115. $F_{cf} = m\omega^2 R\sqrt{5/9} = 8$ N, $F_{cor} = {}^2/_3 m\omega^2 R\sqrt{5 + 8g/3\omega^2 R} = 17$ N.

1.116. (a) $F = 2mv\omega \sin\varphi = 3.8$ kN, on the right rail; (b) along the parallel from the east to the west with the velocity $v = \frac{1}{2}\omega R\cos\varphi \approx 420$ km per hour. Here ω is the angular rotation velocity of the Earth about its axis, R is its radius.

1.117. Will deviate to the east by the distance $x \approx \frac{2}{3}\omega h\sqrt{2h/g} = 24$ cm. Here ω is the angular velocity of the Earth's rotation about its axis.

1.118. $A = \mathbf{F}(\mathbf{r}_2 - \mathbf{r}_1) = -17$ J.

1.119. $A = ma^4t^2/8$.

1.120. $F = 2as\sqrt{1 + (s/R)^2}$.

1.121. $A = mg(h + kl)$.

1.122. $A = -kmgl/(1 - k\cot\alpha) = -0.05$ J

1.123. $F_{min} = (m_1 + m_2/2)\,kg$.

1.124. $A = -(1 - \eta)\,\eta mgl/2 = -1.3$ J.

1.125. $\langle P\rangle = 0$, $P = mg(gt - v_0\sin\alpha)$.

1.126. $P = mRat$, $\langle P\rangle = mRat/2$.

1.127. (a) $\langle P\rangle = -kmgv_0/2 = -2$ W; (b) $P_{max} = -{}^1/_2 mv_0^2\sqrt{\alpha g}$.

1.128. $A = {}^1/_2 m\omega^2(r_2^2 - r_1^2) = 0.20$ J.

1.129. $A_{min} = {}^1/_2 k(\Delta l)^2$, where $k = k_1k_2/(k_1 + k_2)$.

1.130. $A = 3mg/4a$, $\Delta U = mg/2a$.

1.131. (a) $r_0 = 2a/b$, steady; (b) $F_{max} = b^3/27a^2$, see Fig. 7.

1.132. (a) No; (b) ellipses whose ratio of semiaxes is $a/b = \sqrt{\beta/\alpha}$; also ellipses, but with $a/b = \beta/\alpha$.

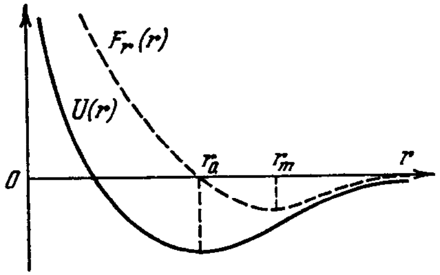


Fig. 7.

1.133. The latter field is potential.

1.134. $s = v_0^2/2g(\sin\alpha + k\cos\alpha)$, $A = -mv_0^2k/2(k + \tan\alpha)$.

1.135. $h = H/2$; $s_{max} = H$.

1.136. $v = {}^2/_3\sqrt{gh/3}$.

1.137. $v_{min} = \sqrt{5gl}$; $T = 3mg$.

1.138. $t = l_0^2/2v_0R$.

1.139. $\Delta l = (1 + \sqrt{1 + 2kl/mg})\,mg/k$.

1.140. $v = \sqrt{19gl_0/32} = 1.7$ m/s.

1.141. $A = \dfrac{kmgl_0}{2}\,\dfrac{1 - \cos\theta}{(\sin\theta + k\cos\theta)\cos\theta} = 0.09$ J.

1.142. $A = \varkappa l_0^2\eta(1 + \eta)/2(1 - \eta)^2$, where $\eta = m\omega^2/\varkappa$.

1.143. $\mathbf{w}_C = \mathbf{g}(m_1 - m_2)^2/(m_1 + m_2)^2$.

1.145. $r = (g/\omega^2)\tan\theta = 0.8$ cm, $T = mg/\cos\theta = 5$ N.

1.146. (a) $F_{fr} = mg[\sin\alpha + (\omega^2 l/g)\cos\alpha] = 6$ N. (b) $\omega < \sqrt{g(k - \tan\alpha)/l(1 + k\tan\alpha)} = 2$ rad/s.

1.147. (a) $\mathbf{V} = (m_1\mathbf{v}_1 + m_2\mathbf{v}_2)/(m_1 + m_2)$; (b) $T = \mu(\mathbf{v}_1 - \mathbf{v}_2)^2/2$, where $\mu = m_1m_2(m_1 + m_2)$.

1.148. $E = \tilde{E} + mV^2/2$.

1.149. $\tilde{E} = \mu(v_1^2 + v_2^2)/2$, where $\mu = m_1m_2/(m_1 + m_2)$.

1.150. $\mathbf{p} = \mathbf{p}_0 + m\mathbf{g}t$, where $\mathbf{p}_0 = m\mathbf{v}_1 + m_2\mathbf{v}_2$, $m = m_1 + m_2$; $\mathbf{r}_C = \mathbf{v}_0 t + \mathbf{g}t^2/2$, where $\mathbf{v}_0 = (m_1\mathbf{v}_1 + m_2\mathbf{v}_2)/(m_1 + m_2)$.

1.151. $v_C = x\sqrt{\varkappa m_2/(m_1 + m_2)}$.

1.152. (a) $l_{max} = l_0 + F/\varkappa$, $l_{min} = l_0$; (b) $l_{max} = l_0 + 2m_1F/\varkappa(m_1 + m_2)$, $l_{min} = l_0$.

1.153. (a) $\Delta l > 3mg/\varkappa$; (b) $h = (1 + \varkappa\Delta l/mg)^2 mg/8\varkappa = 8mg/\varkappa$.

1.154. $\mathbf{v}_1 = -m\mathbf{v}/(M - m)$, $\mathbf{v}_2 = M\mathbf{v}/(M - m)$.

1.155. $\mathbf{v}_{rear} = \mathbf{v}_0 - \dfrac{m}{M + m}\mathbf{u}$; $\mathbf{v}_{form} = \mathbf{v}_0 + \dfrac{mM}{(M + m)^2}\mathbf{u}$.

1.156. (1) $\mathbf{v}_1 = -\dfrac{2m}{M + 2m}\mathbf{u}$; (2) $\mathbf{v}_2 = -\dfrac{m(2M + 3m)}{(M + m)(M + 2m)}\mathbf{u}$, $v_2/v_1 = 1 + m/2(M + m) > 1$.

1.158. $\Delta p = m\sqrt{2gh}\,(\eta + 1)/(\eta - 1) = 0.2$ kg·m/s.

1.159. (a) $\mathbf{l} = -\dfrac{m}{M + m}\mathbf{l}'$; (b) $\mathbf{F} = -\dfrac{mM}{M + m}\dfrac{d\mathbf{v}'}{dt}$.

1.160. $\mathbf{l} = m\mathbf{l}'/2M$.

1.161. $\tau = (p\cos\alpha - M\sqrt{2gl\sin\alpha})/Mg\sin\alpha$.

1.162. (a) $v = (2M/m)\sqrt{gl}\sin(\theta/2)$; (b) $\eta \approx 1 - m/M$.

1.163. $h = Mv^2/2g(M + m)$.

1.164. (1) $A = -\mu gh$, where $\mu = mM/(m + M)$; (2) Yes.

1.166. $\mathbf{v} = 1.0\mathbf{i} + 2.0\mathbf{j} - 4.0\mathbf{k}$, $v \approx 4.6$ m/s.

1.167. $\Delta T = -\mu(\mathbf{v}_1 - \mathbf{v}_2)^2/2$, where $\mu = m_1m_2/(m_1 + m_2)$.

1.168. (a) $\eta = 2m_1/(m_1 + m_2)$; (b) $\eta = 4m_1m_2/(m_1 + m_2)^2$.

1.169. (a) $m_1/m_2 = 1/3$; (b) $m_1/m_2 = 1 + 2\cos\Theta = 2.0$.

1.170. $\eta = {}^1/_2\cos^2\alpha = 0.25$.

1.171. $v_{max} = v(1 + \sqrt{2(\eta - 1)}) = 1.0$ km per second.

1.172. Will continue moving in the same direction, although this time with the velocity $v' = (1 - \sqrt{1 - 2\eta})\,v/2$. For $\eta \ll 1$ the velocity $v' \approx \eta v/2 = 5$ cm/s.

1.173. $\Delta T/T = (1 + m/M)\tan^2\theta + m/M - 1 = -40\%$.

1.174. (a) $p = \mu\sqrt{v_1^2 + v_2^2}$; (b) $T = {}^1/_2\mu(v_1^2 + v_2^2)$. Here $\mu = m_1m_2/(m_1 + m_2)$.

1.175. $\sin\theta_{max} = m_2/m_1$.

1.176. $\mathbf{v}' = -\mathbf{v}(2 - \eta^2)/(6 - \eta^2)$. Respectively at smaller η, equal, or greater than $\sqrt{2}$.

1.178. Suppose that at a certain moment t the rocket has the mass m and the velocity $\mathbf{v}$ relative to the reference frame employed. Consider the inertial reference frame moving with the same velocity as the rocket has at a *given* moment. In this reference frame the momentum increment that the system "rocket-ejected portion of gas"

acquires during the time dt is equal to $d\mathbf{p} = m\,d\mathbf{v} + \mu\,dt\cdot\mathbf{u} = \mathbf{F}\,dt$. What follows is evident.

1.179. $\mathbf{v} = -\mathbf{u}\ln(m_0/m)$.

1.180. $m = m_0 e^{-wt/u}$.

1.181. $\alpha = (u/v_0)\ln(m_0/m)$.

1.182. $\mathbf{v} = \frac{\mathbf{F}}{\mu}\ln\frac{m_0}{m_0-\mu t}$, $\quad \mathbf{w} = \frac{\mathbf{F}}{m_0-\mu t}$.

1.183. $\mathbf{v} = \mathbf{F}t/m_0(1+\mu t/m_0)$, $\mathbf{w} = \mathbf{F}/m_0(1+\mu t/m_0)^2$.

1.184. $v = \sqrt{2gh\ln(l/h)}$.

1.185. $\mathbf{N} = 2\mathbf{b}\sqrt{a/b}$.

1.186. $M = {}^1/_2 mgv_0t^2\cos\alpha$; $\quad M = (mv_0^3/2g)\sin^2\alpha\cos\alpha = 37\ \text{kg}\cdot\text{m}^2/\text{s}$.

1.187. (a) Relative to all points of the straight line drawn at right angles to the wall through the point O;
(b) $|\Delta\mathbf{M}| = 2\ mvl\cos\alpha$.

1.188. Relative to the centre of the circle.
$|\Delta\mathbf{M}| = 2\sqrt{1-(g/\omega^2 l)^2}\,mgl/\omega$.

1.189. $|\Delta\mathbf{M}| = hmV$.

1.190. $M = m\omega v_0^2 t^2$.

1.191. $m = 2kr_1^2/v_2^2$.

1.192. $v_0 = \sqrt{2gl/\cos\theta}$.

1.193. $F = m\omega_0^2 r_0^4/r^3$.

1.194. $M_z = Rmgt$.

1.195. $M = Rmgt\sin\alpha$. Will not change.

1.196. $\mathbf{M}' = \mathbf{M} - [\mathbf{r}_0\mathbf{p}]$. In the case when $\mathbf{p} = 0$, i.e. in the frame of the centre of inertia.

1.198. $\tilde{M} = {}^1/_3\ lmv_0$.

1.199. $\varepsilon_{max} \approx mv_0^2/\varkappa l_0^2$. The problem is easier to solve in the frame of the centre of inertia.

1.200. $T = 2\pi\gamma M/v^3 = 225$ days.

1.201. (a) 5.2 times; (b) 13 km/s, $2.2\cdot10^{-4}$ m/s^2.

1.202. $T = \pi\sqrt{(r+R)^3/2\gamma M}$. It is sufficient to consider the motion along the circle whose radius is equal to the major semi-axis of the given ellipse, i.e. $(r+R)/2$, since in accordance with Kepler's laws the period of revolution is the same.

1.203. Falling of the body on the Sun can be considered as the motion along a very elongated (in the limit, degenerated) ellipse whose major semi-axis is practically equal to the radius R of the Earth's orbit. Then from Kepler's laws, $(2\tau/T)^2 = [(R/2)/R]^3$, where τ is the falling time (the time needed to complete half a revolution along the elongated ellipse), T is the period of the Earth's revolution around the Sun. Hence, $\tau = T/4\sqrt{2} = 65$ days.

1.204. Will not change.

1.205. $l = \sqrt[3]{\gamma M(T/2\pi)^2}$.

1.206. (a) $U = -\gamma m_1 m_2/r$; $\quad$ (b) $U = -\gamma(mM/l)\ln(1+l/a)$; $F = \gamma mM/a(a+l)$.

1.207. $M = m\sqrt{2\gamma m_S r_1 r_2/(r_1+r_2)}$, where m_S is the mass of the Sun.

1.208. $E = T + U = -\gamma m m_S/2a$, where m_S is the mass of the Sun.

1.209. $r_m = \frac{r_0}{2-\eta}\left[1 \pm \sqrt{1-(2-\eta)\eta\sin^2\alpha}\right]$, where $\eta = r_0 v_0^2/\gamma m_S$, m_S being the mass of the Sun.

1.210. $r_{min} = (\gamma m_S/v_0^2)\left[\sqrt{1+(lv_0^2/\gamma m_S)^2}-1\right]$, where m_S is the mass of the Sun.

1.211. (a) First let us consider a thin spherical layer of radius ρ and mass δM. The energy of interaction of the particle with an elementary belt δS of that layer is equal to (Fig. 8)

$$dU = -\gamma\,(m\delta M/2l)\sin\theta\,d\theta. \qquad (*)$$

According to the cosine theorem in the triangle OAP $l^2 = \rho^2 + r^2 - 2\rho r\cos\theta$. Having determined the differential of this expression, we can reduce Eq. (*) to the form that is convenient for integration. After integrating over the whole layer we obtain $\delta U = -\gamma m\,\delta M/r$. And finally, integrating over all layers of the sphere, we obtain $U = -\gamma mM/r$; (b) $F_r = -\partial U/\partial r = -\gamma mM/r^2$.

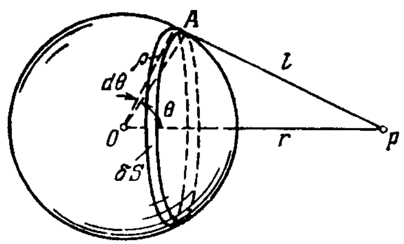


Fig. 8.

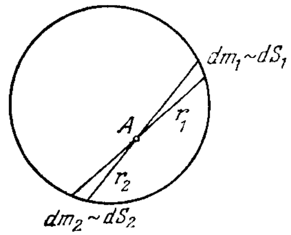


Fig. 9.

1.212. First let us consider a thin spherical layer of substance (Fig. 9). Construct a cone with a small angle of taper and the vertex at the point A. The ratio of the areas cut out by the cone in the layer is $dS_1 : dS_2 = r_1^2 : r_2^2$. The masses of the cut volumes are proportional to their areas. Therefore these volumes will attract the particle A with forces equal in magnitude and opposite in direction. What follows is obvious.

1.213. $A = -{}^3/_2\gamma mM/R$.

1.214. $\mathbf{G} = \begin{cases} -(\gamma M/R^3)\,\mathbf{r} & \text{for } r \leqslant R, \\ -(\gamma M/r^3)\,\mathbf{r} & \text{for } r \geqslant R; \end{cases}$

$\varphi = \begin{cases} -{}^3/_2(1-r^2/3R^2)\,\gamma M/R & \text{for } r \leqslant R, \\ -\gamma M/r & \text{for } r \geqslant R. \end{cases}$ See Fig. 10.

1.215. $\mathbf{G} = -{}^4/_3\pi\gamma\rho\mathbf{l}$. The field inside the cavity is uniform.

1.216. $p = {}^3/_8(1 - r^2/R^2)\gamma M^2/\pi R^4$. About $1.8\cdot10^6$ atmospheres.

1.217. (a) Let us subdivide the spherical layer into small elements, each of mass δm. In this case the energy of interaction of each element with all others is $\delta U = -\gamma m\,\delta m/R$. Summing over all

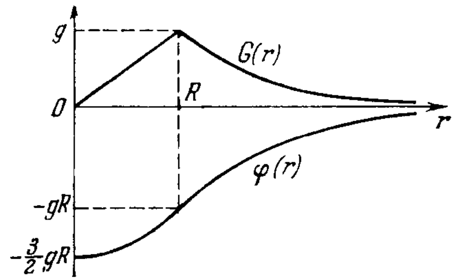


Fig. 10.

elements and taking into account that each pair of interacting elements appears twice in the result, we obtain $U = -\nu m^2/2R$; (b) $U = -3\gamma m^2/5R$.

1.218. $\Delta t \approx \frac{2\pi}{\sqrt{\gamma M}} \frac{r^{3/2}}{3\Delta r/2r+\delta} = \begin{cases} 4.5 \text{ days } (\delta=0), \\ 0.84 \text{ hour } (\delta=2). \end{cases}$

1.219. $w_1 : w_2 : w_3 = 1 : 0.0034 : 0.0006$.

1.220. 32 km; 2650 km.

1.221. $h = R/(2gR/v_0^2 - 1)$.

1.222. $h = R\,(gR/v^2 - 1)$.

1.223. $r = \sqrt[3]{\gamma M\,(T/2\pi)^2} = 4.2\cdot 10^4$ km, where M and T are the mass of the Earth and its period of revolution about its own axis respectively; 3.1 km/s, 0.22 m/s².

1.224. $M = (4\pi^2R^3/\gamma T^2)\,(1 + T/\tau)^2 = 6\cdot 10^{24}$ kg, where T is the period of revolution of the Earth about its own axis.

1.225. $v' = \frac{2\pi R}{T} + \sqrt{\frac{\gamma M}{R}} = 7.0$ km/s, $w' = \frac{\gamma M}{R^2}\left(1 + \frac{2\pi R}{T} \times \sqrt{\frac{R}{\gamma M}}\right) = 4.9$ m/s². Here M is the mass of the Earth, T is its period of revolution about its own axis.

1.226. 1.27 times.

1.227. The decrease in the total energy E of the satellite over the time interval dt is equal to $-dE = Fv\,dt$. Representing E and v as functions of the distance r between the satellite and the centre of the Moon, we can reduce this equation to the form convenient for integration. Finally, we get $\tau \approx (\sqrt{\eta} - 1)\,m/\alpha\sqrt{gR}$

1.228. $v_1 = 1.67$ km/s, $v_2 = 2.37$ km/s.

1.229. $\Delta v = \sqrt{\gamma M/R}\,(1 - \sqrt{2}) = -0.70$ km/s, where M and R are the mass and the radius of the Moon.

1.230. $\Delta v = \sqrt{gR}\,(\sqrt{2} - 1) = 3.27$ km/s, where g is the standard free-fall acceleration, R is the radius of the Earth.

1.231. $r = nR/(1 + \sqrt{\eta}) = 3.8\cdot 10^4$ km.

1.232. $A \approx \gamma m\,(M_1/R_1 + M_2/R_2) = 1.3\cdot 10^8$ kJ, where M and R are the mass and the radius of the Earth and the Moon.

1.233. $v_3 \approx \sqrt{2v_1^2 + (\sqrt{2} - 1)^2\,V_1^2} \approx 17$ km/s. Here $v_1^2 = \gamma M_E/R$, M_E and R are the mass and the radius of the Earth; $V_1^2 = \gamma M_S/r$, M_S is the mass of the Sun, r is the radius of the Earth's orbit.

1.234. $l = 2aF_2/mw = 1.0$ m.

1.235. $\mathbf{N} = (aB - bA)\,\mathbf{k}$, where $\mathbf{k}$ is the unit vector of the z axis; $l = |aB - bA|/\sqrt{A^2 + B^2}$.

1.236. $l = |aA - bB|/\sqrt{A^2 + B^2}$.

1.237. $F_{res} = 2F$. This force is parallel to the diagonal AC and is applied at the midpoint of the side BC.

1.238. (a) $I = {}^1/_3 ml^2$; (b) $I = {}^1/_3 m\,(a^2 + b^2)$.

1.239. (a) $I = {}^1/_2\,\pi\rho b R^4 = 2.8$ g·m²; (b) $I = {}^3/_{10}\,mR^2$.

1.240. $I = {}^1/_4 mR^2$.

1.241. $I = (37/72)\,mR^2 = 0.15$ kg·m².

1.242. $I = {}^2/_3\,mR^2$.

1.243. (a) $\omega = gt/R\,(1 + M/2m)$; (b) $T = mg^2t^2/2(1 + M/2m)$.

1.244. $T = {}^1/_2 mg$, $w_0 = gmr^2/I$.

1.245. $\omega = \sqrt{6F \sin\varphi/ml}$.

1.246. $\beta = \frac{|m_2 - m_1|\,g}{(m_1 + m_2 + m/2)\,R}$, $\frac{T_1}{T_2} = \frac{m_1\,(m + 4m_2)}{m_2\,(m + 4m_1)}$.

1.247. $A = -\frac{(m_2 - km_1)\,km_1g^2t^2}{m + 2\,(m_1 + m_2)}$.

1.248. $n = (1 + k^2)\,\omega_0^2R/8\pi k\,(k + 1)\,g$.

1.249. $t = {}^3/_4\,\omega R/kg$.

1.250. $\langle\omega\rangle = {}^1/_3\,\omega_0$.

1.251. $\beta = 2mgx/Rl\,(M + 2m)$.

1.252. (a) $k \geqslant {}^2/_7 \tan\alpha$; (b) $T = {}^5/_{14}\,mg^2t^2 \sin^2\alpha$.

1.253. (a) $T = {}^1/_6\,mg = 13$ N, $\beta = {}^2/_3\,g/R = 5\cdot 10^2$ rad/s²; (b) $P = {}^2/_3\,mg^2t$.

1.254. $\mathbf{w}' = {}^2/_3\,(\mathbf{g} - \mathbf{w}_0)$, $\mathbf{F} = {}^1/_3\,m\,(\mathbf{g} - \mathbf{w}_0)$.

1.255. $w = g \sin\alpha/(1 + I/mr^2) = 1.6$ m/s².

1.256. $F_{max} = 3kmg/(2 - 3k)$; $w_{max} = 2kg/(2 - 3k)$.

1.257. (a) $w_x = \frac{F\,(\cos\alpha - r/R)}{m\,(1 + \gamma)}$; (b) $A = \frac{F^2t^2\,(\cos\alpha - r/R)^2}{2m\,(1 + \gamma)}$.

1.258. $T = {}^1/_{10}\,mg$.

1.259. $w = 3g\,(M + 3m)/(M + 9m + I/R^2)$.

1.260. (a) $w = \frac{F\,(3m_1 + 2m_2)}{m_1\,(m_1 + m_2)}$; (b) $T = \frac{F^2t^2\,(3m_1 + 2m_2)}{2m_1\,(m_1 + m_2)}$.

1.261. $w_1 = F/(m_1 + {}^2/_7 m_2)$; $w_2 = {}^2/_7\,w_1$.

1.262. (a) $t = {}^1/_3\,\omega_0R/kg$; (b) $A = -{}^1/_6 m\omega_0^2R^2$.

1.263. $\omega = \sqrt{10g\,(R + r)/17r^2}$.

1.264. $v_0 = \sqrt{{}^1/_3 gR\,(7\cos\alpha - 4)} = 1.0$ m/s.

1.265. $v_0 = \sqrt{8gR}$.

1.266. $T = mv^2$.

1.267. $T = {}^7/_{10}\,mv^2\,(1 + {}^2/_7 r^2/R^2)$.

1.269. $N = {}^1/_{24} m\omega^2 l^2 \sin 2\theta$.
1.270. $\cos\theta = {}^3/_2\ g/\omega^2 l$.
1.271. $\Delta x = {}^1/_2\ ka$.
1.272. $v' = \omega_0 l/\sqrt{1 + 3m/M}$.
1.273. $F = {}^9/_2 J^2/ml = 9$ N.
1.274. (a) $v' = \frac{3m - 4M}{3m + 4M} v$; (b) $F = \frac{8Mv^2}{l(1 + 4M/3m)^2}$.
1.275. (a) $v = (M/m)\sqrt{{}^2/_3 gl}\ \sin(\alpha/2)$;
(b) $\Delta p = M\sqrt{{}^1/_6 gl}\ \sin(\alpha/2)$; (c) $x \approx {}^2/_3 l$.
1.276. (a) $\omega = (1 + 2m/M)\,\omega_0$; (b) $A = {}^1/_2 m\omega_0^2 R^2 (1 + 2m/M)$.
1.277. (a) $\varphi = -\frac{2m_1}{2m_1 + m_2}\varphi'$; (b) $N_z = -\frac{m_1 m_2 R}{2m_1 + m_2}\frac{dv'}{dt}$.
1.278. (a) $\omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$; (b) $A = -\frac{I_1 I_2}{2(I_1 + I_2)}(\omega_1 - \omega_2)^2$.
1.279. $v' = v(4 - \eta)/(4 + \eta)$, $\omega = 12v/l(4 + \eta)$. For $\eta = 4$ and $\eta > 4$.
1.280. (a) $A_{90^\circ} = {}^1/_2 I_0^2\omega_0^2/(I + I_0)$, $A_{180^\circ} = 2I_0^2\omega_0^2/I$; (b) $N = I_0^2\omega_0^2/(I + I_0)$.
1.281. $\omega = \sqrt{2g/l} = 6.0$ rad/s; $F = mgl_0/l = 25$ N.
1.282. (a) $M = {}^1/_{12}\ m\omega l^2 \sin\theta$, $M_z = M\sin\theta$. (b) $|\Delta \mathbf{M}| = {}^1/_{12} m\omega l^2 \sin 2\theta$; (c) $N = {}^1/_{24} m\omega^2 l^2 \times \sin 2\theta$.
1.283. (a) $\omega' = mgl/I\omega = 0.7$ rad/s; (b) $F = m\omega'^2 l \sin\theta = 10$ mN. See Fig. 11.
1.284. $\omega = (g + w)\,l/\pi n R^2 = 3 \times 10^2$ rad/s.
1.285. $\omega' = ml\sqrt{g^2 + w^2}/I\omega = 0.8$ rad/s. The vector $\boldsymbol{\omega}'$ forms the angle $\theta = \arctan(w/g) = 6°$ with the vertical.

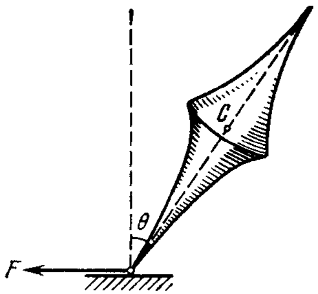


Fig. 11.

1.286. $F' = {}^2/_5\ mR^2\omega\omega'/l = 0.30$ kN.
1.287. $F_{max} = \pi m r^2 \varphi_m \omega/lT = 0.09$ kN.
1.288. $N = 2\pi n I v/R = 6$ kN·m.
1.289. $F_{add} = 2\pi n I v/Rl = 1.4$ kN. The force exerted on the outside rail increases by this value while that exerted on the inside one decreases by the same value.
1.290. $p = \alpha E\Delta T = 2.2 \cdot 10^3$ atm, where α is the thermal expansion coefficient.
1.291. (a) $p \approx \sigma_m\,\Delta r/r = 20$ atm; (b) $p \approx 2\sigma_m\,\Delta r/r = 40$ atm. Here σ_m is the glass strength.
1.292. $n = \sqrt{2\sigma_m/\rho}/\pi l = 0.8 \cdot 10^2$ rps, where σ_m is the tensile strength, and ρ is the density of copper.
1.293. $n = \sqrt{\sigma_m/\rho}/2\pi R = 23$ rps, where σ_m is the tensile strength, and ρ is the density of lead.
1.294. $x \approx l\sqrt[3]{mg/2\pi d^2 E} = 2.5$ cm
1.295. $\varepsilon = {}^1/_2 F_0/ES$.

1.296. $T = {}^1/_2 m\omega^2 l\,(1 - r^2/l^2)$, $\Delta l = {}^1/_3\ \rho\omega^2 l^3/E$, where ρ is the density of copper.
1.297. $\Delta V = (1 - 2\mu)\,Fl/E = 1.6$ mm^3, where μ is Poisson's ratio for copper.
1.298. (a) $\Delta l = {}^1/_2\ \rho g l^2/E$; (b) $\Delta V/V = (1 - 2\mu)\,\Delta l/l$, where ρ is the density, and μ is Poisson's ratio for copper.
1.299. (a) $\Delta V/V = -3(1 - 2\mu)\,p/E$; (b) $\beta = 3(1 - 2\mu)/E$.
1.300. $R = {}^1/_6\ Eh^2/\rho g l^2 = 0.12$ km, where ρ is the density of steel.
1.301. (a) Here N is independent of x and equal to N_0. Integrating twice the initial equation with regard to the boundary conditions $dy/dx\,(0) = 0$ and $y\,(0) = 0$, we obtain $y = (N_0/2EI)\,x^2$. This is the equation of a parabola. The bending deflection is $\lambda = N_0 l^2/2EI$, where $I = a^4/12$.
(b) In this case $N(x) = F(l - x)$ and $y = (F/2EI)(l - x/3)\,x^2$; $\lambda = Fl^3/3EI$, where I is of the same magnitude as in (a).
1.302. $\lambda = Fl^3/48EI$.
1.303. (a) $\lambda = {}^3/_2\ \rho g l^4/Eh^2$; (b) $\lambda = {}^5/_2\ \rho g l^4/Eh^2$. Here ρ is the density of steel.
1.304. $\lambda = {}^9/_5 \beta\rho l^5/Eh^2$, where ρ is the density of steel.
1.305. (a) $\varphi = (l/2\pi r^3\,\Delta r G)\cdot N$; (b) $\varphi = (2l/\pi r^4 G)\cdot N$.
1.306. $N = \pi\,(d_2^4 - d_1^4)\,G\varphi/32l = 0.5$ kN·m.
1.307. $P = {}^1/_2\ \pi r^4 G\varphi\omega = 17$ kW.
1.308. $N = {}^1/_2\ \beta m\,(r_2^4 - r^4)/(r_2^2 - r_1^2)$.
1.309. $U = {}^1/_2 mE\varepsilon^2/\rho = 0.04$ kJ, where ρ is the density of steel.
1.310. (a) $U = {}^1/_6 \pi r^2 l^3 \rho^2 g^2/E$; (b) $U = {}^2/_3\ \pi r^2 lE\,(\Delta l/l)^2$. Here ρ is the density of steel.
1.311. $A \approx {}^1/_6\ \pi^2 h\delta^3 E/l = 0.08$ kJ.
1.312. $U = {}^1/_4\ \pi r^4 G\varphi^2/l = 7$ J.
1.313. $u = {}^1/_2\ G\varphi^2 r^2/l^2$.
1.314. $u = {}^1/_2\beta\,(\rho g h)^2 = 23.5$ kJ/m^3, where β is the compressibility.
1.315. $p_1 > p_2$, $v_1 < v_2$. The density of streamlines grows on transition from point *1* to point *2*.
1.316. $Q = S_1 S_2\sqrt{2g\Delta h/(S_2^2 - S_1^2)}$.
1.317. $Q = S\sqrt{2g\Delta h\rho_0/\rho}$.
1.318. $v = \sqrt{2g\,(h_1 + h_2\rho_2/\rho_1)} = 3$ m/s, where ρ_1 and ρ_2 are the densities of water and kerosene.
1.319. $h = 25$ cm; $l_{max} = 50$ cm.
1.320. $h = {}^1/_2\ v^2/g - h_0 = 20$ cm.
1.321. $p = p_0 + \rho g h\,(1 - R_1^2/r^2)$, where $R_1 < r < R_2$, p_0 is the atmospheric pressure.
1.322. $A = {}^1/_2 \rho V^3/s^2 t^2$, where ρ is the density of water.
1.323. $\tau = \sqrt{2h/g}\ S/s$.
1.324. $v = \omega h\sqrt{2l/h - 1}$.
1.326. $F = 2\rho g S\,\Delta h = 0.50$ N.
1.327. $F = \rho g b l\,(2h - l) = 5$ N.

1.328. $N = \rho l Q^2/\pi r^2 = 0.7$ N·m.

1.329. $F = \rho g h\,(S - s)^2/S = 6$ N

1.330. (a) The paraboloid of revolution: $z = (\omega^2/2g)\,r^2$, where z is the height measured from the surface of the liquid along the axis of the vessel, r is the distance from the rotation axis; (b) $p = p_0 + {}^1/_2\rho\omega^2 r^2$.

1.331. $P = \pi\eta\omega^2 R^4/h = 9$ W.

1.332. $v = v_0 \dfrac{\ln (r/R_2)}{\ln (R_1/R_2)}$.

1.333. (a) $\omega = \omega_2 \dfrac{R_1^2 R_2^2}{R_2^2 - R_1^2}\left(\dfrac{1}{R_1^2} - \dfrac{1}{r^2}\right)$; (b) $N = 4\pi\eta\omega_2 \dfrac{R_1^2 R_2^2}{R_2^2 - R_1^2}$

1.334. (a) $Q = {}^1/_2\,\pi v_0 R^2$; (b) $T = {}^1/_6\,\pi l R^2 \rho v_0^2$; (c) $F_{fr} = 4\pi\eta l v_0$; (d) $\Delta p = 4\eta l v_0/R^2$.

1.335. The additional head $\Delta h = 5$ cm at the left-hand end of the tube imparts kinetic energy to the liquid flowing into the tube. From the condition $\rho v^2/2 = \rho g \Delta h$ we get $v = \sqrt{2g\,\Delta h} = 1.0$ m/s.

1.336. $e^{\alpha\Delta x} = 5$.

1.337. $v_2 = v_1 \dfrac{r_1\rho_1\eta_2}{r_2\rho_2\eta_1} = 5$ μm/s.

1.338. $d = \sqrt[3]{\dfrac{18\,\mathrm{Re}\,\eta^2}{(\rho - \rho_0)\rho_0 g}} = 5$ mm, where ρ_0 and ρ are the densities of glycerin and lead.

1.339. $t = -\dfrac{\rho d^2}{18\eta} \ln n = 0.20$ s.

1.340. $v = c\sqrt{\eta(2-\eta)} = 0.1c$, where c is the velocity of light.

1.341. (a) $P = a\,(1 + \sqrt{4 - 3\beta^2})$; (b) $P = a\,(\sqrt{1-\beta^2} + \sqrt{4-\beta^2})$. Here $\beta = V/c$.

1.342. $l_0 = l\sqrt{(1 - \beta^2\sin^2\theta)/(1-\beta^2)} = 1.08$ m, where $\beta = v/c$.

1.343. (a) $\tan\theta' = \dfrac{\tan\theta}{\sqrt{1-\beta^2}}$. Hence $\theta' = 59°$; (b) $S = S_0\sqrt{1 - \beta^2\cos^2\theta} = 3.3$ m². Here $\beta = v/c$.

1.344. $v = c\sqrt{\left(2 - \dfrac{\Delta t}{t}\right)\dfrac{\Delta t}{t}} = 0.6\cdot 10^8$ m/s.

1.345. $l_0 = c\Delta t'\sqrt{1 - (\Delta t/\Delta t')^2} = 4.5$ m.

1.346. $s = c\Delta t\sqrt{1 - (\Delta t_0/\Delta t)^2} = 5$ m.

1.347. (a) $\Delta t_0 = (l/v)\sqrt{1-(v/c)^2} = 1.4$ μs; (b) $l' = l\sqrt{1 - (v/c)^2} = 0.42$ km.

1.348. $l_0 = v\Delta t/\sqrt{1-(v/c)^2} = 17$ m.

1.349. $l_0 = \sqrt{\Delta x_1 \Delta x_2} = 6.0$ m, $v = c\sqrt{1 - \Delta x_1/\Delta x_2} = 2.2\cdot 10^8$ m/s.

1.350. $v = \dfrac{2l_0/\Delta t}{1 + (l_0/c\Delta t)^2}$.

1.351. The forward particle decayed $\Delta t = l\beta/c\,(1 - \beta^2) = 20$ μs later, where $\beta = v/c$.

1.352. (a) $l_0 = \dfrac{x_A - x_B - v\,(t_A - t_B)}{\sqrt{1-(v/c)^2}}$;

(b) $t_A - t_B = (1 - \sqrt{1-(v/c)^2})\,l_0/v$ or $t_B - t_A = (1 + \sqrt{1-(v/c)^2})\,l_0/v$.

1.353. (a) $t\,(B) = l_0/v$, $t\,(B') = (l_0/v)\sqrt{1-(v/c)^2}$; (b) $t\,(A) = (l_0/v)\sqrt{1-(v/c)^2}$, $t\,(A') = l_0/v$.

1.354. See Fig. 12 showing the positions of hands "in terms of K clocks".

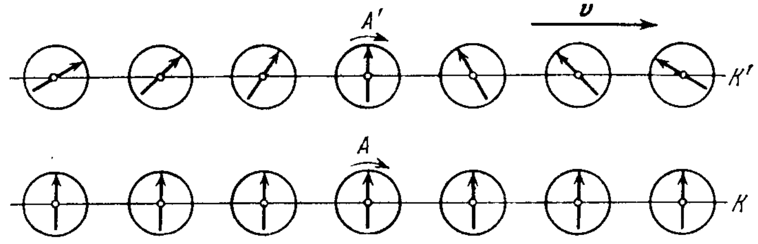


Fig. 12.

1.355. $\dot{x} = (1 - \sqrt{1-\beta^2})\,c/\beta$, where $\beta = V/c$.

1.356. It should be shown first that if $\Delta t = t_2 - t_1 > 0$, then $\Delta t' = t_2' - t_1' > 0$.

1.357. (a) 13 ns; (b) 4.0 m. **Instruction.** Employ the invariance of the interval.

1.358. $v' = \dfrac{\sqrt{(v_x - V)^2 + v_y^2(1 - V^2/c^2)}}{1 - v_x V/c^2}$.

1.359. (a) $v = v_1 + v_2 = 1.25c$; (b) $v = (v_1 + v_2)/(1 + v_1 v_2/c^2) = 0.91c$.

1.360. $l = l_0\,(1 - \beta^2)/(1 + \beta^2)$, where $\beta = v/c$.

1.361. $v = \sqrt{v_1^2 + v_2^2 - (v_1 v_2/c^2)}$.

1.362. $s = \Delta t_0\sqrt{\dfrac{V^2 + (1-\beta^2)\,v'^2}{(1-\beta^2)\,(1 - v'^2/c^2)}}$, where $\beta = V/c$.

1.363. $\tan\theta' = \dfrac{\sqrt{1-\beta^2}\sin\theta}{\cos\theta - V/c}$, where $\beta = V/c$.

1.364. $\tan\theta = v'V/c^2\sqrt{1-(V/c)^2}$.

1.365. (a) $w' = w\,(1-\beta^2)^{3/2}/(1 - \beta v/c)^3$; (b) $w' = w\,(1-\beta^2)$. Here $\beta = V/c$.

1.366. Let us make use of the relation between the acceleration w' and the acceleration w in the reference frame fixed to the Earth:

$$w' = (1 - v^2/c^2)^{-3/2}\,\frac{dv}{dt}.$$

This formula is given in the solution of the foregoing problem (item (a)) where it is necessary to assume $V = v$. Integrating the given equation (for $w' = \text{const}$), we obtain $v = w't/\sqrt{1 + (w't/c)^2}$. The sought distance is $l = (\sqrt{1 + (w't/c)^2} - 1)\,c^2/w' = 0.91$ light-year; $(c - v)/c = 1/2\,(c/w't)^2 = 0.47\,\%$.

1.367. Taking into account that $v = w't/\sqrt{1+(w't/c)^2}$, we get

$$\tau_0 = \int_0^\tau \frac{dt}{\sqrt{1+(w't/c)^2}} = \frac{c}{w'} \ln\left[\frac{w'\tau}{c} + \sqrt{1+\left(\frac{w'\tau}{c}\right)^2}\right] = 3.5 \text{ months.}$$

1.368. $m/m_0 \approx 1/\sqrt{2(1-\beta)} \approx 70$, where $\beta = v/c$.

1.369. $v = c\sqrt{\eta(2+\eta)}/(1+\eta) = 0.6c$, where c is the velocity of light. The definition of density as the ratio of the *rest* mass of a body to its volume is employed here.

1.370. $(c-v)/c = 1 - [1+(m_0c/p)^2]^{-1/2} = 0.44\%$.

1.371. $v = (c/\eta)\sqrt{\eta^2-1} = \frac{1}{2}c\sqrt{3}$.

1.372. $A = 0.42\, m_0c^2$ instead of $0.14\, m_0c^2$.

1.373. $v = \frac{1}{2}c\sqrt{3} = 2.6\cdot 10^8$ m/s.

1.374. For $\varepsilon \ll 1$ the ratio is $T/m_0c^2 \lessapprox \frac{4}{3}\varepsilon \approx 0.013$.

1.375. $p = \sqrt{T(T+2m_0c^2)}/c = 1.09$ GeV/c, where c is the velocity of light.

1.376. $F = (I/ec)\sqrt{T(T+2m_0c^2)}$, $P = TI/e$.

1.377. $p = 2nmv^2/(1-v^2/c^2)$.

1.378. $v = Fct/\sqrt{m_0^2c^2+F^2t^2}$, $l = \sqrt{(m_0c^2/F)^2+c^2t^2} - m_0c^2/F$.

1.379. $F = m_0c^2/a$.

1.380. (a) In two cases: $\mathbf{F} \parallel \mathbf{v}$ and $\mathbf{F} \perp \mathbf{v}$; (b) $\mathbf{F}_\perp = m_0\mathbf{w}\sqrt{1-\beta^2}$, $\mathbf{F}_\parallel = m_0\mathbf{w}/(1-\beta^2)^{3/2}$, where $\beta = v/c$.

1.382. $\varepsilon' = \varepsilon\sqrt{(1-\beta)/(1+\beta)}$, where $\beta = V/c$, $V = \frac{3}{5}c$.

1.383. $E^2 - p^2c^2 = m_0^2c^4$, where m_0 is the rest mass of the particle.

1.384. (a) $\tilde{T} = 2m_0c^2\left(\sqrt{1+T/2m_0c^2}-1\right) = 777$ MeV, $\tilde{p} = \sqrt{\frac{1}{2}m_0T} = 940$ MeV/c; (b) $V = c\sqrt{T/(T+2m_0c^2)} = 2.12\cdot 10^8$ m/s.

1.385. $M_0 = \sqrt{2m_0(T+2m_0c^2)}/c$, $V = c\sqrt{T/(T+2m_0c^2)}$.

1.386. $T' = 2T(T+2m_0c^2)/m_0c^2 = 1.43\cdot 10^3$ GeV.

1.387. $E_{1\,max} = \frac{m_0^2+m_1^2-(m_2+m_3)^2}{2m_0}c^2$. The particle m_1 has the highest energy when the energy of the system of the remaining two particles m_2 and m_3 is the lowest, i.e. when they move as a single whole.

1.388. $v/c = \frac{1-(m/m_0)^{2u/c}}{1+(m/m_0)^{2u/c}}$. Use the momentum conservation law (as in solving Problem 1.178) and the relativistic formula for velocity transformation.

2.1. $m = \rho V\Delta p/p_0 = 30$ g, where p_0 is the standard atmospheric pressure.

2.2. $p = \frac{1}{2}(p_1T_2/T_1 - \Delta p) = 0.10$ atm.

2.3. $m_1/m_2 = (1-a/M_2)/(a/M_1-1) = 0.50$, where $a = mRT/pV$.

2.4. $\rho = \frac{p_0(m_1+m_2)}{RT(m_1/M_1+m_2/M_2)} = 1.5$ g/l.

2.5. (a) $p = (\nu_1+\nu_2+\nu_3)RT/V = 2.0$ atm; (b) $M = (\nu_1M_1+\nu_2M_2+\nu_3M_3)/(\nu_1+\nu_2+\nu_3) = 36.7$ g/mol.

2.6. $T = T_0\eta'(\eta^2-1)/\eta(\eta'^2-1) = 0.42$ kK.

2.7. $n = \frac{\ln\eta}{\ln(1+\Delta V/V)}$.

2.8. $p = p_0e^{-Ct/V}$.

2.9. $t = (V/C)\ln\eta = 1.0$ min.

2.10. $\Delta T = (mg + p_0\Delta S)l/R = 0.9$ K.

2.11. (a) $T_{max} = \frac{2}{3}(p_0/R)\sqrt{p_0/3\alpha}$; (b) $T_{max} = p_0/e\beta R$.

2.12. $p_{min} = 2R\sqrt{\alpha T_0}$.

2.13. $dT/dh = -Mg/R = -33$ mK/m.

2.14. $dT/dh = -Mg(n-1)/nR$.

2.15. 0.5 and 2 atm.

2.16. (a) $h = RT/Mg = 8.0$ km; (b) $h \approx \eta RT/Mg = 0.08$ km.

2.17. $m = (1-e^{-Mgh/RT})p_0S/g$.

2.18. $h_C = \int_0^\infty h\rho\, dh \Big/ \int_0^\infty \rho\, dh = RT/Mg$.

2.19. (a) $p = p_0(1-ah)^n$, $h < 1/a$; (b) $p = p_0/(1+ah)^n$. Here $n = Mg/aRT_0$.

2.20. $p = p_0e^{M\omega^2r^2/2RT}$.

2.21. $p_{id} = \rho RT/M = 280$ atm; $p = \rho RT/(M-\rho b) - a\rho^2/M^2 = 80$ atm.

2.22. (a) $T = a(V-b)(1+\eta)/RV(\eta V+b) = 133$ K; (b) $p = RT/(V-b) - a/V^2 = 9.9$ atm.

2.23. $a = V^2(T_1p_2 - T_2p_1)/(T_2-T_1) = 185$ atm·l²/mol², $b = V - R(T_2-T_1)/(p_2-p_1) = 0.042$ l/mol.

2.24. $\varkappa = V^2(V-b)^2/[RTV^3 - 2a(V-b)^2]$.

2.25. $T > a/bR$.

2.26. $U = pV/(\gamma-1) = 10$ MJ.

2.27. $\Delta T = \frac{1}{2}Mv^2(\gamma-1)/R$.

2.28. $T = T_1T_2(p_1V_1+p_2V_2)/(p_1V_1T_2+p_2V_2T_1)$; $p = (p_1V_1+p_2V_2)/(V_1+V_2)$.

2.29. $\Delta U = -p_0V\Delta T/T_0(\gamma-1) = -0.25$ kJ, $Q' = -\Delta U$.

2.30. $Q = A\gamma/(\gamma-1) = 7$ J.

2.31. $A = R\Delta T = 0.60$ kJ, $\Delta U = Q - R\Delta T = 1.00$ kJ, $\gamma = Q/(Q-R\Delta T) = 1.6$.

2.32. $Q = \nu RT_0(1-1/n) = 2.5$ kJ.

2.33. $\gamma = \frac{\nu_1\gamma_1(\gamma_2-1)+\nu_2\gamma_2(\gamma_1-1)}{\nu_1(\gamma_2-1)+\nu_2(\gamma_1-1)} = 1.33$.

2.34. $c_V = 0.42$ J/(g·K), $c_p = 0.65$ J/(g·K).

2.35. $A = RT(n-1-\ln n)$.

2.36. $A' = p_0V_0\ln[(\eta+1)^2/4\eta]$.

2.37. $\gamma = 1 + (n-1)/(Q/\nu RT_0 - \ln n) = 1.4$.

2.38. See Fig. 13 where V is an isochore, p is an isobaric line, T is an isothermal line, and S is an adiabatic line.

2.39. (a) $T = T_0\eta^{(\gamma-1)/\gamma} = 0.56$ kK; (b) $A' = RT_0(\eta^{(\gamma-1)/\gamma} - 1)/(\gamma-1) = 5.6$ kJ

2.40. The work in the adiabatic process is $n = (\eta^{\gamma-1} - 1)/(\gamma - 1)\ln\eta = 1.4$ times greater.

2.41. $T = T_0\,[(\eta + 1)^2/4\eta]^{(\gamma-1)/2}$.

2.42. $v = \sqrt{2\gamma RT/(\gamma - 1)M} = 3.3$ km/s.

2.43. $Q = R\Delta T\,(2 - \gamma)/(\gamma - 1)$.

2.45. $C_n = R\,(n - \gamma)/(n - 1)\,(\gamma - 1)$; $C_n < 0$ for $1 < n < \gamma$.

2.46. $C = R\,(n - \gamma)/(n - 1)\,(\gamma - 1) = -4.2$ J/(K·mol), where $n = \ln\beta/\ln\alpha$.

2.47. (a) $Q = R\,(n - \gamma)\,\Delta T/(n - 1)\,(\gamma - 1) = 0.11$ kJ; (b) $A = -R\Delta T/(n - 1) = 0.43$ kJ.

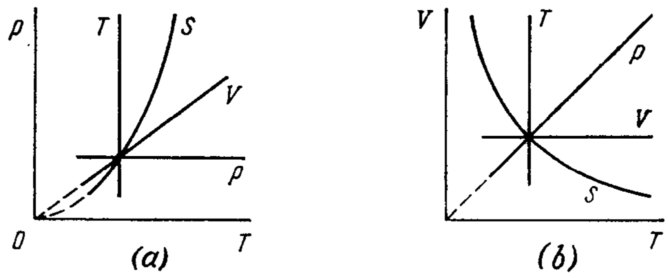


Fig. 13.

2.48. (a) $\Delta U = \alpha V_0^2(\eta^2 - 1)/(\gamma - 1)$; (b) $A = \frac{1}{2}\alpha V_0^2\,(\eta^2 - 1)$; (c) $C = \frac{1}{2}R\,(\gamma + 1)/(\gamma - 1)$.

2.49. (a) $C = -R/(\gamma - 1)$; (b) $TV^{(\gamma-1)/2} = \text{const}$; (c) $A = 2RT_0\,(1 - \eta^{(1-\gamma)/2})/(\gamma - 1)$.

2.50. (a) $A = (1 - \alpha)\,R\Delta T$; (b) $C = R/(\gamma - 1) + R\,(1 - \alpha)$; $C < 0$ for $\alpha > \gamma/(\gamma - 1)$.

2.51. (a) $A = \Delta U\,(\gamma - 1)/\alpha$; $Q = \Delta U\,[1 + (\gamma - 1)/\alpha]$; (b) $C = R/(\gamma - 1) + R/\alpha$.

2.52. (a) $C = C_V + R/\alpha V$; (b) $C = C_V + R/(1 + \alpha V)$.

2.53. (a) $C = \gamma R/(\gamma - 1) + \alpha R/p_0 V$; (b) $\Delta U = p_0\,(V_2 - V_1)/(\gamma - 1)$; $A = p_0\,(V_2 - V_1) + \alpha\ln(V_2/V_1)$; $Q = \gamma p_0\,(V_2 - V_1)/(\gamma - 1) + \alpha\ln(V_2/V_1)$.

2.54. (a) $C = C_p + RT_0/\alpha V$; (b) $Q = \alpha C_p\,(V_2 - V_1) + RT_0\ln(V_2/V_1)$.

2.55. (a) $Ve^{-\alpha T/R} = \text{const}$; (b) $Te^{R/\beta V} = \text{const}$; (c) $V - aT = \text{const}$.

2.56. (a) $A = \alpha\ln\eta - RT_0\,(\eta - 1)/(\gamma - 1)$; (b) $pV^\gamma e^{\alpha(\gamma-1)/pV} = \text{const}$.

2.57. $A = RT\ln\frac{V_2 - b}{V_1 - b} + a\left(\frac{1}{V_2} - \frac{1}{V_1}\right)$, where a and b are Van der Waals constants.

2.58. (a) $\Delta U = a/V_1 - a/V_2 = 0.11$ kJ; (b) $Q = RT\ln\frac{V_2 - b}{V_1 - b} = 3.8$ kJ.

2.59. (a) $T\,(V - b)^{R/C_V} = \text{const}$;
(b) $C_p - C_V = \frac{R}{1 - 2a\,(V - b)^2/RTV^3}$.

2.60. $\Delta T = -\frac{\nu a V_2\,(\gamma - 1)}{RV_1\,(V_1 + V_2)} = -3.0$ K.

2.61. $Q = \nu^2 a\,(V_2 - V_1)/V_1 V_2 = 0.33$ kJ.

2.62. $n = p/kT = 1\cdot 10^5$ cm^{-3}; $\langle l\rangle = 0.2$ mm.

2.63. $p = (1 + \eta)\,mRT/MV = 1.9$ atm, where M is the mass of an N_2 mole.

2.64. $n = (p/kT - \rho/m_2)/(1 - m_1/m_2) = 1.6\cdot 10^{19}$ cm^{-3}, where m_1 and m_2 are the masses of helium and nitrogen molecules.

2.65. $p = 2nmv^2\cos^2\theta = 1.0$ atm, where m is the mass of a nitrogen molecule.

2.66. $i = 2/(\rho v^2/p - 1) = 5$.

2.67. $v/v_{sq} = \sqrt{(i + 2)/3i}$; (a) 0.75; (b) 0.68.

2.68. $\langle\varepsilon\rangle = \begin{cases} (3N - 3)\,kT & \text{for volume molecules.} \\ (3N - 5/2)\,kT & \text{for linear molecules.} \end{cases}$
$1/2(N - 1)$ and $1/(2N - 5/3)$ respectively.

2.69. (a) $C_V = \frac{7}{2}R$, $\gamma = 9/7$; (b) $C_V = (3N - 5/2)\,R$, $\gamma = (6N - 3)/(6N - 5)$; (c) $C_V = 3\,(N - 1)\,R$, $\gamma = (N - \frac{2}{3})/(N - 1)$.

2.70. $A/Q = \begin{cases} 1/(3N - 2) & \text{for volume molecules,} \\ 1/(3N - 3/2) & \text{for linear molecules.} \end{cases}$
For monoatomic molecules $A/Q = 2/5$.

2.71. $M = R/(c_p - c_V) = 32$ g/mol, $i = 2/(c_p/c_V - 1) = 5$.

2.72. (a) $i = 2\,(C_p/R - 1) = 5$; (b) $i = 2\,[C/R + 1/(n - 1)] = 3$, where $n = 1/2$ is the polytropic index.

2.73. $\gamma = (5\nu_1 + 7\nu_2)/(3\nu_1 + 5\nu_2)$.

2.74. Increases by $\Delta p/p = Mv^2/iRT = 2.2\%$, where $i = 5$.

2.75. (a) $v_{sq} = \sqrt{3RT/M} = 0.47$ km/s, $\langle\varepsilon\rangle = \frac{3}{2}kT = 6.0\cdot 10^{-21}$ J; (b) $v_{sq} = 3\sqrt{2kT/\pi\rho d^3} = 0.15$ m/s.

2.76. $\eta^i = 7.6$ times.

2.77. $Q = \frac{1}{2}\,(\eta^2 - 1)\,imRT/M = 10$ kJ.

2.78. $\omega_{sq} = \sqrt{2kT/I} = 6.3\cdot 10^{12}$ rad/sec.

2.79. $\langle\varepsilon\rangle_{rot} = kT_0\eta^{2/i} = 0.7\cdot 10^{-20}$ J.

2.80. Decreases $\eta^{(i+1)/i}$ times, where $i = 5$.

2.81. Decreases $\eta^{(i-1)/(i-2)} = 2.5$ times.

2.82. $C = \frac{1}{2}R\,(i + 1) = 3R$.

2.83. $v_{pr} = \sqrt{2p/\rho} = 0.45$ km/s, $\langle v\rangle = 0.51$ km/s, $v_{sq} = 0.55$ km/s.

2.84. (a) $\delta N/N = (8/\sqrt{\pi})\,e^{-1}\delta\eta = 1.66\%$;
(b) $\delta N/N = 12\sqrt{3/2\pi}\,e^{-3/2}\delta\eta = 1.85\%$.

2.85. (a) $T = \frac{m\,(\Delta v)^2}{k\,(\sqrt{3} - \sqrt{2})^2} = 380$ K; (b) $T = \frac{mv^2}{2k} = 340$ K.

2.86. (a) $T = \frac{m\,(v_2^2 - v_1^2)}{4k\ln(v_2/v_1)} = 330$ K; (b) $v = \sqrt{\frac{3kT_0}{m}\,\frac{\eta\ln\eta}{\eta - 1}}$.

2.87. $T=\frac{m_{\mathrm{N}}(\Delta v)^2}{2k\left(1-\sqrt{m_{\mathrm{N}}/m_0}\right)^2}=0.37$ kK.

2.88. $v=\sqrt{\frac{3kT\ln(m_2/m_1)}{m_2-m_1}}=1.61$ km/s.

2.89. $T={}^1/_3 mv^2/k$.

2.90. $dN/N=\left(\frac{m}{2\pi kT}\right)^{3/2}e^{-mv^2/2kT}\,2\pi v_\perp\,dv_\perp\,dv_x$.

2.91. $\langle v_x\rangle=0$, $\langle|v_x|\rangle=\sqrt{2kT/\pi m}$.

2.92. $\langle v_x^2\rangle=kT/m$.

2.93. $\nu={}^1/_4 n\langle v\rangle$, where $\langle v\rangle=\sqrt{8kT/\pi m}$.

2.94. $p=\int\limits_0^\infty 2mv_x\cdot v_x\,dn(v_x)=nkT$, where $dn(v_x)=$
$=(m/2\pi kT)^{1/2}n\cdot e^{-mv_x^2/2kT}\,dv_x$.

2.95. $\langle 1/v\rangle=\sqrt{2m/\pi kT}=4\pi\langle v\rangle$.

2.96. $dN/N=2\pi(\pi kT)^{-3/2}e^{-\varepsilon/kT}\sqrt{\varepsilon}\,d\varepsilon$; $\varepsilon_{pr}={}^1/_2 kT$; no.

2.97. $\delta N/N=3\sqrt{6\pi}\,e^{-3/2}\delta\eta=0.9\%$.

2.98. $\frac{\Delta N}{N}=\frac{2\pi}{(\pi kT)^{3/2}}\int\limits_{\varepsilon_0}^{\infty}\sqrt{\varepsilon}\,e^{-\varepsilon/kT}\,d\varepsilon$.

The principal contribution to the value of the integral is provided by the smallest values of ε, namely $\varepsilon\approx\varepsilon_0$. The slowly varying factor $\sqrt{\varepsilon}$ can be taken from under the radical sign if ascribed the constant value $\sqrt{\varepsilon_0}$. Then

$$\Delta N/N=2\sqrt{\varepsilon_0/\pi kT}\,e^{-\varepsilon_0/kT}.$$

2.99. (a) $v_{pr}=\sqrt{3kT/m}$; (b) $\varepsilon_{pr}=kT$.

2.100. $d\nu=\int\limits_{v=0}^{\infty}dn\,(d\Omega/4\pi)\,v\cos\theta=n\,(2kT/\pi m)^{1/2}\sin\theta\cos\theta\,d\theta$.

2.101. $d\nu=\int\limits_{\theta=0}^{\pi/2}dn\,(d\Omega/4\pi)\,v\cos\theta=\pi\,(m/2\pi kT)^{3/2}\,e^{-mv^2/2kT}v^3\,dv$.

2.102. $F=(kT/\Delta h)\ln\eta=0.9\cdot10^{-19}$ N.

2.103. $N_A=(6RT/\pi d^3\Delta\rho gh)\ln\eta\approx6.4\cdot10^{23}$ mol^{-1}.

2.104. $\eta/\eta_0=e^{(M_2-M_1)gh/RT}=1.39$.

2.105. $h=\frac{kT\ln(n_2/n_1)}{(m_2-m_1)g}$.

2.106. Will not change.

2.107. $\langle U\rangle=kT$. Does not depend.

2.108. $w\approx\eta RT/Ml\approx70$ g.

2.109. $M=\frac{2RT\rho\ln\eta}{(\rho-\rho_0)(r_2^2-r_1^2)\omega^2}$.

2.110. $\omega=\sqrt{(2RT/Ml^2)\ln\eta}=280$ rad/s.

2.111. (a) $dN=n_0e^{-ar^2/kT}4\pi r^2\,dr$; (b) $r_{pr}=\sqrt{kT/a}$; (c) $dN/N=$
$=(a/\pi kT)^{3/2}e^{-ar^2/kT}4\pi r^2\,dr$; (d) Will increase $\eta^{3/2}$-fold.

2.112. (a) $dN=(2\pi n_0/a^{3/2})\,e^{-U/kT}\sqrt{U}\,dU$; (b) $U_{pr}={}^1/_2 kT$.

2.113. In the latter case.

2.114. (a) $\eta=1-n^{1-\gamma}=0.25$; (b) $\eta=1-n^{1/\gamma-1}=0.18$.

2.115. $\varepsilon=(1-\eta)/\eta=9$.

2.116. $\eta=1-2T_3/(T_1+T_2)$.

2.117. $\eta=1-n^{1-\gamma}=60\%$.

2.118. $\eta=1-n^{-(1-1/\gamma)}$.

2.119. $\eta=1-(n+\gamma)/(1+\gamma n)$

2.120. In both cases $\eta=1-\frac{\ln n}{n-1}$.

2.121. In both cases $\eta=1-\frac{n-1}{n\ln n}$.

2.122. $\eta=1-\frac{n-1}{n\ln n}$.

2.123. (a) $\eta=1-\gamma\frac{n-1}{n^\gamma-1}$; (b) $\eta=1-\frac{n^\gamma-1}{\gamma(n-1)n^{\gamma-1}}$

2.124. (a) $\eta=1-\frac{\gamma(n-1)}{n-1+(\gamma-1)n\ln n}$;
(b) $\eta=1-\frac{n-1+(\gamma-1)\ln n}{\gamma(n-1)}$.

2.125. $\eta=\frac{(\tau-1)\ln\nu}{\tau\ln\nu+(\tau-1)/(\gamma-1)}$.

2.126. $\eta=\frac{(\tau-1)\ln n}{\tau\ln n+(\tau-1)\gamma/(\gamma-1)}$.

2.127. $\eta=1-2\frac{\gamma+\sqrt{\tau}}{(1+\gamma)(1+\sqrt{\tau})}$.

2.128. The inequality $\int\frac{\delta Q_1}{T_1}-\int\frac{\delta Q_2'}{T_2}\leqslant0$ becomes even stronger when T_1 is replaced by T_{max} and T_2 by T_{min}. Then $Q_1/T_{max}-$ $-Q_2'/T_{min}<0$. Hence

$\frac{Q_1-Q_2'}{Q_1}<\frac{T_{max}-T_{min}}{T_{max}}$, or $\eta<\eta_{Carnot}$.

2.129. According to the Carnot theorem $\delta A/\delta Q_1=dT/T$. Let us find the expressions for δA and δQ_1. For an infinitesimal Carnot cycle (e.g. parallelogram *1234* shown in Fig. 14)

$$\delta A=dp\cdot dV=(\partial p/\partial T)_V dT\cdot dV,$$
$$\delta Q_1=dU_{12}+p\,dV=[(\partial U/\partial V)_T+p]\,dV.$$

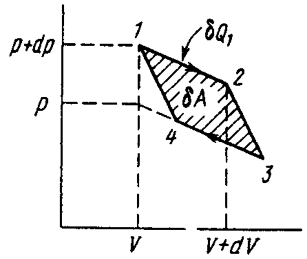


Fig. 14.

It remains to substitute the two latter expressions into the former one.

2.130. (a) $\Delta S=\frac{R\ln n}{\gamma-1}=19$ J/(K·mol); (b) $\Delta S=\frac{\gamma R\ln n}{\gamma-1}=$
$=25$ J/(K·mol).

2.131. $n=e^{\Delta S/\nu R}=2.0$.

2.132. $\Delta S=\nu R\ln n=20$ J/K.

2.133. $\Delta S=-\frac{m}{M}\frac{\gamma R}{\gamma-1}\ln n=-10$ J/K.

2.134. $\Delta S=(\gamma\ln\alpha-\ln\beta)\,\nu R/(\gamma-1)=-11$ J/K.

2.135. $S_2 - S_1 = \nu R\left(\ln\alpha - \frac{\ln\beta}{\gamma-1}\right) = 1.0$ J/K.

2.136. $\Delta S = \frac{(n-\gamma)R}{(n-1)(\gamma-1)}\ln\tau$.

2.137. $\Delta S = \frac{\nu(\gamma+1)R}{\gamma-1}\ln\alpha = 46$ J/K.

2.138. $V_m = \gamma p_0/\alpha(1+\gamma)$.

2.139. $T = T_0 + (R/a)\ln(V/V_0)$.

2.140. $\Delta S = R\ln[(V_2-b)/(V_1-b)]$.

2.141. $\Delta S = C_V\ln(T_2/T_1) + R\ln[(V_2-b)/(V_1-b)]$.

2.142. $S = aT^3/3$.

2.143. $\Delta S = m[a\ln(T_2/T_1) + b(T_2 - T_1)] = 2.0$ kJ/K.

2.144. $C = S/n$; $C < 0$ for $n < 0$.

2.145. $T = T_0 e^{(S-S_0)/C}$. See Fig. 15.

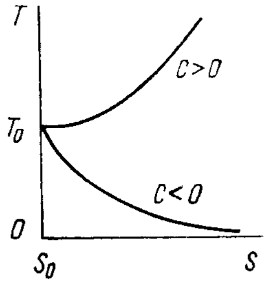


Fig. 15.

2.146. (a) $C = -\alpha/T$; (b) $Q = \alpha\ln(T_1/T_2)$; (c) $A = \alpha\ln(T_1/T_2) + C_V(T_1 - T_2)$.

2.147. (a) $\eta = (n-1)/2n$; (b) $\eta = (n-1)/(n+1)$.

2.148. $\Delta S = \nu R\ln n = 20$ J/K.

2.149. $\Delta U = (2^{\gamma-1} - 1)RT_0/(\gamma-1)$, $\Delta S = R\ln 2$.

2.150. The pressure will be higher after the fast expansion.

2.151. $\Delta S = \nu_1 R\ln(1+n) + \nu_2 R\ln(1+1/n) = 5.1$ J/K.

2.152. $\Delta S = m_1c_1\ln(T/T_1) + m_2c_2\ln(T/T_2) = 4.4$ J/K, where $T = (m_1c_1T_1 + m_2c_2T_2)/(m_1c_1 + m_2c_2)$, c_1 and c_2 are the specific heat capacities of copper and water.

2.153. $\Delta S = C_V\ln\frac{(T_1+T_2)^2}{4T_1T_2} > 0$.

2.154. (a) $P = 1/2^N$; (b) $N = \frac{\log(t/\tau)}{\log 2} \approx 80$, where $\tau \approx 10^{-5}$ s is the mean time which takes a helium atom to cover distances of the order of the vessel's dimensions.

2.155. $\Omega_{pr} = N!/[(N/2)!]^2 = 252$. $P_{N/2} = \Omega_{pr}/2^N = 24.6\%$.

2.156. $P_n = \frac{N!}{n!(N-n)!\,2^N}$; 1/32, 5/32, 10/32, 10/32, 5/32, 1/32 respectively.

2.157. $P_n = \frac{N!}{n!(N-n)!}p^n(1-p)^{N-n}$, where $p = V/V_0$.

2.158. $d = \sqrt[3]{6/\pi n_0\eta^2} = 0.4$ μm, where n_0 is Loschmidt's number; $\langle n\rangle = 1/\eta^2 = 1.0\cdot10^6$.

2.159. Will increase $\Omega/\Omega_0 = (1 + \Delta T/T_0)^{iN_A/2} = 10^{1.31\cdot10^{21}}$ times.

2.160. (a) $\Delta p = 4\alpha/d = 13$ atm; (b) $\Delta p = 8\alpha/d = 1.2\cdot10^{-3}$ atm.

2.161. $h = 4\alpha/\rho g d = 21$ cm.

2.162. $\alpha = {}^1/_8 p_0 d(1-\eta^3/n)/(\eta^2-1)$.

2.163. $p = p_0 + \rho gh + 4\alpha/d = 2.2$ atm.

2.164. $h = [p_0(n^3-1) + 4\alpha(n^2-1)/d]/\rho g = 5$ m.

2.165. $\Delta h = 4\alpha\,|\cos\theta\,|\,(d_2 - d_1)/d_1d_2\rho g = 11$ mm.

2.166. $R = 2\alpha/\rho gh = 0.6$ mm.

2.167. $x = l/(1 + p_0d/4\alpha) = 1.4$ cm.

2.168. $\alpha = [\rho gh + p_0l/(l-h)]\,d/4\cos\theta$.

2.169. $h = 4\alpha/\rho g(d_2 - d_1) = 6$ cm.

2.170. $h = 2a\cos\theta/\rho gx\delta\varphi$.

2.171. $V_1 = {}^1/_4\pi d^2\sqrt{\frac{2gl - 4\alpha(n-1)/\rho d}{n^4-1}} = 0.9$ cm³/s.

2.172. $R_2 - R_1 \approx {}^1/_8\rho gh^3/\alpha = 0.20$ mm.

2.173. $m \approx 2\pi R^2\alpha|\cos\theta|(n^2-1)/gh = 0.7$ kg.

2.174. $F \approx 2\alpha m/\rho h^2 = 1.0$ N.

2.175. $F = 2\pi R^2\alpha/h = 0.6$ kN.

2.176. $F = 2\alpha^2 l/\rho gd^2 = 13$ N.

2.177. $t = 2l\eta R^4/\alpha r^4$.

2.178. $Q = 2\pi\alpha^2/\rho g$.

2.179. (a) $F = \pi\alpha d^2 = 3$ μJ; (b) $F = 2\pi\alpha d^2 = 10$ μJ.

2.180. $\Delta F = 2\pi\alpha d^2(2^{-1/3} - 1) = -1.5$ μJ.

2.181. $A' = F + pV\ln(p/p_0)$, where $F = 8\pi R^2\alpha$, $p = p_0 + 4\alpha/R$, $V = {}^4/_3\pi R^3$.

2.182. $C - C_p = {}^1/_2 R/(1 + {}^3/_8 p_0r/\alpha)$.

2.184. (a) $\Delta S = -2(d\alpha/dT)\Delta\sigma$; (b) $\Delta U = 2(\alpha - T\,d\alpha/dT)\times\Delta\sigma$.

2.185. $A = \Delta mRT/M = 1.2$ J.

2.186. $m_v = (V - mV_l')/(V_v' - V_l') = 20$ g, $V_v = 1.0$ l. Here V_l' is the specific volume of water.

2.187. $m_l \approx Mp_0(V_0 - V)/RT = 2.0$ g, where p_0 is the standard atmospheric pressure.

2.188. $\eta = (n-1)/(N-1)$; $\eta = 1/(N+1)$.

2.189. $\Delta S = mq/T = 6.0$ kJ/K; $\Delta U = m(q - RT/M) = 2.1$ MJ, where $T = 373$ K.

2.190. $h \approx \frac{(Q - mc\Delta T)}{p_0S(1 + qM/RT)} = 20$ cm, where c is the specific heat capacity of water, $\Delta T = 100$ K, q is the specific heat of vaporization of water, T is its boiling temperature.

2.191. $A = mc(T - T_0)RT/qM = 25$ J, where c is the specific heat capacity of water, T is the initial vapour temperature equal to the water boiling temperature, as is seen from the hypothesis, q is the specific heat of vapour condensation.

2.192. $d \approx 4\alpha M/\eta\rho RT = 0.2$ μm, where ρ is the density of water.

2.193. $\mu = \eta p_0\sqrt{M/2\pi RT} = 0.35$ g/(s·cm²), where p_0 is the standard atmospheric pressure.

2.194. $p = \mu\sqrt{2\pi RT/M} = 0.9$ nPa.

2.195. $\Delta p = a/V^2M = 1.7\cdot10^4$ atm.

2.196. $p_i \approx \rho q$. About $2\cdot10^4$ atm.

2.198. $a = {}^{27}/_{64}R^2T_{cr}^2/p_{cr} = 3.6$ atm·l²/mol², $b = {}^1/_8RT_{cr}/p_{cr} = 0.043$ l/mol.

2.199. $V_{cr}' = {}^3/_8RT_{cr}/Mp_{cr} = 4.7$ cm³/g.

2.200. $(\pi + 3/\nu^2)(3\nu - 1) = 8\tau$, $\tau = 1.5$.

2.201. (a) $V_{max} = 3bm/M = 5.0$ l; (b) $p_{max} = a/27b^2 = 230$ atm.
2.202. $T_{cr} = {}^8/_{27}\, a/bR = 0.30$ kK, $\rho_{cr} = {}^1/_3\, M/b = 0.34$ g/cm^3.
2.203. $\eta = {}^8/_3\, Mp_{cr}/\rho RT_{cr} = 0.25$, where ρ is the density of ether at room temperature.
2.204. Let us apply Eq. (2.4e) to the reversible isothermic cycle *1-2-3-4-5-3-1*:

$$T \oint dS = \oint dU + \oint p\,dV.$$

Since the first two integrals are equal to zero, $\oint p\,dV = 0$ as well. The latter equality is possible only when areas *I* and *II* are equal.
Note that this reasoning is inapplicable to the cycle *1-2-3-1*, for example. It is irreversible since it involves the irreversible transition at point *3* from a single-phase to a diphase state.
2.205. $\eta = c\,|t|/q = 0.25$, where q is the specific heat of melting of ice; at $t = -80$°C.
2.206. $\Delta T = -(T\Delta V'/q)\,\Delta p = -7.5$ mK, where q is the specific heat of melting of ice.
2.207. $V'_{sv} \approx q\Delta T/T\Delta p = 1.7$ m^3/kg, where q is the specific heat of vaporization, $T = 373$ K.
2.208. $p_{sv} \approx p_0\,(1 + qM\Delta T/RT^2) = 1.04$ atm where q is the specific heat of vaporization, p_0 is the standard atmospheric pressure, $\Delta T = 1.1$ K.
2.209. $\Delta m/m = (qM/RT - 1)\,\Delta T/T = 5\%$.
2.210. $p = p_0 \exp\left[\frac{qM}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]$. These assumptions are admissible in the case of a vapour narrow temperature interval, far below the critical temperature.
2.211. $\eta \approx cpT\Delta V'/q^2 = 0.03$, where c is the specific heat capacity of ice, $T \approx 273$ K, q is the specific heat of melting.
2.212. (a) 216 *K*, 5.1 atm; (b) 0.78, 0.57, and 0.21 kJ/g.
2.213. $\Delta S \approx m\,[c\ln(T_2/T_1) + q/T_2] = 7.2$ kJ/K.
2.214. $\Delta s \approx q_m/T_1 + c\ln(T_2/T_1) + q_v/T_2 = 8.6$ J/(g·K).
2.215. $\Delta S = mc\ln(T/T_1) = -10$ J/K, where c is the specific heat capacity of copper, $T = 273$ K (under these conditions only a part of the ice will melt).
2.216. (a) When $m_2c_2t_2 < m_1q$, not all the ice will melt and

$$\Delta S = m_2c_2\left(\frac{T_2}{T_1} - 1 - \ln\frac{T_2}{T_1}\right) = 9.2 \text{ J/K};$$

(b) When $m_2c_2t_2 > m_1q$, the ice will melt completely and

$$\Delta S = \frac{m_1q}{T_1} + c_2\left(m_1\ln\frac{T}{T_1} - m_2\ln\frac{T_2}{T}\right) = 18 \text{ J/K},$$

where $T = \dfrac{m_1T_1 + m_2T_2 - m_1q/c_2}{m_1 + m_2}$.
2.217. $\Delta S = mq\left(\frac{1}{T_1} - \frac{1}{T_2}\right) + mc\left(\frac{T_2}{T_1} - 1 - \ln\frac{T_2}{T_1}\right) = 0.48$ J/K.
2.218. $C = C_p - qM/T = -74$ J/(K·mol), where $C_p = R\gamma/(\gamma - 1)$.

2.219. $\Delta S = qM/T_2 + C_p\ln(T_2/T_1)$.
2.220. (a) $\eta \approx 0.37$; (b) $\eta \approx 0.23$.
2.221. $\lambda = \Delta l/\ln\eta$.
2.222. (a) $P = e^{-\alpha t}$; (b) $\langle t\rangle = 1/\alpha$.
2.223. (a) $\lambda = 0.06$ μm, $\tau = 0.13$ ns; (b) $\lambda = 6$ Mm, $\tau = 3.8$ hours.
2.224. 18 times.
2.225. $\lambda = (2\pi N_A/3b)^{2/3}\,(kT_0/\sqrt{2}\pi p_0) = 84$ nm.
2.226. $\nu = \pi d^2 p_0 N_A\sqrt{2\gamma/MRT_0} = 5.5$ GHz.
2.227. (a) 0.7 Pa; (b) $2\cdot10^{14}$ cm^{-3}, 0.2 μm.
2.228. (a) $\nu = \sqrt{2}\pi\, d^2 n\,\langle v\rangle = 0.74\cdot10^{10}$ s^{-1}; (b) $\nu = {}^1/_2\sqrt{2}\pi\, d^2n^2\langle v\rangle = 1.0\times10^{29}$ s^{-1}·cm^{-3}, where $n = p_0/kT_0$, $\langle v\rangle = \sqrt{8RT/\pi M}$.
2.229. (a) $\lambda =$ const, $\nu \propto \sqrt{T}$; (b) $\lambda \propto T$, $\nu \propto 1/\sqrt{T}$.
2.230. (a) $\lambda =$ const, ν increases $\sqrt{n}$ times; (b) λ decreases n times, ν increases n times.
2.231. (a) $\lambda \propto V$, $\nu \propto V^{-6/5}$; (b) $\lambda \propto p^{-5/7}$, $\nu \propto p^{6/7}$; (c) $\lambda \propto T^{-5/2}$, $\nu \propto T^3$.
2.232. (a) $\lambda \propto V$, $\nu \propto V^{-(n+1)/2}$; (b) $\lambda \propto p^{-1/n}$, $\nu \propto p^{(n+1)/2n}$; (c) $\lambda \propto T^{1/(1-n)}$, $\nu \propto T^{(n+1)/2(n-1)}$.
2.233. (a) $C = {}^1/_4 R\,(1 + 2i) = 23$ J/(K·mol); (b) $C = {}^1/_2 R\,(i + 2) = 29$ J/(K·mol).
2.234. $n = n_0e^{-t/\tau}$, where $\tau = 4V/S\langle v\rangle$, $\langle v\rangle = \sqrt{8RT/\pi M}$.
2.235. Increases $(1 + \eta)/(1 + \sqrt{\eta})$ times.
2.236. Increases $\alpha^3/\beta = 2$ times.
2.237. (a) D increases n times, $\eta =$ const; (b) D increases $n^{3/2}$ times, η increases $\sqrt{n}$ times.
2.238. D decreases $n^{4/5} \approx 6.3$ times, η increases $n^{1/5} \approx 1.6$ times.
2.239. (a) $n = 3$; (b) $n = 1$; (c) $n = 1$.
2.240. 0.18 nm.
2.241. $d_{Ar}/d_{He} = 1.7$.
2.242. $N_1 \approx 2\pi\eta\omega R^3/\Delta R$; $p = \sqrt{2}\,kT/\pi d^2 n\Delta R = 0.7$ Pa.
2.243. $\eta = (1/R_1^2 - 1/R_2^2)\,N_1/4\pi\omega$.
2.244. $N = {}^1/_2\pi\eta\omega a^4/h$.
2.245. $N = {}^1/_3\omega a^4 p\,\sqrt{\pi M/2RT}$.
2.246. $\mu = \dfrac{\pi a^4 M}{16\eta RT}\,\dfrac{|p_2^2 - p_1^2|}{l}$.
2.247. $T = (\varkappa_1T_1/l_1 + \varkappa_2T_2/l_2)/(\varkappa_1/l_1 + \varkappa_2/l_2)$.
2.248. $\varkappa = (l_1 + l_2)/(l_1/\varkappa_1 + l_2/\varkappa_2)$.
2.249. $T(x) = T_1\,(T_2/T_1)^{x/l}$; $q = (\alpha/l)\ln(T_2/T_1)$.
2.250. $\Delta T = (\Delta T)_0 e^{-\alpha t}$, where $\alpha = (1/C_1 + 1/C_2)\,S\varkappa/l$.
2.251. $T = T_1\{1 + (x/l)\,[(T_2/T_1)^{3/2} - 1]\}^{2/3}$, where x is the distance from the plate maintained at the temperature T_1.

2.252. $q = \frac{2iR^{3/2}(T_2^{3/2} - T_1^{3/2})}{9\pi^{3/2} l d^2 N_A \sqrt{M}} = 40$ W/m², where $i = 3$, d is the effective diameter of helium atom.

2.253. $\lambda = 23$ mm $> l$, consequently, the gas is ultra-thin; $q = p \langle v \rangle (t_2 - t_1)/6T(\gamma - 1) = 22$ W/m², where $\langle v \rangle = \sqrt{8RT/\pi M}$, $T = {}^1/_2 (T_1 + T_2)$.

2.254. $T = T_1 + \frac{T_2 - T_1}{\ln(R_2/R_1)} \ln \frac{r}{R_1}$.

2.255. $T = T_1 + \frac{T_2 - T_1}{1/R_1 - 1/R_2} \left(\frac{1}{R_1} - \frac{1}{r}\right)$.

2.256. $T = T_0 + (R^2 - r^2) w/4\varkappa$.

2.257. $T = T_0 + (R^2 - r^2) w/6\varkappa$.

3.1. The ratio F_{el}/F_{gr} is equal to $4 \cdot 10^{42}$ and $1 \cdot 10^{36}$ respectively; $q/m = 0.86 \cdot 10^{-10}$ C/kg.

3.2. About $2 \cdot 10^{15}$ N.

3.3. $dq/dt = {}^3/_2 a \sqrt{2\pi\varepsilon_0 mg/l}$.

3.4. $q_3 = -\frac{q_1 q_2}{(\sqrt{q_1} + \sqrt{q_2})^2}$, $\quad \mathbf{r}_3 = \frac{\mathbf{r}_1 \sqrt{q_2} + \mathbf{r}_2 \sqrt{q_1}}{\sqrt{q_1} + \sqrt{q_2}}$.

3.5. $\Delta T = \frac{q q_0}{8\pi^2 \varepsilon_0 r^2}$.

3.6. $\mathbf{E} = 2.7\mathbf{i} - 3.6\mathbf{j}$, $E = 4.5$ kV/m.

3.7. $E = \frac{ql}{\sqrt{2}\pi\varepsilon_0 (l^2 + x^2)^{3/2}}$.

3.8. $E = \frac{q}{2\pi^2 \varepsilon_0 R^2} = 0.10$ kV/m.

3.9. $E = \frac{ql}{4\pi\varepsilon_0 (r^2 + l^2)^{3/2}}$. For $l \gg r$ the strength $E \approx \frac{q}{4\pi\varepsilon_0 l^2}$, as in the case of a point charge. $E_{max} = \frac{q}{6\sqrt{3}\pi\varepsilon_0 r^2}$ for $l = r/\sqrt{2}$.

3.10. $E = \frac{3qR^2}{4\pi\varepsilon_0 x^4}$.

3.11. $F = \frac{q\lambda}{4\pi\varepsilon_0 R}$.

3.12. (a) $E = \frac{\lambda_0}{4\varepsilon_0 R}$; (b) $E = \frac{\lambda_0 R^2}{4\varepsilon_0 (x^2 + R^2)^{3/2}}$. For $x \gg R$ the strength $E \approx \frac{p}{4\pi\varepsilon_0 x^3}$, where $p = \pi R^2 \lambda_0$.

3.13. (a) $E = \frac{q}{4\pi\varepsilon_0 r \sqrt{a^2 + r^2}}$; (b) $E = \frac{q}{4\pi\varepsilon_0 (r^2 - a^2)}$. In both cases $E \approx \frac{q}{4\pi\varepsilon_0 r^2}$ for $r \gg a$.

3.14. $E = \frac{\lambda\sqrt{2}}{4\pi\varepsilon_0 y}$. The vector $\mathbf{E}$ is directed at the angle 45° to the thread.

3.15. (a) $E = \frac{\lambda\sqrt{2}}{4\pi\varepsilon_0 R}$; (b) $E = 0$.

3.16. $\mathbf{E} = -{}^1/_3 \mathbf{a} r/\varepsilon_0$.

3.17. $\mathbf{E} = -{}^1/_3 \mathbf{k} \sigma_0/\varepsilon_0$, where $\mathbf{k}$ is the unit vector of the z axis with respect to which the angle θ is read off. Clearly, the field inside the given sphere is uniform.

3.18. $\mathbf{E} = -{}^1/_6 \mathbf{a} R^2/\varepsilon_0$.

3.19. $|\Phi| = {}^1/_2 \lambda R/\varepsilon_0$. The sign of Φ depends on how the direction of the normal to the circle is chosen.

3.20. $|\Phi| = \frac{q}{\varepsilon_0}\left(1 - \frac{1}{\sqrt{1 + (R/l)^2}}\right)$. The sign of Φ depends on how the direction of the normal to the circle is chosen.

3.21. $|\Phi| = {}^1/_3 \pi \rho r_0 (R^2 - r_0^2)/\varepsilon_0$.

3.22. $E_{max} = \lambda/\pi\varepsilon_0 l$.

3.23. $E = {}^1/_2\, \sigma_0/\varepsilon_0$, with the direction of the vector $\mathbf{E}$ corresponding to the angle $\varphi = \pi$.

3.24. $\Phi = 4\pi R a$.

3.25. (a) $E = \frac{\rho_0 r}{3\varepsilon_0}\left(1 - \frac{3r}{4R}\right)$ for $r \leqslant R$, $E = \frac{\rho_0 R^3}{12\varepsilon_0 r^2}$ for $r \geqslant R$; (b) $E_{max} = {}^1/_9\, \rho_0 R/\varepsilon_0$ for $r_m = {}^2/_3\, R$.

3.26. $q = 2\pi R^2 \alpha$, $E = {}^1/_2 \alpha/\varepsilon_0$.

3.27. $E = \frac{\rho_0}{3\varepsilon_0 \alpha r^2}(1 - e^{-\alpha r^3})$. Accordingly, $E \approx \frac{\rho_0 r}{3\varepsilon_0}$ and $E \approx \frac{\rho_0}{3\varepsilon_0 \alpha r^2}$.

3.28. $\mathbf{E} = {}^1/_3\, \mathbf{a}\rho/\varepsilon_0$.

3.29. $\mathbf{E} = {}^1/_2 \mathbf{a}\rho/\varepsilon_0$, where the vector $\mathbf{a}$ is directed toward the axis of the cavity.

3.30. $\Delta\varphi = \frac{q}{2\pi\varepsilon_0 R}\left(1 - \frac{1}{\sqrt{1 + (a/R)^2}}\right)$.

3.31. $\varphi_1 - \varphi_2 = \frac{\lambda}{2\pi\varepsilon_0} \ln \eta = 5$ kV.

3.32. $\varphi = {}^1/_2 \sigma R/\varepsilon_0$, $E = {}^1/_4\, \sigma/\varepsilon_0$.

3.33. $\varphi = \frac{\sigma l}{2\varepsilon_0}(\sqrt{1 + (R/l)^2} - 1)$, $E = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{l}{\sqrt{l^2 + R^2}}\right)$. When $l \to 0$, then $\varphi = \frac{\sigma R}{2\varepsilon_0}$, $E = \frac{\sigma}{2\varepsilon_0}$; when $l \gg R$, then $\varphi \approx \frac{q}{4\pi\varepsilon_0 l}$, $E \approx \frac{q}{4\pi\varepsilon_0 l^2}$, where $q = \sigma\pi R^2$.

3.34. $\varphi = \sigma R/\pi\varepsilon_0$.

3.35. $\mathbf{E} = -\mathbf{a}$, i.e. the field is uniform.

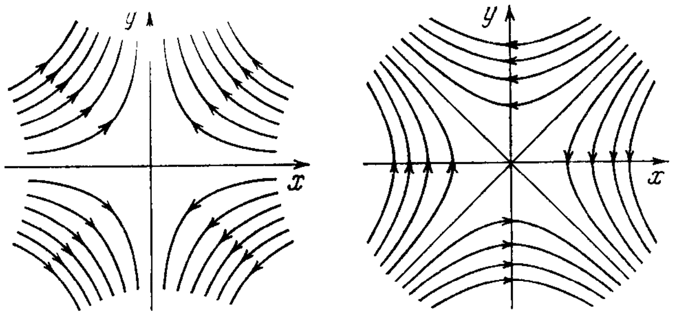
3.36. (a) $\mathbf{E} = -2a(x\mathbf{i} - y\mathbf{j})$; (b) $\mathbf{E} = -a(y\mathbf{i} - x\mathbf{j})$. Here $\mathbf{i}$, $\mathbf{j}$ are the unit vectors of the x and y axes. See Fig. 16 illustrating the case $a > 0$.

3.37. $\mathbf{E} = -2(ax\mathbf{i} + ay\mathbf{j} + bz\mathbf{k})$, $E = 2\sqrt{a^2(x^2 + y^2) + b^2 z^2}$. (a) An ellipsoid of revolution with semiaxes $\sqrt{\varphi/a}$ and $\sqrt{\varphi/b}$. (b) In the case of $\varphi > 0$, a single-cavity hyperboloid of revolution; when $\varphi = 0$, a right round cone; when $\varphi < 0$, a two-cavity hyperboloid of revolution.

3.38. (a) $\varphi_0 = \frac{3q}{8\pi\varepsilon_0 R}$; (b) $\varphi = \varphi_0\left(1 - \frac{r^2}{3R^2}\right)$, $r \leqslant R$.

3.39. $E=\sqrt{E_r^2+E_\theta^2}=\frac{p}{4\pi\varepsilon_0 r^3}\sqrt{1+3\cos^2\theta}$, where E_r is the radial component of the vector $\mathbf{E}$, and E_θ is its component perpendicular to E_r.

3.40. $E_z=\frac{p}{4\pi\varepsilon_0}\frac{3\cos^2\theta-1}{r^3}$, $E_\perp=\frac{p}{4\pi\varepsilon_0}\frac{3\sin\theta\cos\theta}{r^3}$; $\mathbf{E}\perp\mathbf{p}$ at the points located on the lateral surface of a cone whose axis is directed along the z axis and whose semi-vertex angle θ is found



(a) (b)

Fig. 16.

from the relation $\cos\theta = 1/\sqrt{3}$ ($\theta_1 = 54.7°$, $\theta_2 = 123.5°$). At these points $E = E_\perp = \frac{p\sqrt{2}}{4\pi\varepsilon_0 r^3}$.

3.41. $R=\sqrt[3]{\frac{p}{4\pi\varepsilon_0 E_0}}$.

3.42. $\varphi\approx\frac{\lambda l}{2\pi\varepsilon_0 r}\cos\theta$, $E\approx\frac{\lambda l}{2\pi\varepsilon_0 r^2}$.

3.43. $\varphi=\frac{ql}{4\pi\varepsilon_0}\frac{x}{(R^2+x^2)^{3/2}}$, $E_x=-\frac{ql}{4\pi\varepsilon_0}\frac{R^2-2x^2}{(R^2+x^2)^{5/2}}$, where

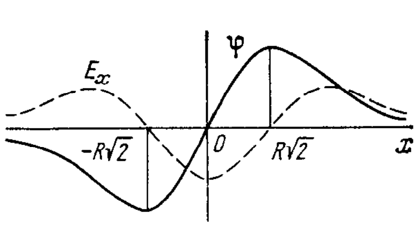


Fig. 17.

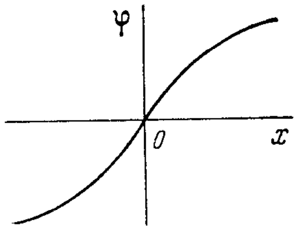


Fig. 18.

E_x is the projection of the vector $\mathbf{E}$ on the x axis. The functions are plotted in Fig. 17. If $|x|\gg R$, then $\varphi\approx\frac{ql}{4\pi\varepsilon_0 x^2}$ and $E_x\approx\frac{ql}{2\pi\varepsilon_0 x^3}$.

3.44. $\varphi=\frac{\sigma l}{2\varepsilon_0}\frac{x}{\sqrt{x^2+R^2}}$, $E_x=-\frac{\sigma l R^2}{2\varepsilon_0(x^2+R^2)^{3/2}}$. See Fig. 18.

3.45. $\varphi\approx\pm\frac{\sigma l}{2\varepsilon_0}\left(1-\frac{x}{\sqrt{x^2+R^2}}\right)$, $E\approx\frac{\sigma l R^2}{2\varepsilon_0(x^2+R^2)^{3/2}}$. If $x\gg R$, then $\varphi\approx\pm\frac{p}{4\sigma\varepsilon_0 x^2}$ and $E\approx\frac{p}{2\pi\varepsilon_0 x^3}$, where $p=\pi R^2\sigma l$. In the formulas for the potential φ the plus sign corresponds to the space adjoining the positively charged plate and the minus sign to the space adjoining the negatively charged plate.

3.46. (a) $\mathbf{F}=0$; (b) $\mathbf{F}=-\frac{\lambda\mathbf{p}}{2\pi\varepsilon_0 r^2}$; (c) $\mathbf{F}=\frac{\lambda\mathbf{p}}{2\pi\varepsilon_0 r^2}$.

3.47. $F=\frac{3p^2}{2\pi\varepsilon_0 l^4}=2.1\cdot10^{-16}$ N.

3.48. $\varphi=-axy+\text{const}$.

3.49. $\varphi=ay\left(\frac{y^2}{3}-x^2\right)+\text{const}$.

3.50. $\varphi=-y(ax+bz)+\text{const}$.

3.51. $\rho=6\varepsilon_0 ax$.

3.52. $\rho=2\varepsilon_0\Delta\varphi/d^2$; $E=\rho d/\varepsilon_0$.

3.53. $\rho=-6\varepsilon_0 a$.

3.54. $q=4l\sqrt{\pi\varepsilon_0 kx}$.

3.55. $A=\frac{q^2}{16\pi\varepsilon_0 l}$.

3.56. (a) $F=\frac{(2\sqrt{2}-1)q^2}{8\pi\varepsilon_0 l^2}$; (b) $E=2\left(1-\frac{1}{5\sqrt{5}}\right)\frac{q}{\pi\varepsilon_0 l^2}$.

3.57. $F=\frac{(2\sqrt{2}-1)q^2}{32\pi\varepsilon_0 l^2}$.

3.58. $F=\frac{3p^2}{32\pi\varepsilon_0 l^4}$.

3.59. $\sigma=-\frac{ql}{2\pi(l^2+r^2)^{3/2}}$, $q_{ind}=-q$.

3.60. (a) $F_1=\frac{\lambda^2}{4\pi\varepsilon_0 l}$; (b) $\sigma=\frac{l\lambda}{\pi(l^2+x^2)}$.

3.61. (a) $\sigma=\frac{\lambda}{2\pi l}$; (b) $\sigma(r)=\frac{\lambda}{2\pi\sqrt{l^2+r^2}}$.

3.62. (a) $\sigma=\frac{lq}{2\pi(l^2+R^2)^{3/2}}$; (b) $E=\frac{1}{4\pi\varepsilon_0}\frac{q}{4l^2[1+{}^1/_4(R/l)^2]^{3/2}}$, $\varphi=\frac{1}{4\pi\varepsilon_0}\frac{q}{R}\left(1-\frac{1}{\sqrt{1+4(l/R)^2}}\right)$.

3.63. $\varphi=\frac{q}{4\pi\varepsilon_0 l}$.

3.64. $\varphi=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{1}{R_1}+\frac{1}{R_2}\right)$.

3.65. $q_2=-\frac{b}{a}q_1$; $\varphi=\frac{q_1}{4\pi\varepsilon_0}\times\begin{cases}1/r-1/a & \text{if } a\leqslant r\leqslant b,\\ (1-b/a)r & \text{if } r\geqslant b.\end{cases}$

3.66. (a) $E_{23}=\Delta\varphi/d$, $E_{12}=E_{34}={}^1/_2 E_{23}$; (b) $|\sigma_1|=\sigma_4={}^1/_2\varepsilon_0\Delta\varphi/d$, $\sigma_2=|\sigma_3|={}^3/_2\varepsilon_0\Delta\varphi/d$.

3.67. $q_1=-q(l-x)/l$, $q_2=-qx/l$. **Instruction.** If the charge q is imagined to be uniformly spread over the plane passing through

that charge and parallel to the plates, the charges q_1 and q_2 remain, obviously, unchanged. What changes is only their distribution, and the electric field becomes easy to calculate.

3.68. $dF/dS = {}^1/_2\sigma^2/\varepsilon_0$.

3.69. $F = \frac{q^2}{32\pi\varepsilon_0 R^2} = 0.5$ kN.

3.70. $F = {}^1/_4\pi R^2\sigma_0^2/\varepsilon_0$.

3.71. $N = \frac{n_0 p}{(\varepsilon-1)\,\varepsilon_0 E} = 3\cdot 10^3$, where n_0 is the concentration of molecules.

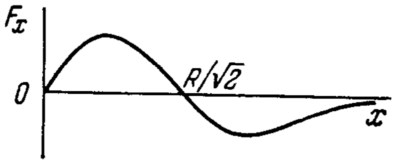


Fig. 19.

3.72. $F = \frac{3\beta p^2}{4\pi^2\varepsilon_0 l^7}$.

3.73. (a) $x = R/\sqrt{2}$; (b) $x = \begin{cases} 1.1R & \text{(attraction)} \\ 0.29R & \text{(repulsion).} \end{cases}$ See Fig. 19.

3.74. $\mathbf{P} = \frac{\varepsilon-1}{\varepsilon}\frac{q}{4\pi r^3}\mathbf{r}$, $q' = -\frac{\varepsilon-1}{\varepsilon}q$.

3.76. $q'_{inn} = -q(\varepsilon-1)/\varepsilon$, $q'_{out} = q(\varepsilon-1)/\varepsilon$.

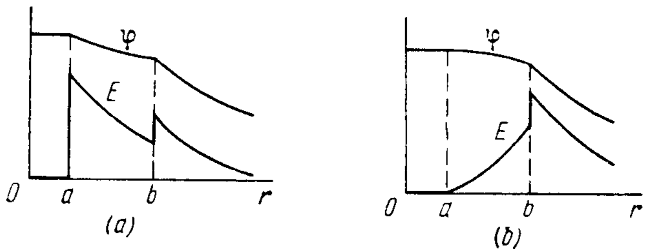


Fig. 20.

3.77. See Fig. 20.

3.78. $E = \frac{E_0}{\varepsilon}\sqrt{\cos^2\alpha_0 + \varepsilon^2\sin^2\alpha_0} = 5.2$ V/m;

$\tan\alpha = \varepsilon\tan\alpha_0$, hence, $\alpha = 74°$; $\sigma' = \frac{\varepsilon_0(\varepsilon-1)}{\varepsilon}E_0\cos\alpha_0 = 64$ pC/m^2.

3.79. (a) $\oint \mathbf{E}\,d\mathbf{S} = \frac{\varepsilon-1}{\varepsilon}\pi R^2 E_0\cos\theta$; (b) $\oint \mathbf{D}\,d\mathbf{r} = -\varepsilon_0(\varepsilon-1)\times$ $\times\, l E_0\sin\theta$.

3.80. (a) $E = \begin{cases} \rho l/\varepsilon\varepsilon_0 & \text{for } l < d, \\ \rho d/\varepsilon_0 & \text{for } l > d, \end{cases}$ $\varphi = \begin{cases} -\rho l^2/2\varepsilon\varepsilon_0 & \text{for } l \leqslant d, \\ -(d/2\varepsilon + l - d)\,\rho d/\varepsilon_0 & \text{for } l \geqslant d. \end{cases}$

The plots $E_x(x)$ and $\varphi(x)$ are shown in Fig. 21. (b) $\sigma' = \rho d(\varepsilon-1)/\varepsilon$, $\rho' = -\rho(\varepsilon-1)/\varepsilon$.

3.81. (a) $E = \begin{cases} \rho r/3\varepsilon_0\varepsilon & \text{for } r < R, \\ \rho R^3/3\varepsilon_0 r^2 & \text{for } r > R; \end{cases}$

(b) $\rho' = -\rho(\varepsilon-1)/\varepsilon$, $\sigma' = \rho R(\varepsilon-1)/3\varepsilon$. See Fig. 22.

3.82. $\mathbf{E} = -d\mathbf{P}/4\varepsilon_0 R$.

3.83. $\mathbf{E} = -\mathbf{P}_0(1 - x^2/d^2)/\varepsilon_0$, $U = 4dP_0/3\varepsilon_0$.

3.84. (a) $E_1 = 2\varepsilon E_0/(\varepsilon+1)$, $E_2 = 2E_0/(\varepsilon+1)$, $D_1 = D_2 = 2\varepsilon\varepsilon_0 E_0/(\varepsilon+1)$; (b) $E_1 = E_0$, $E_2 = E_0/\varepsilon$, $D_1 = D_2 = \varepsilon_0 E_0$.

3.85. (a) $E_1 = E_2 = E_0$, $D_1 = \varepsilon_0 E_0$, $D_2 = \varepsilon D_1$; (b) $E_1 = E_2 = 2E_0/(\varepsilon+1)$, $D_1 = 2\varepsilon_0 E_0/(\varepsilon+1)$, $D_2 = \varepsilon D_1$.

3.86. $E = q/2\pi\varepsilon_0(\varepsilon+1)\,r^2$.

3.87. $\rho = \rho_0\varepsilon/(\varepsilon-1) = 1.6$ g/cm^3, where ε and ρ_0 are the permittivity and density of kerosene.

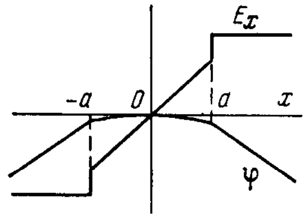


Fig. 21.

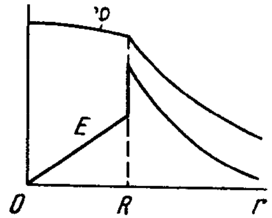


Fig. 22.

3.88. $\sigma'_{max} = (\varepsilon-1)\,\varepsilon_0 E = 3.5$ nC/m^2, $q' = \pi R^2(\varepsilon-1)\,\varepsilon_0 E = 10$ pC.

3.89. (a) Since the normal component of the vector **D** is continuous at the dielectric interface, we obtain
$\sigma' = -ql(\varepsilon-1)/2\pi r^3(\varepsilon+1)$, for $l \to 0$ and $\sigma' \to 0$;
(b) $q' = -q(\varepsilon-1)/(\varepsilon+1)$.

3.90. $F = q^2(\varepsilon-1)/16\pi\varepsilon_0 l^2(\varepsilon+1)$.

3.91. $D = \begin{cases} q/2\pi(1+\varepsilon)\,r^2 & \text{in vacuum,} \\ \varepsilon q/2\pi(1+\varepsilon)\,r^2 & \text{in dielectric;} \end{cases}$

$\left.\begin{array}{l} E = q/2\pi\varepsilon_0(1+\varepsilon)\,r^2 \\ \varphi = q/2\pi\varepsilon_0(1+\varepsilon)\,r \end{array}\right\}$ both in vacuum and in dielectric.

3.92. $\sigma' = ql(\varepsilon-1)/2\pi r^3\varepsilon(\varepsilon+1)$; for $l \to 0$ and $\sigma' \to 0$.

3.93. $\sigma' = ql(\varepsilon-1)/2\pi r^3\varepsilon$.

3.94. $\mathbf{E}_1 = \mathbf{P}h/\varepsilon_0 d$ (between the plates), $\mathbf{E}_2 = -(1-h/d)\mathbf{P}/\varepsilon_0$, $\mathbf{D}_1 = \mathbf{D}_2 = \mathbf{P}h/d$.

3.95. $\rho' = -2\alpha$, i.e. is independent of r.

3.96. (a) $\mathbf{E} = -\mathbf{P}/3\varepsilon_0$.

3.97. $\mathbf{E}_0 = \mathbf{E} - \mathbf{P}/3\varepsilon_0$.

3.98. $\mathbf{E} = 3\mathbf{E}_0/(\varepsilon+2)$, $\mathbf{P} = 3\varepsilon_0\mathbf{E}_0(\varepsilon-1)/(\varepsilon+2)$.

3.99. $\mathbf{E} = -\mathbf{P}/2\varepsilon_0$.

3.100. $\mathbf{E} = 2\mathbf{E}_0/(\varepsilon+1)$; $\mathbf{P} = 2\varepsilon_0\mathbf{E}_0(\varepsilon-1)/(\varepsilon+1)$.

3.101. $C = \frac{4\pi\varepsilon_0\varepsilon R_1}{1+(\varepsilon-1)\,R_1/R_2}$.

3.102. The strength decreased ${}^1/_2(\varepsilon+1)$ times; $q = {}^1/_2 C\mathscr{E}(\varepsilon-1)/(\varepsilon+1)$.

3.103. (a) $C = \frac{\varepsilon_0 S}{d_1/\varepsilon_1 + d_2/\varepsilon_2}$; (b) $\sigma' = \varepsilon_0 V\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1 d_2 + \varepsilon_2 d_1}$.

3.104. (a) $C = \varepsilon_0(\varepsilon_2-\varepsilon_1)\,S/d\ln(\varepsilon_2/\varepsilon_1)$; (b) $\rho' = -q(\varepsilon_2-\varepsilon_1)/dS\varepsilon^2$.

3.105. $C = 4\pi\varepsilon_0 a/\ln(R_2/R_1)$.
3.106. When $\varepsilon_1 R_1 E_{1m} = \varepsilon_2 R_2 E_{2m}$.
3.107. $V = R_1 E_1 [\ln(R_2/R_1) + (\varepsilon_1/\varepsilon_2)\ln(R_3/R_2)]$.
3.108. $C \approx \pi\varepsilon_0 \ln(b/a)$.
3.109. $C \approx 2\pi\varepsilon_0/\ln(2b/a)$.
3.110. $C \approx 2\pi\varepsilon_0 \varepsilon a$. **Instruction.** When $b \gg a$, the charges can be assumed to be distributed practically uniformly over the surfaces of the balls.
3.111. $C \approx 4\pi\varepsilon_0 a$.
3.112. (a) $C_{total} = C_1 + C_2 + C_3$; (b) $C_{total} = C$.
3.113. (a) $C = 2\varepsilon_0 S/3d$; (b) $C = 3\varepsilon_0 S/2d$.
3.114. $V \leqslant V_1(1 + C_1/C_2) = 9$ kV.
3.115. $U = \mathscr{E}/(1 + 3\eta + \eta^2) = 10$ V.
3.116. $C_x = C(\sqrt{5} - 1)/2 = 0.62C$. Since the chain is infinite, all the links beginning with the second can be replaced by the capacitance C_x equal to the sought one.
3.117. $V_1 = q/C_1 = 10$ V, $V_2 = q/C_2 = 5$ V, where $q = (\varphi_A - \varphi_B + \mathscr{E})C_1C_2/(C_1 + C_2)$.
3.118. $V_1 = (\mathscr{E}_2 - \mathscr{E}_1)/(1 + C_1/C_2)$, $V_2 = (\mathscr{E}_1 - \mathscr{E}_2)/(1 + C_2/C_1)$.
3.119. $q = |\mathscr{E}_1 - \mathscr{E}_2| C_1C_2/(C_1 + C_2)$.
3.120. $\varphi_A - \varphi_B = \mathscr{E}\dfrac{C_2C_3 - C_1C_4}{(C_1+C_2)(C_3+C_4)}$. In the case when $C_1/C_2 = C_3/C_4$.
3.121. $q = \dfrac{V}{1/C_1 + 1/C_2 + 1/C_3} = 0.06$ mC.
3.122. $q_1 = \mathscr{E}C_2$, $q_2 = -\mathscr{E}C_1C_2/(C_1 + C_2)$.
3.123. $q_1 = \mathscr{E}C_1(C_1 - C_2)/(C_1 + C_2) = -24$ μC, $q_2 = \mathscr{E}C_2(C_1 - C_2)/(C_1 + C_2) = -36$ μC, $q_3 = \mathscr{E}(C_2 - C_1) = +60$μ C.
3.124. $\varphi_A - \varphi_B = (C_2\mathscr{E}_2 - C_1\mathscr{E}_1)/(C_1 + C_2 + C_3)$.
3.125. $\varphi_1 = \dfrac{\mathscr{E}_2C_2 + \mathscr{E}_3C_3 - \mathscr{E}_1(C_2+C_3)}{C_1+C_2+C_3}$,
$$\varphi_2 = \frac{\mathscr{E}_1C_1 + \mathscr{E}_3C_3 - \mathscr{E}_2(C_1+C_3)}{C_1+C_2+C_3}, \quad \varphi_3 = \frac{\mathscr{E}_1C_1 + \mathscr{E}_2C_2 - \mathscr{E}_3(C_1+C_2)}{C_1+C_2+C_3}.$$
3.126. $C_{total} = \dfrac{2C_1C_2 + C_3(C_1 + C_2)}{C_1 + C_2 + 2C_3}$.
3.127. (a) $W = (\sqrt{2} + 4)q^2/4\pi\varepsilon_0 a$; (b) $W = (\sqrt{2} - 4)q^2/4\pi\varepsilon_0 a$; (c) $W = -\sqrt{2}q^2/4\pi\varepsilon_0 a$.
3.128. $W = -\dfrac{2\ln 2}{4\pi\varepsilon_0}\dfrac{q^2}{a}$.
3.129. $W = -q^2/8\pi\varepsilon_0 l$.
3.130. $W = q_1q_2/4\pi\varepsilon_0 l$.
3.131. $\Delta W = -{}^1/_2 V^2 C_1C_2/(C_1 + C_2) = -0.03$ mJ.
3.132. $Q = \mathscr{E}^2CC_0/(2C + C_0)$.
3.133. $Q = {}^1/_2 C\mathscr{E}_2^2$. It is remarkable that the result obtained is independent of $\mathscr{E}_1$.
3.134. $W = W_1 + W_2 + W_{12} = \dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{q_1^2}{2R_1} + \dfrac{q_2^2}{2R_2} + \dfrac{q_1q_2}{R_2}\right)$.

3.135. (a) $W = 3q^2/20\pi\varepsilon_0 R$; (b) $W_1/W_2 = 1/5$.
3.136. $W = (q^2/8\pi\varepsilon_0\varepsilon)(1/a - 1/b) = 27$ mJ.
3.137. $A = (q^2/8\pi\varepsilon_0)(1/R_1 - 1/R_2)$.
3.138. $A = \dfrac{q(q_0 + q/2)}{4\pi\varepsilon_0}\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$.
3.139. $F_1 = \sigma^2/2\varepsilon_0$.
3.140. $A = (q^2/8\pi\varepsilon_0)(1/a - 1/b)$.
3.141. (a) $A = q^2(x_2 - x_1)/2\varepsilon_0 S$; (b) $A = \varepsilon_0 SV^2(x_2 - x_1)/2x_1x_2$.
3.142. (a) $A = {}^1/_2 CV^2\eta/(1 - \eta)^2 = 1.5$ mJ; (b) $A = {}^1/_2 CV^2\eta\varepsilon(\varepsilon - 1)/[\varepsilon - \eta(\varepsilon - 1)]^2 = 0.8$ mJ.
3.143. $\Delta p = \varepsilon_0\varepsilon(\varepsilon - 1)V^2/2d^2 = 7$ kPa $= 0.07$ atm.
3.144. $h = (\varepsilon - 1)\sigma^2/2\varepsilon_0\varepsilon\rho g$.
3.145. $F = \pi R\varepsilon_0(\varepsilon - 1)V^2/d$.
3.146. $N = (\varepsilon - 1)\varepsilon_0 R^2V^2/4d$.
3.147. $I = 2\pi\varepsilon_0 aEv = 0.5$ μA.
3.148. $I \approx 2\pi\varepsilon_0(\varepsilon - 1)rvV/d = 0.11$ μA.
3.149. (a) $\alpha = (\alpha_1 + \eta\alpha_2)/(1 + \eta)$; (b) $\alpha \approx (\alpha_2 + \eta\alpha_1)/(1 + \eta)$.
3.150. (a) ${}^5/_6 R$; (b) ${}^7/_{12} R$; (c) ${}^3/_4 R$.
3.151. $R_x = R(\sqrt{3} - 1)$.
3.152. $R = (1 + \sqrt{1 + 4R_2/R_1})R_1/2 = 6\ \Omega$. **Instruction.** Since the chain is infinite, all the links beginning with the second can be replaced by the resistance equal to the sought resistance R.
3.153. Imagine the voltage V to be applied across the points A and B. Then $V = IR = I_0R_0$, where I is the current carried by the lead wires, I_0 is the current carried by the conductor AB.

The current I_0 can be represented as a superposition of two currents. If the current I flowed into point A and spread all over the infinite wire grid, the conductor AB would carry (because of symmetry) the current $I/4$. Similarly, if the current I flowed into the grid from infinity and left the grid through point B, the conductor AB would also carry the current $I/4$. Superposing both of these solutions, we obtain $I_0 = I/2$. Therefore, $R = R_0/2$.
3.154. $R = (\rho/2\pi l)\ln(b/a)$.
3.155. $R = \rho(b - a)/4\pi ab$. In the case of $b \to \infty$ $R = \rho/4\pi a$.
3.156. $\rho = 4\pi\Delta tab/(b - a)C\ln\eta$.
3.157. $R = \rho/2\pi a$.
3.158. (a) $j = 2alV/\rho r^3$; (b) $R = \rho/4\pi a$.
3.159. (a) $j = lV/2\rho r^2\ln(l/a)$; (b) $R_1 = (\rho/\pi)\ln(l/a)$.
3.160. $I = VC/\rho\varepsilon\varepsilon_0 = 1.5$ μA.
3.161. $RC = \rho\varepsilon\varepsilon_0$.
3.162. $\sigma = D_n = D\cos\alpha$; $j = D\sin\alpha/\varepsilon\varepsilon_0\rho$.
3.163. $I = VS(\sigma_2 - \sigma_1)/d\ln(\sigma_2/\sigma_1) = 5$ nA.
3.165. $q = \varepsilon_0(\rho_2 - \rho_1)I$.
3.166. $\sigma = \varepsilon_0 V(\varepsilon_2\rho_2 - \varepsilon_1\rho_1)/(\rho_1 d_1 + \rho_2 d_2)$, $\sigma = 0$ if $\varepsilon_1\rho_1 = \varepsilon_2\rho_2$.
3.167. $q = \varepsilon_0 I(\varepsilon_2\rho_2 - \varepsilon_1\rho_1)$.
3.168. $\rho = 2\varepsilon_0 V(\eta - 1)/d^2(\eta + 1)$.

3.169. (a) $R_1 = 2\pi\alpha/S^2$; (b) $E = 2\pi\alpha I/S^2$.
3.170. $t = -RC \ln (I - V/V_0) = 0.6$ μs.
3.171. $\rho = \tau/\varepsilon_0\varepsilon \ln 2 = 1.4\cdot10^{13}$ Ω·m.
3.172. $I = [(\eta - 1)\,\mathscr{E}/R]\,\mathrm{e}^{-\eta t/RC}$.
3.173. $V = \mathscr{E}/(\eta + 1) = 2.0$ V.
3.174. $\varphi_1 - \varphi_2 = (\mathscr{E}_1 - \mathscr{E}_2)\,R_1/(R_1 + R_2) - \mathscr{E}_1 = -4$ V.
3.175. $R = R_2 - R_1$, $\Delta\varphi = 0$ in the source of current with internal resistance R_2.
3.176. (a) $I = \alpha$; (b) $\varphi_A - \varphi_B = 0$.
3.177. $\varphi_A - \varphi_B = (\mathscr{E}_1 - \mathscr{E}_2)\,R_1/(R_1 + R_2) = -0.5$ V.
3.178. $I_1 = \mathscr{E}R_2/(RR_1 + R_1R_2 + R_2R) = 1.2$ A, $I_2 = I_1R_1/R_2 = 0.8$ A.
3.179. $V = V_0Rx/[Rl + R_0\,(l - x)\,x/l]$; for $R \gg R_0$ $V \approx V_0x/l$.
3.180. $\mathscr{E} = (\mathscr{E}_1R_2 + \mathscr{E}_2R_1)/(R_1 + R_2)$, $R_i = R_1R_2/(R_1 + R_2)$.
3.181. $I = (R_1\mathscr{E}_2 - R_2\mathscr{E}_1)/(RR_1 + R_1R_2 + R_2R) = 0.02$ A, the current is directed from the left to the right (see Fig. 3.44).
3.182. (a) $I_1 = [R_3\,(\mathscr{E}_1 - \mathscr{E}_2) + R_2(\mathscr{E}_1 + \mathscr{E}_3)]/(R_1R_2 + R_2R_3 + R_3R_1) = 0.06$ A; (b) $\varphi_A - \varphi_B = \mathscr{E}_1 - I_1R_1 = 0.9$ V.
3.183. $I = [\mathscr{E}\,(R_2 + R_3) + \mathscr{E}_0R_3]/[R\,(R_2 + R_3) + R_2R_3]$.
3.184. $\varphi_A - \varphi_B = [\mathscr{E}_2R_3\,(R_1 + R_2) - \mathscr{E}_1R_1\,(R_2 + R_3)]/(R_1R_2 + R_2R_3 + R_3R_1) = -1.0$ V.
3.185. $I_1 = [R_3\,(\varphi_1 - \varphi_2) + R_2\,(\varphi_1 - \varphi_3)]/(R_1R_2 + R_2R_3 + R_3R_1) = 0.2$ A.
3.186. $I = \dfrac{V}{R_2}\left(\dfrac{R_1 + R_2}{R_1\,[1 + R_2R_4\,(R_1 + R_3)/R_1R_3\,(R_2 + R_4)]} - 1\right) = 1.0$ A.
The current flows from point C to point D.
3.187. $R_{AB} = r\,(r + 3R)/(R + 3r)$.
3.188. $V = {}^1/_2\mathscr{E}\,(1 - \mathrm{e}^{-2t/RC})$.
3.189. (a) $Q = {}^4/_3\,q^2R/\Delta t$; (b) $Q = {}^1/_2 \ln 2\cdot q^2R/\Delta t$.
3.190. $R = 3R_0$.
3.192. $Q = I\,(\mathscr{E} - V) = 0.6$ W, $P = -IV = -2.0$ W.
3.193. $I = V/2R$; $P_{max} = V^2/4R$; $\eta = 1/2$.
3.194. By $2\eta = 2\%$.
3.195. $T - T_0 = (1 - \mathrm{e}^{-kt/C})\,V^2/kR$.
3.196. $R_x = R_1R_2/(R_1 + R_2) = 12$ Ω.
3.197. $R = R_1R_2/(R_1 + R_2)$;
$Q_{max} = (\mathscr{E}_1R_2 + \mathscr{E}_2R_1)^2/4R_1R_2\,(R_1 + R_2)$.
3.198. $n = \sqrt{Nr/R} = 3$.
3.199. $Q = {}^1/_2C\mathscr{E}^2R_1/(R_1 + R_2) = 60$ mJ.
3.200. (a) $\Delta W = -{}^1/_2CV^2\eta/(1 - \eta) = -0.15$ mJ; (b) $A = {}^1/_2CV^2\eta/(1 - \eta) = 0.15$ mJ.
3.201. $\Delta W = -{}^1/_2\,(\varepsilon - 1)\,CV^2 = -0.5$ mJ, $A_{mech} = {}^1/_2\,(\varepsilon - 1)\,CV^2 = 0.5$ mJ.
3.202. $h \approx {}^1/_2\varepsilon_0\,(\varepsilon - 1)\,V^2/\rho gd^2$, where ρ is the density of water.
3.203. (a) $q = q_0\mathrm{e}^{-t/\varepsilon_0\varepsilon\rho}$; (b) $Q = (1/a - 1/b)\,q_0^2/8\pi\varepsilon_0\varepsilon$.
3.204. (a) $q = q_0\,(1 - \mathrm{e}^{-\tau/RC}) = 0.18$ mC; (b) $Q = (1 - \mathrm{e}^{-2\tau/RC})q_0^2/2C = 82$ mJ.

3.205. (a) $I = (V_0/R)\,\mathrm{e}^{-2t/RC}$; (b) $Q = {}^1/_4\,CV_0^2$.
3.206. $e/m = l\omega r/qR = 1.8\cdot10^{11}$ C/kg.
3.207. $p = lIm/e = 0.40$ μN·s.
3.208. $s = enl\,\langle v\rangle/j \sim 10^7$ m, where n is the concentration of free electrons, $\langle v\rangle$ is the mean velocity of thermal motion of an electron.
3.209. (a) $t = enlS/I = 3$ Ms; (b) $F = enl\rho I = 1.0$ MN, where ρ is the resistivity of copper.
3.210. $E = (I/2\pi\varepsilon_0r)\sqrt{m/2eV} = 32$ V/m, $\Delta\varphi = (I/4\pi\varepsilon_0)\sqrt{m/2eV} = 0.80$ V.
3.211. (a) $\rho\,(x) = -{}^4/_9\varepsilon_0ax^{-2/3}$; (b) $j = {}^4/_9\varepsilon_0a^{3/2}\sqrt{2e/m}$.
3.212. $n = Id/e\,(u_0^+ + u_0^-)\,VS = 2.3\cdot10^8$ cm^{-3}.
3.213. $u_0 = \omega_0l^2/2V_0$.
3.214. (a) $\dot{n}_i = I_{sat}/eV = 6\cdot10^9$ cm$^{-3}\cdot$s^{-1}; (b) $n = \sqrt{n_i/r} = 6\cdot10^7$ cm^{-3}.
3.215. $t = (\eta - 1)\Big/\sqrt{r\dot{n}_i} = 13$ ms.
3.216. $t = \varepsilon_0\eta U/e\dot{n}_id^2 = 4.6$ days.
3.217. $I = e\nu_0\mathrm{e}^{\alpha d}$.
3.218. $j = (\mathrm{e}^{\alpha d} - 1)\,e\dot{n}_i/\alpha$.
3.219. (a) $B = \mu_0I/2R = 6.3$ μT; (b) $B = \mu_0R^2I/2(R^2 + x^2)^{3/2} = 2.3$ μT.
3.220. $B = n\mu_0I\tan(\pi/n)/2\pi R$, for $n \to \infty$ $B = \mu_0I/2R$
3.221. $B = 4\mu_0I/\pi d \sin\varphi = 0.10$ mT.
3.222. $B = (\pi - \varphi + \tan\varphi)\,\mu_0I/2\pi R = 28$μT.
3.223. (a) $B = \dfrac{\mu_0I}{4\pi}\left(\dfrac{2\pi - \varphi}{a} + \dfrac{\varphi}{b}\right)$; (b) $B = \dfrac{\mu_0I}{4\pi}\left(\dfrac{3\pi}{4a} + \dfrac{\sqrt{2}}{b}\right)$.
3.224. $B \approx \mu_0hI/4\pi^2Rr$, where r is the distance from the cut.
3.225. $B = \mu_0I/\pi^2R$.
3.226. (a) $B = (\mu_0/4\pi)\,(\pi I/R)$; (b) $B = (\mu_0/4\pi)\,(1 + 3\pi/2)\,I/R$; (c) $B = (\mu_0/4\pi)\,(2 + \pi)\,I/R$.
3.227. $B = (\mu_0/4\pi)\,I\sqrt{2}/l = 2.0$μT.
3.228. (a) $B = (\mu_0/4\pi)\sqrt{4 + \pi^2}\,I/R = 0.30$ μT; (b) $B = (\mu_0/4\pi)\times\sqrt{2 + 2\pi + \pi^2}\,I/R = 0.34$μT; (c) $B = (\mu_0/4\pi)\sqrt{2}\,I/R = 0.11$ μT.
3.229. (a) $B = \mu_0i/2$; (b) $B = \mu_0i$ between the planes and $B = 0$ outside the planes.
3.230. $B = \begin{cases} \mu_0jx & \text{inside the plate,} \\ \mu_0jd & \text{outside the plate.} \end{cases}$
3.231. In the half-space with the straight wire, $B = \mu_0I/2\pi r$, r is the distance from the wire. In the other half-space $B \equiv 0$.
3.232. The given integral is equal to μ_0I.
3.233. $\mathbf{B} = \begin{cases} {}^1/_2\mu_0\,[\mathbf{jr}] & \text{for } r \leqslant R, \\ {}^1/_2\mu_0\,[\mathbf{jr}]\,R^2/r^2 & \text{for } r \geqslant R. \end{cases}$
3.234. $\mathbf{B} = {}^1/_2\mu_0\,[\mathbf{jl}]$, i.e. field inside the cavity is uniform.
3.235. $j\,(r) = (b/\mu_0)\,(1 + \alpha)\,r^{\alpha-1}$.

3.236. $B=\mu_0 nI/\sqrt{1+(2R/l)^2}$.

3.237. (a) $B={}^1/_2\mu_0 nI\,(1-x/\sqrt{x^2+R^2})$, where $x>0$ outside the solenoid and $x<0$ inside the solenoid; see Fig. 23; (b) $x_0=$ $=R\,(1-2\eta)/2\sqrt{\eta\,(1-\eta)}\approx 5R$.

3.238. $B=\begin{cases}(\mu_0 I/h)\sqrt{(1-(h/2\pi R)^2}=0.3\ \text{mT},\ r<R,\\ (\mu_0/4\pi)\,2I/r,\ r>R.\end{cases}$

3.239. $\eta\approx N/\pi=8\cdot 10^2$.

3.240. $\Phi=(\mu_0/4\pi)\ I=$ $=1.0\ \mu\text{Wb/m}$.

3.241. $\Phi=\Phi_0/2=\mu_0 nIS/2$, where Φ_0 is the flux of the vector **B** through the cross-section of the solenoid far from its ends.

3.242. $\Phi=(\mu_0/4\pi)\ 2INh\ln\eta=$ $=8\ \mu\text{Wb}$.

3.243. $p_m=2\pi R^3B/\mu_0=$ $=30\ \text{mA}\cdot\text{m}^2$.

3.244. $p_m={}^1/_2NId^2=$ $=0.5\ \text{A}\cdot\text{m}^2$.

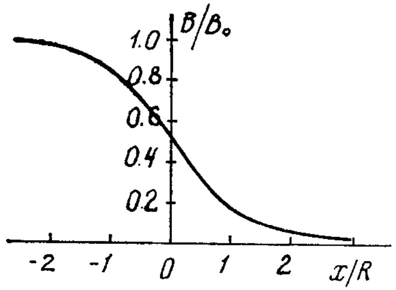


Fig. 23.

3.245. (a) $B=\dfrac{\mu_0 IN\ln(b/a)}{2\,(b-a)}=7\ \mu\text{T}$; (b) $p_m={}^1/_3\pi IN\ (a^2+ab+b^2)=15\ \text{mA}\cdot\text{m}^2$.

3.246. (a) $B={}^1/_2\mu_0\sigma\omega R$; (b) $p_m={}^1/_4\pi\sigma\omega R^4$.

3.247. $B={}^2/_3\mu_0\sigma\omega R=29\ \text{pT}$.

3.248. $p_m={}^1/_5qR^2\omega$; $p_m/M=q/2m$.

3.249. $\mathbf{B}=0$.

3.250. $F_m/F_e=\mu_0\varepsilon_0v^2=(v/c)^2=1.00\cdot 10^{-6}$.

3.251. (a) $F_1=\mu_0I^2/4R=0.20\ \text{mN/m}$; (b) $F_1=\mu_0I^2/\pi l=$ $=0.13\ \text{mN/m}$.

3.252. $B=\pi d^2\sigma_m/4RI=8\ \text{kT}$, where σ_m is the strength of copper.

3.253. $B=(2\rho gS/I)\tan\theta=10\ \text{mT}$, where ρ is the density of copper.

3.254. $B=\Delta mgl/NIS=0.4\ \text{T}$.

3.255. (a) $F=2\mu_0II_0/\pi\ (4\eta^2-1)=0.40\ \mu\text{N}$; (b) $A=$ $=(\mu_0aII_0/\pi)\ \ln\,[(2\eta+1)/(2\eta-1)]=0.10\ \mu\text{J}$.

3.256. $R\approx\sqrt{\mu_0/\varepsilon_0}\ (\ln\eta)/\pi=0.36\ \text{k}\Omega$.

3.257. $F_1=\mu_0I^2/\pi^2R$.

3.258. $F_1=\dfrac{\mu_0}{4\pi}\,\dfrac{2I_1I_2}{b}\ln\,(1+b/a)$.

3.259. $F_1=B^2/2\mu_0$.

3.260. In all three cases $F_1=(B_1^2-B_2^2)/2\mu_0$. The force is directed to the right. The current in the conducting plane is directed beyond the drawing.

3.261. $\Delta p=IB/a=0.5\ \text{kPa}$.

3.262. $p=\mu_0I^2/8\pi^2R^2$.

3.263. $p={}^1/_2\mu_0n^2I^2$.

3.264. $I_{lim}=\sqrt{2F_{lim}/\mu_0 nR}$.

3.265. $P=v^2B^2d^2R/(R+\rho d/S)^2$; when $R=\rho d/S$, the power is $P=P_{max}={}^1/_4v^2B^2dS/\rho$.

3.266. $U={}^1/_4\mu_0I^2/\pi^2R^2ne=2\text{pV}$.

3.267. $n=jB/eE=2.5\cdot 10^{28}\ \text{m}^{-1}$; almost 1 : 1.

3.268. $u_0=1/\eta B=3.2\cdot 10^{-3}\ \text{m}^2/(\text{V}\cdot\text{s})$.

3.269. (a) $F=0$; (b) $F=(\mu_0/4\pi)\ 2Ip_m/r^2$, $\mathbf{F}\uparrow\downarrow\mathbf{B}$; (c) $F=$ $=(\mu_0/4\pi)\ 2Ip_m/r^2$, $\mathbf{F}\uparrow\downarrow\mathbf{r}$.

3.270. $F=(\mu_0/4\pi)\ 6\pi R^2Ip_mx/(R^2+x^2)^{5/2}$.

3.271. $F={}^3/_2\mu_0p_{1m}p_{2m}/\pi l^4=9\ \text{nN}$.

3.272. $I'\approx 2Bx^3/\mu_0R^2=0.5\ \text{kA}$.

3.273. $B'=B\sqrt{\mu^2\sin^2\alpha+\cos^2\alpha}$.

3.274. (a) $\oint\mathbf{H}\,d\mathbf{S}=\pi R^2B\cos\theta\cdot(\mu-1)/\mu\mu_0$;

(b) $\oint\mathbf{B}\,d\mathbf{r}=(1-\mu)\ Bl\sin\theta$.

3.275. (a) $I'_{sur}=\chi I$; (b) $I'_{vol}=\chi I$; in opposite directions.

3.276. See Fig. 24.

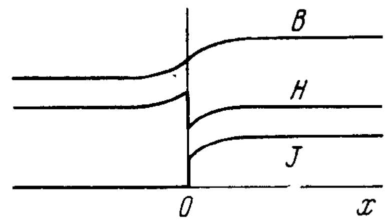


Fig. 24.

3.277. $B=\dfrac{\mu_0\mu_1\mu_2}{\mu_1+\mu_2}\,\dfrac{I}{\pi r}$.

3.278. $\mathbf{B}=2\mathbf{B}_0\mu/(1+\mu)$.

3.279. $\mathbf{B}=3\mathbf{B}_0\mu/(2+\mu)$.

3.280. $H_c=NI/l=6\ \text{kA/m}$.

3.281. $H\approx bB/\mu_0\pi d=$ $=0.10\ \text{kA/m}$.

3.282. When $b\ll R$, the permeability is $\mu\approx 2\pi RB/(\mu_0NI-$ $-bB)=3.7\cdot 10^3$.

3.283. $H=0.06\ \text{kA/m}$, $\mu_{max}\approx 1.0\cdot 10^4$.

3.284. From the theorem on circulation of the vector **H** we obtain

$$B\approx\frac{\mu_0NI}{b}-\frac{\mu_0\pi d}{b}H=1.51-0.987H\ (\text{kA/m}).$$

Besides, B and H are interrelated as shown in Fig. 3.76. The required values of H and B must simultaneously satisfy both relations. Solving this system of equations by means of plotting, we obtain $H\approx 0.26\ \text{kA/m}$, $B\approx 1.25\ \text{T}$, and $\mu=B/\mu_0H\approx 4\cdot 10^3$.

3.285. $F\approx{}^1/_2\chi SB^2/\mu_0$.

3.286. (a) $x_m=1/\sqrt{4a}$; (b) $\chi=\mu_0F_{max}\sqrt{e/a}/VB_0^2=3.6\cdot 10^{-4}$.

3.287. $A\approx{}^1/_2\chi VB^2/\mu_0$.

3.288. $\mathscr{E}_i=By\sqrt{8w/a}$.

3.289. $I=Bvl/(R+R_\mu)$, where $R_\mu=R_1R_2/(R_1+R_2)$.

3.290. (a) $\Delta\varphi={}^1/_2\omega^2a^2m/e=3.0\ \text{nV}$; (b) $\Delta\varphi\approx{}^1/_2\omega Ba^2=$ $=20\ \text{mV}$.

3.291. $\displaystyle\int_A^C\mathbf{E}\,d\mathbf{r}=-{}^1/_2\omega Bd^2=-10\ \text{mV}$.

3.292. $\mathscr{E}_i = {}^1/_2(-1)^n\, Ba\beta t$, where $n = 1, 2, \ldots$ is the number of the half-revolution that the loop performs at the given moment t. The plot $\mathscr{E}_i(t)$ is shown in Fig. 25 where $t_n = \sqrt{2\pi n/\beta}$.

3.293. $I_{\text{ind}} = \alpha/r$, where $\alpha = {}^1/_2\mu_0 lvI/\pi R$.

3.294. $\mathscr{E}_i = \dfrac{\mu_0}{4\pi}\,\dfrac{2Ia^2v}{x(x+a)}$.

3.295. $\mathscr{E}_i = {}^1/_2\,(\omega a^3B^3 + 2mg\sin\omega t)/aB$.

3.296. $v = \dfrac{mgR\sin\alpha}{B^2l^2}$.

3.297. $w = \dfrac{g\sin\alpha}{1+l^2B^2C/m}$.

3.298. $\langle P\rangle = {}^1/_2\,(\pi\omega a^2B)^2/R$.

3.299. $B = {}^1/_2 qR/NS = 0.5$ T.

3.300. $q = \dfrac{\mu_0 aI}{2\pi R}\ln\dfrac{b+a}{b-a}$, i.e. is independent of L.

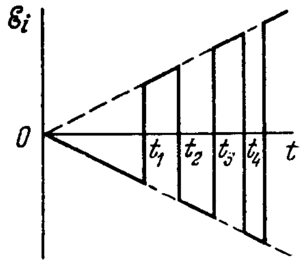


Fig. 25.

3.301. (a) $I = \dfrac{\mu_0 I_0 v}{2\pi R}\ln\dfrac{b}{a}$; (b) $F = \dfrac{v}{R}\left(\dfrac{\mu_0 I_0}{2\pi}\ln\dfrac{b}{a}\right)^2$.

3.302. (a) $s = v_0 mR/l^2B^2$, (b) $Q = {}^1/_2 mv_0^2$.

3.303. $v = \dfrac{F}{\alpha m}\,(1-e^{-\alpha t})$, where $\alpha = B^2l^2/mR$.

3.304. (a) In the round conductor the current flows clockwise, there is no current in the connector; (b) in the outside conductor, clockwise; (c) in both round conductors, clockwise; no current in the connector, (d) in the left-hand side of the figure eight, clockwise.

3.305. $I = {}^1/_4\omega B_0\,(a-b)/\rho = 0.5$ A.

3.306. $\mathscr{E}_{im} = {}^1/_3\pi a^2 N\omega B_0$.

3.307. $\mathscr{E}_i = {}^3/_2 wl\,\dot{B}t^2 = 12$ mV.

3.308. $E = \begin{cases} {}^1/_2\mu_0 n\dot{I}r & \text{for } r<a, \\ {}^1/_2\mu_0 n\dot{I}a^2/r & \text{for } r>a. \end{cases}$

3.309. $I = {}^1/_4\mu_0 nSd\dot{I}/\rho = 2$ mA, where ρ is the resistivity of copper.

3.310. $E = {}^1/_2 ab\,(\eta-1)/(\eta+1)$.

3.311. $\boldsymbol{\omega} = -\dfrac{q}{2m}\mathbf{B}(t)$.

3.312. $F_{1\,max} = \dfrac{\mu_0 a^2V^2}{4rRlb^2}$.

3.313. $Q = {}^1/_3 a^2\tau^3/R$.

3.314. $I = {}^1/_4\,(b^2-a^2)\,\beta h/\rho$.

3.315. $l = \sqrt{4\pi l_0 L/\mu_0} = 0.10$ km.

3.316. $L = \dfrac{\mu_0}{4\pi}\,\dfrac{mR}{l\rho\rho_0}$, where ρ and ρ_0 are the resistivity and the density of copper.

3.317. $t = -\dfrac{L}{R}\ln(1-\eta) = 1.5$ s.

3.318. $\tau = \dfrac{\mu_0}{4\pi}\,\dfrac{m}{l\rho\rho_0} = 0.7$ ms, where ρ is the resistivity, ρ_0 is the density of copper.

3.319. $L_1 = \dfrac{\mu\mu_0}{2\pi}\ln\eta = 0.26$ μH/m.

3.320. $L = \dfrac{\mu_0}{2\pi}\,\mu N^2 a\ln\left(1+\dfrac{a}{b}\right)$.

3.321. $L_1 = \mu_0 h/b = 25$ nH/m.

3.322. $L_1 \approx \dfrac{\mu_0}{\pi}\ln\eta$.

3.323. (a) $I = \pi a^2B/L$; (b) $A = {}^1/_2\pi^2a^4B^2/L$.

3.324. $I = I_0\,(1+\eta) = 2$ A.

3.325. $I = \dfrac{\pi aB}{\mu_0\left(\ln\dfrac{8a}{b}-2\right)} = 50$ A.

3.326. $I = \dfrac{\mathscr{E}}{R}\,[1+(\eta-1)\,e^{-t\eta R/L}]$.

3.327. $I = \dfrac{\mathscr{E}}{R}\,(1-e^{-tR/2L})$.

3.328. $I_1 = \dfrac{\mathscr{E}L_2}{R(L_1+L_2)}$, $\quad I_2 = \dfrac{\mathscr{E}L_1}{R(L_1+L_2)}$.

3.329. $L_{12} = \dfrac{\mu_0 b}{2\pi}\ln\left(1+\dfrac{a}{l}\right)$.

3.330. $L_{12} = \dfrac{\mu_0\mu hN}{2\pi}\ln\dfrac{b}{a}$.

3.331. (a) $L_{12} \approx {}^1/_2\mu_0\pi a^2/b$; (b) $\Phi_{21} = {}^1/_2\mu_0\pi a^2I/b$.

3.332. $p_m = 2aRq/\mu_0 N$.

3.333. $L_{12} \approx {}^1/_2\mu_0\pi a^4/l^3$.

3.334. $I_2 = \dfrac{\alpha L_{12}}{R}\,(1-e^{-tR/L_2})$.

3.335. $Q = \dfrac{L\mathscr{E}^2}{2R^2(1+R_0/R)} = 3$ μJ.

3.336. $W = {}^1/_2 N\Phi I = 0.5$ J.

3.337. $W = BH\pi^2a^2b = 2.0$ J, where $H = {}^1/_2\,NI/\pi b$.

3.338. (a) $W_{gap}/W_m \approx \mu b/\pi d = 3.0$; (b) $L \approx \dfrac{\mu_0 SN^2}{b+\pi d/\mu} = 0.15$ H.

3.339. $W_1 = \mu_0\lambda^2\omega^2a^2/8\pi$.

3.340. $E = B/\sqrt{\varepsilon_0\mu_0} = 3\cdot10^8$ V/m.

3.341. $w_m/w_e = \varepsilon_0\mu_0\omega^2a^4/l^2 = 1.1\cdot10^{-15}$.

3.343. (a) $L_{total} = 2L$; (b) $L_{total} = L/2$.

3.344. $L_{12} = \sqrt{L_1L_2}$.

3.346. $W_{12} = \dfrac{\mu_0\pi a^2}{2b}\,I_1I_2\cos\theta$.

3.347. (a) $\mathbf{j}_d = -\mathbf{j}$; (b) $I_d = q/\varepsilon_0\varepsilon\rho$.

3.348. The displacement current should be taken into account in addition to the conduction current.

3.349. $E_m = I_m/\varepsilon_0\omega S = 7$ V/cm.

3.350. $H = H_m\cos(\omega t+\alpha)$, where $H_m = \dfrac{rV_m}{2d}\sqrt{\sigma^2+(\varepsilon_0\varepsilon\omega)^2}$ and α is determined from the formula $\tan\alpha = \varepsilon_0\varepsilon\omega/\sigma$.

3.351. $j_d = \begin{cases} \frac{1}{2}\ddot{B}r & \text{for } r < R, \\ \frac{1}{2}\ddot{B}R^2/r & \text{for } r > R. \end{cases}$

Here $\ddot{B} = \mu_0 n I_m \omega^2 \sin \omega t$.

3.352. (a) $\mathbf{j}_d = \frac{2q\mathbf{v}}{4\pi r^3}$; (b) $\mathbf{j}_d = -\frac{q\mathbf{v}}{4\pi r^3}$.

3.353. $x_m = 0$, $j_{d\,max} = \frac{qv}{4\pi a^3}$.

3.354. $\mathbf{H} = \frac{q\,[\mathbf{vr}]}{4\pi r^3}$.

3.355. (a) If $\mathbf{B}(t)$, then $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t \neq 0$. The spatial derivatives of the field $\mathbf{E}$, however, may not be equal to zero ($\nabla \times \mathbf{E} \neq 0$) only in the presence of an electric field.

(b) If $\mathbf{B}(t)$, then $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t \neq 0$. But in the uniform field $\nabla \times \mathbf{E} = 0$.

(c) It is assumed that $\mathbf{E} = \mathbf{a}f(t)$, where $\mathbf{a}$ is a vector which is independent of the coordinates, $f(t)$ is an arbitrary function of time. Then $-\partial\mathbf{B}/\partial t = \nabla \times \mathbf{E} = 0$, that is the field $\mathbf{B}$ does not vary with time. Generally speaking, this contradicts the equation $\nabla \times \mathbf{H} = \partial\mathbf{D}/\partial t$ for in this case its left-hand side does not depend on time whereas its right-hand side does. The only exception is the case when $f(t)$ is a linear function. In this case the uniform field $\mathbf{E}$ can be time-dependent.

3.356. Let us find the divergence of the two sides of the equation $\nabla \times \mathbf{H} = \mathbf{j} + \partial\mathbf{D}/\partial t$. Since the divergence of a rotor is always equal to zero, we get $0 = \nabla\cdot\mathbf{j} + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{D})$. It remains to take into account that $\nabla\cdot\mathbf{D} = \rho$.

3.357. Let us consider the divergence of the two sides of the first equation. Since the divergence of a rotor is always equal to zero, $\nabla\cdot(\partial\mathbf{B}/\partial t) = 0$ or $\frac{\partial}{\partial t}(\nabla\cdot\mathbf{B}) = 0$. Hence, $\nabla\cdot\mathbf{B} = \text{const}$ which does not contradict the second equation.

3.358. $\nabla \times \mathbf{E} = -[\boldsymbol{\omega}\mathbf{B}]$.

3.359. $\mathbf{E}' = [\mathbf{vB}]$.

3.360. $\sigma = \varepsilon_0 vB = 0.40$ pC/m².

3.361. $\rho = -2\varepsilon\boldsymbol{\omega}\mathbf{B} = -0.08$ nC/m³, $\sigma = \varepsilon_0 a\boldsymbol{\omega}\mathbf{B} = 2$ pC/m².

3.362. $\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q\,[\mathbf{vr}]}{r^3}$.

3.364. $\mathbf{E}' = b\mathbf{r}/r^2$, where r is the distance from the z' axis.

3.365. $\mathbf{B}' = \frac{a\,[\mathbf{rv}]}{c^2 r^2}$, where r is the distance from the z' axis.

3.367. (a) $E' = E\sqrt{\frac{1-\beta^2\cos^2\alpha}{1-\beta^2}} = 9$ kV/m; $\tan\alpha' = \frac{\tan\alpha}{\sqrt{1-\beta^2}}$, whence $\alpha \approx 51°$; (b) $B' = \frac{\beta E\sin\alpha}{c\sqrt{1-\beta^2}} = 14$ μT.

3.368. (a) $E' = \frac{\beta B\sin\alpha}{c\sqrt{1-\beta^2}} = 1.4$ nV/m;

(b) $B' = B\sqrt{\frac{1-\beta^2\cos^2\alpha}{1-\beta^2}} = 0.9$ T, $\alpha' \approx 51°$.

3.370. $B' = B\sqrt{1-(E/cB)^2} \approx 0.15$ mT.

3.371. Suppose the charge q moves in the positive direction of the x axis of the reference frame K. Let us pass into the frame K' at whose origin of coordinates this charge is at rest (the x and x' axes of the two frames coincide and the y and y' axes are parallel). In the frame K' the field of the charge has the simplest form: $\mathbf{E}' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'^3}\mathbf{r}'$, with the following components in the plane x, y

$$E'_x = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'^3}x', \quad E'_y = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'^3}y'.$$

Now let us make the reverse transition to the initial frame K. At the moment when the charge q passes through the origin of coordinates of the frame K, the x and y projections of the vector $\mathbf{r}$ are related to the x' and y' projections of the vector $\mathbf{r}'$ as

$$x = r\cos\theta = x'\sqrt{1-(v/c)^2}, \quad y = r\sin\theta = y'.$$

Besides, in accordance with the formulas that are reciprocal to Eqs. (3.6i),

$$E_x = E'_x, \quad E_y = E'_y/\sqrt{1-(v/c)^2}.$$

Solving simultaneously all these equations, we obtain

$$\mathbf{E} = E_x\mathbf{i} + E_y\mathbf{j} = \frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{r}}{r^3}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}.$$

Note that in this case ($\mathbf{v} = \text{const}$) the vector $\mathbf{E}$ is collinear with the vector $\mathbf{r}$.

3.372. $v = \sqrt[3]{\frac{9}{2}ale/m} = 16$ km/s.

3.373. $\tan\alpha = \frac{al^2}{4}\sqrt{\frac{m}{2eV^3}}$.

3.374. (a) $x = 2E_0/a$; (b) $w = qE_0/m$.

3.375. $t = \frac{\sqrt{T(T+2m_0c^2)}}{ceE} = 3.0$ ns.

3.376. $w = \frac{eE}{m_0(1+T/m_0c^2)^3}$.

3.377. (a) $\tan\theta = \frac{eEt}{m_0v_0}\sqrt{1-(v_0/c)^2}$, where e and m_0 are the charge and the mass of a proton; (b) $v_x = v_0/\sqrt{1+(1-v_0^2/c^2)(eEt/m_0c^2)^2}$.

3.378. $\alpha = \arcsin\left(dB\sqrt{\frac{q}{2mV}}\right) = 30°$.

3.379. (a) $v = reB/m = 100$ km/s, $T = 2\pi m/eB = 6.5$ μs; (b) $v = c/\sqrt{1+(m_0c/reB)^2} = 0.51$ c, $T = \frac{2\pi m_0}{eB\sqrt{1-(v/c^2)}} = 4.1$ ns.

3.380. (a) $p = qrB$; $T = m_0c^2\left(\sqrt{1+(qrB/m_0c)^2} - 1\right)$; (c) $w = \dfrac{c^2}{r\,[1+(m_0c/qrB)^2]}$.

3.381. $T = \eta m_0 c^2$, 5 keV and 9 MeV respectively.

3.382. $\Delta l = 2\pi\sqrt{2mV/eB^2}\cos\alpha = 2.0$ cm.

3.383. $q/m = \dfrac{8\pi^2 V}{l^2(B_2 - B_1)^2}$.

3.384. $r = 2\rho\,|\sin(\varphi/2)|$, where $\rho = \dfrac{mv}{eB}\sin\alpha$, $\varphi = \dfrac{leB}{mv\cos\alpha}$.

3.385. $r_{max} = ae^{v_0/b}$, where $b = \dfrac{\mu_0}{2\pi}\dfrac{e}{m}I$.

3.386. $v = \dfrac{V}{rB\ln(b/a)}$, $q/m = \dfrac{V}{r^2B^2\ln(b/a)}$.

3.387. (a) $y_n = \dfrac{2\pi^2 mEn^2}{qB^2}$; (b) $\tan\alpha = \dfrac{v_0B}{2\pi En}$.

3.388. $z = l\tan\sqrt{\dfrac{qB^2}{2mE}y}$; for $z \ll 1$ this equation reduces to $y = (2mE/ql^2B^2)\,z^2$.

3.389. $F = mEI/qB = 20$ μN.

3.390. $\Delta l = \dfrac{2\pi mE}{eB^2}\tan\varphi = 6$ cm.

3.391. $q/m = \dfrac{a(a+2b)B^2}{2E\Delta x}$.

3.392. (a) $x = a(\omega t - \sin\omega t)$; $y = a(1 - \cos\omega t)$, where $a = mE/qB^2$, $\omega = qB/m$. The trajectory is a cycloid (Fig. 26). The

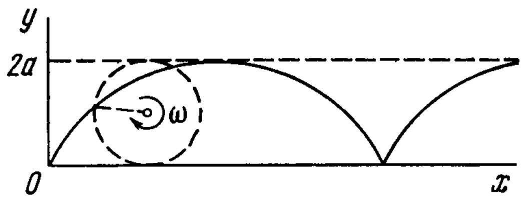


Fig. 26.

motion of the particle is the motion of a point located at the rim of a circle of radius a rolling without slipping along the x axis so that its centre travels with the velocity $v = E/B$; (b) $s = 8mE/gB^2$; (c) $\langle v_x\rangle = E/B$.

3.393. $V = 2\,\dfrac{e}{m}\left(\dfrac{\mu_0 I}{4\pi}\right)^2\ln\dfrac{a}{b}$.

3.394. $B \leqslant \dfrac{2b}{b^2 - a^2}\sqrt{\dfrac{2m}{e}V}$.

3.395. $y = \dfrac{a}{2\omega}t\sin\omega t$, $x = \dfrac{a}{2\omega^2}(\sin\omega t - \omega t\cos\omega t)$, where $a = qE_m/m$. The trajectory has the form of unwinding spiral.

3.396. $V \gtrsim 2\pi^2\nu^2 mr\Delta r/e = 0.10$ MV.

3.397. (a) $T = \dfrac{(erB)^2}{2m} = 12$ MeV; (b) $\nu_{min} = \dfrac{1}{\pi r}\sqrt{\dfrac{T}{2m}} = 20$ MHz.

3.398. (a) $t = \dfrac{\pi^2\nu mr^2}{eV} = 17$ μs; (b) $s \approx \dfrac{4\pi^3\nu^2 mr^2}{3eV} = 0.74$ km.

Instruction. Here $s \sim \sum\limits_{n=1}^{N} v_n \sim \sum\sqrt{n}$, where v_n is the velocity of the particle after the nth passage across the accelerating gap. Since N is large, $\sum\limits_{1}^{N}\sqrt{n} \approx \int\limits_0^N \sqrt{n}\,dn$.

3.399. $n = 2\pi\nu W/eBc^2 = 9$.

3.400. $\omega = \omega_0/\sqrt{1+at}$, where $\omega_0 = qB/m$, $a = qB\Delta W/\pi m^2c^2$.

3.401. $v = {}^1\!/_2 rqB/m$, $\rho = r/2$.

3.402. $N = W/e\dot{\Phi} = 5\cdot10^6$ revolutions, $s = 2\pi rN = 8\cdot10^3$ km.

3.403. On the one hand,

$$\frac{dp}{dt} = eE = \frac{e}{2\pi r}\frac{d\Phi}{dt},$$

where p is the momentum of the electron, r is the radius of the orbit, Φ is the magnetic flux acting inside the orbit.

On the other hand, dp/dt can be found after differentiating the relation $p = erB$ for $r =$ const. It follows from the comparison of the expressions obtained that $dB_0/dt = {}^1\!/_2\, d\langle B\rangle/dt$. In particular, this condition will be satisfied if $B_0 = {}^1\!/_2\langle B\rangle$.

3.404. $r_0 = \sqrt{2B_0/3a}$.

3.405. $dE/dr = \dot{B}(r_0) - {}^1\!/_2\langle\dot{B}\rangle = 0$.

3.406. $\Delta W = 2\pi r^2 eB/\Delta t = 0.10$ keV.

3.407. (a) $W = \left(\sqrt{1+(reB/m_0c)^2} - 1\right)m_0c^2$; (b) $s = W\Delta t/reB$.

4.1. (a) See Fig. 27; (b) $(v_x/a\omega)^2 + (x/a)^2 = 1$ and $w_x = -\omega^2 x$.

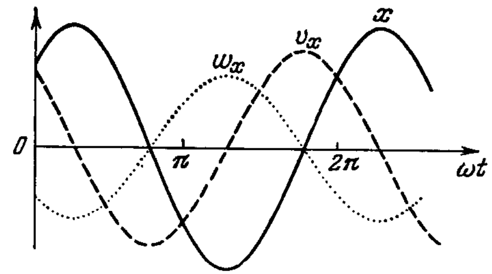


Fig. 27.

4.2. (a) The amplitude is equal to $a/2$, and the period is $T = \pi/\omega$, see Fig. 28*a*; (b) $v_x^2 = 4\omega^2 x(a - x)$, see Fig. 28*b*.

4.3. $x = a\cos(\omega t + \alpha) = -29$ cm, $v_x = -81$ cm/s, where $a = \sqrt{x_0^2 + (v_{x0}/\omega)^2}$, $\alpha = \arctan(-v_{x0}/\omega x_0)$.

4.4. $\omega=\sqrt{(v_1^2-v_2^2)/(x_2^2-x_1^2)}$; $a=\sqrt{(v_1^2x_2^2-v_2^2x_1^2)/(v_1^2-v_2^2)}$.

4.5. (a) $\langle v\rangle = 3a/T = 0.50$ m/s; (b) $\langle v\rangle = 6a/T = 1.0$ m/s.

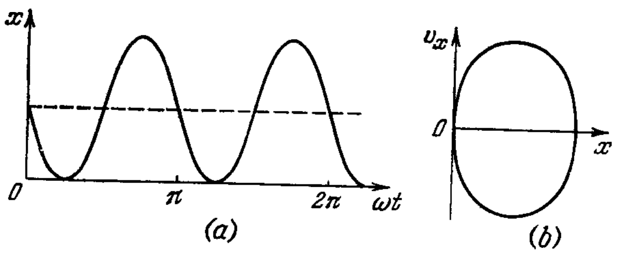


Fig. 28.

4.6. (a) $\langle v_x\rangle = \frac{2\sqrt{2}}{3\pi}a\omega$; (b) $|\langle \mathbf{v}\rangle| = \frac{2\sqrt{2}}{3\pi}a\omega$; (c) $\langle v\rangle = \frac{2(4-\sqrt{2})}{3\pi}a\omega$.

4.7. $s=\begin{cases} a[n+1-\cos(\omega t-n\pi/2)], & n \text{ is even,}\\ a[n+\sin(\omega t-n\pi/2)], & n \text{ is odd.}\end{cases}$

Here n is a whole number of the ratio $2\omega t/\pi$.

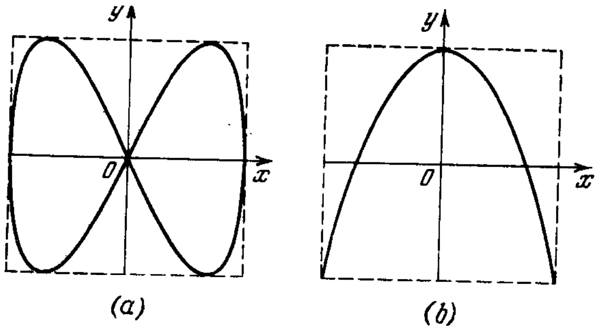


Fig. 29.

4.8. $s = 0.6$ m.

4.9. $dP/dx = 1/\pi\sqrt{a^2-x^2}$

4.10. In both cases $a = 7$.

4.11. $v_{max} = 2.73a\omega$.

4.12. 47.9 and 52.1 s^{-1}, 1.5 s.

4.13. 18 or 26 Hz.

4.14. (a) $x^2/a^2 + y^2/b^2 = 1$, clockwise; (b) $\mathbf{w} = -\omega^2\mathbf{r}$.

4.15. (a) $y^2 = 4x^2(1-x^2/a^2)$; (b) $y = a(1-2x^2/a^2)$. See Fig. 29.

4.16. $T=2\pi\sqrt{m/a^2U_0}$.

4.17. $T=4\pi a\sqrt{ma/b^2}$.

4.18. $T=\pi\sqrt{ml/F}=0.2$ s.

4.19. $T=2\pi\sqrt{\eta l/g(\eta-1)}=1.1$ s.

4.20. $T=2\sqrt{l/g}\,[\pi/2+\arcsin(\alpha/\beta)]$.

4.21. $t=\sqrt{\frac{2h}{w}}\,\frac{\sqrt{1+\eta}-\sqrt{1-\eta}}{1-\sqrt{1-\eta}}$ where $\eta=w/g$.

4.22. $T=\sqrt{4\pi m/\rho g r^2}=2.5$ s.

4.23. $T=2\pi\sqrt{\eta(1-\eta)m/\varkappa}=0.13$ s.

4.24. $T=2\pi\sqrt{m/(\varkappa_1+\varkappa_2)}$.

4.25. $T=2\pi\sqrt{m/\varkappa}$, where $\varkappa=\varkappa_1\varkappa_2/(\varkappa_1+\varkappa_2)$.

4.26. $\omega=\sqrt{2T_0/ml}$.

4.27. $T=2\pi\sqrt{m/S\rho g(1+\cos\theta)}=0.8$ s.

4.28. $T=\pi\sqrt{2l/kg}=1.5$ s.

4.29. (a) $\ddot{x} + (g/R)x = 0$, where x is the displacement of the body relative to the centre of the Earth, R is its radius, g is the standard free-fall acceleration; (b) $\tau = \pi\sqrt{R/g} = 42$ min, (c) $v = \sqrt{gR} = 7.9$ km/s.

4.30. $T = 2\pi\sqrt{l/|\mathbf{g}-\mathbf{w}|} = 0.8$ s, where $|\mathbf{g}-\mathbf{w}| = \sqrt{g^2+w^2-2gw\cos\beta}$.

4.31. $T=2\pi/\sqrt{\varkappa/m-\omega^2}=0.7$ s, $\omega\geqslant\sqrt{\varkappa/m}=10$ rad/s.

4.32. $k = 4\pi^2a/gT^2 = 0.4$.

4.33. (a) $\theta = 3.0°\cos 3.5t$; (b) $\theta = 4.5°\sin 3.5t$; (c) $\theta = 5.4°\cos(3.5t + 1.0)$. Here t is expressed in seconds.

4.34. $F = (m_1+m_2)g \pm m_1a\omega^2 = 60$ and 40 N.

4.35. (a) $F=mg\left(1+\frac{a\omega^2}{g}\cos\omega t\right)$, see Fig. 30; (b) $a_{min}=g/\omega^2=8$ cm; (c) $a=\left(\omega\sqrt{2h/g}-1\right)g/\omega^2=20$ cm.

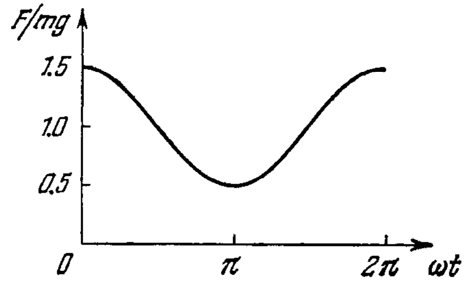


Fig. 30.

4.36. (a) $y = (1-\cos\omega t)\,mg/\varkappa$, where $\omega = \sqrt{\varkappa/m}$; (b) $T_{max} = 2mg$, $T_{min} = 0$.

4.37. $(x/r_0)^2 + \alpha(y/v_0)^2 = 1$.

4.38. (a) $y = (1-\cos\omega t)\,w/\omega^2$; (b) $y = (\omega t-\sin\omega t)\,\alpha/\omega^3$. Here $\omega = \sqrt{\varkappa/m}$.

4.39. $\Delta h_{max} = mg/k = 10$ cm, $E = m^2g^2/2k = 4.8$ mJ.

4.40. $a=(mg/\varkappa)\sqrt{1+2h\varkappa/mg}$, $E=mgh+m^2g^2/2\varkappa$.

4.41. $a=(mg/\varkappa)\sqrt{1+2h\varkappa/(m+M)g}$.

4.42. Let us write the motion equation in projections on the x and y axes:

$$\ddot{x} = \omega\dot{y},\quad \ddot{y} = -\omega\dot{x},\quad \text{where } \omega = a/m.$$

Integrating these equations, with the initial conditions taken into account, we get $x = (v_0/\omega)(1 - \cos \omega t)$, $y = (v_0/\omega) \sin \omega t$. Hence $(x - v_0/\omega)^2 + y^2 = (v_0/\omega)^2$. This is the equation of a circle of radius v_0/ω with the centre at the point $x_0 = v_0/\omega$, $y_0 = 0$.

4.43. Will increase $\sqrt{1 + {}^2/_5 (R/l)^2}$ times. It is taken into account here that the water (when in liquid phase) moves translationwise, and the system behaves as a *mathematical* pendulum.

4.44. $\omega = \sqrt{\frac{3g}{2l}\left(1 + \frac{2\varkappa l}{mg}\right)}$.

4.45. (a) $T = 2\pi\sqrt{l/3g} = 1.1$ s; (b) $E = {}^1/_2 mgl\alpha^2 = 0.05$ J.

4.46. $\varphi_m = \varphi_0\sqrt{1 + mR^2\dot{\varphi}_0^2/2k\varphi_0^2}$, $E = {}^1/_2 k\varphi_m^2$.

4.47. $\langle T \rangle = {}^1/_8 mgl\theta_0^2 + {}^1/_{12} ml^2\dot{\theta}_0^2$.

4.48. $T = 4\pi/\omega$.

4.49. $I = ml^2 (\omega_2^2 - g/l)/(\omega_1^2 - \omega_2^2) = 0.8\ \text{g}\cdot\text{m}^2$.

4.50. $\omega = \sqrt{(I_1\omega_1^2 + I_2\omega_2^2)/(I_1 + I_2)}$.

4.51. $x = l/2\sqrt{3}$, $T_{min} = 2\pi\sqrt{l/g\sqrt{3}}$.

4.52. $T = \pi\sqrt{2h/g}$, $l_{red} = h/2$.

4.53. $\omega_0 = \sqrt{3a\omega^2/2l}$.

4.54. $\omega_0 = \sqrt{\varkappa/(m + I/R^2)}$.

4.55. $\omega_0 = \sqrt{\frac{2mg\cos\alpha}{MR + 2mR(1 + \sin\alpha)}}$.

4.56. $T = 2\pi\sqrt{3(R - r)/2g}$.

4.57. $T = \pi\sqrt{3m/2\varkappa}$.

4.58. $\omega_0 = \sqrt{\varkappa/\mu}$, where $\mu = m_1m_2/(m_1 + m_2)$.

4.59. (a) $\omega = \sqrt{\varkappa/\mu} = 6\ \text{s}^{-1}$; (b) $E = {}^1/_2 \mu v_1^2 = 5$ mJ, $a = v_1/\omega = 2$ cm. Here $\mu = m_1m_2/(m_1 + m_2)$.

4.60. $T = 2\pi\sqrt{I'/k}$, where $I' = I_1I_2/(I_1 + I_2)$.

4.61. $\omega_2/\omega_1 = \sqrt{1 + 2m_O/m_C} \approx 1.9$, where m_O and m_C are the masses of oxygen and carbon atoms.

4.62. $\omega = S\sqrt{2\gamma p_0/mV_0}$, where γ is the adiabatic exponent.

4.63. $q = 4h\sqrt{\pi\varepsilon_0 mg(\eta^2 - 1)} = 2.0\ \mu\text{C}$.

4.64. The induction of the field increased $\eta^2 = 25$ times.

4.65. $x = (v_0/\omega)\sin\omega t$, where $\omega = lB/\sqrt{mL}$.

4.66. $x = (1 - \cos\omega t)\, g/\omega^2$, where $\omega = lB/\sqrt{mL}$.

4.67. (a) a_0 and $a_0\omega$; (b) $t_n = \frac{1}{\omega}\left(\arctan\frac{\omega}{\beta} + n\pi\right)$, where $n = 0, 1, 2, \dots$

4.68. (a) $\dot{\varphi}(0) = -\beta\varphi_0$, $\ddot{\varphi}(0) = (\beta^2 - \omega^2)\varphi_0$; (b) $t_n = \frac{1}{\omega}\left(\arctan\frac{\omega^2 - \beta^2}{2\beta\omega} + n\pi\right)$, where $n = 0, 1, 2, \dots$

4.69. (a) $a_0 = \frac{|\dot{x}_0|}{\omega}$, $\alpha = \begin{cases} -\pi/2, & \text{when } \dot{x}_0 > 0, \\ +\pi/2, & \text{when } \dot{x}_0 < 0; \end{cases}$ (b) $a_0 = |x_0|\sqrt{1 + (\beta/\omega)^2}$, $\alpha = \arctan(-\beta/\omega)$, with $-\pi/2 < \alpha < 0$, if $x_0 > 0$ and $\pi/2 < \alpha < \pi$, if $x_0 < 0$.

4.70. $\beta = \omega\sqrt{\eta^2 - 1} = 5\ \text{s}^{-1}$.

4.71. (a) $v(t) = a_0\sqrt{\omega^2 + \beta^2}\, e^{-\beta t}$; (b) $v(t) = |\dot{x}_0|\sqrt{1 + (\beta/\omega)^2}\, e^{-\beta t}$.

4.72. The answer depends on what is meant by the given question. The first oscillation attenuates faster in time. But if one takes the natural time scale, the period T, for each oscillation, the second oscillation attenuates faster during that period.

4.73. $\lambda = n\lambda_0/\sqrt{1 + (1 - n^2)(\lambda_0/2\pi)^2} = 3.3$, $n' = \sqrt{1 + (2\pi/\lambda_c)^2} = 4.3$ times.

4.74. $T = \sqrt{(4\pi^2 + \lambda^2)\Delta x/g} = 0.70$ s.

4.75. $Q = \pi n/\ln\eta = 5\cdot10^2$.

4.76. $s \approx l(1 + e^{-\lambda/2})/(1 - e^{-\lambda/2}) = 2$ m.

4.77. $Q = {}^1/_2\sqrt{\frac{4g\tau^2}{l\ln^2\eta} - 1} = 1.3\cdot10^2$.

4.78. $T = \sqrt{{}^3/_2(4\pi^2 + \lambda^2)R/g} = 0.9$ s.

4.79. $\omega = \sqrt{\frac{2\alpha}{mR^2} - \left(\frac{\pi\eta R^2}{m}\right)^2}$.

4.80. $\eta = 2\lambda hI/\pi R^4 T$.

4.81. $\tau = 2RI/a^4B^2$.

4.82. (a) $T = 2\pi\sqrt{m/\varkappa} = 0.28$ s; (b) $n = (x_0 - \Delta)/4\Delta = 3.5$ oscillations, here $\Delta = kmg/\varkappa$.

4.83. $x = \frac{F_0/m}{\omega^2 - \omega_0^2}(\cos\omega_0 t - \cos\omega t)$.

4.84. The motion equations and their solutions:

$$t \leqslant \tau,\quad \ddot{x} + \omega_0^2 x = F/m,\quad x = (1 - \cos\omega_0 t)\, F/k,$$
$$t \geqslant \tau,\quad \ddot{x} + \omega_0^2 x = 0,\quad x = a\cos[\omega_0(t - \tau) + \alpha],$$

where $\omega_0^2 = k/m$, a and α are arbitrary constants. From the continuity of x and $\dot{x}$ at the moment $t = \tau$ we find the sought amplitude:

$$a = (2F/k)|\sin(\omega_0 t/2)|.$$

4.85. $\omega_{res} = \sqrt{\frac{1 - (\lambda/2\pi)^2}{1 + (\lambda/2\pi)^2}\frac{g}{\Delta l}}$, $a_{res} = \frac{\lambda F_0 \Delta l}{4\pi mg}\left(1 + \frac{4\pi^2}{\lambda^2}\right)$.

4.86. $\omega_{res} = \sqrt{(\omega_1^2 + \omega_2^2)/2} = 5.1\cdot10^2\ \text{s}^{-1}$.

4.87. (a) $\omega_0 = \sqrt{\omega_1\omega_2}$; (b) $\beta = |\omega_2 - \omega_1|/2\sqrt{3}$, $\omega = \sqrt{\omega_1\omega_2 - (\omega_2 - \omega_1)^2/12}$.

4.88. $\eta = (1 + \lambda^2/4\pi^2)\pi/\lambda = 2.1$.

4.89. $A = \pi a F_0 \sin\varphi$.

4.90. (a) $Q = {}^1/_2 \sqrt{\frac{4\omega^2\omega_0^2}{(\omega^2-\omega_0^2)^2 \tan^2\varphi} - 1} = 2.2$; (b) $A = \pi m a^2 (\omega_0^2 - \omega^2)\tan\varphi = 6$ mJ. Here $\omega_0 = \sqrt{\varkappa/m}$.

4.91. (a) $\langle P\rangle = \frac{F_0^2 \beta\omega^2/m}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$; (b) $\omega = \omega_0$, $\langle P\rangle_{max} = F_0^2/4\beta m$.

4.92. $\frac{\langle P\rangle_{max} - \langle P\rangle}{\langle P\rangle_{max}} = \frac{100}{\eta^2-1}$ %.

4.43. (a) $A = -\pi\varphi_m N_m \sin\alpha$; (b) $Q = \frac{\sqrt{(\cos\alpha + 2\omega^2 I\varphi_m/N_m)^2 - 1}}{2\sin\alpha}$.

4.94. $\omega = \sqrt{ne^2/\varepsilon_0 m} = 1.65\cdot10^{16}$ s^{-1}.

4.95. $V^2 + I^2 L/C = V_m^2$.

4.96. (a) $I = I_m \sin\omega_0 t$, where $I_m = V_m\sqrt{C/L}$, $\omega_0 = 1/\sqrt{LC}$; (b) $\mathscr{E}_s = V_m/\sqrt{2}$.

4.97. $A = (\eta^2 - 1)\, W$.

4.98. (a) $T = 2\pi\sqrt{L(C_1+C_2)} = 0.7$ ms; (b) $I_m = V\sqrt{(C_1+C_2)/L} = 8$ A.

4.99. $V = {}^1/_2 (1 \pm \cos\omega t)\, V_0$, where the plus sign refers to the left-hand capacitor, and the minus sign to the right-hand one; $\omega = \sqrt{2/LC}$.

4.100. $I = \frac{\Phi}{L}\cos(t/\sqrt{LC})$.

4.101. (a) $t_n = \frac{\pi n}{\omega}$; (b) $t_n = \frac{1}{\omega}\left[\arctan\left(-\frac{\beta}{\omega}\right) + \pi n\right]$. Here $n = 0, 1, 2, \ldots$

4.102. $V_0/V_m = \sqrt{1 - \frac{R^2 C}{4L}}$.

4.103. $V_C = I_m\sqrt{L/C}\, e^{-\beta t}\sin(\omega t + \alpha)$ with $\tan\alpha = \omega/\beta$; $V_C(0) = I_m\sqrt{\frac{L}{C(1+\beta^2/\omega^2)}}$.

4.104. $W_L/W_C = L/CR^2 = 5$.

4.105. $L = L_1 + L_2$, $R = R_1 + R_2$.

4.106. $t = \frac{Q}{\pi\nu}\ln\eta = 0.5$ s.

4.107. $n = \frac{1}{2\pi}\sqrt{\frac{4L}{CR^2} - 1} = 16$.

4.108. $\frac{\omega_0 - \omega}{\omega_0} = 1 - \frac{1}{\sqrt{1 + 1/(2Q)^2}} \approx \frac{1}{8Q^2} = 0.5\%$.

4.109. (a) $W_0 = {}^1/_2 \mathscr{E}^2 (L + CR^2)/(r + R)^2 = 2.0$ mJ; (b) $W = W_0 e^{-tR/L} = 0.10$ mJ.

4.110. $t \approx \frac{Q}{2\pi\nu_0}\ln\eta = 1.0$ ms.

4.111. (a) $\omega = \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$; (b) $Q = \frac{1}{2}\sqrt{\frac{4R^2C}{L} - 1}$. When solving the problem, it should be taken into account that $dq/dt = I - I'$, where q is the charge of the capacitor, I is the current in the coil winding, I' is the leakage current ($I' = V/R$).

4.112. $Q = \frac{V^2 m}{2\langle P\rangle}\sqrt{\frac{C}{L}} = 1.0\cdot10^2$.

4.113. $\langle P\rangle = R\langle I^2\rangle = {}^1/_2 R I_m^2 = 20$ mW.

4.114. $\langle P\rangle = {}^1/_2 RCV_m^2/L = 5$ mW.

4.115. $\omega = \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$; $R < \frac{1}{2}\sqrt{\frac{L}{C}}$.

4.116. $\frac{1}{L_1} + \frac{1}{L_2} = \frac{1}{L}$ and $\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R}$.

4.117. $I = \frac{V_0}{L} t e^{-t/\sqrt{LC}}$; $I = I_{max} = \frac{V_0}{e}\sqrt{\frac{C}{L}}$ at the moment $t_m = \sqrt{LC}$.

4.118. $I = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}[\cos(\omega t - \varphi) - \cos\varphi\cdot e^{-tR/L}]$, $\tan\varphi = \omega L/R$.

4.119. $I = \frac{V_m}{\sqrt{R^2 + 1/(\omega C)^2}}[\cos(\omega t - \varphi) - \cos\varphi\cdot e^{-t/RC}]$, $\tan\varphi = -\frac{1}{\omega RC}$.

4.120. The current lags behind the voltage by phase angle φ, defined by the equation $\tan\varphi = \frac{\mu_0 \pi^2 \nu a}{4n\rho}$.

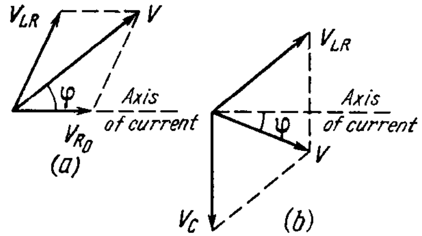


Fig. 31.

4.121. The current is ahead of the voltage by the phase angle $\varphi = 60°$, defined by the equation $\tan\varphi = \sqrt{(V_m/RI_m)^2 - 1}$.

4.122. (a) $V' = V_0 + V_m\cos(\omega t - \alpha)$, where $V_m = V_0/\sqrt{1 + (\omega RC)^2}$, $\alpha = \arctan(\omega RC)$; (b) $RC = \sqrt{\eta^2 - 1}/\omega = 22$ ms.

4.123. See Fig. 31.

4.124. (a) $I_m = V_m/\sqrt{R^2 + (\omega L - 1/\omega C)^2} = 4.5$ A; (b) $\tan\varphi = \frac{\omega L - 1/\omega C}{R}$, $\varphi = -60°$ (the current is ahead of the voltage); (c) $V_C = I_m/\omega C = 0.65$ kV, $V_L = I_m\sqrt{R^2 + \omega^2 L^2} = 0.50$ kV.

4.125. (a) $\omega = \sqrt{\omega_0^2 - 2\beta^2}$; (b) $\omega = \omega_0^2/\sqrt{\omega_0^2 - 2\beta^2}$, where $\omega_0^2 = 1/LC$, $\beta = R/2L$.

4.126. For $C = \frac{1}{\omega^2 L} = 28$ μF; $V_L = V_m\sqrt{1 + (\omega L/R)^2} = 0.54$ kV; $V_C = V_m\omega L/R = 0.51$ kV.

4.127. $I = I_m\cos(\omega t + \varphi)$, where $I_m = \frac{V_m}{R}\sqrt{1 + (\omega RC)^2}$ and $\tan\varphi = \omega RC$.

4.128. $\omega_0 = \sqrt{\frac{L_2}{C(L_1L_2 - L_{12}^2)}}$.

4.129. $Q = \sqrt{n^2 - 1/4}$.

4.130. $Q = \sqrt{\frac{\eta^2 - 1}{(n-1)^2} - \frac{1}{4}}$.

4.131. (a) $\omega_0 = \sqrt{\omega_1\omega_2}$; (b) $Q = \sqrt{\frac{\omega_1\omega_2(n^2-1)}{(\omega_2 - \omega_1)^2} - \frac{1}{4}}$.

4.133. $I_0/I = \sqrt{1 + (Q^2 + 1/4)(\eta^2 - 1)^2/\eta^2}$, 2.2 and 19 respectively.

4.134. $t = {}^1/_2 \pi t_0$.

4.135. (a) $I = \frac{2}{\sqrt{3}} I_0 \approx 1.15 I_0$; (b) $I = \frac{\pi}{\sqrt{8}} I_0 \approx 1.11 I_0$.

4.136. $\nu = \frac{R}{2\pi L}\sqrt{\eta - 1} = 2$ kHz.

4.137. The current lags behind the voltage by the phase angle $\varphi = \arccos\sqrt{1 - (X_L/Z)^2} \approx 37°$, $P = \frac{V^2}{Z^2}\sqrt{Z^2 - X_L^2} = 0.16$ kW.

4.138. For $R = \omega L - r = 0.20$ kΩ; $P_{max} = \frac{V^2}{2\omega L} = 0.11$ kW.

4.139. Increased by $\sqrt{n} - 1 \approx 30\%$.

4.140. For $Q \gg 1$ the ratio is $\Delta\omega/\omega_0 \approx {}^1/_2 \sqrt{n - 1}/Q = 0.5\%$.

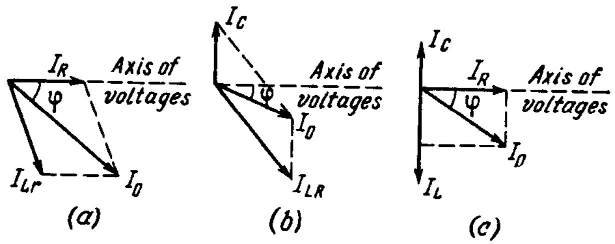


Fig. 32.

4.141. $P_2 = {}^1/_2 (V^2 - V_1^2 - V_2^2)/R = 30$ W.

4.142. $P_1 = {}^1/_2 (I^2 - I_1^2 - I_2^2) R = 2.5$ W.

4.143. $Z = R/\sqrt{1 + (\omega C R)^2} = 40\ \Omega$.

4.144. See Fig. 32.

4.145. (a) $\omega_{res} = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} = 3 \cdot 10^4$ rad/s; (b) $I = VRC/L = 3$ mA, $I_L = V\sqrt{C/L} = 1.0$ A, $I_C = V\sqrt{\frac{C}{L} - \left(\frac{RC}{L}\right)^2} = 1.0$ A.

4.146. $\tan\varphi = \frac{\omega C(R^2 + \omega^2 L^2) - \omega L}{R}$.

4.147. $Z = \sqrt{\frac{R^2 + \omega^2 L^2}{(\omega C R)^2 + (1 - \omega^2 C L)^2}}$.

4.149. $\langle F_x \rangle = \frac{\omega^2 L_2 L_{12} I_0^2}{2(R^2 + \omega^2 L_2^2)} \frac{\partial L_{12}}{\partial x}$.

4.150. $t = \frac{2l}{\alpha(\sqrt{T_1} + \sqrt{T_2})}$.

4.151. $\Delta\varphi = \frac{\omega}{v} \mid (x_1 - x_2)\cos\alpha + (y_1 - y_2)\cos\beta + (z_1 - z_2)\cos\gamma \mid$.

4.152. $\mathbf{k} = \omega\left(\frac{\mathbf{e}_x}{v_1} + \frac{\mathbf{e}_y}{v_2} + \frac{\mathbf{e}_z}{v_3}\right)$.

4.153. $\xi = a\cos[(1 - V/v)\omega t - kx']$, where $v = \omega/k$.

4.155. (a) $a/\lambda = 5.1 \cdot 10^{-5}$; (b) $v_m = 11$ cm/s, $3.2 \cdot 10^{-4}$; (c) $(\partial\xi/\partial x)_m = 3.2 \cdot 10^{-4}$, $(\partial\xi/\partial t)_m = v(\partial\xi/\partial x)_m$, where $v = 0.34$ km/s is the velocity of the wave.

4.156. See Fig. 33.

4.157. $\Delta\varphi = -\frac{2\pi}{\gamma\lambda}\ln(1 - \eta) \approx \frac{2\pi\eta}{\gamma\lambda} = 0.3$ rad.

4.158. $\mathbf{r} = (a_1\mathbf{r}_1 + a_2\mathbf{r}_2)/(a_1 + a_2)$.

4.159. (a) $\gamma = \frac{\ln(\eta r_0/r)}{r - r_0} = 0.08$ m^{-1}; (b) $v_m = \frac{2\pi\nu a_0}{\eta} = 15$ cm/s.

4.160. (a) See Fig. 34*a*. The particles of the medium at the points lying on the solid straight lines ($y = x \pm n\lambda$, $n = 0, 1, 2, \ldots$)

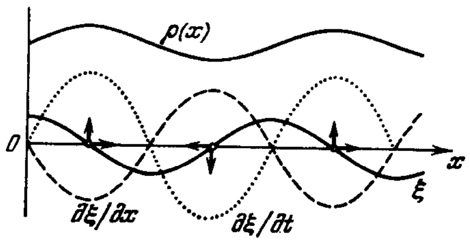


Fig. 33.

oscillate with maximum amplitude, those on the dotted lines do not oscillate at all.

(b) See Fig. 34*b*. The particles of the medium at the points lying on the straight lines $y = x \pm n\lambda$, $y = x \pm (n \pm 1/2)\lambda$ and $y = x \pm (n \pm 1/4)\lambda$ oscillate respectively along those lines, at

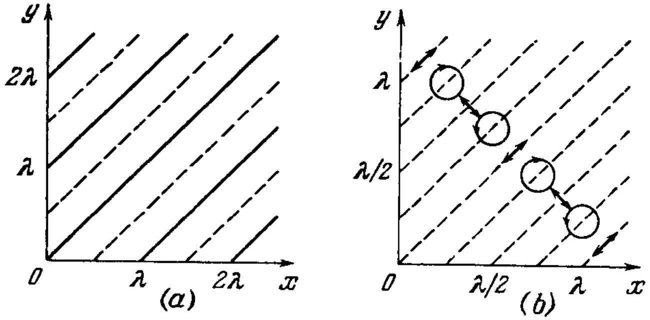


Fig. 34.

right angles to them, or move along the circles (here $n = 0, 1, 2, \ldots$). At all other points the particles move along the ellipses.

4.161. $\langle w \rangle = {}^2/_3 w_0$.

4.162. $\langle \Phi \rangle = 2\pi l^2 I_0\left(1 - \frac{1}{\sqrt{1 + (R/l)^2}}\right) = 20\ \mu$W.

4.163. $\langle \Phi \rangle = P/\sqrt{1 + (2R/h)^2} = 0.07$ W.

4.164. Fee Fig. 35, for (a) and (b); see Fig. 36 for (c).

4.165. (a) $w_p = {}^1/_2\rho a^2\omega^2 \sin^2 kx\cdot\cos^2 \omega t$; (b) $w_k = {}^1/_2\rho a^2\omega^2 \times$ $\times \cos^2 kx\cdot\sin^2 \omega t$. See Fig. 37.

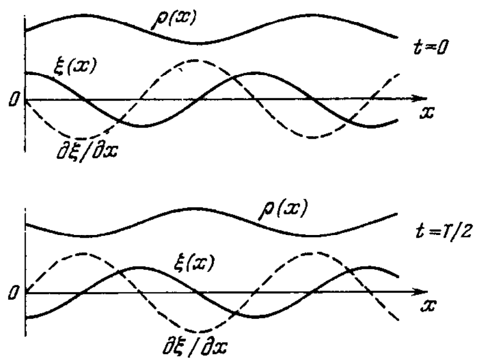


Fig. 35.

4.166. $a_{max} = 5$ mm; to the third overtone.

4.167. $\frac{\nu_2}{\nu_1} = \sqrt{\frac{\eta_2(1+\eta_1)}{\eta_1(1+\eta_2)}} = 1.4.$

4.168. Will increase $\eta = \frac{\sqrt{1-\Delta T/T}}{1+\Delta l/l} = 2$ times.

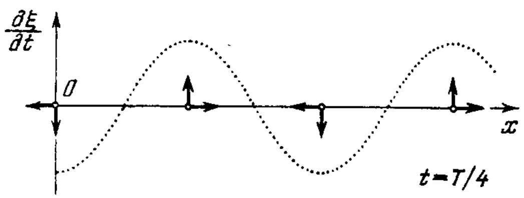


Fig. 36.

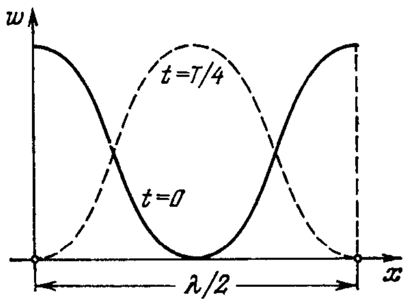


Fig. 37.

4.169. $v = 2l\nu = 0.34$ km/s.

4.170. (a) $\nu_n = \frac{v}{4l}(2n+1)$, six oscillations; (b) $\nu_n = \frac{v}{2l}(n+1)$, also six oscillations. Here $n = 0, 1, 2, \ldots$

4.171. $\nu_n = \frac{2n+1}{2l}\sqrt{\frac{E}{\rho}} = 3.8(2n+1)$ kHz; four oscillations with frequencies 26.6, 34.2, 41.8, and 49.4 kHz.

4.172. (a) $T_{max} = {}^1/_4 m\omega^2 a^2_{max}$; (b) $\langle T\rangle = {}^1/_8 m\omega^2 a^2_{max}$.

4.173. $W = {}^1/_4 \pi S\rho\omega^2 a^2/k$.

4.174. $\nu = 2\nu_0 vu/(v^2 - u^2) \approx 2\nu_0 u/v = 1.0$ Hz.

4.175. $u = \frac{v\nu_0}{\nu}\left(\sqrt{1+(\nu/\nu_0)^2} - 1\right) \approx \frac{v\nu}{2\nu_0} = 0.5$ m/s.

4.176. $\omega = \frac{\nu_0 v}{a\Delta\nu}\left(\sqrt{1+(\Delta\nu/\nu_0)^2} - 1\right) = 34\ \text{s}^{-1}$.

4.177. $\nu = \nu_0/\sqrt{1+2wt/v} = 1.35$ kHz.

4.178. (a) $\nu = \nu_0/(1-\eta^2) = 5$ kHz; (b) $r = l\sqrt{1+\eta^2} = 0.32$ km.

4.179. Decreases by $2u/(v+u) = 2.0\%$.

4.180. $\nu = 2\nu_0 u/(v+u) = 0.60$ Hz.

4.181. $\gamma = \frac{\ln(\eta r_1^2/r_2^2)}{2(r_2 - r_1)} = 6\cdot 10^{-3}\ \text{m}^{-1}$.

4.182. (a) $L' = L - 20\gamma x \log e = 50$ dB; (b) $x = 0.30$ km.

4.183. (a) $L = L_0 + 20\log(r_0/r) = 36$ dB; (b) $r > 0.63$ km.

4.184. $\beta = \ln(r_B/r_A)/[\tau + (r_B - r_A)/v] = 0.12\ \text{s}^{-1}$.

4.185. (a) Let us consider the motion of a plane element of the medium of thickness dx and unit area of cross-section. In accordance with Newton's second law $\rho\, dx\ddot{\xi} = -dp$, where dp is the pressure increment over the length dx. Recalling the wave equation $\ddot{\xi} = v^2(\partial^2\xi/\partial x^2)$, we can write the foregoing equation as

$$\rho v^2 \frac{\partial^2\xi}{\partial x^2}dx = -dp.$$

Integrating this equation, we get

$$\Delta p = -\rho v^2\frac{\partial\xi}{\partial x} + \text{const}.$$

In the absence of a deformation (a wave) the surplus pressure is $\Delta p = 0$. Hence, const $= 0$.

4.186. $\langle\Phi\rangle = \pi R^2(\Delta p)_m^2/2\rho v\lambda = 11$ mW.

4.187. (a) $(\Delta p)_m = \sqrt{\rho v P/2\pi r^2} = 5$ Pa, $(\Delta p)_m/p = 5\cdot 10^{-5}$; (b) $a = (\Delta p)_m/2\pi\nu\rho v = 3\,\mu$m, $a/\lambda = 5\cdot 10^{-6}$.

4.188. $P = 4\pi r^2 e^{2\gamma r} I_0\cdot 10^L = 1.4$ W, where L is expressed in bels.

4.189. $\Delta\lambda = (1/\sqrt{\varepsilon} - 1)c/\nu = -50$ m.

4.190. $t = 2(\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2})\, l/c \ln(\varepsilon_1/\varepsilon_2)$.

4.191. $j/j_{dis} = \sigma/2\pi\nu\varepsilon\varepsilon_0 = 2$.

4.192. $\mathbf{H} = \frac{1}{k}\sqrt{\varepsilon_0/\mu_0}\,[\mathbf{k}\mathbf{E}_m]\cos(ckt)$, where c is the velocity of the wave in vacuum.

4.193. (a) $\mathbf{H} = \mathbf{e}_z E_m\sqrt{\varepsilon_0/\mu_0}\cos kx = -0.30\mathbf{e}_z$; (b) $\mathbf{H} = \mathbf{e}_z E_m\sqrt{\varepsilon_0/\mu_0}\cos(ckt_0 - kx) = 0.18\mathbf{e}_z$. Here $\mathbf{e}_z$ is the unit vector of the z axis, H is expressed in A/m.

4.194. $\varepsilon_m = 2\pi\nu l^2 E_m/c = 13$ mV.

4.196. $\langle\mathbf{S}\rangle = {}^1/_2\mathbf{k}\varepsilon_0 c^2 E_m^2/\omega$.

4.197. (a) $j_{dis} = \pi\sqrt{2}\varepsilon_0\nu E_m = 0.20$ mA/m²; (b) $\langle S\rangle = {}^1/_2\varepsilon_0 c E_m^2 = 3.3\ \mu\text{W/m}^2$.

4.198. Since $t \gg T$, where T is the period of oscillations, $W = {}^1/_2 \sqrt{\varepsilon\varepsilon_0/\mu_0} E^2 m\pi R^2 t = 5$ kJ.
4.199. $\mathbf{B} = \mathbf{B}_m \sin kx \cdot \sin \omega t$, where $\mathbf{B}_m \perp \mathbf{E}_m$, with $B_m = E_m/c$.
4.200. $S_x = {}^1/_4 \varepsilon_0 c E_m^2 \sin 2kx \cdot \sin 2\omega t$, $\langle S_x \rangle = 0$.
4.201. $W_m/W_e = {}^1/_8 \varepsilon_0 \mu_0 \omega^2 R^2 = 5.0 \cdot 10^{-15}$.
4.202. $W_e/W_m = {}^1/_8 \varepsilon_0 \mu_0 \omega^2 R^2 = 5.0 \cdot 10^{-15}$.
4.204. $\Phi_S = I^2 R$.
4.205. $S = I^2 \sqrt{m/2eU} / 4\pi^2 \varepsilon_0 r^2$.
4.207. To the left.
4.208. $\Phi = VI$.
4.209. $\langle \Phi \rangle = {}^1/_2 V_0 I_0 \cos \varphi$.

4.211. The electric dipole moment of the system is $\mathbf{p} = \sum e\mathbf{r}_i = (e/m) M \mathbf{r}_C$, where M is the mass of the system, $\mathbf{r}_C$ is the radius vector of its centre of inertia. Since the radiation power $P \propto \ddot{\mathbf{p}}^2 \propto \ddot{\mathbf{r}}_C^2$, and in our case $\ddot{\mathbf{r}}_C = 0$, $P = 0$ too.

4.212. $\langle P \rangle = \frac{1}{4\pi\varepsilon_0} \frac{e^2 a^2 \omega^4}{3c^3} = 5 \cdot 10^{-15}$ W.

4.213. $P = \frac{1}{(4\pi\varepsilon_0)^3} \frac{2}{3c^3} \left(\frac{qe^2}{mR^2} \right)^2$.

4.214. $\Delta W \approx \frac{1}{(4\pi\varepsilon_0)^3} \frac{\pi e^4 q^2}{3c^3 m^2 v b^3}$.

4.215. $\Delta W/T = {}^1/_3 e^3 B/\varepsilon_0 c^3 m^2 = 2 \cdot 10^{-18}$.

4.216. $T = T_0 e^{-\alpha t}$, where $\alpha = {}^1/_3 e^4 B^2/\pi\varepsilon_0 c^3 m^3$. After $t_0 = \frac{1}{\alpha} =$

$$= \begin{cases} 2.5 \text{ s} & \text{for the electron,} \\ 1.6 \cdot 10^{10} \text{ s} = 0.5 \cdot 10^3 & \text{years for the proton.} \end{cases}$$

4.217. $S_1/S_2 = \tan^2 (\omega l/c) = 3$.

4.218. (a) Suppose that t is the moment of time when the particle is at a definite point x, y of the circle, and t' is the moment when the information about that reaches the point P. Denoting the observed values of the y coordinate at the point P by y' (see Fig. 4.40), we shall write

$$t' = t + \frac{l - x(t)}{c}, \quad y'(t') = y(t).$$

The sought acceleration is found by means of the double differentiation of y' with respect to t':

$$\frac{dy'}{dt'} = \frac{dy}{dt'} = \frac{dy}{dt} \frac{dt}{dt'}, \quad \frac{d^2 y}{dt'^2} = \frac{dt}{dt'} \frac{d}{dt} \left(\frac{dy'}{dt'} \right) = \frac{v^2}{R} \frac{v/c - y/R}{(1 - vy/cR)^3},$$

where the following relations are taken into account: $x = R \sin \omega t$, $y = R \cos \omega t$, and $\omega = v/R$.

(b) Energy flow density of electromagnetic radiation S is proportional to the square of the y projection of the observed acceleration of the particle. Consequently, $S_1/S_2 = (1 + v/c)^4/(1 - v/c)^4$.

4.219. $\langle P \rangle = {}^8/_3 \pi r^2 S_0$.
4.220. $\langle w \rangle = {}^3/_8 P_0/\pi r^2 c$.
4.221. $P = {}^1/_6 p^2 \omega^4/\pi\varepsilon_0 c^3$.

4.222. $\langle P \rangle / \langle S \rangle = (e^2/m)^2 \mu_0^2/6\pi$.

4.223. $\langle P \rangle / \langle S \rangle = \frac{\mu_0^2}{6\pi} \frac{(e^2/m)^2 \omega^4}{(\omega_0^2 - \omega^2)^2}$.

4.224. $R = 3P/16\pi c\gamma\rho M_C \approx 0.6$ μm.

5.1. (a) 3 and 9 mW; (b) $\Phi = {}^1/_2 (V_1 + V_2) \Phi_e / A = 1.6$ lm, where $A = 1.6$ mW/lm, V_1 and V_2 are the values of relative spectral response of an eye for the given wavelengths.

5.2. $E_m^2 = \sqrt{\mu_0/\varepsilon_0} A\Phi/2\pi r^2 V_\lambda$, hence $E_m = 1.1$ V/m, $H_m = 3.0$ mA/m. Here $A = 1.6$ mW/lm, V_λ is the relative spectral response of an eye for the given wavelength.

5.3. (a) $\langle E \rangle = {}^1/_2 E_0$; (b) $\langle E \rangle = \frac{1 - \sqrt{1 - (R/l)^2}}{1 - R/l} \frac{I}{R^2} = 50$ lx.

5.4. $M = {}^2/_3 \pi L_0$.
5.5. (a) $\Phi = \pi L \Delta S \sin^2 \theta$; (b) $M = \pi L$.
5.6. $h \approx R$, $E = LS/4R^2 = 40$ lx.
5.7. $I = I_0/\cos^3 \theta$, $\Phi = \pi I_0 R^2/h^2 = 3 \cdot 10^2$ lm.

5.8. $E_{max} = (9/16\pi\sqrt{3}) \rho ES/R^2 = 0.21$ lx, at the distance $R/\sqrt{3}$ from the ceiling.

5.9. $E = \pi L$.
5.10. $E = \pi L$.
5.11. $M = E_0 (1 + h^2/R^2) = 7 \cdot 10^2$ lm/m².
5.12. $E_0 = \pi L R^2/h^2 = 25$ lx.
5.13. $\mathbf{e}' = \mathbf{e} - 2(\mathbf{en})\,\mathbf{n}$.

5.14. Suppose $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are the unit vectors of the normals to the planes of the given mirrors, and $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the unit vectors of the incident ray and the rays reflected from the first, second, and the third mirror. Then (see the answer to the foregoing problem): $\mathbf{e}_1 = \mathbf{e}_0 - 2(\mathbf{e}_0\mathbf{n}_1)\,\mathbf{n}_1$, $\mathbf{e}_2 = \mathbf{e}_1 - 2(\mathbf{e}_1\mathbf{n}_2)\,\mathbf{n}_2$, $\mathbf{e}_3 = \mathbf{e}_2 - 2(\mathbf{e}_2\mathbf{n}_3)\,\mathbf{n}_3$. Summing termwise the left-hand and right-hand sides of these expressions, it can be readily shown that $\mathbf{e}_3 = -\mathbf{e}_0$.

5.15. $\theta_1 = \arctan n = 53°$.

5.16. $n_1/n_2 = 1/\sqrt{\eta^2 - 1} = 1.25$.

5.17. $x = [1 - \sqrt{(1 - \sin^2\theta)/(n^2 - \sin^2\theta)}]\, d \sin\theta = 3.1$ cm.

5.18. $h' = (hn^2 \cos^3 \theta)/(n^2 - \sin^2 \theta)^{3/2}$.
5.21. $\Theta = 83°$.
5.22. From 37 to 58°.
5.23. $\alpha = 8.7°$.

5.24. $\Delta\alpha = \frac{2 \sin(\Theta/2)}{\sqrt{1 - n^2 \sin^2(\Theta/2)}} \Delta n = 0.44°$.

5.27. (a) $f = l\beta/(1 - \beta^2) = 10$ cm; (b) $f = l\beta_1\beta_2/(\beta_2 - \beta_1) = 2.5$ cm.

5.28. $I' = \rho I_0 f^2 (f - s)^2 = 2.0 \cdot 10^3$ cd.

5.29. Suppose S is a point source of light and S' its image (Fig. 38). According to Fermat's principle the optical paths of all rays originating at S and converging at S' are equal. Let us draw

circles with the centres at S and S' and radii SO and $S'M$. Consequently, the optical paths (DM) and (OB) must be equal:

$$n \cdot DM = n' \cdot OB. \qquad (*)$$

However, in the case of paraxial rays $DM \approx AO + OC$, where $AO \approx h^2/(-2s)$ and $OC \approx h'^2/2R$. Besides, $OB = OC - BC \approx \approx h'^2/2R - h'^2/2s'$. Substituting these expressions into $(*)$ and taking into account that $h' \approx h$, we obtain $n'/s' - n/s = (n'-n)/R$.

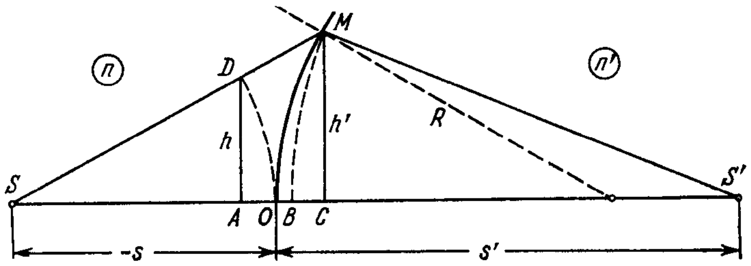


Fig. 38.

5.30. $x = \frac{nf}{n+1}\left(1 - \sqrt{1 - \frac{(n+1)\,r^2}{(n-1)\,f^2}}\right)$, $r_{max} = f\sqrt{(n-1)/(n+1)}$.

5.31. 6.3 cm.

5.32. (a) $\beta = 1 - d\,(n-1)/nR = -0.20$; (b) $E = \pi n^2 D^2 L/4d^2 = 42$ lx.

5.33. (a) $\Phi = \Phi_0\,(n - n_0)/(n-1) = 2.0$ D, $f' = -f = n_0/\Phi = 85$ cm; (b) $\Phi = {}^1/_2\Phi_0\,(2n - n_0 - 1)/(n-1) = 6.7$ D, $f = 1/\Phi \approx 15$ cm. $f' = n_0/\Phi \approx 20$ cm. Here n and n_0 are the refractive indices of glass and water.

5.35. $\Delta x \approx \Delta l f^2/(l-f)^2 = 0.5$ mm.

5.36. (a) $f = [l^2 - (\Delta l)^2]/4l = 20$ cm;
(b) $f = l\sqrt{\eta}/(1+\sqrt{\eta})^2 = 20$ cm.

5.37. $h = \sqrt{h'h''} = 3.0$ mm.

5.38. $E = (1-\alpha)\,\pi L D^2/4f^2 = 15$ lx.

5.39. (a) Is independent of D; (b) is proportional to D^2.

5.40. $f = n_0 R/2(n_1 - n_2) = 35$ cm, where n_0 is refractive index of water.

5.41. $f = R/2(2n-1) = 10$ cm.

5.42. (a) To the right of the last lens at the distance 3.3 cm from it; (b) $l = 17$ cm.

5.43. (a) 50 and 5 cm; (b) by a distance of 0.5 cm.

5.44. $\Gamma = D/d$.

5.45. $\psi = \psi'/\sqrt{\eta} = 0.6'$.

5.46. $\Gamma' = (\Gamma + 1)\frac{n - n_0}{n_0\,(n-1)} - 1 = 3.1$, where n_0 is the refractive index of water.

5.47. $\Gamma \leqslant D/d_0 = 20$.

5.48. $\Gamma = 60$.

5.49. (a) $\Gamma = 2\alpha l_0/d_0 = 15$, where l_0 is the distance of the best vision (25 cm); (b) $\Gamma \leqslant 2\alpha l_0/d_0$.

5.50. The principal planes coincide with the centre of the lens. The focal lengths in air and water: $f = -1/\Phi = -11$ cm, $f' = n_0/\Phi = +15$ cm. Here $\Phi = (2n - n_0 - 1)/R$, where n and n_0 are the refractive indices of glass and water. The nodal points coincide and are located in water at the distance $x = f' + f = 3.7$ cm from the lens.

5.51. See Fig. 39.

5.54. (a) The optical power of the system is $\Phi = \Phi_1 + \Phi_2 - d\Phi_1\Phi_2 = +4$ D, the focal length is 25 cm. Both principal planes

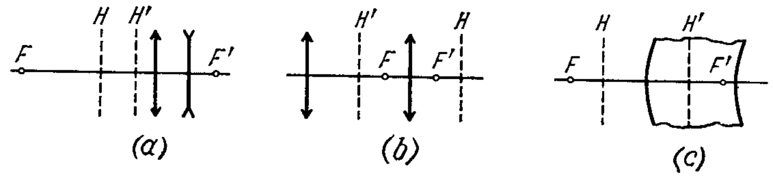


Fig. 39.

are located in front of the converging lens: the front one at a distance of 10 cm from the converging lens, and the rear one at a distance of 10 cm from the diverging lens ($x = d\Phi_2/\Phi$ and $x' = -d\Phi_1/\Phi$); (b) $d = 5$ cm; about 4/3.

5.55. The optical power of the given lens is $\Phi = \Phi_1 + \Phi_2 - (d/n)\,\Phi_1\Phi_2$, $x = d\Phi_2/n\Phi = 5.0$ cm, $x' = -d\Phi_1/n\Phi = 2.5$ cm, i.e. both principal planes are located outside the lens from the side of its convex surface.

5.56. $f = \frac{f_1 f_2}{f_1 + f_2 - d}$. The lens should be positioned in the front principal plane of the system, i.e. at a distance of $x = f_1 d/(f_1 + f_2 - d)$ from the first lens.

5.57. $\Phi = 2\Phi' - 2\Phi'^2 l/n_0 = 3.0$ D, where $\Phi' = (2n - n_0 - 1)/R$, n and n_0 are the refractive indices of glass and water.

5.58. (a) $d = n\Delta R/(n-1) = 4.5$ cm; (b) $d = 3.0$ cm.

5.59. (a) $\Phi = d\,(n-1)^2/nR^2 > 0$, the principal planes are located on the side of the convex surface at a distance of d from each other, with the front principal plane being removed from the convex surface of the lens by a distance of $R/(n-1)$; (b) $\Phi = (1/R_2 - 1/R_1) \times (n-1)/n < 0$; both principal planes pass through the common curvature centre of the surfaces of the lens.

5.60. $d = {}^1/_2 n\,(R_1 + R_2)/(n-1) = 9.0$ cm, $\Gamma = R_1/R_2 = 5.0$.

5.61. $\Phi = 2(n^2 - 1)/n^2 R = 37$ D.

5.63. $\rho = 3 \cdot 10^7$ m; $|\nabla n| = 1.6 \cdot 10^{-7}$ m^{-1}.

5.65. 1.9a.

5.66. Let us represent the kth oscillation in the complex form

$$\xi_k = ae^{i[\omega t+(k-1)\varphi]} = a_k^* e^{i\omega t},$$

where $a_k^* = ae^{i(k-1)\varphi}$ is the complex amplitude. Then the complex amplitude of the resulting oscillation is

$$A^* = \sum_{k=1}^{N} ae^{i(k-1)\varphi} = a\,[1+e^{i\varphi}+e^{i2\varphi}+\ldots+e^{i(N-1)\varphi}] =$$

$$= a\,(e^{i\varphi N}-1)/(e^{i\varphi}-1).$$

Multiplying A^* by the complex conjugate value and extracting the square root, we obtain the real amplitude

$$A = a\sqrt{\frac{1-\cos N\varphi}{1-\cos\varphi}} = a\,\frac{\sin(N\varphi/2)}{\sin(\varphi/2)}.$$

5.67. (a) $\cos\theta = (k - \varphi/2\pi)\,\lambda/d$, $\quad k = 0, \pm1, \pm2, \ldots$; (b) $\varphi = \pi/2$, $d/\lambda = k + 1/4$, $k = 0, 1, 2, \ldots$

5.68. $\Delta\varphi = 2\pi\,[k - (d/\lambda)\sin(\omega t + \alpha)]$, where $k = 0, \pm1, \pm2, \ldots$

5.69. $\lambda = 2\Delta x\Delta h/l\,(\eta - 1) = 0.6$ μm.

5.71. (a) $\Delta x = \lambda\,(b + r)/2\alpha r = 1.1$ mm, 9 maxima; (b) the shift is $\delta x = (b/r)\,\delta l = 13$ mm; (c) the fringe pattern is still sharp when $\delta x \leqslant \Delta x/2$, hence $\delta_{max} = (1 + r/b)\lambda/4\alpha = 43$ μm.

5.72. $\lambda = 2\alpha\Delta x = 0.64$ μm.

5.73. (a) $\Delta x = \lambda f/a = 0.15$ mm, 13 maxima; (b) the fringes are still sufficiently sharp when $\delta x \leqslant \Delta x/2$, where δx is the shift of the fringes from the extreme elements of the slit, hence, $\delta_{max} = \lambda f^2/2ab = 37$ μm.

5.74. $\lambda = 2a\,\Theta(n - 1)\,\Delta x/(a + b) = 0.6$ μm.

5.75. $\Delta x \approx \lambda/2\Theta\,(n - n') = 0.20$ mm.

5.76. The fringes are displaced toward the covered slit over the distance $\Delta x = hl\,(n - 1)/d = 2.0$ mm.

5.77. $n' = n + N\lambda/l = 1.000377$.

5.78. (a) Let **E**, **E**′, and **E**′′ be the electric field vectors in the incident, reflected and transmitted waves. Select the x-, y-axes at the interface so that they coincide in direction with **E** and **H** in the incident wave.

The continuity of the tangential components across the interface yields

$$\mathbf{E} + \mathbf{E}' = \mathbf{E}''.$$

The minus sign before **H**′ appears because $\mathbf{H}' \downarrow\uparrow \mathbf{H}$.

Rewrite the second equation taking into account that $\mathbf{H} \propto nE$. Solving the obtained and the first equation find:

$$\mathbf{E}'' = 2\mathbf{E}n_1/(n_1 + n_2).$$

Hence, we see that **E**′′ and **E** are collinear, that is, cophasal.

(b) $\mathbf{E}' = \mathbf{E}(n_1 - n_2)/(n_1 + n_2)$,

that is at $n_2 > n_1$ and $\mathbf{E}' \downarrow\uparrow \mathbf{E}$ the phase abruptly changes by π at the interface. If $n_2 < n_1$ the phase jump does not occur.

5.79. $d = {}^1/_4\lambda\,(1+2k)\,/\sqrt{n^2-\sin^2\theta_1} = 0.14\,(1+2k)$ μm, where $k = 0, 1, 2, \ldots$

5.80. $d_{min} = 0.65$ μm.

5.81. $d = {}^1/_4\lambda\,(1+2k)\,/\sqrt{n}$, where $k = 0, 1, 2, \ldots$

5.82. $d = \lambda\,\dfrac{\sqrt{n^2-\sin^2\theta}}{\sin 2\theta\cdot\delta\theta} = 15$ μm.

5.83. $\lambda \approx \dfrac{d\,(r_i^2-r_k^2)}{4nl^2\,(i-k)}$.

5.84. $\Delta x = \dfrac{\lambda\cos\theta_1}{2\alpha\sqrt{n^2-\sin^2\theta_1}}$.

5.85. (a) $\Theta = {}^1/_2\lambda/n\Delta x = 3'$; (b) $\Delta\lambda/\lambda \approx \Delta x/l = 0.014$.

5.86. $\Delta r \approx {}^1/_4\lambda R/r$.

5.87. $r' = \sqrt{r^2-2R\Delta h} = 1.5$ mm.

5.88. $r = \sqrt{r_0^2+(k-1/2)\,\lambda R} = 3.8$ mm, where $k = 6$.

5.89. $\lambda = {}^1/_4\,(d_2^2 - d_1^2)/R\,(k_2 - k_1) = 0.50$ μm, where k_1 and k_2 are the numbers of the dark rings.

5.90. $\Phi = 2(n - 1)(2k - 1)\lambda/d^2 = 2.4$ D, where k is the number of the bright ring.

5.91. (a) $r = \sqrt{2k\lambda\,(n-1)/\Phi} = 3.5$ mm, where $k = 10$; (b) $r' = r/\sqrt{n_0} = 3.0$ mm, where n_0 is the refractive index of water.

5.92. $r = \sqrt{{}^1/_2\,(1+2k)\,\lambda R/n_2} = 1.3$ mm, where $k = 5$.

5.93. $k_{min} = {}^1/_2\lambda_1/(\lambda_2 - \lambda_1) = 140$.

5.94. The transition from one sharp pattern to another occurs if the following condition is met:

$$(k + 1)\,\lambda_1 = k\lambda_2,$$

where k is a certain integer. The corresponding displacement Δh of the mirror is determined from the equation $2\Delta h = k\lambda_2$. From these two equations we get

$$\Delta h = \frac{\lambda_1\lambda_2}{2\,(\lambda_2-\lambda_1)} \approx \frac{\lambda^2}{2\Delta\lambda} = 0.3 \text{ mm}.$$

5.95. (a) The condition for maxima: $2d\cos\theta = k\lambda$; hence, the order of interference k diminishes as the angle θ, i.e. the radius of the rings, increases (see Fig. 5.18). (b) Differenting both sides of the foregoing equation and taking into account that on transition from one maximum to another the value of k changes by unity, we obtain $\delta\theta = {}^1/_2\lambda/d\sin\theta$; this shows that the angular width of the fringes decreases with an increase of the angle θ, i.e. with a decrease in the order of interference.

5.96. (a) $k_{max} = 2d/\lambda = 1.0 \cdot 10^5$, (b) $\Delta\lambda = \lambda/k = \lambda^2/2d = 5$ pm.

5.97. $I_0 = \frac{2}{bN\lambda} \int\limits_0^\infty I(r)\, r\, dr.$

5.98. $b = ar^2/(k\lambda a - r^2) = 2.0$ m.

5.99. $\lambda = (r_2^2 - r_1^2)(a + b)/2ab = 0.60$ μm.

5.100. (a) $I \approx 4I_0$, $I \approx 2I_0$; (b) $I \approx I_0$.

5.101. (a) $I \approx 0$; (b) $I \approx I_0/2$.

5.102. (a) $I_1 \approx {}^9/_{16}\, I_0$, $I_2 = {}^1/_4\, I_0$, $I_3 = {}^1/_{16}\, I_0$, $I_4 = I_2$, $I \approx (1 - \varphi/2\pi)^2 I_0$; (b) $I_5 \approx {}^{25}/_{16}\, I_0$, $I_6 \approx {}^9/_4\, I_0$, $I_7 \approx {}^{49}/_{16}\, I_0$, $I_8 = I_6$, $I \approx (1 + \varphi/2\pi)^2 I_0$. Here φ is the angle covered by the screen.

5.103. (a) $h = \lambda(k + 3/8)/(n - 1) = 1.2\,(k + 3/8)$ μm; (b) $h = 1.2\,(k + 7/8)$ μm, (c) $h = 1.2k$ or $1.2\,(k + 3/4)$ μm. Here $k = 0, 1, 2, \ldots$

5.104. $h = \lambda(k + 3/4)/(n - 1)$, where $k = 0, 1, 2, \ldots$, (b) $I_{max} \approx 8I_0$.

5.105. $h_{min} \approx \lambda(k + 5/8)/(n - 1) = 2.5$ μm, where $k = 2$.

5.106. $r = \sqrt{k\lambda fb/(b - f)} = 0.90\sqrt{k}$ mm, where $k = 1, 3, 5, \ldots$

5.107. $b' = b/\eta^2 = 1.0$ m.

5.108. (a) $y' = yb/a = 9$ mm; (b) $h_{min} \approx ab\lambda/D\,(a + b) = 0.10$ mm.

5.109. $f = ab/(a + b) = 0.6$ m. This value corresponds to the principal focal point, apart from which there are other points as well.

5.110. (a) $h = 0.60\,(2k + 1)$ μm; (b) $h = 0.30\,(2k + 1)$ μm. Here $k = 0, 1, 2, \ldots$

5.111. (a) $I_{max}/I_{min} \approx 1.7$, (b) $\lambda = 2\,(\Delta x)^2/b(v_2 - v_1)^2 = 0.7$ μm, where v_1 and v_2 are the corresponding values of the parameter along Cornu's spiral.

5.112. $I_{centr.}/I_{edge.} \approx 2.6$.

5.113. $\lambda = (\Delta h)^2/2b\,(v_2 - v_1)^2 = 0.55$ μm, where v_1 and v_2 are the corresponding values of the parameter along Cornu's spiral.

5.114. $h \approx \lambda(k + 3/4)/(n - 1)$, where $k = 0, 1, 2, \ldots$

5.115. $I_2/I_1 \approx 1.9$.

5.116. $I \approx 2.8 I_0$.

5.117. $I_1 : I_2 : I_3 \approx 1 : 4 : 7$.

5.118. $I \approx I_0$.

5.119. $I_\theta \propto (\sin^2\alpha)/\alpha^2$, where $\alpha = (\pi b/\lambda)\sin\theta$; $b\sin\theta = k\lambda$, $k = 1, 2, 3, \ldots$

5.120. The condition for a maximum leads to the transcendental equation $\tan\alpha = \alpha$, where $\alpha = (\pi b/\lambda)\sin\theta$. The solution of this equation (by means of plotting or selection) provides the following root values: $\alpha_1 = 1.43\pi$, $\alpha_2 = 2.46\pi$, $\alpha_3 = 3.47\pi$. Hence $b\sin\theta_1 = 1.43\lambda$, $b\sin\theta_2 = 2.46\lambda$, $b\sin\theta_3 = 3.47\lambda$.

5.121. $b\,(\sin\theta - \sin\theta_0) = k\lambda$; for $k = +1$ and $k = -1$ the angles θ are equal to 33° and 27° respectively.

5.122. (a) $\Delta\theta = \arcsin(n\sin\theta) - \theta = 7.9°$; (b) from the condition $b\,(\sin\theta_1 - n\sin\theta) = \pm\lambda$ we obtain $\Delta\theta = \theta_{+1} - \theta_{-1} = 7.3°$.

5.123. $\lambda \approx (\alpha^2 - \alpha_0^2)\, d/2k = 0.6$ μm.

5.125. 55°.

5.126. $d = 2.8$ μm.

5.127. $\lambda = (d\sin\Delta\theta)/\sqrt{5 - 4\cos\Delta\theta} = 0.54$ μm.

5.128. (a) 45°; (b) -64°.

5.129. $x = 2R/(n - 1)\sqrt{(d/\lambda)^2 - 1} = 8$ cm.

5.130. From the condition $d\,[n\sin\theta - \sin(\theta + \theta_k)] = k\lambda$ we obtain $\theta_0 = -18.5°$, $\theta_{+1} = 0°$; $k_{max} = +6$, $\theta_{+6} = +78.5°$. See Fig. 40.

5.131. $h_k = \lambda(k - 1/2)/(n - 1)$, where $k = 1, 2, \ldots$; $a\sin\theta_1 = \lambda/2$.

5.132. $v = \lambda\nu f/\Delta x = 1.5$ km/s.

5.133. Each star produces its own diffraction pattern in the objective's focal plane, with their zeroth maxima being separated

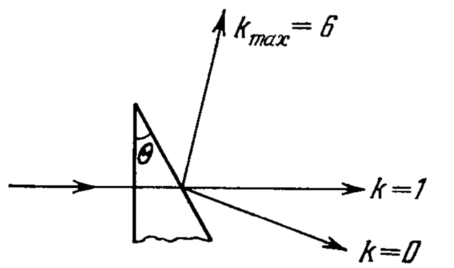


Fig. 40.

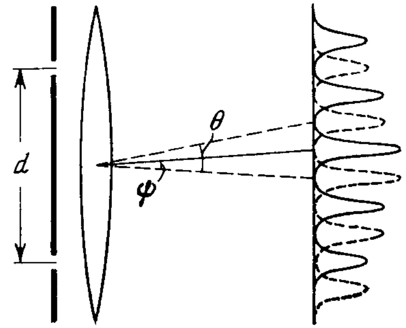


Fig. 41.

by an angle ψ (Fig. 41). As the distance d decreases the angle θ between the neighbouring maxima in each diffraction pattern increases, and when θ becomes equal to 2ψ, the first deterioration of visibility occurs: the maxima of one system of fringes coincide with the minima of the other system. Thus, from the condition $\theta = 2\psi$ and the formula $\sin\theta = \lambda/d$ we obtain $\psi = \lambda/2d \approx 0.06''$.

5.134. (a) $D = k/d\sqrt{1 - (k\lambda/d)^2} = 6.5$ ang. min/nm, where $k = 2$; (b) $D = k/d\sqrt{1 - (k\lambda/d - \sin\theta_0)^2} = 13$ ang. min/nm, where $k = 4$.

5.135. $d\theta/d\lambda = (\tan\theta)/\lambda$.

5.136. $\Delta\theta = 2\lambda/Nd\sqrt{1 - (k\lambda/d)^2} = 11''$.

5.139. $\theta = 46°$.

5.140. (a) In the fourth order; (b) $\delta\lambda_{min} \approx \lambda^2/l = 7$ pm.

5.141. (a) $d = 0.05$ mm; (b) $l = 6$ cm.

5.142. (a) 6 and 12 μm; (b) not in the first order, yes in the second order.

5.143. According to Rayleigh's criterion the maximum of the line of wavelength λ must coincide with the first minimum of the line of wavelength $\lambda + \delta\lambda$. Let us write both conditions for the least deviation angle in terms of the optical path differences for the extreme rays (see Fig. 5.28):

$$bn - (DC + CE) = 0, \quad b(n + \delta n) - (DC + CE) = \lambda + \delta\lambda.$$

Hence, $b\delta n \approx \lambda$. What follows is obvious.

5.144. (a) $\lambda/\delta\lambda = 2bB/\lambda^3$; $1.2\cdot 10^4$ and $0.35\cdot 10^4$ (b) 1.0 cm.

5.145. About 20 cm.

5.146. $R = 7\cdot 10^4$, $\Delta y_{min} \approx 4$ cm.

5.147. About 50 m.

5.148. Suppose $\Delta\psi$ and $\Delta\psi'$ are the minimum angular separations resolved by the telescope's objective and the eye respectively ($\Delta\psi = 1.22\lambda/D$, $\Delta\psi' = 1.22\lambda/d_0$). Then the sought magnification of the telescope is $\Gamma_{min} = \Delta\psi'/\Delta\psi = D/d_0 = 13$.

5.149. $d_{min} = 0.61\lambda/\sin\alpha = 1.4$ µm.

5.150. Suppose d_{min} is the minimum separation resolved by the microscope's objective, $\Delta\psi$ is the angle subtended by the eye at the object over the distance of the best visibility l_0 (25 cm), and $\Delta\psi'$ is the minimum angular separation resolved by the eye ($\Delta\psi' = 1.22\lambda/d_0$). Then the sought magnification of the microscope is $\Gamma_{min} = \Delta\psi'/\Delta\psi = 2(l_0/d_0)\sin\alpha = 30$.

5.151. 26, 60, 84, 107 and 134°.

5.152. $a = 0.28$ nm, $b = 0.41$ nm.

5.153. Suppose α, β, and γ are the angles between the direction to the diffraction maximum and the directions of the array along the periods a, b, and c respectively. Then the values of these angles can be found from the following conditions: $a(1 - \cos\alpha) = k_1\lambda$, $b\cos\beta = k_2\lambda$, and $c\cos\gamma = k_3\lambda$. Recalling that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we obtain

$$\lambda = \frac{2k_1/a}{(k_1/a)^2 + (k_2/b)^2 + (k_3/c)^2}.$$

5.154. $\lambda = \frac{2}{k}\sqrt[3]{\frac{m}{2\rho}}\sin\alpha = 244$ pm, where $k = 2$, m is the mass of a NaCl molecule.

5.155. $d = \frac{\lambda}{2\sin(\alpha/2)}\sqrt{k_1^2 + k_2^2 - 2k_1k_2\cos(\alpha/2)} = 0.28$ pm, where k_1 and k_2 are the orders of reflection.

5.156. $r = l\tan 2\alpha = 3.5$ cm, where α is the glancing angle found from the condition $2d\sin\alpha = k\lambda$.

5.157. $I_0/4$.

5.158. (a) I_0; (b) $2I_0$.

5.159. $E = \pi\Phi_0/\omega = 0.6$ mJ.

5.160. $\eta = {}^1/_2(\cos\varphi)^{2(N-1)} = 0.12$.

5.161. $I_0/I = \frac{2}{\tau^3\cos^4\varphi} \approx 60$.

5.162. $I_{pol}/I_{nat} = P/(1 - P) = 0.3$.

5.163. $P = (\eta - 1)/(1 - \eta\cos 2\varphi) = 0.8$.

5.164. (a) Let us represent the natural light as a sum of two mutually perpendicular components with intensities I_0. Suppose that each polarizer transmits in its plane the fraction α_1 of the light with oscillation plane parallel to the polarizer's plane, and the fraction α_2 with oscillation plane perpendicular to the polarizer's plane. The intensity of light transmitted through the system of two polarizers is then equal to

$$I_\parallel = \alpha_1^2 I_0 + \alpha_2^2 I_0,$$

when their planes are parallel, and to

$$I_\perp = \alpha_1\alpha_2 I_0 + \alpha_2\alpha_1 I_0,$$

when their planes are perpendicular; according to the condition, $I_\parallel/I_\perp = \eta$.

On the other hand, the degree of polarization produced separately by each polarizer is

$$P_0 = (\alpha_1 - \alpha_2)/(\alpha_1 + \alpha_2).$$

Eliminating α_1 and α_2 from these equations, we get

$$P_0 = \sqrt{(\eta - 1)/(\eta + 1)} = 0.905.$$

(b) $P = \sqrt{1 - 1/\eta^2} = 0.995$.

5.165. The relative intensity variations of both beams in the cases A and B are

$$(\Delta I/I)_A = 4\cot(\varphi/2)\cdot\delta\varphi, \quad (\Delta I/I)_B = 4\tan(\varphi/2)\cdot\delta\varphi.$$

Hence

$$\eta = (\Delta I/I)_A/(\Delta I/I)_B = \cot^2(\varphi/2), \quad \varphi = 11.5°.$$

5.166. 90°.

5.167. (a) $\rho = {}^1/_2(n^2 - 1)^2/(n^2 + 1)^2 = 0.074$;

(b) $P = \rho/(1 - \rho) = \frac{(1+n^2)^2 - 4n^2}{(1+n^2)^2 + 4n^2} = 0.080$. Here n is the refractive index of glass.

5.168. $I = I_0(1 - \rho)/n = 0.721 I_0$, where n is the refractive index of water.

5.169. $\rho = [(n^2 - 1)/(n^2 + 1)]\sin^2\varphi = 0.038$, where n is the refractive index of water.

5.170. $P_1 = P_3 = 1$, $P_2 = \frac{\rho}{1-\rho} = 0.087$, $P_4 = \frac{2\rho(1-\rho)}{1 - 2\rho(1-\rho)} = 0.17$.

5.171. (a) In this case the coefficient of reflection from each surface of the plate is equal to $\rho = (n^2 - 1)^2/(n^2 + 1)^2$, and therefore $I_4 = I_0(1 - \rho)^2 = 16I_0n^4/(1 + n^2)^2 = 0.725I_0$;

(b) $P = \frac{1 - (1-\rho')^2}{1 + (1-\rho')^2} = \frac{(1+n^2)^4 - 16n^4}{(1+n^2)^4 + 16n^4} \approx 0.16$, where ρ' is the coefficient of reflection for the component of light whose electric vector oscillates at right angles to the incidence plane.

5.172. (a) $P = (1 - \alpha^{4N})/(1 + \alpha^{4N})$, where $\alpha = 2n/(1 + n^2)$, n is the refractive index of glass; (b) 0.16, 0.31, 0.67, and 0.92 respectively.

5.173. (a) $\rho = (n-1)^2/(n+1)^2 = 0.040$; (b) $\Delta\Phi/\Phi = 1 - (1-\rho)^{2N} = 0.34$, where N is the number of lenses.

5.175. (a) 0.83; (b) 0.044.

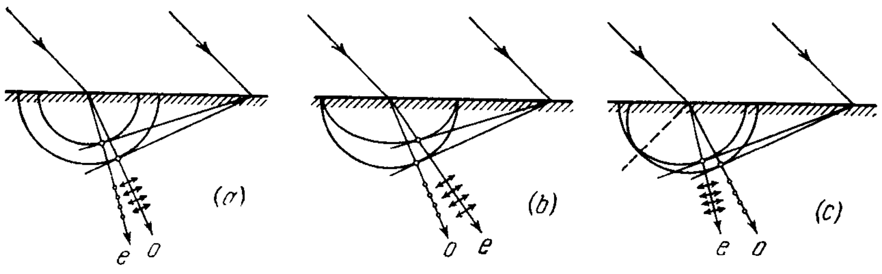


Fig. 42.

5.176. See Fig. 42, where o and e are the ordinary and extraordinary rays.

5.177. $\delta \approx 11°$.

5.178. For the right-handed system of coordinates:

(1) circular anticlockwise polarization, when observed toward the incoming wave;

(2) elliptical clockwise polarization, when observed toward the incoming wave; the major axis of the ellipse coincides with the straight line $y = x$;

(3) plane polarization, along the straight line $y = -x$.

5.179. (a) 0.490 mm; (b) 0.475 mm.

5.180. $\lambda = 4d\Delta n/(2k+1)$; 0.58, 0.55 and 0.51 μm respectively at $k = 15$, 16 and 17.

5.181. Four.

5.182. 0.69 and 0.43 μm.

5.183. $d = (k - 1/2)\,\lambda_1/\Delta n = 0.25$ mm, where $k = 4$.

5.184. $\Delta n = \lambda/\Theta\Delta x = 0.009$.

5.185. Let us denote the intensity of transmitted light by $I_\perp$ in the case of the crossed Polaroids, and by $I_\parallel$ in the case of the parallel Polaroids. Then

$$I_\perp = {}^1/_2 I_0 \sin^2 2\varphi \cdot \sin^2(\delta/2),$$
$$I_\parallel = {}^1/_2 I_0 [1 - \sin^2 2\varphi \cdot \sin^2(\delta/2)].$$

The conditions for the maximum and the minimum:

Polaroids	I_{max}	I_{min}
$\perp$	$\Delta = (k + 1/2)\lambda$, $\varphi = \pi/4$	$\Delta = k\lambda$, for any φ
$\parallel$	$\Delta = k\lambda$, for any φ	$\Delta = (k+1/2)\lambda$, $\varphi = \pi/4$

Here Δ is the optical path difference for the ordinary and extraordinary rays, $k = 0, 1, 2, \ldots$

5.187. (a) The light with right-hand circular polarization (from the observer's viewpoint) becomes plane polarized on passing through a quarter-wave plate. In this case the direction of oscillations of the electric vector of the electromagnetic wave forms an angle of $+45°$ with the axis OO' of the crystal (Fig. 43*a*); in the case of left-hand polarization this angle will be equal to $-45°$ (Fig. 43*b*).

(b) If for any position of the plate the rotation of the Polaroid (located behind the plate) does not bring about any variation in the intensity of the transmitted light, the initial light is natural;

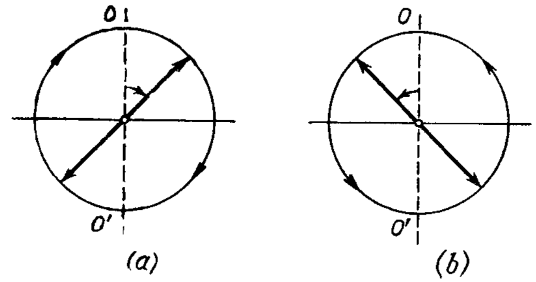


Fig. 43.

if the intensity of the transmitted light varies and drops to zero, the initial light is circularly polarized; if it varies but does not drop to zero, then the initial light is composed of natural and circularly polarized light.

5.188. (a) $\Delta x = {}^1/_2\lambda\,(n_e - n_o)\,\Theta$, (b) $d\,(n_o' - n_e') = -2\,(n_e - n_o)\,\Theta\delta x < 0$.

5.189. $\Delta n = \alpha\lambda/\pi = 0.71 \cdot 10^{-4}$, where α is the rotational constant.

5.190. $\alpha = \pi/\Delta x \tan\Theta = 21$ ang. deg./mm, $I(x) \sim \cos^2(\pi x/\Delta x)$, where x is the distance from the maximum.

5.191. $d_{min} = (1/\alpha) \arcsin\sqrt{2\eta} = 3.0$ mm.

5.192. 8.7 mm.

5.193. $[\alpha] = 72$ ang. deg./(dm·g/cm³).

5.194. (a) $E_{min} = 1/\sqrt{4Bl} = 10.6$ kV/cm; (b) $2.2 \cdot 10^8$ interruptions per second.

5.195. $\Delta n = 2cHV/\omega$, where c is the velocity of light in vacuum.

5.196. $V = {}^1/_2\,(\varphi_1 - \varphi_2)/lH = 0.015$ ang. min/A.

5.197. If one looks toward the transmitted beam and counts the positive direction clockwise, then $\varphi = (\alpha - VNH)\,l$, where N is the number of times the beam passes through the substance (in Fig. 5.35 the number is $N = 5$).

5.198. $H_{min} = \pi/4Vl = 4.0$ kA/m, where V is the Verdet constant. The direction along which the light is transmitted changes to the opposite.

5.199. $t = mc\omega_0/\lambda I = 12$ hours. Although the effect is very small, it was observed both for visible light and for SHF radiation.

5.200. (a) $a = eE_0/m\omega^2 = 5\cdot10^{-16}$ cm, where $E_0 = \sqrt{2I/\varepsilon_0 c}$, $v = a\omega = 1.7$ cm/s; (b) $F_m/F_e = 2.9\cdot10^{-11}$.

5.201. (a) $\varepsilon = 1 - n_0e^2/\varepsilon_0 m\omega^2$, $v = c\sqrt{1+(n_0e^2/4\pi^2\varepsilon_0 mc^2)\lambda^2}$.

5.202. $n_0 = (4\pi^2\nu^2 m\varepsilon_0/e^2)(1-n^2) = 2.4\cdot10^7$ cm^{-3}.

5.203. $n-1 = -n_0e^2\lambda^2/8\pi^2\varepsilon_0 mc^2 = -5.4\cdot10^{-7}$, where n_0 is the concentration of electrons in carbon.

5.204. (a) $x = a\cos(\omega t + \varphi)$, where a and φ are defined by the formulas

$$a = \frac{eE_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}, \quad \tan\varphi = \frac{2\beta\omega}{\omega^2-\omega_0^2}.$$

Here $\beta = \gamma/2m$, $\omega_0^2 = k/m$, m is the mass of an electron. (b) $\langle P\rangle = \frac{m\beta(eE_0/m)^2\omega^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$, $\langle P\rangle_{max} = \frac{m}{4\beta}\left(\frac{eE_0}{m}\right)^2$ for $\omega = \omega_0$.

5.205. Let us write the wave equation in the form $A = A_0 e^{i(\omega t - kx)}$, where $k = 2\pi/\lambda$. If $n' = n + i\varkappa$, then $k = (2\pi/\lambda_0)n'$ and

$$A = A_0 e^{2\pi\varkappa x/\lambda_0} e^{i(\omega t - 2\pi n x/\lambda_0)},$$

or in the real form

$$A = A_0 e^{\varkappa' x}\cos(\omega t - k'x),$$

i.e. the light propagates as a plane wave whose amplitude depends on x. When $\varkappa < 0$, the amplitude diminishes (the attenuation of the wave due to absorption). When $n' = i\varkappa$, then

$$A = A_0 e^{\varkappa' x}\cos\omega t.$$

This is a standing wave whose amplitude decreases exponentially (if $\varkappa < 0$). In this case the light experiences total internal reflection in the medium (without absorption).

5.206. $n_0 = 4\pi^2\varepsilon_0 mc^2/e^2\lambda_0^2 = 2.0\cdot10^9$ cm^{-3}.

5.208. (a) $u = {}^3/_2\, v$; (b) $u = 2v$; (c) $u = {}^1/_3\, v$.

5.209. $\varepsilon = 1 + A/\omega^2$, where A is a constant.

5.210. $v = c/n = 1.83\cdot10^8$ m/s, $u = [1 + (\lambda/n)(dn/d\lambda)]\,c/n = 1.70\cdot10^8$ m/s.

5.211. It is sufficient to discuss three harmonic components of the train of waves (most easily with the help of a plot).

5.212. $I = {}^1/_2 I_0 e^{-\varkappa l}\sin^2\varphi$, where $\varphi = VlH$.

5.213. (a) $I = I_0(1-\rho)^2(1+\rho^2+\rho^4+\ldots) = I_0(1-\rho)^2/(1-\rho^2)$; (b) $I = I_0(1-\rho)^2\sigma(1+\sigma^2\rho^2+\sigma^4\rho^4+\ldots) = I_0\sigma(1-\rho)^2/(1-\sigma^2\rho^2)$, where $\sigma = \exp(-\varkappa d)$.

5.214. $\varkappa = \frac{\ln(\tau_1/\tau_2)}{d_2-d_1} = 0.35$ cm^{-1}.

5.215. $\varkappa = \frac{1}{lN}\ln\frac{(1-\rho)^{2N}}{\tau} = 0.034$ cm^{-1}.

5.216. $\tau = (1-\rho)^2\exp[-{}^1/_2(\varkappa_1+\varkappa_2)\,l]$.

5.217. $I = I_0(1-\rho)^2\frac{e^{-\varkappa_1 l} - e^{-\varkappa_2 l}}{(\varkappa_2-\varkappa_1)\,l}$.

5.218. $\Delta\lambda = 2\lambda_0\sqrt{(\ln\eta)/\alpha d}$.

5.219. $I = \frac{\Phi}{4\pi b^2}(1-\rho)^2 e^{-\varkappa(b-a)}$.

5.220. Will decrease $\exp(\mu d) = 0.6\cdot10^2$ times.

5.221. $d = 0.3$ mm.

5.222. $d = (\ln 2)/\mu = 8$ mm.

5.223. $N = (\ln\eta)/\ln 2 = 5.6$.

5.224. $c = 2lz(n_2 - n_1) = 3.0\cdot10^8$ m/s.

5.225. First of all note that when $v \ll c$, the time rate is practically identical in the reference frames fixed to the source and to the receiver. Suppose that the source emits short pulses with the intervals T_0. Then in the reference frame fixed to the receiver the distance between two successive pulses is equal to $\lambda = cT_0 - v_rT_0$, when measured along the observation line. Here v_r is the projection of the source velocity on the observation line ($v_r = v\cos\theta$). The frequency of received pulses $\nu = c/\lambda = \nu_0/(1 - v_r/c)$, where $\nu_0 = 1/T_0$. Hence $(\nu - \nu_0)/\nu_0 = (v/c)\cos\theta$.

5.226. $\Delta\lambda = -\lambda\sqrt{2T/mc^2}\cos\theta = -26$ nm.

5.227. $T = 4\pi R\lambda\, c\delta\lambda = 25$ days, where R is the radius of the Sun.

5.228. $d = (\Delta\lambda/\lambda)_m c\tau/\pi = 3\cdot10^7$ km, $m = (\Delta\lambda/\lambda)_m^3 c^3\tau/2\pi\gamma = 2.9\cdot10^{29}$ kg, where γ is the gravitational constant.

5.229. $\omega = \omega_0(1+\beta)/(1-\beta)$, where $\beta = V/c$; $\omega \approx \omega_0(1+2V/c)$.

5.230. $v = {}^1/_2\lambda\Delta\nu \approx 900$ km per hour.

5.231. Substituting the expressions for t' and x' (from the Lorentz transformation) into the equation $\omega t - kx = \omega' t' - k'x'$, we obtain

$$\omega = \omega'(1+\beta)/\sqrt{1-\beta^2}, \quad k = k'(1+\beta)/\sqrt{1-\beta^2},$$

where $\beta = V/c$. Here it is taken into account that $\omega' = ck'$.

5.232. From the formula $\omega' = \omega\sqrt{(1-\beta)/(1+\beta)}$ we get $\beta = v/c = 0.26$.

5.233. $v = c\frac{(\lambda/\lambda')^2-1}{(\lambda/\lambda')^2+1} = 7.1\cdot10^4$ km/s.

5.234. $\omega = \omega_0\sqrt{3/7}$.

5.235. $\Delta\lambda = \lambda T/m_0c^2 = 0.70$ nm, where m_0 is the mass of the atom.

5.236. (a) $\omega = \omega_0/\sqrt{1-\beta^2} = 5.0\cdot10^{10}$ s^{-1}; (b) $\omega = \omega_0\sqrt{1-\beta^2} = 1.8\cdot10^{10}$ s^{-1}. Here $\beta = v/c$.

5.237. The charge of an electron and the positive charge induced in the metal form a dipole. In the reference frame fixed to the electron the electric dipole moment varies with a period $T' = d'/v$, where $d' = d\sqrt{1-(v/c)^2}$. The corresponding "natural" frequency

is $\nu' = v/d'$. Due to the Doppler effect the observed frequency is

$$\nu = \nu' \frac{\sqrt{1-(v/c)^2}}{1-(v/c)\cos\theta} = \frac{v/d}{1-(v/c)\cos\theta}.$$

The corresponding wavelength is $\lambda = c/\nu = d\,(c/v - \cos\theta)$. When $\theta = 45°$ and $v \approx c$ the wavelength is $\lambda \approx 0.6$ μm.

5.238. (a) Let v_x be the projection of the velocity vector of the radiating atom on the observation direction. The number of atoms with projections falling within the interval v_x, $v_x + dv_x$ is

$$n\,(v_x)\,dv_x \sim \exp\,(-mv_x^2/2kT)\cdot dv_x.$$

The frequency of light emitted by the atoms moving with velocity v_x is $\omega = \omega_0\,(1 + v_x/c)$. From the expression the frequency distribution of atoms can be found: $n\,(\omega)\,d\omega = n\,(v_x)\,dv_x$. And finally it should be taken into account that the spectral radiation intensity $I_\omega \sim n\,(\omega)$. (b) $\Delta\omega/\omega_0 = 2\sqrt{(2\ln 2)\,kT/mc^2}$.

5.239. $u = \frac{c/n+V}{1+V/cn}$. If $V \ll c$, then $u \approx \frac{c}{n} + V\left(1 - \frac{1}{n^2}\right)$.

5.240. $v = {}^1/_2 c\delta\theta = 30$ km/s.

5.242. $\theta' = 8°$.

5.243. The field induced by a charged particle moving with velocity V excites the atoms of the medium turning them into sources of light waves. Let us consider two arbitrary points A and B along the path of the particle. The light waves emitted from these points when the particle passes them reach the point P (Fig. 44) simultaneously and amplify each other provided the time taken by the light wave to propagate from the point A to the point C is equal to that taken by the particle to fly over the distance AB. Hence, we obtain $\cos\theta = v/V$, where $v = c/n$ is the phase velocity of light. It is evident that the radiation is possible only if $V > v$, i.e. when the velocity of the particle exceeds the phase velocity of light in the medium.

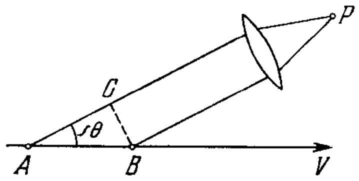


Fig. 44.

5.244. $T_{min} = (n/\sqrt{n^2-1} - 1)\,mc^2$; 0.14 MeV and 0.26 GeV respectively. For muons.

5.245. $T = \left(\frac{n\cos\theta}{\sqrt{n^2\cos^2\theta - 1}} - 1\right) mc^2 = 0.23$ MeV.

5.247. $T_2 = bT_1/(b + T_1\Delta\lambda) = 1.75$ kK.

5.248. $\lambda_m = 3.4$ μm.

5.249. $5\cdot 10^9$ kg/s, about 10^{11} years.

5.250. $T = \sqrt[3]{3cR\rho/\sigma M} = 2\cdot 10^7$ K, where R is the universal gas constant, M is the molar mass of hydrogen.

5.251. $t = (\eta^3 - 1)\,c\rho d/18\sigma T_0^3 = 3$ hours, where c is the specific heat capacity of copper, ρ is its density.

5 252. $T_2 = T_1\sqrt{d/2l} = 0.4$ kK.

5.253. (a) $C_V = (\partial U/\partial T)_V = 16\,\sigma T^3 V/c = 3$ nJ/K, where $U = 4\sigma T^4 V/c$; (b) $S = 16\sigma T^3 V/3c = 1.0$ nJ/K.

5.254. (a) $\omega_{pr} = 3T/a = 7.85\cdot 10^{14}$ s^{-1}; (b) $\lambda_{pr} = 2\pi ca/5T = 1.44$ μm.

5.255. (a) $u_\omega = (kT/\pi^2c^3)\,\omega^2$; (b) $u_\omega = (\hbar/\pi^2c^3)\,\omega^3\,e^{-\hbar\omega/kT}$.

5.256. $u_\nu = \frac{16\pi^2\hbar}{c^3}\,\frac{\nu^3}{e^{2\pi\hbar\nu/kT}-1}$, $u_\lambda = \frac{16\pi^2 c\hbar\lambda^{-5}}{e^{2\pi\hbar c/kT\lambda}-1}$.

5.257. $\Delta P = 4\pi^2c^2\hbar T^5\Delta\lambda/b^5\,(e^{2\pi\hbar c/kb} - 1) = 0.31$ W/cm^2, where b is the constant in Wien's displacement law.

5.258. (a) 1.1 μm; (b) 0.37; (c) $P_2/P_1 = (T_2/T_1)^4\,(1-y_2)/(1-y_1) = 4.9$.

5.259. $n_\omega\,d\omega = \frac{1}{\pi^2c^3}\,\frac{\omega^2\,d\omega}{e^{\hbar\omega/kT}-1}$, $n_\lambda\,d\lambda = \frac{8\pi\lambda^{-4}\,d\lambda}{e^{2\pi\hbar c/kT\lambda}-1}$.

5.260. (a) $\langle j\rangle = P\lambda/8\pi^2c\hbar r^2 = 6\cdot 10^{13}$ cm^{-2}s^{-1}; (b) $r = \sqrt{P\lambda/2\hbar n}/2\pi c = 9$ m.

5.261. $dp/dt = \Phi_e/c$.

5.262. $\langle p\rangle = 4\,(1+\rho)\,E/\pi d^2c\tau \approx 50$ atm.

5.263. $p = (E/c)\sqrt{1+\rho^2+2\rho\cos 2\theta} = 35$ nN·s.

5.264. $p = (I/c)\,(1+\rho)\cos^2\theta = 0.6$ nN/cm^2.

5.265. $F = \pi R^2 I/c = 0.18$ μN.

5.266. $F = P/2c\,(1+\eta^2)$.

5.267. (a) $\Delta p = \frac{2\hbar\omega}{c}\,\frac{\sqrt{1-\beta^2}}{1-\beta}$; (b) $\Delta p = \frac{2\hbar\omega}{c}\,\frac{1}{1-\beta}$. Here $\beta = V/c$. It is evident that in the reference frame fixed to the mirror the latter obtains the smaller momentum.

5.268. $\sin(\theta/2) \approx E/mc\sqrt{gl}$, $\theta = 0.5°$.

5.269. $\Delta\omega/\omega_0 = -(1 - e^{-\gamma M/Rc^2}) < 0$, i.e. the frequency of the photon decreases.

5.270. $V = 2\pi\hbar c\,(1 - 1/\eta)/e\Delta\lambda = 16$ kV.

5.271. $V = \pi\hbar c/ed\,\sin\alpha = 31$ kV.

5.272. $\lambda_{min} = 2\pi\hbar/mc\,(\gamma - 1) = 2.8$ pm, where $\gamma = 1/\sqrt{1-(v/c)^2}$.

5.273. 332 nm, $6.6\cdot 10^5$ m/s.

5.274. $A = 2\pi c\hbar\,\frac{(\eta^2 - \lambda_2/\lambda_1)}{\lambda_2\,(\eta^2-1)} = 1.9$ eV.

5.275. $\varphi_{max} = 4.4$ V.

5.276. $T_{max} = \hbar\,(\omega_0 + \omega) - A_f = 0.38$ eV.

5.277. $w = 2\pi c\hbar J/e\lambda = 0.020$.

5.278. $v_{max} = 6.4\cdot 10^5$ m/s.

5.279. 0.5 V; the polarity of the contact potential difference is opposite to that of external voltage.

5.280. $\hbar/mc$, the Compton wavelength for the given particle.

5.281. Let us write the energy and momentum conservation laws in the reference frame fixed to the electron for the moment preceding the collision with the photon: $\hbar\omega + m_0c^2 = mc^2$, $\hbar\omega/c = mv$, where $m = m_0\sqrt{1-(v/c)^2}$. From this it follows that $v = 0$ or $v = c$. The results have no physical meaning.

5.282. (a) Light is scattered by the free electrons; (b) the increase of the number of electrons that turn free (the free electrons have the binding energy much lower than the energy transferred to them by the photons); (c) the presence of a non-displaced component is due to scattering by the strongly bound electrons and the nuclei.

5.283. $\lambda = 4\pi\lambda\!\!\!^{-}_C\,[\sin(\theta_2/2) - \eta\sin(\theta_1/2)]/(\eta - 1) = 1.2$ pm.

5.284. $T = \hbar\omega\eta/(1+\eta) = 0.20$ MeV.

5.285. (a) $\omega' = 2\pi c/(\lambda + 2\pi\hbar/mc) = 2.2\cdot 10^{20}$ rad/s;
(b) $T = \dfrac{2\pi c\hbar/\lambda}{1+\lambda mc/2\pi\hbar} = 60$ keV.

5.286. $\hbar\omega' = \dfrac{\hbar\omega}{1+2(\hbar\omega/mc^2)\sin(\theta/2)} = 0.144$ MeV.

5.287. $\sin(\theta/2) = \sqrt{mc(p-p')/2pp'}$. Hence $\theta = 120°$.

5.288. $\hbar\omega = \left[1+\sqrt{1+2mc^2/T\sin^2(\theta/2)}\right]T/2 = 0.68$ MeV.

5.289. $\lambda = (2\pi\hbar/mc)(\sqrt{1+2mc^2/T_{max}} - 1) = 3.7$ pm.

5.290. $\tan\varphi = \dfrac{\sqrt{4\pi\hbar/mc\Delta\lambda - 1}}{1+\hbar\omega/mc^2}$, $\varphi = 31°$.

5.291. $\rho = \dfrac{2\eta(1+\eta)\,mc}{(1+2\eta)\,eB} = 3.4$ cm.

5.292. $\Delta\lambda = (4\hbar/mc)\sin^2(\theta/2) = 1.2$ pm.

6.1. $r = 3e^2/2E = 0.16$ nm, $\lambda = (2\pi c/e)\sqrt{mr^3} = 0.24$ μm.

6.2. $b = 0.73$ pm.

6.3. (a) $r_{min} = 0.59$ pm; (b) $r_{min} = (2Ze^2/T)(1 + m_\alpha/m_{Li}) = 0.034$ pm.

6.4. (a) $\rho_{min} = (Ze^2/T)\cot^2(\theta/2) = 0.23$ pm; (b) $r_{min} = [1 + \operatorname{cosec}(\theta/2)]\,Ze^2/T = 0.56$ pm.

6.5. $p \approx 2\sqrt{2mT/[1+(2bT/Ze^2)^2]}$.

6.6. $T_e = m_pe^4/m_eb^2T = 4$ eV.

6.7. $b = \dfrac{Rn\sin(\theta/2)}{\sqrt{1+n^2-2n\cos(\theta/2)}}$, where $n = \sqrt{1+U_0/T}$.

6.8. (a) $\cos(\theta/2) = b/(R+r)$; (b) $dP = {}^1\!/_2\sin\theta\,d\theta$; (c) $P = 1/2$.

6.9. $3.3\cdot 10^{-5}$.

6.10. $d = (4Jr^2T^2/nIZ^2e^4)\sin^4(\theta/2) = 1.5$ μm, where n is the concentration of nuclei.

6.11. $Z_{Pt} = Z_{Ag}\sqrt{\eta A_{Pt}/A_{Ag}} = 78$.

6.12. (a) $1.6\cdot 10^6$; (b) $N = \pi nd\,(Ze^2/T)^2\cot^2(\theta_0/2)\,I_0\tau = 2.0\cdot 10^7$, where n is the concentration of nuclei.

6.13. $P = \pi nd\,(Ze^2/mv^2)^2 = 0.006$, where n is the concentration of nuclei.

6.14. $\Delta N/N = 1 - \pi nZ^2e^4/T^2\tan^2(\theta_0/2) = 0.6$.

6.15. $\Delta N/N = \dfrac{\pi e^4}{4T^2}\left(0.7\dfrac{Z_1^2}{M_1} + 0.3\dfrac{Z_2^2}{M_2}\right)\rho dN_A\cot^2\dfrac{\theta}{2} = 1.4\cdot 10^{-3}$, where Z_1 and Z_2 are the atomic numbers of copper and zinc, M_1 and M_2 are their molar masses, N_A is Avogadro's number.

6.16. $\Delta\sigma = \pi(Ze^2/T)^2\cot^2(\theta_0/2) = 0.73$ kb.

6.17. (a) 0.9 MeV; (b) $d\sigma/d\Omega = \Delta\sigma/4\pi\sin^4(\theta/2) = 0.64$ kb/sp.

6.18. $t = (3mc^3/2e^2\omega^2)\ln\eta = 15$ ns.

6.19. $t \approx m^2c^3r^3/4e^4 \approx 13$ ps.

6.21. $r_n = \sqrt{n\hbar/m\omega}$, $E_n = n\hbar\omega$, where $n = 1, 2, \ldots$, $\omega = \sqrt{k/m}$.

6.22.

	r_1, pm	v, 10^6 m/s	T, eV	E_b, eV	φ_i, V,	φ_1, V	λ, nm
H	52.9	2.18	13.6	13.6	13.6	10.2	121.5
He^+	26.5	4.36	54.5	54.5	54.5	40.8	30.4

6.23. $\omega = me^4Z^2/\hbar^3n^3 = 2.07\cdot 10^{16}$ s^{-1}.

6.24. $\mu_n = ne\hbar/2mc$, $\mu_n/M_n = e/2mc$, $\mu_1 = \mu_B$.

6.25. $B = m^2e^7/c\hbar^5 = 125$ kG.

6.27. The Brackett series, $\lambda_{6\to 4} = 2.63$ μm.

6.28. (a) 657, 487 and 434 nm; (b) $\lambda/\delta\lambda \approx 1.5\cdot 10^3$.

6.29. For $n \gg 1$ $\sin\theta \approx n^3\pi c/lR$, whence $\theta \approx 60°$.

6.30. He^+.

6.31. $N = {}^1\!/_2 n(n-1)$.

6.32. 97.3, 102.6 and 121.6 nm.

6.33. $n = 5$.

6.34. $R = \dfrac{176\pi c}{15Z^2\Delta\lambda} = 2.07\cdot 10^{16}$ s^{-1}.

6.35. $Z = \sqrt{(176/15)\,\pi c/R\Delta\lambda} = 3$, Li^{++}.

6.36. $\lambda = (2\pi c/\Delta\omega)(Z\sqrt{R/\Delta\omega} - 1)/(2Z\sqrt{R/\Delta\omega} - 1) = 0.47$ μm.

6.37. $E_b = 54.4$ eV (He^+).

6.38. $E = E_0 + 4\hbar R = 79$ eV.

6.39. $v = \sqrt{2(\hbar\omega - 4\hbar R)/m} = 2.3\cdot 10^6$ m/s, where $\omega = 2\pi c/\lambda$.

6.40. $T_{min} = {}^3\!/_2\hbar R = 20.5$ eV.

6.41. $v = 3\hbar R/4mc = 3.25$ m/s, where m is the mass of the atom.

6.42. $(\varepsilon - \varepsilon')/\varepsilon \approx 3\hbar R/8mc^2 = 0.55\cdot 10^{-6}\,\%$, where m is the mass of the atom.

6.43. $v = 2\sqrt{\hbar R/m} = 3.1\cdot 10^6$ m/s, where m is the mass of the electron.

6.44. $v = 3R\Delta\lambda/8\pi\cos\theta = 0.7\cdot 10^6$ m/s.

6.45. (a) $E_n = n^2\pi^2\hbar^2/2ml^2$; (b) $E_n = n^2\hbar^2/2mr^2$; (c) $E_n = n\hbar\sqrt{\alpha/m}$; (d) $E_n = -m\alpha^2/2\hbar^2n^2$.

6.46. $E_b = \mu e^4/2\hbar^2$, $R = \mu e^4/2\hbar^3$, where μ is the reduced mass of the system. If the motion of the nucleus is not taken into account, these values (in the case of a hydrogen atom) are greater by $m/M \approx \approx 0.055\%$, where m and M are the masses of an electron and a proton.

6.47. $E_D - E_H = 3.7$ meV, $\lambda_H - \lambda_D = 33$ pm.

6.48. (a) 0.285 pm, 2.53 keV, 0.65 nm; (b) 106 pm, 6.8 eV, 0.243 μm.

6.49. 123, 2.86 and 0.186 pm.

6.50. 0.45 keV.

6.51. For both particles $\lambda = 2\pi\hbar\,(1 + m_n/m_d)/\sqrt{2m_nT} = 8.6$ pm.

6.52. $\tilde{\lambda} = 2\lambda_1\lambda_2/\sqrt{\lambda_1^2 + \lambda_2^2}$.

6.53. $\lambda = 2\pi\hbar/\sqrt{2mkT} = 128$ pm.

6.54. First, let us find the distribution of molecules over de Broglie wavelengths. From the relation $f(v)\,dv = -\varphi(\lambda)\,d\lambda$ where $f(v)$ is Maxwell's distribution of velocities, we obtain

$$\varphi(\lambda) = A\lambda^{-4}e^{-a/\lambda^2}, \quad a = 2\pi^2\hbar^2/nkT.$$

The condition $d\varphi/d\lambda = 0$ provides $\lambda_{pr} = \pi\hbar/\sqrt{mkT} = 0.09$ nm.

6.55. $\lambda = 2\pi\hbar/\sqrt{2mT\,(1 + T/2mc^2)}$, $T \leqslant 4mc^2\Delta\lambda/\lambda = 20.4$ keV (for an electron) and 37.5 MeV (for a proton).

6.56. $T = (\sqrt{2} - 1)\,mc^2 = 0.21$ MeV.

6.57. $\lambda = \lambda_{sh}/\sqrt{1 + mc\lambda_{sh}/\pi\hbar} = 3.3$ pm.

6.58. $v = 4\pi\hbar l/mb\Delta x = 2.0 \cdot 10^6$ m/s.

6.59. $\Delta x = 2\pi\hbar l/d\sqrt{2meV} = 4.9$ μm.

6.60. $V_0 = \pi^2\hbar^2/2me\,(\sqrt{\eta} - 1)^2 d^2 \sin^2\theta = 0.15$ keV.

6.61. $d = \pi\hbar k/\sqrt{2mT}\cos(\theta/2) = 0.21$ nm, where $k = 4$.

6.62. $d = \pi\hbar k/\sqrt{2mT}\sin\theta = 0.23 \pm 0.04$ nm, where $k = 3$ and the angle θ is determined by the formula $\tan 2\theta = D/2l$.

6.63. (a) $n = \sqrt{1 + V_i/V} = 1.05$; (b) $V/V_i \geqslant 1/\eta\,(2 + \eta) = 50$.

6.64. $E_n = n^2\pi^2\hbar^2/2ml^2$, where $n = 1, 2, \ldots$

6.66. $1 \cdot 10^4$, $1 \cdot 10$ and $1 \cdot 10^{-20}$ cm/s.

6.67. $\Delta v \approx \hbar/ml = 1 \cdot 10^6$ m/s; $v_1 = 2.2 \cdot 10^6$ m/s.

6.69. $\Delta t \approx \eta ml^2/\hbar \approx 10^{-16}$ s.

6.70. $T_{min} \approx \hbar^2/2ml^2 = 1$ eV. Here we assumed that $p \approx \Delta p$ and $\Delta x = l$.

6.71. $\Delta v/v \sim \hbar/l\sqrt{2mT} = 1 \cdot 10^{-4}$.

6.72. $F \approx \hbar^2/ml^3$.

6.73. Taking into account that $p \sim \Delta p \sim \hbar/\Delta x \sim \hbar/x$, we get $E = T + U \approx \hbar^2/2mx^2 + kx^2/2$. From the condition $dE/dx = 0$ we find x_0 and then $E_{min} \approx \hbar\sqrt{k/m} = \hbar\omega$, where ω is the oscillator's angular frequency. The rigorous calculations furnish the value $\hbar\omega/2$.

6.74. Taking into account that $p \sim \Delta p \sim \hbar/\Delta r$ and $\Delta r \sim r$, we get $E = p^2/2m - e^2/r \approx \hbar^2/2mr^2 - e^2/r$. From the condition $dE/dr = 0$ we find $r_{eff} \approx \hbar^2/me^2 = 53$ pm, $E_{min} \approx -me^4/2\hbar^2 = -13.6$ eV.

6.75. The width of the image is $\Delta \approx \delta + \Delta' \approx \delta + \hbar l/p\delta$, where Δ' is an additional widening associated with the uncertainty of the momentum Δp_y (when the hydrogen atoms pass through the slit), p is the momentum of the incident hydrogen atoms. The function $\Delta(\delta)$ has the minimum when $\delta \approx \sqrt{\hbar l/mv} = 0.01$ mm.

6.76. The solution of the Schrödinger equation should be sought in the form $\Psi = \psi(x) \cdot f(t)$. The substitution of this function into the initial equation with subsequent separation of the variables x and t results in two equations. Their solutions are $\psi(x) \sim e^{ikx}$, where $k = \sqrt{2mE}/\hbar$, E is the energy of the particle, and $f(t) \sim e^{-i\omega t}$, where $\omega = E/\hbar$. Finally, $\Psi = ae^{i(kx - \omega t)}$, where a is a certain constant.

6.77. $P = 1/3 + \sqrt{3}/2\pi = 0.61$.

6.78. $\psi = \begin{cases} A\cos(\pi nx/l), & \text{if } n = 1, 3, 5, \ldots, \\ A\sin(\pi nx/l), & \text{if } n = 2, 4, 6, \ldots \end{cases}$

Here $A = \sqrt{2/l}$.

6.80. $dN/dE = (l/\pi\hbar)\sqrt{m/2E}$; if $E = 1$ eV, then $dN/dE = 0.8 \cdot 10^7$ levels per eV.

6.81. (a) In this case the Schrödinger equation takes the form

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + k^2\psi = 0, \quad k^2 = 2mE/\hbar^2.$$

Let us take the origin of coordinates at one of the corners of the well. On the sides of the well the function $\psi(x, y)$ must turn into zero (according to the condition), and therefore it is convenient to seek this function inside the well in the form $\psi(x, y) = a\sin k_1x \times \sin k_2y$, since on the two sides ($x = 0$ and $y = 0$) $\psi = 0$ automatically. The possible values of k_1 and k_2 are found from the condition of ψ turning into zero on the opposite sides of the well:

$$\psi(l_1, y) = 0, \quad k_1 = \pm(\pi/l_1)\,n_1, \quad n_1 = 1, 2, 3, \ldots,$$
$$\psi(x, l_2) = 0, \quad k_2 = \pm(\pi/l_2)\,n_2, \quad n_2 = 1, 2, 3, \ldots$$

The substitution of the wave function into the Schrödinger equation leads to the relation $k_1^2 + k_2^2 = k^2$, whence

$$E_{n_1n_2} = (n_1^2/l_1^2 + n_2^2/l_2^2)\,\pi^2\hbar^2/2m.$$

(b) 9.87, 24.7, 39.5, and 49.4 units of $\hbar^2/ml^2$.

6.82. $P = 1/3 - \sqrt{3}/4\pi = 19.5\%$.

6.83. (a) $E = (n_1^2 + n_2^2 + n_3^2)\,\pi^2\hbar^2/2ma^2$, where n_1, n_2, n_3 are integers not equal to zero: (b) $\Delta E = \pi^2\hbar^2/ma^2$; (c) for the 6-th level $n_1^2 + n_2^2 + n_3^2 = 14$ and $E = 7\pi^2\hbar^2/ma^2$; the number of states is equal o six (it is equal to the number of permutations of a triad 1, 2, 3.)

6.84. Let us integrate the Schrödinger equation over a small interval of the coordinate x within which there is a discontinuity in $U(x)$, for example at the point $x = 0$:

$$\frac{\partial\psi}{\partial x}(+\delta) - \frac{\partial\psi}{\partial x}(-\delta) = \int_{-\delta}^{+\delta} \frac{2m}{\hbar^2}(E-U)\,\psi\,dx.$$

Since the discontinuity U is finite the integral tends to zero as $\delta \to 0$. What follows is obvious.

6.85. (a) Let us write the Schrödinger equation for two regions

$$0 < x < l, \quad \psi_1'' + k^2\psi_1 = 0, \quad k^2 = 2mE/\hbar^2,$$
$$x > l, \quad \psi_2'' - \varkappa^2\psi_2 = 0, \quad \varkappa^2 = 2m(U_0 - E)/\hbar^2.$$

Their common solutions

$$\psi_1(x) = a\sin(kx + \alpha), \quad \psi_2(x) = be^{-\varkappa x} + ce^{\varkappa x}$$

must satisfy the standard and boundary conditions. From the condition $\psi_1(0) = 0$ and the requirement for the finiteness of the wave

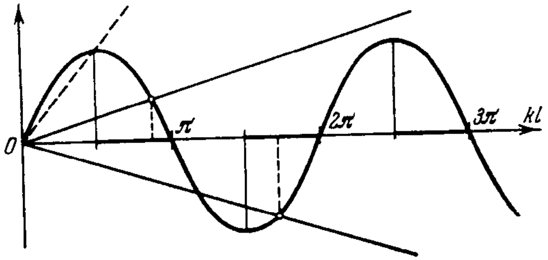


Fig. 45.

function it follows that $\alpha = 0$ and $c = 0$. And finally, from the continuity of $\psi(x)$ and its derivative at the point $x = l$ we obtain $\tan kl = -k/\varkappa$, whence

$$\sin kl = \pm kl\sqrt{\hbar^2/2ml^2U_0}.$$

Plotting the left-hand and right-hand sides of the last equation (Fig. 45), we can find the points at which the straight line crosses the sine curve. The roots of the equation corresponding to the eigenvalues of energy E are found from those intersection points $(kl)_i$ for which $\tan(kl)_i < 0$, i.e. the roots of that equation are located in the even quadrants (these segments of the abscissa axis are shown heavy in the figure). It is seen from the plot that the roots of the equation, i.e. the bound states of the particle, do not always exist. The dotted line indicates the ultimate position of the straight line.
(b) $(l^2U_0)_{1\,min} = \pi^2\hbar^2/8m$, $(l^2U_0)_{n\,min} = (2n-1)\,\pi^2\hbar^2/8m$.

6.86. Suppose that P_a and P_i are the probabilities of the particle being outside and inside the well. Then

$$\frac{P_a}{P_i} = \frac{\int_l^\infty b^2e^{-2\varkappa x}\,dx}{\int_0^l a^2\sin^2 kx\,dx} = \frac{2}{2+3\pi},$$

where the ratio b/a can be found from the condition $\psi_1(l) = \psi_2(l)$. Now it remains to take into account that $P_a + P_i = 1$; then $P_a = 2/(4+3\pi) = 14.9\%$.

The penetration of the particle into the region where its energy $E < U$ is a purely quantum phenomenon. It occurs owing to the wave properties of the particle ruling out the simultaneous precise magnitudes of the coordinate and the momentum, and consequently the precise division of the total energy of the particle into the potential and the kinetic energy. The latter could be done only within the limits set by the uncertainty principle.

6.87. Utilizing the substitution indicated, we get

$$\chi'' + k^2\chi = 0, \quad \text{where} \quad k^2 = 2mE/\hbar^2.$$

We shall seek the solution of this equation in the form $\chi = a\sin(kr + \alpha)$. From the finiteness of the wave function ψ at the point $r = 0$ it follows that $\alpha = 0$. Thus, $\psi = (a/r)\sin kr$. From the boundary condition $\psi(r_0) = 0$ we obtain $kr_0 = n\pi$, where $n = 1, 2, \ldots$ Hence, $E_n = n^2\pi^2\hbar^2/2mr_0^2$.

6.88. (a) $\psi(r) = \frac{1}{\sqrt{2\pi r_0}}\frac{\sin(n\pi r/r_0)}{r}$, $n = 1, 2, \ldots$; (b) $r_{pr} = r_0/2$; 50%.

6.89. (a) The solutions of the Schrödinger equation for the function $\chi(r)$:

$$r < r_0, \quad \chi_1 = A\sin(kr + \alpha), \quad \text{where } k = \sqrt{2mE}/\hbar,$$
$$r > r_0, \quad \chi_2 = Be^{\varkappa r} + Ce^{-\varkappa r}, \quad \text{where } \varkappa = \sqrt{2m(U_0 - E)}/\hbar.$$

Since the function $\psi(r)$ is finite throughout the space, $\alpha = 0$ and $B = 0$. Thus,

$$\psi_1 = A\frac{\sin kr}{r}, \quad \psi_2 = C\frac{e^{-\varkappa r}}{r}.$$

From the continuity of the function ψ and its derivative at the point $r = r_0$ we get $\tan kr_0 = -k/\varkappa$, or

$$\sin kr_0 = \pm\sqrt{\hbar^2/2mr_0^2U_0}\,kr_0.$$

As it was demonstrated in the solution of Problem 6.85, this equation determines the discontinuous spectrum of energy eigenvalues.
(b) $r_0^2U_0 = \pi^2\hbar^2/8m$.

6.90. $\alpha = m\omega/2\hbar$, $E = \hbar\omega/2$, where $\omega = \sqrt{k/m}$.

6.91. $E = -me^4/8\hbar^2$, i.e. the level with principal quantum number $n = 2$.

6.92. (a) The probability of the electron being at the interval r, $r + dr$ from the nucleus is $dP = \psi^2(r)\, 4\pi r^2 dr$. From the condition for the maximum of the function dP/dr we get $r_{pr} = r_1$; (b) $\langle F \rangle = 2e^2/r_1^2$; (c) $\langle U \rangle = -e^2/r_1$.

6.93. $\varphi_0 = \int (\rho/r)\, 4\pi r^2\, dr = -e/r_1$, where $\rho = -e\psi^2$ is the space charge density, ψ is the normalized wave function.

6.94. (a) Let us write the solutions of the Schrödinger equation to the left and to the right of the barrier in the following form:

$$x < 0,\quad \psi_1(x) = a_1 e^{ik_1x} + b_1 e^{-ik_1x},\quad \text{where } k_1 = \sqrt{2mE}/\hbar,$$

$$x > 0,\quad \psi_2(x) = a_2 e^{ik_2x} + b_2 e^{-ik_2x},\quad \text{where } k_2 = \sqrt{2m(E-U_0)}/\hbar.$$

Let us assume that the incident wave has an amplitude a_1 and the reflected wave an amplitude b_2. Since in the region $x > 0$ there is only a travelling wave, $b_2 = 0$. The reflection coefficient R is the ratio of the reflected stream of particles to the incident stream, or, in other words, the ratio of the squares of amplitudes of corresponding waves. Due to the continuity of ψ and its derivative at the point $x = 0$ we have $a_1 + b_1 = a_2$ and $a_1 - b_1 = (k_2/k_1)\, a_2$, whence

$$R = (b_1/a_1)^2 = (k_1 - k_2)^2/(k_1 + k_2)^2.$$

(b) In the case of $E < U_0$ the solution of the Schrödinger equation to the right of the barrier takes the form

$$\psi_2(x) = a_2 e^{\varkappa x} + b_2 e^{-\varkappa x},\quad \text{where } \varkappa = \sqrt{2m(U_0-E)}/\hbar.$$

From the finiteness of $\psi(x)$ it follows that $a_2 = 0$. The probability of finding the particle under the barrier has the density $P_2(x) = \psi_2^2(x) \sim e^{-2\varkappa x}$. Hence, $x_{eff} = 1/2\varkappa$.

6.95. (a) $D \approx \exp\left[-\frac{2l}{\hbar}\sqrt{2m(U_0-E)}\right]$;

(b) $D \approx \exp\left[-\frac{8l\sqrt{2m}}{3\hbar U_0}(U_0-E)^{3/2}\right]$.

6.96. $D \approx \exp\left[-\frac{\pi l}{\hbar}\sqrt{\frac{2m}{U_0}}\,(U_0-E)\right]$.

6.97. -0.41 for an S term and -0.04 for a P term.

6.98. $\alpha = \sqrt{\hbar R/(E_0 - e\varphi_1)} - 3 = -0.88$.

6.99. $E_b = \hbar R/(\sqrt{R\lambda_1\lambda_2/2\pi c\Delta\lambda} - 1)^2 = 5.3$ eV.

6.100. 0.82 μm ($3S \to 2P$) and 0.68 μm ($2P \to 2S$).

6.101. $\Delta E = 2\pi\hbar c\Delta\lambda/\lambda^2 = 2.0$ meV.

6.102. $\Delta\omega = 1.05 \cdot 10^{14}$ rad/s.

6.103. $3S_{1/2}$, $3P_{1/2}$, $3P_{3/2}$, $3D_{3/2}$, $3D_{5/2}$.

6.104. (a) 1, 2, 3, 4, 5; (b) 0, 1, 2, 3, 4, 5, 6; (c) 1/2, 3/2, 5/2, 7/2, 9/2.

6.105. For the state ^4P: $\hbar\sqrt{3}/2$, $\hbar\sqrt{15}/2$, and $\hbar\sqrt{32}/2$; for the state ^5D: 0, $\hbar\sqrt{2}$; $\hbar\sqrt{6}$, $\hbar\sqrt{12}$, $\hbar\sqrt{20}$.

6.106. (a) $^2F_{7/2}$, $M_{max} = \hbar\sqrt{63}/2$, (b) $^3F_4$, $M_{max} = 2\hbar\sqrt{5}$.

6.107. In the F state $M_s = \hbar\sqrt{6}$; for the D state it can be only found that $M_s \geqslant \hbar\sqrt{6}$.

6.108. 3, 4, 5.

6.109. (a) 1, 3, 5, 7, 9; (b) 2, 4, 6; (c) 5, 7, 9.

6.110. 31°.

6.111. $^3D_2$.

6.112. $^1P_1$, $^1D_2$, $^1F_3$, $^3P_{0,1,2}$, $^3D_{1,2,3}$, $^3F_{2,3,4}$.

6.113. The same as in the foregoing problem.

6.114. The second and the third term.

6.115. $g = 4 + 6 = 10$.

6.116. 4, 7 and 10.

6.117. $^3F_3$.

6.118. As.

6.119. (a) $^4S_{3/2}$; (b) $^3P_2$.

6.120. (a) $^4F_{3/2}$, $\hbar\sqrt{15}/2$; (b) $^4F_{9/2}$, $\hbar 3\sqrt{11}/2$.

6.121. (a) Two d electrons; (b) five p electrons; (c) five d electrons.

6.122. (a) $^3P_0$, (b) $^4F_{9/2}$.

6.123. $^4F_{3/2}$.

6.124. $\mu = \mu_B\sqrt{35}$ ($^6S_{5/2}$).

6.125. $\eta = n^2 e^{-\hbar\omega/kT} = 3 \cdot 10^{-17}$, where $\omega = R(1 - 1/n^2)$.

6.126. $N/N_0 = (g/g_0)\, e^{-\hbar\omega/kT} = 1.14 \cdot 10^{-4}$, where g and g_0 are the statistical weights (degeneracy ratios) of the levels $3P$ and $3S$ respectively ($g = 6$, $g_0 = 2$).

6.127. $\tau = l/v \ln \eta = 1.3$ μs.

6.128. $N = \lambda\tau P/2\pi c\hbar = 7 \cdot 10^9$.

6.129. $\tau = (n\hbar\omega/P)\,(g/g_0)\, e^{-\hbar\omega/kT} = 65$ ns, where g and g_0 are the degeneracy ratios of the resonant and the basic level.

6.130. (a) $P_{ind}/P_{sp} = 1/(e^{\hbar\omega/kT} - 1) \approx 10^{-34}$, where $\omega = {}^3/_4 R$; (b) $T = 1.7 \cdot 10^5$ K.

6.131. Suppose that I is the intensity of the passing ray. The decrease in this value on passing through the layer of the substance of thickness dx is equal to

$$-dI = \varkappa I\, dx = (N_1B_{12} - N_2B_{21})\,(I/c)\,\hbar\omega\, dx,$$

where N_1 and N_2 are the concentrations of atoms on the lower and upper levels, B_{12} and B_{21} are the Einstein coefficients. Hence

$$\varkappa = (\hbar\omega/c)\, N_1B_{12}\,(1 - g_1N_2/g_2N_1).$$

Next, the Boltzmann distribution should be taken into consideration, as well as the fact that $\hbar\omega \gg kT$ (in this case N_1 is approximately equal to N_0, the total concentration of the atoms).

6.132. $\Delta\lambda_{Dop}/\Delta\lambda_{nat} \approx 4\pi\tau v_{pr}/\lambda \approx 10^3$, where $v_{pr} = \sqrt{2RT/M}$.

6.133. $\lambda = 154$ pm.
6.134. (a) 843 pm for Al, 180 pm for Co; (b) ≈ 5 keV.
6.135. Three.
6.136. $V = 15$ kV.
6.137. Yes.
6.138. $Z = 1 + 2\sqrt{(n-1)eV_1/3\hbar R(n - V_1/V_2)} = 29$.
6.139. $Z = 1 + \sqrt{4\Delta\omega/3R} = 22$, titanium.
6.140. $E_b = {}^3/_4\hbar R(Z-1)^2 + 2\pi c\hbar/\lambda_L = 5.5$ keV.
6.141. $E_L = \hbar\omega(2\pi c/\omega\Delta\lambda - 1) \approx 0.5$ keV, where $\omega = {}^3/_4 R(Z-1)^2$.
6.142. $T = {}^3/_4\hbar R(Z-1)^2 - 2\pi c\hbar/\lambda_K = 1.45$ keV, $v = 2.26\cdot10^7$ m/s.
6.143. (a) $g = 2$, with the exception of the singlet state, where $g = 0/0$; (b) $g = 1$.
6.144. (a) $-2/3$; (b) 0; (c) 1; (d) 5/2; (e) 0/0.
6.145. (a) $\sqrt{12}\mu_B$; (b) $2\sqrt{3/5}\mu_B$; (c) $(8/\sqrt{3})\mu_B$.
6.146. $M_s = 2\sqrt{3}\hbar$.
6.147. $\mu = (8/\sqrt{3})\mu_B$.
6.148. $\mu = 3\sqrt{7/5}\mu_B$.
6.149. $\mu - (5\sqrt{5}/2)\mu_B$.
6.150. $M = \hbar\sqrt{3/2}$.
6.151. ${}^5F_1$.
6.152. $\omega = \mu_B gB/\hbar = 1.2\cdot10^{10}$ rad/s, where g is the Landé factor.
6.153. $F_{max} = \mu_{B\ max}\cdot|\partial B/\partial z| = (3/\sqrt{8})\pi IgJ\mu_B/cr^2 = 4\cdot10^{-27}$ N.
6.154. $F = 2I\mu_B/cr^2 = 3\cdot10^{-26}$ N.
6.155. $\partial B/\partial z = 2T\delta/gJ\mu_B l_1(l_1 + 2l_2) = 15$ kG/cm.
6.156. (a) It does not split; (b) splits into 6 sublevels; (c) does not split ($g = 0$).
6.157. (a) 58 μeV; (b) $\Delta E = 2gJ\mu_B B = 145$ μeV.
6.158. (a) Normal; (b) anomalous; (c) normal; (d) normal (both terms have identical Landé factors).
6.159. $L = \Delta E/2\mu_B B = 3$; ${}^1F_3$.
6.160. $\Delta\lambda = \lambda^2 eB/2\pi mc^2 = 35$ pm.
6.161. $B_{min} = 4.0$ kG.
6.162. $B = \hbar\Delta\omega/g\mu_B = 3$ kG.
6.163. (a) 2 .1 (the ratio of the corresponding Landé factors); (b) $B = 2\pi c\hbar\Delta\lambda/g\mu_B\eta\lambda^2 = 5.5$ kG.
6.164. $\Delta\omega = (\pm1.3, \pm4.0, \pm6.6)\cdot10^{10}$ s^{-1}, six components.
6.165. (a) Six (*1*) and four (*2*); (b) nine (*1*) and six (*2*).
6.166. $\Delta\omega = (m_1g_1 - m_2g_2)_{max}eB/mc = 1.0\cdot10^{11}$ s^{-1}.
6.167. $\omega = 4\sqrt{2\hbar/md^2} = 1.57\cdot10^{11}$ s^{-1}, where m is the mass of the molecule.
6.168. 2 and 3.

6.169. $M = \sqrt{md^2E/2} = 3.5\hbar$, where m is the mass of the molecule.
6.170. $I = \hbar/\Delta\omega = 1.93\cdot10^{-40}$ g$\cdot$cm^2, $d = 112$ pm.
6.171. 13 levels.
6.172. $N \approx \sqrt{2I\omega/\hbar} = 33$ lines.
6.173. $dN/dE \approx \sqrt{I/2\hbar^2E}$, where I is the moment of inertia of the molecule. In the case of $J = 10$ $dN/dE = 1.0\cdot10^4$ levels per eV.
6.174. $E_{vib}/E_{rot} = \omega\mu d^2/\hbar$, where μ is the reduced mass of the molecule; (a) 36; (b) $1.7\cdot10^2$; (c) $2.9\cdot10^3$.
6.175. $N_{vib}/N_{rot} = {}^1/_3 e^{-\hbar(\omega-2B)/kT} = 3.1\cdot10^{-4}$, where $B = \hbar/2I$, I is the moment of inertia of the molecule.
6.176. According to the definition

$$\langle E\rangle = \frac{\sum E_v \exp(-E_v/kT)}{\sum \exp(-E_v/kT)} = \frac{\sum E_v \exp(-\alpha E_v)}{\sum \exp(-\alpha E_v)},$$

where $E_v = \hbar\omega(v + 1/2)$, $\alpha = 1/kT$. The summation is carried out over v taking the values from 0 to ∞ as follows:

$$\langle E\rangle = -\frac{\partial}{\partial\alpha}\ln\sum\exp(-\alpha E_v) = -\frac{\partial}{\partial\alpha}\ln\frac{\exp(-\alpha\hbar\omega/2)}{1-\exp(-\alpha\hbar\omega)} = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp(\hbar\omega/kT)-1};$$

$$C_{vib} = N\frac{\partial\langle E\rangle}{\partial T} = \frac{R(\hbar\omega/kT)^2 e^{\hbar\omega/kT}}{(e^{\hbar\omega/kT}-1)^2} = 0.56R,$$

where R is the universal gas constant.
6.177. $d = \sqrt{2\hbar/\mu\Delta\omega} = 0.13$ nm, where μ is the reduced mass of the molecule.
6.178. $\lambda = \lambda_0/(1 \mp \omega\lambda_0/2\pi c) = 423$ and 387 nm.
6.179. $\omega = \pi c(\lambda_r - \lambda_v)/\lambda_r\lambda_v = 1.37\cdot10^{14}$ rad/s, $\varkappa = 4.96$ N/cm.
6.180. $I_v/I_r = \exp(-\hbar\omega/kT) = 0.067$. Will increase 3.9 times.
6.181. (a) See Fig. 46*a* in which the arrows indicate the motion directions of the nuclei in the molecule at the same moment. The oscillation frequencies are ω_1, ω_2, ω_3, with ω_3 being the frequency of two independent oscillations in mutually perpendicular planes. Thus, there are four different oscillations. (b) See Fig. 46*b*; there are seven different oscillations: three longitudinal ones (ω_1, ω_2, ω_3) and four transversal ones (ω_4, ω_5), two oscillations for each frequency.
6.182. $dN_\omega = (l/\pi v)\,d\omega$.
6.183. $dN_\omega = (S/2\pi v^2)\,\omega\,d\omega$.
6.184. $dN_\omega = (V/\pi^2v^3)\,\omega^2\,d\omega$.
6.185. (a) $\Theta = (\hbar/k)\pi v n_0$; (b) $\Theta = (\hbar/k)v\sqrt{4\pi n_0}$; (c) $\Theta = (\hbar/k)v\sqrt[3]{6\pi^2 n_0}$.

6.186. $\Theta = (\hbar/k)\sqrt[3]{18\pi^2 n_0/(v_{\parallel}^{-3} + 2v_{\perp}^{-3})} = 470$ K, where n_0 is the concentration of the atoms.

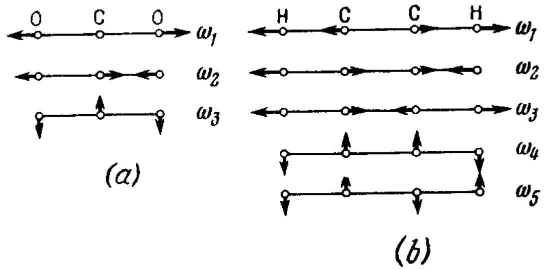


Fig. 46.

6.187. $v \approx k\Theta/\hbar\sqrt[3]{6\pi^2 n_0} = 3.4$ km/s, where n_0 is the concentration of the atoms. The tabulated values are: $v_{\parallel} = 6.3$ km/s, $v_{\perp} = 3.1$ km/s.

6.188. The oscillation energy of a mole of a "crystal" is

$$U = R\Theta\left[\frac{1}{4} + \left(\frac{T}{\Theta}\right)^2 \int\limits_0^{\Theta/T} \frac{x\,dx}{e^x - 1}\right],$$

where $x = \hbar\omega/kT$. Hence the molar heat capacity is

$$C = R\left(\frac{2T}{\Theta}\int\limits_0^{\Theta/T} \frac{x\,dx}{e^x - 1} - \frac{\Theta/T}{e^{\Theta/T} - 1}\right).$$

When $T \gg \Theta$, the heat capacity $C \approx R$.

6.189. (a) $dN/d\omega = 2l/\pi a\sqrt{\omega_{max}^2 - \omega^2}$; (b) $N = l/a$, i.e. is equal to the number of the atoms in the chain.

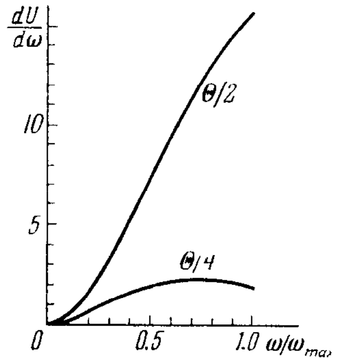


Fig. 47.

6.190. $U_0 = 9R\Theta/8\mu = 48.6$ J/g, where μ is the molar mass of copper.

6.191. (a) $\Theta \approx 220$ K; (b) $C \approx 10$ J/(mol·K); (c) $\omega_{max} = 4.1 \times 10^{13}$ rad/s.

6.193. Yes, because the heat capacity is proportional to T^3 at these temperatures.

6.194. $\langle E\rangle = {}^3/_8\, k\Theta$.

6.195. See Fig. 47.

6.196. $\hbar\omega_{max} = 28$ meV, $\hbar k_{max} \sim 10^{-19}$ g·cm/s.

6.197. (a) $T_{max} = (3\pi^2 n)^{2/3}\hbar^2/2m$; (b) $\langle T\rangle = {}^3/_5 T_{max}$.

6.198. $\eta = 1 - 2^{-3/2} \approx 65\%$.

6.199. 0.93.

6.200. $\approx 3\cdot 10^4$ K.

6.201. $\Delta E = 2\pi^2\hbar^2/mV\,(3\pi^2 n)^{1/3} = 2\cdot 10^{-22}$ eV.

6.202. (a) $dn_v = (m^3/\pi^2\hbar^3)\, v^2\, dv$; (b) $\langle v\rangle/v_{max} = 3/4$.

6.203. $dn_\lambda = 8\pi\lambda^{-4}\, d\lambda$.

6.204. $p = {}^2/_3 n\, \langle T\rangle = (\pi\sqrt[3]{9\pi}\hbar^2/5m)\, n^{5/3} \approx 5\cdot 10^4$ atm.

6.205. $A = kT\,(\eta T/\Delta T - 2) = 4.5$ eV.

6.206. $n = \sqrt{1 + U_0/T} = 1.02$, where $U_0 = T_{max} + A$, $T_{max} = (3\pi^2 n)^{2/3}\hbar^2/2m$. A is the work function.

6.207. $E_{min} = \frac{2kT_1T_2}{T_2 - T_1}\ln\eta = 0.33$ eV.

6.208. $\alpha = \frac{1}{\rho}\frac{\partial\rho}{\partial T} = -\frac{\pi c\hbar}{kT^2\lambda_r} = -0.05\ \text{K}^{-1}$, where $\rho \sim e^{\Delta E_0/2kT}$, ΔE_0 is the forbidden band width.

6.209. $\Delta E = -2k\frac{\Delta \ln\sigma}{\Delta(T^{-1})} = 1.2$ and 0.06 eV.

6.210. $\tau = t/\ln\frac{(\rho - \rho_1)\rho_2}{(\rho - \rho_2)\rho_1} = 0.01$ s.

6.211. $n = hBV/el\rho V_H = 5\cdot 10^{15}\ \text{cm}^{-3}$, $u_0 = lV_H/hBV = 0.05\ \text{m}^2/(\text{V}\cdot\text{s})$.

6.212. $|u_0^- - u_0^+| = 1/\eta B = 0.20\ \text{m}^2/(\text{V}\cdot\text{s})$.

6.213. $n^+/n^- = \eta^2 = 4.0$.

6.214. (a) $P = 1 - \exp(-\lambda t)$; (b) $\tau = 1/\lambda$.

6.215. About 1/4.

6.216. $1.2\cdot 10^{15}$.

6.217. $\tau \approx 16$ s.

6.218. $T = 5.3$ days.

6.219. $4.6\cdot 10^2$ part./min.

6.220. $\lambda = -(1/t)\ln(1 - \eta) \approx \eta/t = 1.1\cdot 10^{-5}\ \text{s}^{-1}$, $\tau = 1/\lambda = 1.0$ years.

6.221. $T = 4.5\cdot 10^9$ years, $A = 1.2\cdot 10^4$ dis./s.

6.222. $4.1\cdot 10^3$ years.

6.223. About $2.0\cdot 10^9$ years.

6.224. $3.2\cdot 10^{17}$ and $0.8\cdot 10^5$ dis/(s·g) respectively.

6.225. $V = (A/A')\exp(-t\ln 2/T) = 6$ l.

6.226. 0.19%.

6.227. $T_1 = 1.6$ hours, $T_2 = 9.8$ hours; $N_2/N_1 = (T_2/T_1) \times \exp(\ln A_2 - \ln A_1) = 10$.

6.228. $t = -(T/\ln 2)\ln(1 - A/q) = 9.5$ days.

6.229. (a) $N_2(t) = N_{10}\frac{\lambda_1}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t})$; (b) $t_m = \frac{\ln(\lambda_1/\lambda_2)}{\lambda_1 - \lambda_2}$.

6.230. (a) $N_2(t) = \lambda N_{10} t\exp(-\lambda t)$; (b) $t_m = 1/\lambda$.

6.231. $N_3(t) = N_{10}\left(1 + \frac{\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1}\right)$.

6.232. $\dot N_\beta = N_0\lambda_1\exp(-\lambda_1 t) = 0.72\cdot 10^{11}$ part./s, $\dot N_\alpha = N_0(e^{-\lambda_1 t} - e^{-\lambda_2 t})\lambda_1\lambda_2/(\lambda_2 - \lambda_1) = 1.46\cdot 10^{11}$ part./s. Here N_0 is the initial number of Bi^{210} nuclei.

6.233. (a) Pb^{206}; (b) eight alpha decays and six beta decays.

6.234. $v = \sqrt{2m_\alpha T_\alpha}/m = 3.4 \cdot 10^5$ m/s; 0.020.
6.235. 1.6 MJ.
6.236. 0.82 MeV.
6.237. (a) 6.1 cm; (b) $2.1 \cdot 10^5$ and $0.77 \cdot 10^5$ respectively.
6.238. $Q = \begin{cases} (M_p - M_d)c^2 \text{ for } \beta^- \text{ decay and } K\text{-capture,} \\ (M_p - M_d - 2m)\,c^2 \text{ for } \beta^+ \text{ decay.} \end{cases}$
6.239. 0.56 MeV and 47.5 eV.
6.240. 5 MJ.
6.241. 0.32 and 0.65 MeV.
6.242. $T \approx {}^1/_2 Q\,(Q + 2mc^2)/M_{\mathrm{N}}c^2 = 0.11$ keV, where $Q = (M_{\mathrm{N}} - M_{\mathrm{C}} - 2m)\,c^2$, m is the mass of an electron.
6.243. 40 km/s.
6.244. 0.45 c, where c is the velocity of light.
6.245. $\Delta\varepsilon/\varepsilon = E/2mc^2 = 3.6 \cdot 10^{-7}$, where m is the mass of the nucleus.
6.246. $v \approx \varepsilon/mc = 0.22$ km/s, where m is the mass of the nucleus.
6.247. $v = gh/c = 65$ μm/s.
6.248. $h_{min} = \hbar c^2/g\varepsilon\tau = 4.6$ m.
6.249. $T = T_\alpha/[1 + (M - m)^2/4mM \cos^2\Theta] = 6.0$ MeV, where m and M are the masses of an alpha particle and a lithium nucleus.
6.250. (a) $\eta = 4mM/(m + M)^2 = 0.89$; (b) $\eta = 2m/(m + M) = 2/3$. Here m and M are the masses of a neutron and a deuteron.
6.251. $\theta_{max} = \arcsin\,(m_1/m_2) = 30°$, where m_1 and m_2 are the masses of a proton and a deuteron.
6.252. $2 \cdot 10^{11}$ kg/cm³, $1 \cdot 10^{38}$ nucl./cm³.
6.253. (a) d; (b) F^{17}; (c) α; (d) Cl^{37}.
6.255. Be^8, $E_b = 56.5$ MeV.
6.256. (a) 8.0 MeV; (b) 11.5 and 8.7 MeV; (c) 14.5 MeV.
6.257. $E_n - E_p = 0.22$ MeV.
6.258. $E = 20\varepsilon_{\mathrm{Ne}} - 2 \cdot 4\varepsilon_\alpha - 12\varepsilon_{\mathrm{C}} = 11.9$ MeV, where ε is the binding energy per nucleon in the corresponding nucleus.
6.259. (a) 8.0225 a.m.u.; (b) 10.0135 a.m.u.
6.260. $Q = (E_3 + E_4) - (E_1 + E_2)$.
6.261. (a) $8.2 \cdot 10^{10}$ kJ, $2.7 \cdot 10^6$ kg; (b) 1.5 kg.
6.262. $5.74 \cdot 10^7$ kJ, $2 \cdot 10^4$ kg.
6.263. 2.79 MeV; 0.85 MeV.
6.264. $Q = 8\varepsilon_\alpha - 7\varepsilon_{\mathrm{Li}} = 17.3$ MeV.
6.265. $Q = (1 + \eta_p)\,T_p - (1 - \eta_\alpha)\,T_\alpha - 2\sqrt{\eta_p\eta_\alpha T_p T_\alpha} \times \cos\theta = -1.2$ MeV, where $\eta_p = m_p/m_{\mathrm{O}}$, $\eta_\alpha = m_\alpha/m_{\mathrm{O}}$.
6.266. (a) -1.65 MeV; (b) 6.82 MeV; (c) -2.79 MeV; (d) 3.11 MeV.
6.267. $v_\alpha = 0.92 \cdot 10^7$ m/s, $v_{\mathrm{Li}} = 0.53 \cdot 10^7$ m/s.
6.268. 1.9 MeV.
6.269. $T_n = \dfrac{Q + (1 - m_\alpha/m_{\mathrm{C}})\,T}{1 + m_n/m_{\mathrm{C}}} = 8.5$ MeV.
6.270. 9.1 MeV, 170.5°.

6.272. $T \geqslant E_b\,(m_p + m_d)/m_d = 3.3$ MeV.
6.273. Between 1.89 and 2.06 MeV.
6.274. $Q = -{}^{11}/_{12}\,T_{th} = -3.7$ MeV.
6.275. 1.88 and 5.75 MeV respectively.
6.276. 4.4 MeV; $5.3 \cdot 10^6$ m/s.
6.277. $T_\alpha = \dfrac{1}{m_3 + m_4}\left[(m_4 - m_1)\,T - \dfrac{m_2 m_4}{m_1 + m_2} T_{th}\right] = 2.2$ MeV, where m_1, m_2, m_3, m_4 are the masses of neutron, a C^{12} nucleus, an alpha particle, and a Be^9 nucleus.
6.278. By $E_b/2mc^2 = 0.06\%$, where m is the mass of a deuteron.
6.279. $E = Q + {}^2/_3 T = 6.5$ MeV.
6.280. $E_i = E_b + \dfrac{m_{\mathrm{C}}}{m_d + m_{\mathrm{C}}} T_i = 16.7$, 16.9, 17.5 and 17.7 MeV, where E_b is the binding energy of a deuteron in the transitional nucleus.
6.281. $\sigma = (M/N\rho\, d)\ln\eta = 2.5$ kb, where M is the molar mass of cadmium, N is the Avogadro number, ρ is the density of cadmium.
6.282. $I_0/I = \exp\,[(2\sigma_1 + \sigma_2)\,nd] = 20$, where n is the concentration of heavy water molecules.
6.283. $w = \{1 - \exp\,[-(\sigma_s + \sigma_a)\,nd]\}\;\sigma_s/(\sigma_s + \sigma_a) = 0.35$, where n is the concentration of Fe nuclei.
6.284. (a) $T = (w/k)\ln 2$; (b) $w = ATe/It \ln 2 = 2 \cdot 10^{-3}$.
6.285. (a) $t = \eta/\sigma J = 3 \cdot 10^6$ years; (b) $N_{max} = J\sigma N_0 T/\ln 2 = 1.0 \cdot 10^{13}$, where N_0 is the number of Au^{197} nuclei in the foil.
6.286. $N = (1 - e^{-\lambda t})\; Jn\sigma/\lambda$.
6.287. $J = Ae^{\lambda t}/\sigma N_0\,(1 - e^{-\lambda t}) = 6 \cdot 10^9$ part./(cm²·s), where λ is the decay constant, N_0 is the number of Au nuclei in the foil.
6.288. $N = N_0 k^{i-1} = 1.3 \cdot 10^5$, where i is the number of generations.
6.289. $N = \nu P/E = 0.8 \cdot 10^{19}$ s⁻¹.
6.290. (a) $N/N_0 = 4 \cdot 10^2$; (b) $T = \tau/(k - 1) = 10$ s.
6.291. 0.05, 0.4, and 9 GeV respectively.
6.292. $\langle l \rangle = c\tau_0\sqrt{\eta(\eta + 2)} = 15$ m.
6.293. $\tau_0 = lmc/\sqrt{T\,(T + 2mc^2)} = 26$ ns, where m is the rest mass of a pion.
6.294. $J/J_0 = \exp\,[-lmc/\tau_0\sqrt{T\,(T + 2mc^2)}] = 0.22$, where m is the rest mass of a negative pion.
6.295*. $T_\mu = (m_\pi - m_\mu)^2/2m_\pi = 4.1$ MeV, $E_\nu = 29.8$ MeV.
6.296*. $T = [(m_\Sigma - m_n)^2 - m_\pi^2]/2m_\Sigma = 19.5$ MeV.
6.297*. $T_{max} = (m_\mu - m_e)^2/2m_\mu = 52.5$ MeV.
6.298*. $m = m_p + T + \sqrt{m_\pi^2 + T\,(T + 2m_p)} = 1115$ MeV, a Λ particle.
6.299*. $E_\nu = {}^1/_2\,(m_\pi^2 - m_\mu^2)/(m_\pi + T) = 22$ MeV.

* In the answers to Problems 6.295 - 6.299 marked by an asterisk the quantity mc^2 is abbreviated as m.

6.300*. $m = \sqrt{m_\Sigma^2 + m_\pi^2 - 2(m_\Sigma + T_\Sigma)(m_\pi + T_\pi)} = 0.94$ GeV, neutron.

6.301*. $T_\pi = m_\pi\,[\operatorname{cosec}(\Theta/2) - 1]$, $E_\gamma = m_\pi/2 \sin(\Theta/2)$. For $\Theta = 60°$ the energy $T_\pi = E_\gamma = m_\pi$.

6.303*. $\cos(\Theta/2) = 1/\sqrt{1 + 2m/T}$, whence $\Theta = 99°$.

6.304*. (a) $\varepsilon_{th} = 4m_e = 2.04$ MeV; (b) $\varepsilon_{th} = 2m_\pi(1 + m_\pi/m_p) = 320$ MeV.

6.305*. (a) $T_{th} = 6m_p = 5.6$ GeV; (b) $T_{th} = m_\pi(4m_p + m_\pi)/2m_p = 0.28$ GeV.

6.306. (a) 0.90 GeV; (b) 0.77 GeV.

6.307. $S = -2$, $Y = -1$, Ξ^0 particle.

6.308. Processes 1, 2, and 3 are forbidden.

6.309. Processes 2, 4, and 5 are forbidden.

6.310. Process 1 is forbidden in terms of energy; in other processes the following laws of conservation are broken: of baryon charge (2), of electric charge (3), of strangeness (4), of lepton charge (5), and of electron and muon charge (6).

---

* In the answers to Problems 6.300-6.305 marked by an asterisk the quantity mc^2 is abbreviated as m.

APPENDICES

1. Basic Trigonometrical formulas

$\sin^2\alpha + \cos^2\alpha = 1$ $\sec^2\alpha - \tan^2\alpha = 1$ $\operatorname{cosec}^2\alpha - \cot^2\alpha = 1$ $\sin\alpha \cdot \operatorname{cosec}\alpha = 1$ $\cos\alpha \cdot \sec\alpha = 1$ $\tan\alpha \cdot \cot\alpha = 1$	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \cdot \tan\beta}$ $\cot(\alpha \pm \beta) = \dfrac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$
$\sin\alpha = \dfrac{1}{\sqrt{1+\cot^2\alpha}}$ $\cos\alpha = \dfrac{1}{\sqrt{1+\tan^2\alpha}}$ $\sin 2\alpha = 2\sin\alpha\cos\alpha$ $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$ $\cot 2\alpha = \dfrac{\cot^2\alpha - 1}{2\cot\alpha}$	$\sin\alpha + \sin\beta = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$ $\sin\alpha - \sin\beta = 2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$ $\cos\alpha + \cos\beta = 2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$ $\cos\alpha - \cos\beta = -2\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$ $\tan\alpha \pm \tan\beta = \dfrac{\sin(\alpha \pm \beta)}{\cos\alpha\cos\beta}$ $\cot\alpha \pm \cot\beta = \pm\dfrac{\sin(\alpha \pm \beta)}{\sin\alpha\sin\beta}$
$\sin\dfrac{\alpha}{2} = \sqrt{\dfrac{1-\cos\alpha}{2}}$ $\cos\dfrac{\alpha}{2} = \sqrt{\dfrac{1+\cos\alpha}{2}}$	$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$ $2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$ $2\sin\alpha\cos\beta = \sin(\alpha-\beta) + \sin(\alpha+\beta)$
$\sinh\alpha = \dfrac{e^\alpha - e^{-\alpha}}{2}$ $\cosh\alpha = \dfrac{e^\alpha + e^{-\alpha}}{2}$	$\tanh\alpha = \dfrac{e^\alpha - e^{-\alpha}}{e^\alpha + e^{-\alpha}}$ $\coth\alpha = \dfrac{e^\alpha + e^{-\alpha}}{e^\alpha - e^{-\alpha}}$

PART ONE

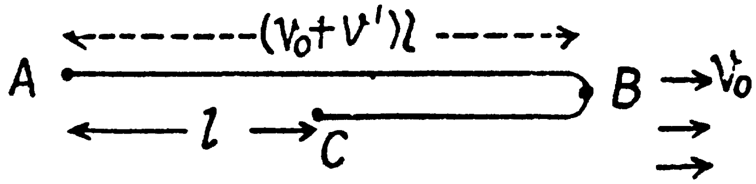
PHYSICAL FUNDAMENTALS OF MECHANICS

1.1 KINEMATICS

1.1 Let v_0 be the stream velocity and v' the velocity of motorboat with respect to water. The motorboat reached point B while going downstream with velocity $(v_0 + v')$ and then returned with velocity $(v' - v_0)$ and passed the raft at point C. Let t be the time for the raft (which flows with stream with velocity v_0) to move from point A to C, during which the motorboat moves from A to B and then from B to C.

Therefore

$$\frac{l}{v_0} = \tau + \frac{(v_0 + v')\tau - l}{(v' - v_0)}$$



On solving we get $v_0 = \dfrac{l}{2\tau}$

1.2 Let s be the total distance traversed by the point and t_1 the time taken to cover half the distance. Further let $2t$ be the time to cover the rest half of the distance.

Therefore $$\frac{s}{2} = v_0 t_1 \quad \text{or} \quad t_1 = \frac{s}{2 v_0} \tag{1}$$

and $$\frac{s}{2} = (v_1 + v_2)\, t \quad \text{or} \quad 2t = \frac{s}{v_1 + v_2} \tag{2}$$

Hence the sought average velocity

$$\langle v \rangle = \frac{s}{t_1 + 2t} = \frac{s}{[s/2 v_0] + [s/(v_1 + v_2)]} = \frac{2 v_0 (v_1 + v_2)}{v_1 + v_2 + 2 v_0}$$

1.3 As the car starts from rest and finally comes to a stop, and the rate of acceleration and deceleration are equal, the distances as well as the times taken are same in these phases of motion.

Let Δt be the time for which the car moves uniformly. Then the acceleration / deceleration time is $\dfrac{\tau - \Delta t}{2}$ each. So,

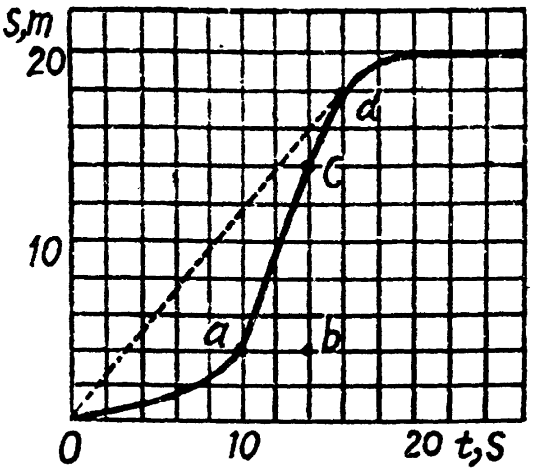
$$<v>\tau = 2\left\{\frac{1}{2}w\frac{(\tau-\Delta t)^2}{4}\right\} + w\frac{(\tau-\Delta t)}{2}\Delta t$$

or
$$\Delta t^2 = \tau^2 - \frac{4<v>\tau}{w}$$

Hence
$$\Delta t = \tau\sqrt{1-\frac{4<v>}{w\tau}} = 15\text{ s.}$$

1.4 (a) Sought average velocity

$$<v> = \frac{s}{t} = \frac{200\text{ cm}}{20\text{s}} = 10\text{ cm/s}$$



(b) For the maximum velocity, $\frac{ds}{dt}$ should be maximum. From the figure $\frac{ds}{dt}$ is maximum for all points on the line ac, thus the sought maximum velocity becomes average velocity for the line ac and is equal to :

$$\frac{bc}{ab} = \frac{100\text{ cm}}{4\text{s}} = 25\text{ cm/s}$$

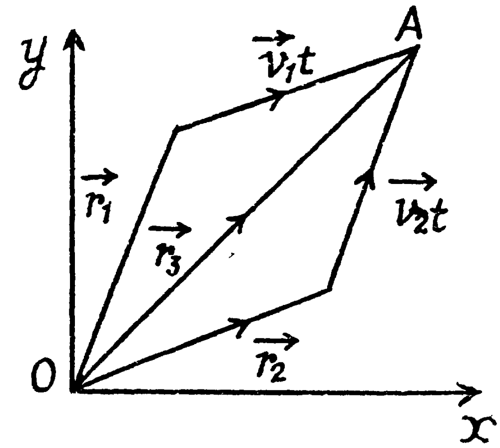
(c) Time t_0 should be such that corresponding to it the slope $\frac{ds}{dt}$ should pass through the point O (origin), to satisfy the relationship $\frac{ds}{dt} = \frac{s}{t_0}$. From figure the tangent at point d passes through the origin and thus corresponding time $t = t_0 = 16$ s.

1.5 Let the particles collide at the point A (Fig.), whose position vector is $\vec{r}_3$ (say). If t be the time taken by each particle to reach at point A, from triangle law of vector addition :

$$\vec{r}_3 = \vec{r}_1 + \vec{v}_1 t = \vec{r}_2 + \vec{v}_2 t$$

so, $$\vec{r}_1 - \vec{r}_2 = (\vec{v}_2 - \vec{v}_1)\,t \qquad (1)$$

therefore, $$t = \frac{|\vec{r}_1 - \vec{r}_2|}{|\vec{v}_2 - \vec{v}_1|} \qquad (2)$$



From Eqs. (1) and (2)

$$\vec{r}_1 = \vec{r}_2 - (\vec{v}_2 - \vec{v}_1)\frac{|\vec{r}_1 - \vec{r}_2|}{|\vec{v}_2 - \vec{v}_1|}$$

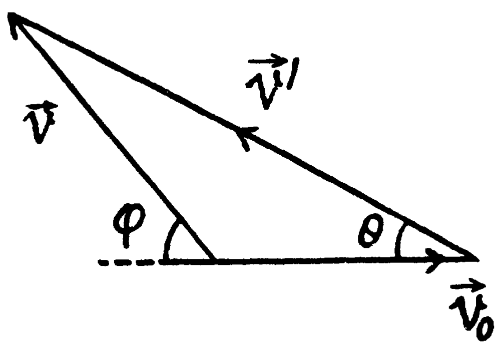
or, $$\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} = \frac{\vec{v}_2 - \vec{v}_1}{|\vec{v}_2 - \vec{v}_1|},$$ which is the sought relationship.

1.6 We have

$$\vec{v}' = \vec{v} - \vec{v}_0 \qquad (1)$$

From the vector diagram [of Eq. (1)] and using properties of triangle

$$v' = \sqrt{v_0^2 + v^2 + 2\,v_0\,v\cos\varphi} = 39.7 \text{ km/hr} \qquad (2)$$



and $$\frac{v'}{\sin(\pi - \varphi)} = \frac{v}{\sin\theta} \quad \text{or,} \quad \sin\theta = \frac{v\sin\varphi}{v'}$$

or $$\theta = \sin^{-1}\left(\frac{v\sin\varphi}{v'}\right)$$

Using (2) and putting the values of v and d

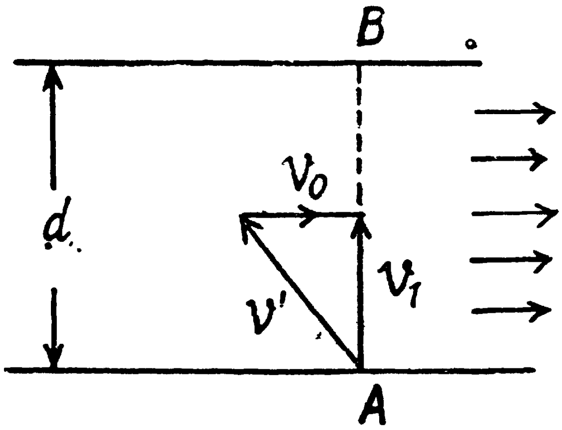
$$\theta = 19.1°$$

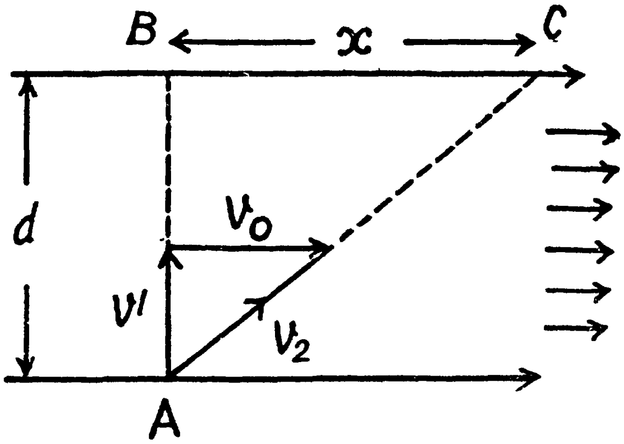
1.7 Let one of the swimmer (say 1) cross the river along AB, which is obviously the shortest path. Time taken to cross the river by the swimmer 1.

$$t_1 = \frac{d}{\sqrt{v'^2 - v_0^2}}, \text{ (where } AB = d \text{ is the width of the river)} \qquad (1)$$

For the other swimmer (say 2), which follows the quickest path, the time taken to cross the river.

$$t_2 = \frac{d}{v'} \qquad (2)$$





In the time t_2, drifting of the swimmer 2, becomes

$$x = v_0\,t_2 = \frac{v_0}{v'}d, \text{ (using Eq. 2)} \qquad (3)$$

If t_3 be the time for swimmer 2 to walk the distance x to come from C to B (Fig.), then

$$t_3 = \frac{x}{u} = \frac{v_0\,d}{v'\,u} \quad \text{(using Eq. 3)} \qquad (4)$$

According to the problem $t_1 = t_2 + t_3$

or, $$\frac{d}{\sqrt{v'^2 - v_0^2}} = \frac{d}{v'} + \frac{v_0\,d}{v'\,u}$$

On solving we get

$$u = \frac{v_0}{\left(\dfrac{1 - v_0^2}{v'^2}\right)^{-\frac{1}{2}} - 1} = 3 \text{ km/hr.}$$

1.8 Let l be the distance covered by the boat A along the river as well as by the boat B acr the river. Let v_0 be the stream velocity and v' the velocity of each boat with respect water. Therefore time taken by the boat A in its journey

$$t_A = \frac{l}{v'+v_0} + \frac{l}{v'-v_0}$$

and for the boat B
$$t_B = \frac{l}{\sqrt{v'^2 - v_0^2}} + \frac{l}{\sqrt{v'^2 - v_0^2}} = \frac{2l}{\sqrt{v'^2 - v_0^2}}$$

Hence,
$$\frac{t_A}{t_B} = \frac{v'}{\sqrt{v'^2 - v_0^2}} = \frac{\eta}{\sqrt{\eta^2 - 1}} \quad \left(\text{where } \eta = \frac{v'}{v}\right)$$

On substitution $\quad t_A/t_B = 1{\cdot}8$

1.9 Let v_0 be the stream velocity and v' the velocity of boat with respect to water. A $\frac{v_0}{v'} = \eta = 2 > 0$, some drifting of boat is inevitable.

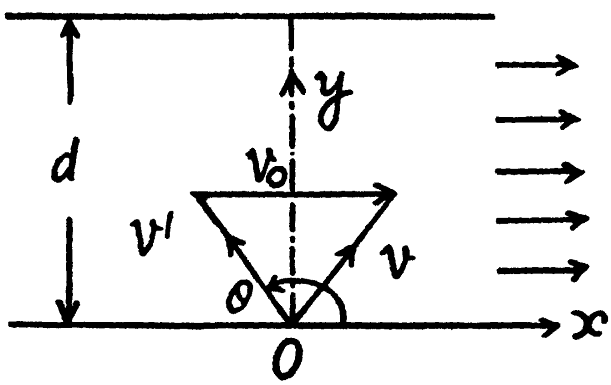
Let $\vec{v}'$ make an angle θ with flow direction. (Fig.), then the time taken to cross the rive

$$t = \frac{d}{v' \sin\theta} \quad \text{(where } d \text{ is the width of the river)}$$

In this time interval, the drifting of the boat

$$x = (v' \cos\theta + v_0)\, t$$

$$= (v' \cos\theta + v_0) \frac{d}{v' \sin\theta} = (\cot\theta + \eta \operatorname{cosec}\theta)\, d$$



For $x_{\min}$ (minimum drifting)

$$\frac{d}{d\theta}(\cot\theta + \eta \operatorname{cosec}\theta) = 0, \text{ which yields}$$

$$\cos\theta = -\frac{1}{\eta} = -\frac{1}{2}$$

Hence, $\quad \theta = 120°$

1.10 The solution of this problem becomes simple in the frame attached with one of the bodies. Let the body thrown straight up be 1 and the other body be 2, then for the body 1 in the frame of 2 from the kinematic equation for constant acceleration :

$$\vec{r}_{12} = \vec{r}_{0(12)} + \vec{v}_{0(12)}\, t + \frac{1}{2} \vec{w}_{12}\, t^2$$

So, $\quad \vec{r}_{12} = \vec{v}_{0(12)}\, t$, (because $\vec{w}_{12} = 0$ and $\vec{r}_{0(12)} = 0$)

or, $\quad |\vec{r}_{12}| = |\vec{v}_{0(12)}|\, t \qquad (1)$

But $\quad |\vec{v}_{01}| = |\vec{v}_{02}| = v_0$

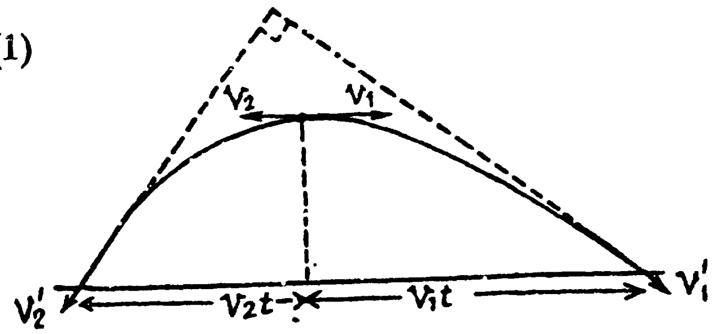
So, from properties of triangle

$$v_{0(12)} = \sqrt{v_0^2 + v_0^2 - 2\, v_0\, v_0 \cos(\pi/2 - \theta_0)}$$

Hence, the sought distance

$$|\vec{r}_{12}| = v_0 \sqrt{2(1 - \sin\theta)}\; t = 22 \text{ m}.$$

1.11 Let the velocities of the paricles (say $\vec{v}_1'$ and $\vec{v}_2'$) becomes mutually perpendicular after time t. Then their velocitis become



$$\vec{v}_1' = \vec{v}_1 + \vec{g}\,t\,;\ \vec{v}_2' = \vec{v}_2 + \vec{g}\,t \qquad (1)$$

As $\vec{v}_1' \perp \vec{v}_2'$ so, $\vec{v}_1' \cdot \vec{v}_2' = 0$

or, $(\vec{v}_1 + \vec{g}\,t) \cdot (\vec{v}_2 + \vec{g}\,t) = 0$

or $\quad -v_1 v_2 + g^2 t^2 = 0$

Hence, $\quad t = \dfrac{\sqrt{v_1 v_2}}{g} \qquad (3)$

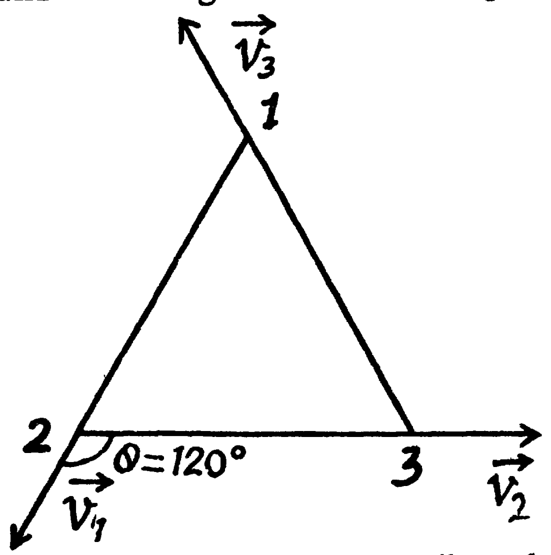
Now form the Eq. $\vec{r}_{12} = \vec{r}_{0(12)} + \vec{v}_{0(12)}\,t + \frac{1}{2}\vec{w}_{12}\,t^2$

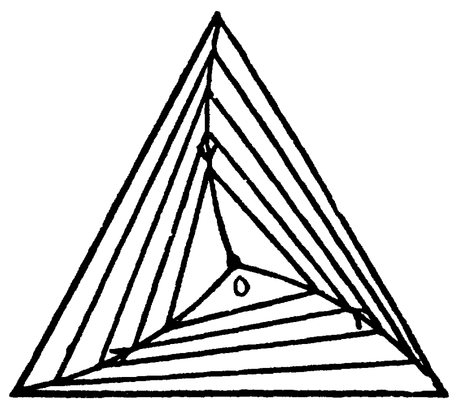
$|\vec{r}_{12}| = |\vec{v}_{0(12)}|\,t$, (because here $\vec{w}_{12} = 0$ and $\vec{r}_{0(12)} = 0$)

Hence the sought distance

$$|\vec{r}_{12}| = \frac{v_1 + v_2}{g}\sqrt{v_1 v_2} \quad (\text{as } |\vec{v}_{0(12)}| = v_1 + v_2)$$

1.12 From the symmetry of the problem all the three points are always located at the vertices of equilateral triangles of varying side length and finally meet at the centriod of the initial equilateral triangle whose side length is a, in the sought time interval (say t).





Let us consider an arbitrary equilateral triangle of edge length l (say).

Then the rate by which 1 approaches 2, 2 approches 3, and 3 approches 1, becomes :

$$\frac{-dl}{dt} = v - v\cos\left(\frac{2\pi}{3}\right)$$

On integrating :
$$-\int_a^0 dl = \frac{3v}{2}\int_0^t dt$$

$$a = \frac{3}{2}vt \quad \text{so} \quad t = \frac{2a}{3v}$$

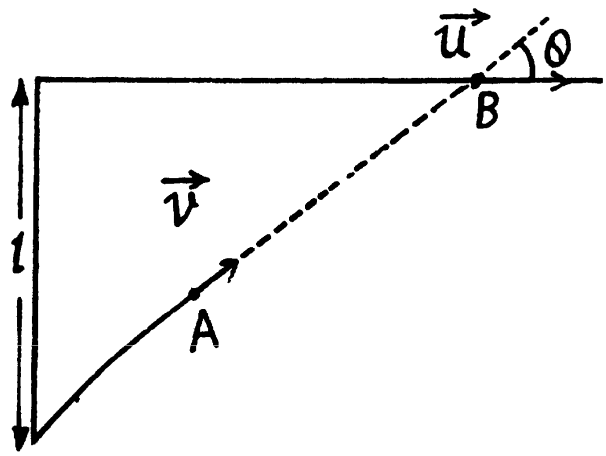
1.13 Let us locate the points A and B at an arbitrary instant of time (Fig.).
If A and B are separated by the distance s at this moment, then the points converge or point A approaches B with velocity $\frac{-ds}{dt} = v - u\cos\alpha$ where angle α varies with time.

On intergating,

$$-\int_l^0 ds = \int_0^T (v - u\cos\alpha)\,dt,$$

(where T is the sought time.)

or $$l = \int_0^T (v - u\cos\alpha)\,dt \qquad (1)$$



As both A and B cover the same distance in x-direction during the sought time interval, so the other condition which is required, can be obtained by the equation

$$\Delta x = \int v_x\,dt$$

So, $$uT = \int_0^T v\cos\alpha\,dt \qquad (2)$$

Solving (1) and (2), we get $T = \frac{ul}{v^2 - u^2}$

One can see that if $u = v$, or $u < v$, point A cannot catch B.

1.14 In the reference frame fixed to the train, the distance between the two events is obviously equal to l. Suppose the train starts moving at time $t = 0$ in the positive x direction and take the origin ($x = 0$) at the head-light of the train at $t = 0$. Then the coordinate of first event in the earth's frame is

$$x_1 = \frac{1}{2}wt^2$$

and similarly the coordinate of the second event is

$$x_2 = \frac{1}{2}w(t+\tau)^2 - l$$

The distance between the two events is obviously.

$$x_1 - x_2 = l - w\tau\,(t + \tau/2) = 0{\cdot}242 \text{ km}$$

in the reference frame fixed on the earth..

For the two events to occur at the same point in the reference frame K, moving with constant velocity V relative to the earth, the distance travelled by the frame in the time interval T must be equal to the above distance.

Thus $$V\tau = l - w\tau(t + \tau/2)$$

So, $$V = \frac{l}{\tau} - w\,(t + \tau/2) = 4{\cdot}03 \text{ m/s}$$

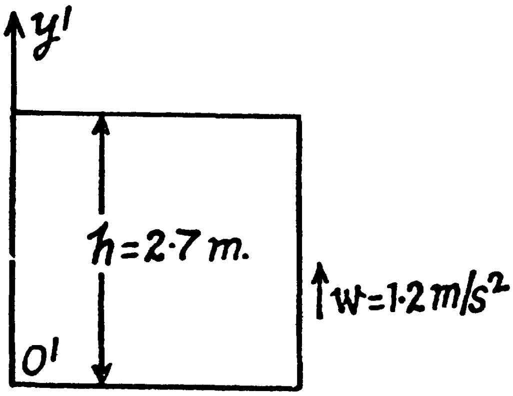
The frame K must clearly be moving in a direction opposite to the train so that if (for example) the origin of the frame coincides with the point x_1 on the earth at time t, it coincides with the point x_2 at time $t + \tau$.

1.15 (a) One good way to solve the problem is to work in the elevator's frame having the observer at its bottom (Fig.).

Let us denote the separation between floor and celing by $h = 2{\cdot}7$ m. and the acceleration of the elevator by $w = 1{\cdot}2\ \text{m/s}^2$

From the kinematical formula

$$y = y_0 + v_{0y}\,t + \frac{1}{2} w_y\, t^2 \qquad (1)$$



Here $\quad y = 0,\ y_0 = +h,\ v_{0y} = 0$

and $\quad w_y = w_{\text{bolt}\,(y)} - w_{\text{ele}\,(y)}$

$$= (-g) - (w) = -(g+w)$$

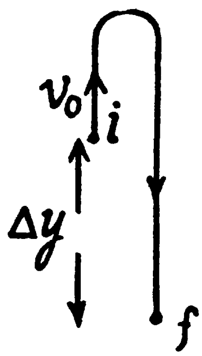
So, $$0 = h + \frac{1}{2}\left\{-(g+w)\right\} t^2$$

or, $$t = \sqrt{\frac{2h}{g+w}} = 0{\cdot}7\ \text{s}.$$

(b) At the moment the bolt loses contact with the elevator, it has already aquired the velocity equal to elevator, given by :

$$v_0 = (1{\cdot}2)\,(2) = 2{\cdot}4\ \text{m/s}$$

In the reference frame attached with the elevator shaft (ground) and pointing the y-axis upward, we have for the displacement of the bolt,



$$\Delta y = v_{0y}\, t + \frac{1}{2} w_y\, t^2$$

$$= v_0\, t + \frac{1}{2}(-g)\, t^2$$

or, $$\Delta y = (2{\cdot}4)\,(0{\cdot}7) + \frac{1}{2}(-9{\cdot}8)\,(0{\cdot}7)^2 = -0{\cdot}7\ \text{m}.$$

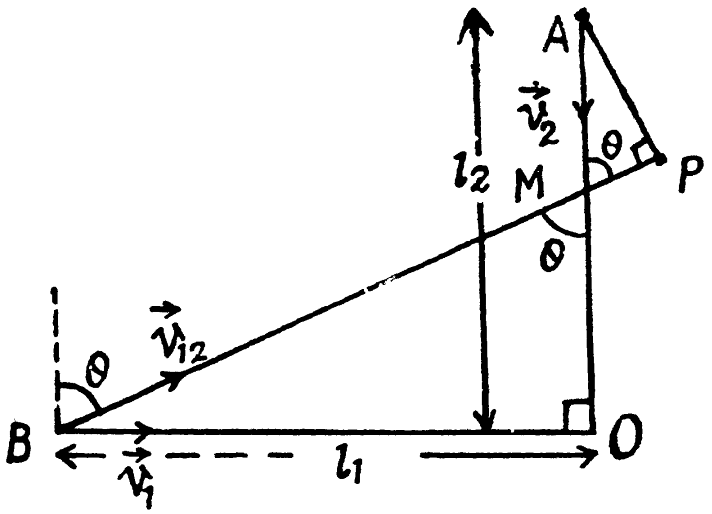
Hence the bolt comes down or displaces downward relative to the point, when it loses contact with the elevator by the amount 0·7 m (Fig.).

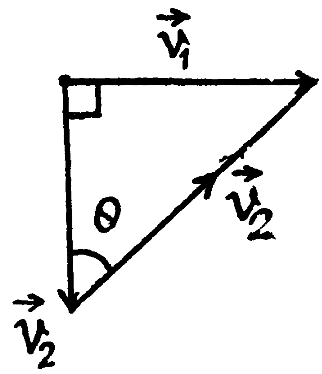
Obviously the total distance covered by the bolt during its free fall time

$$s = |\Delta y| + 2\left(\frac{v_0^2}{2g}\right) = 0{\cdot}7\ \text{m} + \frac{(2{\cdot}4)^2}{(9{\cdot}8)}\ \text{m} = 1{\cdot}3\ \text{m}.$$

1.16 Let the particle 1 and 2 be at points B and A at $t = 0$ at the distances l_1 and l_2 from intersection point O.

Let us fix the inertial frame with the particle 2. Now the particle 1 moves in relative to this reference frame with a relative velocity $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$ and its trajectory is the straight line BP. Obviously, the minimum distance between the particles is equal to the length of the perpendicular AP dropped from point A on to the straight line BP (Fig.).





From Fig. (b), $\quad v_{12} = \sqrt{v_1^2 + v_2^2}$, and $\tan\theta = \dfrac{v_1}{v_2}$ (1)

The shortest distance

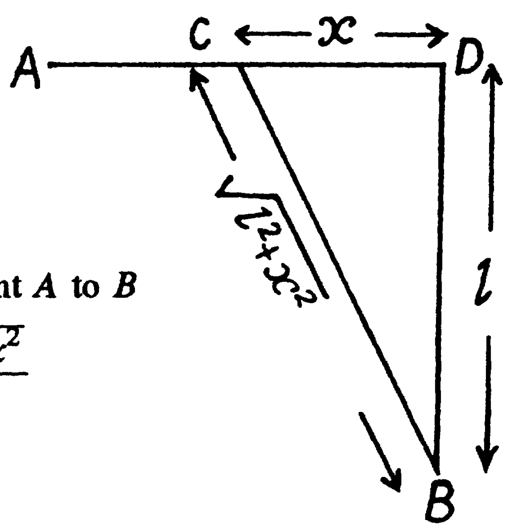
$$AP = AM\sin\theta = (OA - OM)\sin\theta = (l_2 - l_1\cot\theta)\sin\theta$$

or
$$AP = \left(l_2 - l_1\frac{v_2}{v_1}\right)\frac{v_1}{\sqrt{v_1^2+v_2^2}} = \frac{v_1 l_2 - v_2 l_1}{\sqrt{v_1^2+v_2^2}} \qquad \text{(using 1)}$$

The sought time can be obtained directly from the condition that $(l_1 - v_1 t)^2 + (l_2 - v_2 t)^2$ is minimum. This gives $t = \dfrac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}$.

1.17 Let the car turn off the highway at a distance x from the point D.
So, $CD = x$, and if the speed of the car in the field is v, then the time taken by the car to cover the distance $AC = AD - x$ on the highway

$$t_1 = \frac{AD - x}{\eta v} \qquad (1)$$



and the time taken to travel the distance CB in the field

$$t_2 = \frac{\sqrt{l^2+x^2}}{v} \qquad (2)$$

So, the total time elapsed to move the car from point A to B

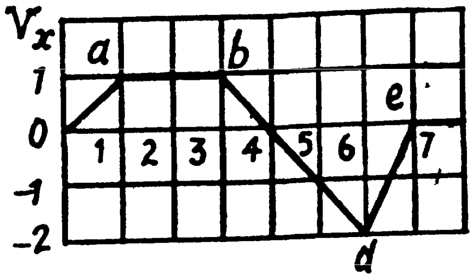
$$t = t_1 + t_2 = \frac{AD - x}{\eta v} + \frac{\sqrt{l^2+x^2}}{v}$$

For t to be minimum

$$\frac{dt}{dx} = 0 \quad \text{or} \quad \frac{1}{v}\left[-\frac{1}{\eta} + \frac{x}{\sqrt{l^2+x^2}}\right] = 0$$

or
$$\eta^2 x^2 = l^2 + x^2 \quad \text{or} \quad x = \frac{l}{\sqrt{\eta^2 - 1}}$$

1.18 To plot $x(t)$, $s(t)$ and $w_x(t)$ let us partion the given plot $v_x(t)$ into five segments (for detailed analysis) as shown in the figure.



For the part oa : $w_x = 1$ and $v_x = t = v$

Thus, $\Delta x_1(t) = \int v_x\,dt = \int_0^t dt = \frac{t^2}{2} = s_1(t)$

Putting $t = 1$, we get, $\Delta x_1 = s = \frac{1}{2}$ unit

For the part ab :

$w_x = O$ and $v_x = v =$ constant $= 1$

Thus $\Delta x_2(t) = \int v_x\,dt = \int_1^t dt = (t-1) = s_2(t)$

Putting $t = 3$, $\Delta x_2 = s_2 = 2$ unit

For the part $b4$: $w_x = 1$ and $v_x = 1-(t-3) = 4-t) = v$

Thus $\Delta x_3(t) = \int_3^t (4-t)\,dt = 4t - \frac{t^2}{2} - \frac{15}{2} = s_3(t)$

Putting $t = 4$, $\Delta x_3 = x_3 = \frac{1}{2}$ unit

For the part $4d$: $v_x = -1$ and $v_x = -(1-4) = 4-1$

So, $v = |v_x| = t-4$ for $t > 4$

Thus $\Delta x_4(t) = \int_4^t (1-t)\,dt = 4t - \frac{t^2}{2} - 8$

Putting $t = 6$, $\Delta x_4 = -1$

Similarly $s_4(t) = \int |v_x|\,dt = \int_4^t (t-4)\,dt = \frac{t^2}{2} - 4t + 8$

Putting $t = 6$, $s_4 = 2$ unit

For the part $d\,7$: $w_x = 2$ and $v_x = -2+2(t-6) = 2(t-7)$

$v = |v_x| = 2(7-t)$ for $t \leftarrow 7$

Now, $\Delta x(t) = \int_t^6 2\ (t-7)\,dt = t^2 - 14t + 48$

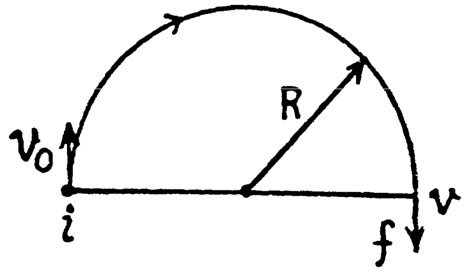
Putting $t = 4$, $\Delta x_5 = -1$

Similarly $s_5(t) = \int_t^6 2(7-t)\,dt = 14t - t^2 - 48$

Putting $t = 7$, $s_5 = 1$

On the basis of these obtained expressions $w_x(t)$, $x(t)$ and $s(t)$ plots can be easily plotted as shown in the figure of answersheet.

1.19 (a) Mean velocity



$$\langle v \rangle = \frac{\text{Total distance covered}}{\text{Time elapsed}}$$

$$= \frac{s}{t} = \frac{\pi R}{\tau} = 50 \text{ cm/s} \qquad (1)$$

(b) Modulus of mean velocity vector

$$|\langle \vec{v} \rangle| = \frac{|\Delta \vec{r}|}{\Delta t} = \frac{2R}{\tau} = 32 \text{ cm/s} \qquad (2)$$

(c) Let the point moves from i to f along the half circle (Fig.) and v_0 and v be the spe at the points respectively.

We have $\qquad \frac{dv}{dt} = w_t$

or, $\qquad v = v_0 + w_t t$ (as w_t is constant, according to the problem)

So, $$\langle v \rangle = \frac{\int_0^t (v_0 + w_t t)\, dt}{\int_0^t dt} = \frac{v_0 + (v_0 + w_t t)}{2} = \frac{v_0 + v}{2} \qquad ($$

So, from (1) and (3)

$$\frac{v_0 + v}{2} = \frac{\pi R}{\tau} \qquad (3$$

Now the modulus of the mean vector of total acceleration

$$|\langle \vec{w} \rangle| = \frac{|\Delta \vec{v}|}{\Delta t} = \frac{|\vec{v} - \vec{v}_0|}{\tau} = \frac{v_0 + v}{\tau} \text{ (see Fig.)} \qquad (5$$

Using (4) in (5), we get :

$$|\langle \vec{w} \rangle| = \frac{2\pi R}{\tau^2}$$

1.20 (a) we have $\qquad \vec{r} = \vec{a}\, t\,(1 - \alpha t)$

So, $\qquad \vec{v} = \frac{d\vec{r}}{dt} = \vec{a}(1 - 2\alpha t)$

and $\qquad \vec{w} = \frac{d\vec{v}}{dt} = -2\alpha \vec{a}$

(b) From the equation

$$\vec{r} = \vec{a}\, t\,(1 - \alpha t),$$

$$\vec{r} = 0, \text{ at } t = 0 \text{ and also at } t = \Delta t = \frac{1}{\alpha}$$

So, the sought time $\Delta t = \frac{1}{\alpha}$

As $\qquad \vec{v} = \vec{a}(1 - 2\alpha t)$

So, $$v = |\vec{v}| = \begin{cases} a(1 - 2\alpha t) & \text{for } t \le \frac{1}{2\alpha} \\ a(2\alpha t - 1) & \text{for } t > \frac{1}{2\alpha} \end{cases}$$

Hence, the sought distance

$$s = \int v\,dt = \int_0^{1/2\alpha} a\,(1 - 2\,\alpha\,t)\,dt + \int_{1/2\alpha}^{1/\alpha} a\,(2\,\alpha\,t - 1)\,dt$$

Simplifying, we get, $s = \dfrac{a}{2\,\alpha}$

1.21 (a) As the particle leaves the origin at $t = 0$

So, $$\Delta x = x = \int v_x\,dt \qquad (1)$$

As $$\vec{v} = \vec{v_0}\left(1 - \frac{t}{\tau}\right),$$

where $\vec{v_0}$ is directed towards the +ve x-axis

So, $$v_x = v_0\left(1 - \frac{t}{\tau}\right) \qquad (2)$$

From (1) and (2),

$$x = \int_0^t v_0\left(1 - \frac{t}{\tau}\right)dt = v_0\,t\left(1 - \frac{t}{2\,\tau}\right) \qquad (3)$$

Hence x coordinate of the particle at $t = 6$ s.

$$x = 10 \times 6\left(1 - \frac{6}{2 \times 5}\right) = 24 \text{ cm} = 0{\cdot}24 \text{ m}$$

Similarly at $$t = 10 \text{ s}$$

$$x = 10 \times 10\left(1 - \frac{10}{2 \times 5}\right) = 0$$

and at $$t = 20 \text{ s}$$

$$x = 10 \times 20\left(1 - \frac{20}{2 \times 5}\right) = -200 \text{ cm} = -2 \text{ m}$$

(b) At the moments the particle is at a distance of 10 cm from the origin, $x = \pm\, 10$ cm.

Putting $x = +10$ in Eq. (3)

$$10 = 10t\left(1 - \frac{t}{10}\right) \text{ or, } t^2 - 10t + 10 = 0,$$

So, $$t = t = \frac{10 \pm \sqrt{100 - 40}}{2} = 5 \pm \sqrt{15} \text{ s}$$

Now putting $x = -10$ in Eqn (3)

$$-10 = 10\left(1 - \frac{t}{10}\right),$$

On solving, $$t = 5 \pm \sqrt{35} \text{ s}$$

As t cannot be negative, so,

$$t = (5 + \sqrt{35}\,) \text{ s}$$

Hence the particle is at a distance of 10 cm from the origin at three moments of time :

$$t = 5 \pm \sqrt{15}\ \text{s},\ 5 + \sqrt{35}\ \text{s}$$

(c) We have $\vec{v} = \vec{v}_0\left(1 - \frac{t}{\tau}\right)$

So,
$$v = |\vec{v}| = \left.\begin{matrix} v_0\left(1 - \frac{t}{\tau}\right) & \text{for } t \le \tau \\ v_0\left(\frac{t}{\tau} - 1\right) & \text{for } t > \tau \end{matrix}\right\}$$

So
$$s = \int_0^t v_0\left(1 - \frac{t}{\tau}\right) dt \text{ for } t \le \tau = v_0 t\,(1 - t/2\tau)$$

and
$$s = \int_0^\tau v_0\left(1 - \frac{t}{2}\right) dt + \int_\tau^t v_0\left(\frac{t}{\tau} - 1\right) dt \text{ for } t > \tau$$

$$= v_0\tau\,[1 + (1 - t/\tau)^2]/2 \text{ for } t > \tau \tag{A}$$

$$s = \int_0^4 v_0\left(1 - \frac{t}{\tau}\right) dt = \int_0^4 10\left(1 - \frac{t}{5}\right) dt = 24 \text{ cm.}$$

And for $t = 8$ s

$$s = \int_0^5 10\left(1 - \frac{t}{5}\right) dt + \int_5^8 10\left(\frac{t}{5} - 1\right) dt$$

On integrating and simplifying, we get

$$s = 34 \text{ cm.}$$

On the basis of Eqs. (3) and (4), $x(t)$ and $s(t)$ plots can be drawn as shown in the answer sheet.

1.22 As particle is in unidirectional motion it is directed along the x-axis all the time. As at $t = 0,\ x = 0$

So, $\Delta x = x = s$, and $\frac{dv}{dt} = w$

Therefore, $v = \alpha\sqrt{x} = \alpha\sqrt{s}$

or,
$$w = \frac{dv}{dt} = \frac{\alpha}{2\sqrt{s}}\frac{ds}{dt} = \frac{\alpha}{2\sqrt{s}}$$

$$= \frac{\alpha v}{2\sqrt{s}} = \frac{\alpha\,\alpha\sqrt{s}}{2\sqrt{s}} = \frac{\alpha^2}{2} \tag{1}$$

As, $w = \frac{dv}{dt} = \frac{\alpha^2}{2}$

On integrating,
$$\int_0^v dv = \int_0^t \frac{\alpha^2}{2} dt \quad \text{or,} \quad v = \frac{\alpha^2}{2} t \tag{2}$$

(b) Let s be the time to cover first s m of the path. From the Eq.

$$s = \int v\,dt$$

$$s = \int_0^t \frac{\alpha^2}{2}\,dt = \frac{\alpha^2}{2}\frac{t^2}{2} \qquad \text{(using 2)}$$

or $$t = \frac{2}{\alpha}\sqrt{s} \qquad (3)$$

The mean velocity of particle

$$<v> = \frac{\int v(t)\,dt}{\int dt} = \frac{\int_0^{2\sqrt{s}/\alpha} \frac{\alpha^2}{2}\,t\,dt}{2\sqrt{s}/\alpha} = \frac{\alpha\sqrt{s}}{2}$$

1.23 According to the problem

$$-\frac{v\,dv}{ds} = a\sqrt{v} \text{ (as } v \text{ decreases with time)}$$

or, $$-\int_{v_0}^{0} \sqrt{v}\,dv = a\int_0^s ds$$

On integrating we get $s = \frac{2}{3a} v_0^{3/2}$

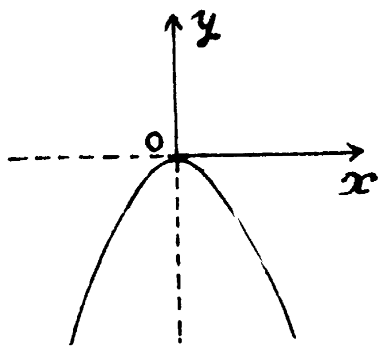
Again according to the problem

$$-\frac{dv}{dt} = a\sqrt{v} \text{ or } -\frac{dv}{\sqrt{v}} = a\,dt$$

or, $$\int_{v_0}^{0} \frac{dv}{\sqrt{v}} = a\int_0^t dt$$

Thus $$t = \frac{2\sqrt{v_0}}{a}$$

1.24 (a) As $$\vec{r} = a\,t\,\vec{i} - b\,t^2\,\vec{j}$$



So, $$x = a\,t,\; y = -b\,t^2$$

and therefore $$y = \frac{-b\,x^2}{a^2}$$

which is Eq. of a parabola, whose graph is shown in the Fig.

(b) As $$\vec{r} = a t \vec{i} - b t^2 \vec{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = a\vec{i} - 2bt\vec{j} \tag{1}$$

So, $$v = \sqrt{a^2 (-2bt)^2} = \sqrt{a^2 + 4b^2 t^2}$$

Diff. Eq. (1) w.r.t. time, we get

$$\vec{w} = \frac{d\vec{v}}{dt} = -2b\vec{j}$$

So, $$|\vec{w}| = w = 2b$$

(c) $$\cos\alpha = \frac{\vec{v} \cdot \vec{w}}{v\,w} = \frac{(a\vec{i} - 2bt\vec{j}) \cdot (-2b\vec{j})}{\left(\sqrt{a^2 + 4b^2 t^2}\right) 2b}$$

or, $$\cos\alpha = \frac{2bt}{\sqrt{a^2 + 4b^2 t^2}},$$

so, $$\tan\alpha = \frac{a}{2bt}$$

or, $\alpha = \tan^{-1}\left(\frac{a}{2bt}\right)$

(d) The mean velocity vector

$$\langle \vec{v} \rangle = \frac{\int \vec{v}\,dt}{\int dt} = \frac{\int_0^t (a\vec{i} - 2bt\vec{j})\,dt}{t} = a\vec{i} - bt\vec{j}$$

Hence, $$|\langle \vec{v} \rangle| = \sqrt{a^2 + (-bt)^2} = \sqrt{a^2 + b^2 t^2}$$

1.25 (a) We have

$$x = at \text{ and } y = at(1 - \alpha t) \tag{1}$$

Hence, $y(x)$ becomes,

$$y = \frac{ax}{a}\left(1 - \frac{\alpha x}{a}\right) = x - \frac{\alpha}{a}x^2 \text{ (parabola)}$$

(b) Differentiating Eq. (1) we get

$$v_x = a \text{ and } v_y = a(1 - 2\alpha t) \tag{2}$$

So, $$v = \sqrt{v_x^2 + v_y^2} = a\sqrt{1 + (1 - 2\alpha t)^2}$$

Diff. Eq. (2) with respect to time

$$w_x = 0 \text{ and } w_y = -2a\alpha$$

So, $$w = \sqrt{w_x^2 + w_y^2} = 2a\alpha$$

(c) From Eqs. (2) and (3)

We have $$\vec{v} = a\vec{i} + a(1 - 2\alpha t)\vec{j} \text{ and } \vec{w} = 2a\alpha\vec{j}$$

So, $$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\vec{v}\cdot\vec{w}}{vw} = \frac{-a(1 - 2\alpha t_0)2a\alpha}{a\sqrt{1 + (1 - 2\alpha t_0)^2}\,2a\alpha}$$

On simplifying. $$1 - 2\alpha t_0 = \pm 1$$

As, $$t_0 \neq 0, \quad t_0 = \frac{1}{\alpha}$$

1.26 Differentiating motion law : $x = a\sin\omega t$, $y = a(1 - \cos\omega t)$, with respect to time, $v_x = a\omega\cos\omega t$, $v_y = a\omega\sin\omega t$

So, $$\vec{v} = a\omega\cos\omega t\,\vec{i} + a\omega\sin\omega t\,\vec{j} \tag{1}$$

and $$v = a\omega = \text{Const.} \tag{2}$$

Differentiating Eq. (1) with respect to time

$$\vec{w} = \frac{d\vec{v}}{dt} = -a\omega^2\sin\omega t\,\vec{i} + a\omega^2\cos\omega t\,\vec{j} \tag{3}$$

(a) The distance s traversed by the point during the time τ is given by

$$s = \int_0^\tau v\,dt = \int_0^\tau a\omega\,dt = a\omega\tau \quad (\text{using } 2)$$

(b) Taking inner product of $\vec{v}$ and $\vec{w}$

We get, $\vec{v}\cdot\vec{w} = (a\omega\cos\omega t\,\vec{i} + a\omega\sin\omega t\,\vec{j})\cdot(a\omega^2\sin\omega t(-i) + a\omega^2\cos\omega t - \vec{j})$

So, $$\vec{v}\cdot\vec{w} = -a^2\omega^2\sin\omega t\cos\omega t + a^2\omega^3\sin\omega t\,\cos\omega t = 0$$

Thus, $\vec{v} \perp \vec{w}$, i.e., the angle between velocity vector and acceleration vector equals $\frac{\pi}{2}$.

1.27 Accordiing to the problem

$$\vec{w} = w(-\vec{j})$$

So, $$w_x = \frac{dv_x}{dt} = 0 \text{ and } w_y = \frac{dv_y}{dt} = -w \tag{1}$$

Differentiating Eq. of trajectory, $y = ax - bx^2$, with respect to time

$$\frac{dy}{dt} = \frac{a\,dx}{dt} - 2bx\,\frac{dx}{dt} \tag{2}$$

So, $$\left.\frac{dy}{dt}\right|_{x=0} = a \left.\frac{dx}{dt}\right|_{x=0}$$

Again differentiating with respect to time

$$\frac{d^2 y}{dt^2} = \frac{a\, d^2 x}{dt^2} - 2b\left(\frac{dx}{dt}\right)^2 - 2bx\,\frac{d^2 x}{dt^2}$$

or, $$-w = a\,(0) - 2b\left(\frac{dx}{dt}\right)^2 - 2bx\,(0) \text{ (using 1)}$$

or, $$\frac{dx}{dt} = \sqrt{\frac{w}{2b}} \text{ (using 1)} \qquad (3$$

Using (3) in (2) $$\left.\frac{dy}{dt}\right|_{x=0} = a\sqrt{\frac{w}{2b}} \qquad (4$$

Hence, the velocity of the particle at the origin

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2_{x=0} + \left(\frac{dy}{dt}\right)^2_{x=0}} = \sqrt{\frac{w}{2b} + a^2\frac{w}{2b}} \text{ (using Eqns (3) and (4))}$$

Hence, $$v = \sqrt{\frac{w}{2b}(1+a^2)}$$

1.28 As the body is under gravity of constant accelration $\vec{g}$, it's velocity vector and displacemen vectors are:

$$\vec{v} = \vec{v_0} + \vec{g}\,t \qquad (1)$$

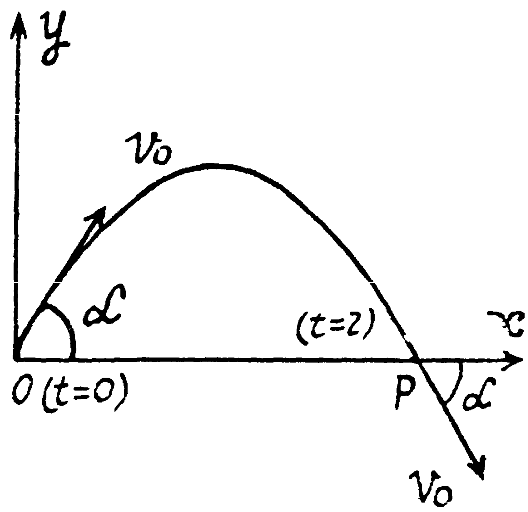
and $$\Delta\vec{r} = \vec{r} = \vec{v_0}\,t + \frac{1}{2}gt^2 \quad (\vec{r} = 0 \text{ at } t = 0) \qquad (2)$$

So, $<\vec{v}>$ over the first t seconds

$$<\vec{v}> = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}}{t} = \vec{v_0} + \frac{\vec{g}t}{2} \qquad (3)$$

Hence from Eq. (3), $<\vec{v}>$ over the first t seconds

$$<\vec{v}> = \vec{v_0} + \frac{\vec{g}}{2}\tau \qquad (4)$$



For evaluating t, take

$$\vec{v}\cdot\vec{v} = (\vec{v_0} + \vec{g}\,t)\cdot(\vec{v_0} + \vec{g}\,t) = v_0^2 + 2\,(\vec{v_0}\cdot\vec{g})\,t + g^2 t^2$$

or, $$v^2 = v_0^2 + (\vec{v_0}\cdot\vec{g})\,t + g^2 t^2$$

But we have $v = v_0$ at $t = 0$ and

Also at $t = \tau$ (Fig.) (also from energy conservation)

Hence using this propety in Eq. (5)

$$v_0^2 = v_0^2 + 2\,(\vec{v}_0 \cdot \vec{g})\,\tau + g^2 \tau^2$$

As $\qquad \tau \neq 0, \quad \text{so,}\ \tau = -\dfrac{2\,(\vec{v}_o \cdot \vec{g})}{g^2}$

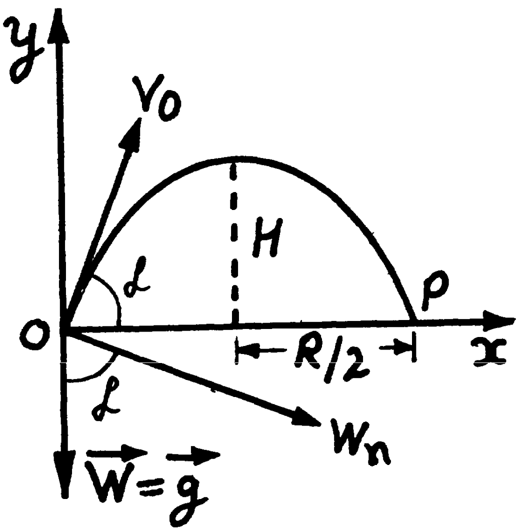
Putting this value of τ in Eq. (4), the average velocity over the time of flight

$$\langle \vec{v} \rangle = \vec{v}_0 - \vec{g}\,\frac{(\vec{v}_0 \cdot \vec{g})}{g^2}$$

1.29 The body thrown in air with velocity v_0 at an angle α from the horizontal lands at point P on the Earth's surface at same horizontal level (Fig.). The point of projection is taken as origin, so, $\Delta x = x$ and $\Delta y = y$

(a) From the Eq. $\quad \Delta y = v_{0y} t + \dfrac{1}{2} w_y t^2$

$$0 = v_0 \sin\alpha\,\tau - \frac{1}{2} g \tau^2$$



As $\tau \neq 0$, so, time of motion $\tau = \dfrac{2\,v_0 \sin\alpha}{g}$

(b) At the maximum height of ascent, $v_y = 0$

so, from the Eq. $\qquad v_y^2 = v_{0y}^2 + 2\,w_y\,\Delta y$

$$0 = (v_0 \sin\alpha)^2 - 2\,g\,H$$

Hence maximum height $H = \dfrac{v_0^2 \sin^2\alpha}{2g}$

During the time of motion the net horizontal displacement or horizontal range, will be obtained by the equation

$$\Delta x = v_{0x}\, t + \frac{1}{2} w_x \tau^2$$

or, $\qquad R = v_0 \cos\alpha\,\tau - \dfrac{1}{2}(0)\,\tau^2 = v_0 \cos\alpha\,\tau = \dfrac{v_0^2 \sin 2\alpha}{g}$

when $\qquad R = H$

$$\frac{v_0^2 \sin^2\alpha}{g} = \frac{v_0^2 \sin^2\alpha}{2g} \quad \text{or} \quad \tan\alpha = 4, \quad \text{so,} \quad \alpha = \tan^{-1} 4$$

(c) For the body, $x\,(t)$ and $y\,(t)$ are

$$x = v_0 \cos\alpha\, t \qquad (1)$$

and $$y = v_0 \sin\alpha\, t - \frac{1}{2} g t^2 \tag{2}$$

Hence putting the value of t from (1) into (2) we get,

$$y = v_0 \sin\alpha \left(\frac{x}{v_0 \cos\alpha}\right) - \frac{1}{2} g \left(\frac{x}{v_0 \cos\alpha}\right)^2 = x \tan\alpha - \frac{g x^2}{2 v_0^2 \cos^2\alpha},$$

Which is the sought equation of trajectory i.e. $y(x)$

(d) As the body thrown in air follows a curve, it has some normal acceleration at all the moments of time during it's motion in air.

At the initial point $(x = 0, y = 0)$, from the equation :

$$w_n = \frac{v^2}{R}, \text{ (where } R \text{ is the radius of curvature)}$$

$$g \cos\alpha = \frac{v_0^2}{R_0} \text{ (see Fig.) or } R_0 = \frac{v_0^2}{g \cos\alpha}$$

At the peak point $v_y = 0$, $v = v_x = v_0 \cos\alpha$ and the angential acceleration is zero.

Now from the Eq. $$w_n = \frac{v^2}{R}$$

$$g = \frac{v_0^2 \cos^2\alpha}{R}, \text{ or } R = \frac{v_0^2 \cos^2\alpha}{g}$$

Note : *We may use the formula of curvature radius of a trajectory $y(x)$, to solve part (d),*

$$R = \frac{\left[1 + (dy/dx)^2\right]^{\frac{3}{2}}}{\left| d^2 y / dx^2 \right|}$$

1.30 We have, $v_x = v_0 \cos\alpha$, $v_y = v_0 \sin\alpha - gt$

As $\vec{v} \uparrow\uparrow \hat{u}_t$ all the moments of time.

Thus $$v^2 = v_t^2 - 2 g t v_0 \sin\alpha + g^2 t^2$$

Now, $$w_t = \frac{dv_t}{dt} = \frac{1}{2 v_t} \frac{d}{dt}(v_t^2) = \frac{1}{v_t}(g^2 t - g v_0 \sin\alpha)$$

$$= -\frac{g}{v_t}(v_0 \sin\alpha - g t) = -g \frac{v_y}{v_t}$$

Hence $$|w_t| = g \frac{|v_y|}{v}$$

Now $$w_n = \sqrt{w^2 - w_t^2} = \sqrt{g^2 - g^2 \frac{v_y^2}{v_t^2}}$$

or $$w_n = g \frac{v_x}{v_t} \left(\text{where } v_x = \sqrt{v_t^2 - v_y^2}\right)$$

As $\qquad \vec{v} \uparrow\uparrow \hat{v}_t$, during time of motion

$$w_v = w_t = -g\frac{v_y}{v}$$

On the basis of obtained expressions or facts the sought plots can be drawn as shown in the figure of answer sheet.

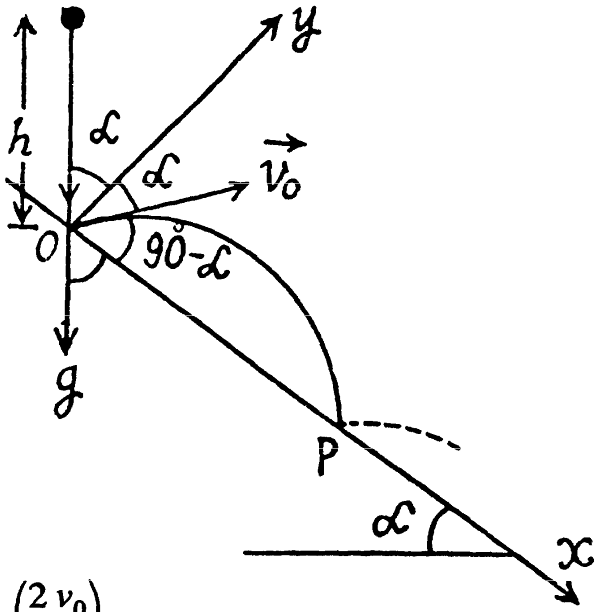
1.31 The ball strikes the inclined plane (Ox) at point O (origin) with velocity $v_0 = \sqrt{2gh}$ (1)

As the ball elastically rebounds, it recalls with same velocity v_0, at the same angle α from the normal or y axis (Fig.). Let the ball strikes the incline second time at P, which is at a distance l (say) from the point O, along the incline. From the equation

$$y = v_{0y}\, t + \frac{1}{2} w_y\, t^2$$

$$0 = v_0 \cos\alpha\, \tau - \frac{1}{2} g \cos\alpha\, t^2$$

where τ is the time of motion of ball in air while moving from O to P.



As $\quad \tau \neq 0$, so, $\tau = \dfrac{2v_0}{g}$ $\qquad$ (2)

Now from the equation.

$$x = v_{0x}\, t + \frac{1}{2} w_x\, t^2$$

$$l = v_0 \sin\alpha\, \tau + \frac{1}{2} g \sin\alpha\, \tau^2$$

so,
$$l = v_0 \sin\alpha \left(\frac{2v_0}{g}\right) + \frac{1}{2} g \sin\alpha \left(\frac{2v_0}{g}\right)$$

$$= \frac{4v_0^2 \sin\alpha}{g} \quad \text{(using 2)}$$

Hence the sought distance, $l = \dfrac{4(2gh)\sin\alpha}{g} = 8h\sin\alpha$ (Using Eq. 1)

1.32 Total time of motion

$$\tau = \frac{2v_0 \sin\alpha}{g} \quad \text{or} \quad \sin\alpha = \frac{\tau g}{2v_0} = \frac{9{\cdot}8\,\tau}{2\times 240} \qquad (1)$$

and horizontal range

$$R = v_0 \cos\alpha\, \tau \quad \text{or} \quad \cos\alpha = \frac{R}{v_0\tau} = \frac{5100}{240\,\tau} = \frac{85}{4\tau} \qquad (2)$$

From Eqs. (1) and (2)

$$\frac{(9{\cdot}8)^2\tau^2}{(480)^2} + \frac{(85)^2}{(4\tau^2)^2} = 1$$

On simplifying $\qquad \tau^4 - 2400\,\tau^2 + 1083750 = 0$

Solving for τ^2 we get :

$$\tau^2 = \frac{2400 \pm \sqrt{1425000}}{2} = \frac{2400 \pm 1194}{2}$$

Thus $\tau = 42{\cdot}39 \text{ s} = 0{\cdot}71 \text{ min}$ and

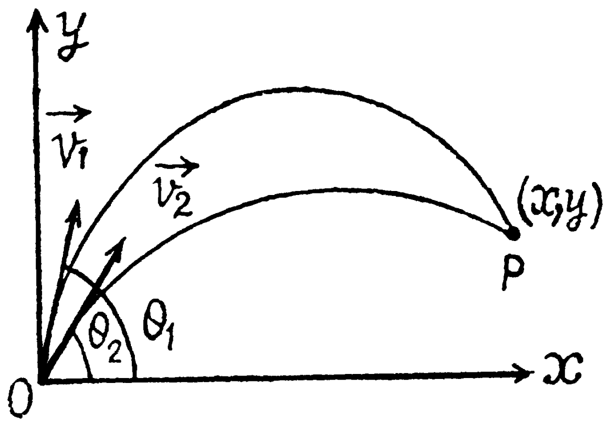
$\tau = 24{\cdot}55 \text{ s} = 0{\cdot}41 \text{ min}$ depending on the angle α.

1.33 Let the shells collide at the point $P(x, y)$. If the first shell takes t s to collide with second and Δt be the time interval between the firings, then

$$x = v_0 \cos\theta_1\, t = v_0 \cos\theta_2\,(t - \Delta t) \quad (1)$$

and
$$y = v_0 \sin\theta_1\, t - \frac{1}{2} g t^2$$

$$= v_0 \sin\theta_2\,(t - \Delta t) - \frac{1}{2} g\,(t - \Delta t)^2 \quad (2)$$



From Eq. (1)
$$t = \frac{\Delta t \cos\theta_2}{\cos\theta_2 - \cos\theta_1} \quad (3)$$

From Eqs. (2) and (3)

$$\Delta t = \frac{2 v_0 \sin(\theta_1 - \theta_2)}{g(\cos\theta_2 + \cos\theta_1)} \quad \text{as } \Delta t \neq 0$$

1.34 According to the problem

(a) $\dfrac{dy}{dt} = v_0$ or $dy = v_0\, dt$

Integrating
$$\int_0^y dy = v_0 \int_0^t dt \quad \text{or} \quad y = v_0 t \quad (1)$$

And also we have $\dfrac{dx}{dt} = ay$ or $dx = a\,y\,dt = a\,v_0\,t\,dt$ (using 1)

So,
$$\int_0^x dx = a v_0 \int_0^t t\,dt, \text{ or, } x = \frac{1}{2} a v_0 t^2 = \frac{1}{2} \frac{a y^2}{v_0} \text{ (using 1)}$$

(b) According to the problem

$$v_y = v_0 \text{ and } v_x = a\,y \quad (2)$$

So,
$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{v_0^2 + a^2 y^2}$$

Therefore
$$w_t = \frac{dv}{dt} = \frac{a^2 y}{\sqrt{v_0^2 + a\,y^2}} \frac{dy}{dt} = \frac{a^2 y}{\sqrt{1 + \left(ay / v_0\right)^2}}$$

Diff. Eq. (2) with respect to time.

$$\frac{d v_y}{dt} = w_y = 0 \text{ and } \frac{d v_x}{dt} = w_x = a \frac{dy}{dt} = a\,v_0$$

So,
$$w = |w_x| = a\,v_0$$

Hence $w_n = \sqrt{w^2 - w_t^2} = \sqrt{a^2 v_0^2 - \dfrac{a^4 y^2}{1 + \left(ay / v_0\right)^2}} = \dfrac{a v_0}{\sqrt{1 + \left(ay / v_0\right)^2}}$

1.35 (a) The velocity vector of the particle

$$\vec{v} = a\,\vec{i} + bx\,\vec{j}$$

So,
$$\frac{dx}{dt} = a : \frac{dy}{dt} = bx \tag{1}$$

From (1)
$$\int_0^x dx = a\int_0^t dt \quad \text{or,} \quad x = at \tag{2}$$

And
$$dy = bx\,dt = bat\,dt$$

Integrating
$$\int_0^y dy = ab\int_0^t t\,dt \quad \text{or,} \quad y = \frac{1}{2}ab\,t^2 \tag{3}$$

From Eqs. (2) and (3), we get, $y = \dfrac{b}{2a}x^2$ (4)

(b) The curvature radius of trajectory $y(x)$ is :

$$R = \frac{\left[1 + (dy/dx)^2\right]^{\frac{3}{2}}}{\left|d^2 y / dx^2\right|} \tag{5}$$

Let us differentiate the path Eq. $y = \dfrac{b}{2a}x^2$ with respect to x,

$$\frac{dy}{dx} = \frac{b}{a}x \quad \text{and} \quad \frac{d^2 y}{dx^2} = \frac{b}{a} \tag{6}$$

From Eqs. (5) and (6), the sought curvature radius :

$$R = \frac{a}{b}\left[1 + \left(\frac{b}{a}x\right)^2\right]^{\frac{3}{2}}$$

1.36 In accordance with the problem

$$w_t = \vec{a}\cdot\vec{\tau}$$

But
$$w_t = \frac{v\,dv}{ds} \quad \text{or} \quad v\,dv = w_t\,ds$$

So,
$$v\,dv = (\vec{a}\cdot\vec{\tau})\,ds = \vec{a}\cdot d\vec{r}$$

or, $v\,dv = a\,\vec{i}\cdot d\vec{r} = a\,dx$ (because $\vec{a}$ is directed towards the x-axis)

So,
$$\int_0^v v\,dv = a\int_0^x dx$$

Hence
$$v^2 = 2\,ax \quad \text{or,} \quad v = \sqrt{2\,ax}$$

1.37 The velocity of the particle $v = at$

So, $$\frac{dv}{dt} = w_t = a \tag{1}$$

And $$w_n = \frac{v^2}{R} = \frac{a^2t^2}{R} \quad \text{(using } v = at) \tag{2}$$

From $$s = \int v\, dt$$

$$2\pi R \eta = a\int_0^t v\, dt = \frac{1}{2}at^2$$

So, $$\frac{4\pi\eta}{a} = \frac{t^2}{R} \tag{3}$$

From Eqs. (2) and (3) $w_n = 4\pi a\eta$

Hence $w = \sqrt{w_t^2 + w_n^2}$

$$= \sqrt{a^2 + (4\pi a\eta)^2} = a\sqrt{1 + 16\pi^2\eta^2} = 0{\cdot}8\ \text{m/s}^2$$

1.38 According to the problem

$$|w_t| = |w_n|$$

For $v(t)$, $$\frac{-dv}{dt} = \frac{v^2}{R}$$

Integrating this equation from $v_0 \le v \le v$ and $0 \le t \le t$

$$-\int_{v_0}^{v} \frac{dv}{v^2} = \frac{1}{R}\int_0^t dt \quad \text{or,} \quad v = \frac{v_0}{\left(1 + \frac{v_0 t}{R}\right)}$$

Now for $v(s)$, $-\frac{v\,dv}{ds} = \frac{v^2}{R}$, Integrating this equation from $v_0 \le v \le v$ and $0 \le s \le s$

SO, $$\int_{v_0}^{v} \frac{dv}{v} = -\frac{1}{R}\int_0^s ds \quad \text{or,} \quad \ln\frac{v}{v_0} = -\frac{s}{R}$$

Hence $$v = v_0 e^{-s/R} \tag{2}$$

(b) The normal acceleration of the point

$$w_n = \frac{v^2}{R} = \frac{v^2 e^{-2s/R}}{R} \quad \text{(using 2)}$$

And as accordance with the problem

$$|w_t| = |w_n| \quad \text{and} \quad w_t\hat{u}_t \perp w_n\hat{u}_n$$

so, $$w = \sqrt{2}\, w_n = \sqrt{2}\,\frac{v_0^2}{R}e^{-2s/R} = \sqrt{2}\,\frac{v^2}{R}$$

1.39 From the equation $v = a\sqrt{s}$

$$w_t = \frac{dv}{dt} = \frac{a}{2\sqrt{s}}\frac{ds}{dt} = \frac{a}{2\sqrt{s}}a\sqrt{s} = \frac{a^2}{2}, \text{ and}$$

$$w_n = \frac{v^2}{R} = \frac{a^2 s}{R}$$

As w_t is a positive constant, the speed of the particle increases with time, and the tangential acceleration vector and velocity vector coincides in direction.
Hence the angle between $\vec{v}$ and $\vec{w}$ is equal to between $w_t \hat{u}_t$ an $\vec{w}$, and α can be found by means of the formula : $\tan\alpha = \frac{|w_n|}{|w_t|} = \frac{a^2 s/R}{a^2/2} = \frac{2s}{R}$

1.40 From the equation $l = a\sin\omega t$

$$\frac{dl}{dt} = v = a\,\omega\cos\omega t$$

So, $$w_t = \frac{dv}{dt} = -a\,\omega^2\sin\omega t, \text{ and} \tag{1}$$

$$w_n = \frac{v^2}{R} = \frac{a^2\omega^2\cos^2\omega t}{R} \tag{2}$$

(a) At the point $l = 0$, $\sin\omega t = 0$ and $\cos\omega t = \pm 1$ so, $\omega t = 0, \pi$ etc.

Hence $$w = w_n = \frac{a^2\omega^2}{R}$$

Similarly at $l = \pm a$, $\sin\omega t = \pm 1$ and $\cos\omega t = 0$, so, $w_n = 0$

Hence $$w = |w_t| = a\,\omega^2$$

1.41 As $w_t = a$ and at $t = 0$, the point is at rest

So, $v(t)$ and $s(t)$ are, $v = at$ and $s = \frac{1}{2}at^2$ (1)

Let R be the curvature radius, then

$$w_n = \frac{v^2}{R} = \frac{a^2t^2}{R} = \frac{2\,as}{R} \text{ (using 1)}$$

But according to the problem

$$w_n = bt^4$$

So, $bt^4 = \frac{a^2t^2}{R}$ or, $R = \frac{a^2}{bt^2} = \frac{a^2}{2\,bs}$ (using 1) (2)

Therefore $w = \sqrt{w_t^2 + w_n^2} = \sqrt{a^2 + (2\,as/R)^2} = \sqrt{a^2 + \left(4\,bs^2/a^2\right)^2}$ (using 2)

Hence $w = a\sqrt{1 + \left(4\,bs^2/a^3\right)^2}$

1.42 (a) Let us differentiate twice the path equation $y(x)$ with respect to time.

$$\frac{dy}{dt} = 2\,ax\frac{dx}{dt};\ \frac{d^2y}{dt^2} = 2a\left[\left(\frac{dx}{dt}\right)^2 + x\frac{d^2x}{dt^2}\right]$$

Since the particle moves uniformly, its acceleration at all points of the path is normal and at the point $x = 0$ it coincides with the direction of derivative d^2y/dt^2. Keeping in mind that at the point $x = 0$, $\left|\frac{dx}{dt}\right| = v$,

We get
$$w = \left|\frac{d^2y}{dt^2}\right|_{x=0} = 2\,a\,v^2 = w_n$$

So,
$$w_n = 2\,a\,v^2 = \frac{v^2}{R},\quad \text{or}\quad R = \frac{1}{2a}$$

Note that we can also calculate it from the formula of problem (1.35 b)

(b) Differentiating the equation of the trajectory with respect to time we see that

$$b^2x\,\frac{dx}{dt} + a^2y\,\frac{dy}{dt} = 0 \tag{1}$$

which implies that the vector $(b^2x\vec{i} + a^2y\vec{j})$ is normal to the velocity vector $\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$ which, of course, is along the tangent. Thus the former vactor is along the normal and the normal component of acceleration is clearly

$$w_n = \frac{b^2x\frac{d^2x}{dt^2} + a^2y\frac{d^2y}{dt^2}}{(b^4x^2 + a^4y^2)^{1/2}}$$

on using $w_n = \vec{w}\cdot\vec{n}/|\vec{n}|$. At $x = 0$, $y = \pm\, b$ and so at $x = 0$

$$w_n = \pm\left.\frac{d^2y}{dt^2}\right|_{x=0}$$

Differentiating (1)

$$b^2\left(\frac{dx}{dt}\right)^2 + b^2x\left(\frac{d^2x}{dt^2}\right) + a^2\left(\frac{dy}{dt}\right)^2 + a^2y\left(\frac{d^2y}{dt^2}\right) = 0$$

Also from (1) $\qquad \frac{dy}{dt} = 0$ at $x = 0$

So $\left(\frac{dx}{dt}\right) = \pm\, v$ (since tangential velocity is constant $= v$)

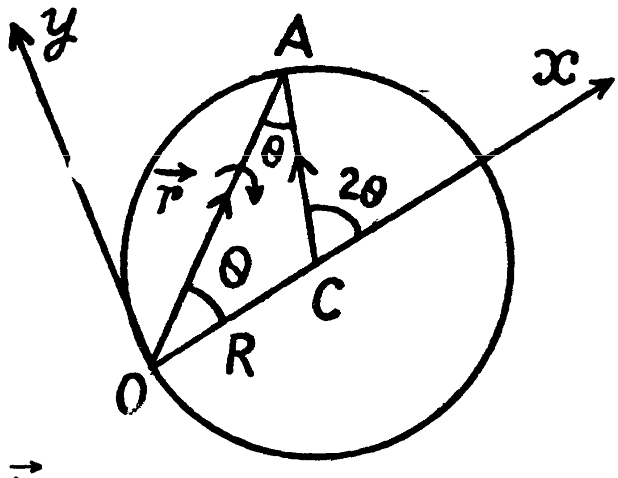
Thus $\left(\frac{d^2y}{dt^2}\right) = \pm\,\frac{b}{a^2}v^2$

and
$$|w_n| = \frac{bv^2}{a^2} = \frac{v^2}{R}$$

This gives $R = a^2/b$.

1.43 Let us fix the co-ordinate system at the point O as shown in the figure, such that the radius vector $\vec{r}$ of point A makes an angle θ with x axis at the moment shown.

Note that the radius vector of the particle A rotates clockwise and we here take line ox as reference line, so in this case obviously the angular velocity $\omega = \left(-\frac{d\theta}{dt}\right)$ taking anticlockwise sense of angular displacement as positive.



Also from the geometry of the triangle OAC

$$\frac{R}{\sin\theta} = \frac{r}{\sin(\pi - 2\theta)} \text{ or, } r = 2R\cos\theta.$$

Let us write,

$$\vec{r} = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} = 2R\cos^2\theta\,\vec{i} + R\sin 2\theta\,\vec{j}$$

Differentiating with respect to time.

$$\frac{d\vec{r}}{dt} \text{ or } \vec{v} = 2R\,2\cos\theta\,(-\sin\theta)\frac{d\theta}{dt}\vec{i} + 2R\cos 2\theta\frac{d\theta}{dt}\vec{j}$$

$$\text{or, } \vec{v} = 2R\left(\frac{-d\theta}{dt}\right)[\sin 2\theta\,\vec{i} - \cos 2\theta\,\vec{j}]$$

$$\text{or, } \vec{v} = 2R\,\omega\,(\sin 2\theta\,\vec{i} - \cos^2\theta\,\vec{j})$$

So, $\qquad |\vec{v}|$ or $v = 2\omega R = 0{\cdot}4\text{ m}/s.$

As ω is constant, v is also constant and $w_t = \frac{dv}{dt} = 0,$

So, $$w = w_n = \frac{v^2}{R} = \frac{(2\omega R)^2}{R} = 4\omega^2 R = 0{\cdot}32\ \text{m/s}^2$$

Alternate : From the Fig. the angular velocity of the point A, with respect to centre of the circle C becomes

$$-\frac{d(2\theta)}{dt} = 2\left(\frac{-d\theta}{dt}\right) = 2\omega = \text{constant}$$

Thus we have the problem of finding the velocity and acceleration of a particle moving along a circle of radius R with constant angular velocity 2ω.

Hence $\qquad v = 2\omega R$ and

$$w = w_n = \frac{v^2}{R} = \frac{(2\omega R)^2}{R} = 4\omega^2 R$$

1.44 Differentiating $\varphi(t)$ with respect to time

$$\frac{d\varphi}{dt} = \omega_z = 2at \qquad (1)$$

For fixed axis rotation, the speed of the point A:

$$v = \omega R = 2atR \text{ or } R = \frac{v}{2at} \qquad (2)$$

Differentiating with respect to time

$$w_t = \frac{dv}{dt} = 2\,a\,R = \frac{v}{t}, \text{ (using 1)}$$

But $$w_n = \frac{v^2}{R} = \frac{v^2}{v/2\,a\,t} = 2\,a\,t\,v \quad \text{(using 2)}$$

So, $$w = \sqrt{w_t^2 + w_n^2} = \sqrt{(v/t)^2 + (2\,a\,t\,v)^2}$$

$$= \frac{v}{t}\sqrt{1 + 4\,a^2 t^4}$$

1.45 The shell acquires a constant angular acceleration at the same time as it accelerates linearly. The two are related by (assuming both are constant)

$$\frac{w}{l} = \frac{\beta}{2\pi n}$$

Where w = linear acceleration and β = angular acceleration

Then, $\omega = \sqrt{2 \cdot \beta\, 2\pi n} = \sqrt{2 \cdot \frac{w}{l}(2\pi n)^2}$

But $v^2 = 2\,w\,l$, hence finally

$$\omega = \frac{2\pi n v}{l}$$

1.46 Let us take the rotation axis as z-axis whose positive direction is associated with the positive direction of the cordinate φ, the rotation angle, in accordance with the right-hand screw rule (Fig.)

(a) Defferentiating $\varphi(t)$ with respect to time.

$$\frac{d\varphi}{dt} = a - 3\,b\,t^2 = \omega_z \qquad (1) \text{ and}$$

$$\frac{d^2\varphi}{dt^2} = \frac{d\omega_z}{dt} = \beta_z = -6\,b\,t \qquad (2)$$

From (1) the solid comes to stop at $\Delta t = t = \sqrt{\frac{a}{3b}}$

The angular velocity $\omega = a - 3\,b\,t^2$, for $0 \le t \le \sqrt{a/3b}$

So, $$<\omega> = \frac{\int \omega\, dt}{\int dt} = \frac{\int_0^{\sqrt{a/3b}} (a - 3b\,t^2)\,dt}{\int_0^{\sqrt{a/3b}} dt} = [at - b\,t^3]_0^{\sqrt{a/3b}} \Big/ \sqrt{a/3b} = 2a/3$$

Similarly $\beta = |\beta_z| = 6b\,t$ for all values of t.

So, $$<\beta> = \frac{\int \beta\, dt}{\int dt} = \frac{\int_0^{\sqrt{a/3b}} 6bt\, dt}{\int_0^{\sqrt{a/3b}} dt} = \sqrt{3ab}$$

(b) From Eq. (2) $\beta_z = -6bt$

So, $$(\beta_z)_{t = \sqrt{a/3b}} = -6b\sqrt{\frac{a}{3b}} = -2\sqrt{ab}$$

Hence $$\beta = \left|(\beta_z)_{t = \sqrt{a/3b}}\right| = 2\sqrt{3ab}$$

1.47 Angle α is related with $|w_t|$ and w_n by means of the fomula :

$$\tan\alpha = \frac{w_n}{|w_t|}, \text{ where } w_n = \omega^2 R \text{ and } |w_t| = \beta R \qquad (1)$$

where R is the radius of the circle which an arbitrary point of the body circumscribes.

From the given equation $\beta = \frac{d\omega}{dt} = at$ (here $\beta = \frac{d\omega}{dt}$, as β is positive for all values of t)

Integrating within the limit $\int_0^{\omega} d\omega = a\int_0^t t\, dt$ or, $\omega = \frac{1}{2}at^2$

So, $$w_n = \omega^2 R = \left(\frac{at^2}{2}\right)^2 R = \frac{a^2t^4}{4}R$$

and $$|w_t| = \beta R = atR$$

Putting the values of $|w_t|$ and w_n in Eq. (1), we get,

$$\tan\alpha = \frac{a^2t^4R/4}{atR} = \frac{at^3}{4} \text{ or, } t = \left[\left(\frac{4}{a}\right)\tan\alpha\right]^{1/3}$$

1.48 In accordance with the problem, $\beta_z < 0$

Thus $-\frac{d\omega}{dt} = k\sqrt{\omega}$, where k is proportionality constant

or, $$-\int_{\omega_0}^{\omega}\frac{d\omega}{\sqrt{\omega}} = k\int_0^t dt \text{ or, } \sqrt{\omega} = \sqrt{\omega_0} - \frac{kt}{2} \qquad (1)$$

When $\omega = 0$, total time of rotation $t = \tau = \frac{2\sqrt{\omega_0}}{k}$

$$\text{Average angular velocity } \langle \omega \rangle = \frac{\int \omega\, dt}{\int dt} = \frac{\int_0^{2\sqrt{\omega_0}/k} \left(\omega_0 + \frac{k^2 t^2}{4} - k t \sqrt{\omega_0} \right) dt}{2\sqrt{\omega_0}\,/\,k}$$

$$\text{Hence } \langle \omega \rangle = \left[\omega_0 t + \frac{k^2 t^3}{12} - \frac{k}{2} \sqrt{\omega_0}\; t^2 \right]_0^{2\sqrt{\omega_0}/k} \Big/ \; 2 \frac{\sqrt{\omega_0}}{k} = \omega_0/3$$

1.49 We have $\omega = \omega_0 - a\,\varphi = \dfrac{d\varphi}{dt}$

Integratin this Eq. within its limit for $(\varphi)\, t$

$$\int_0^{\varphi} \frac{d\varphi}{\omega_0 - k\varphi} = \int_0^{t} dt \text{ or, } \ln \frac{\omega_0 - k\varphi}{\omega_0} = -kt$$

Hence $$\varphi = \frac{\omega_0}{k} (1 - e^{-kt}) \qquad (1)$$

(b) From the Eq., $\omega = \omega_0 - k\varphi$ and Eq. (1) or by differentiating Eq. (1)

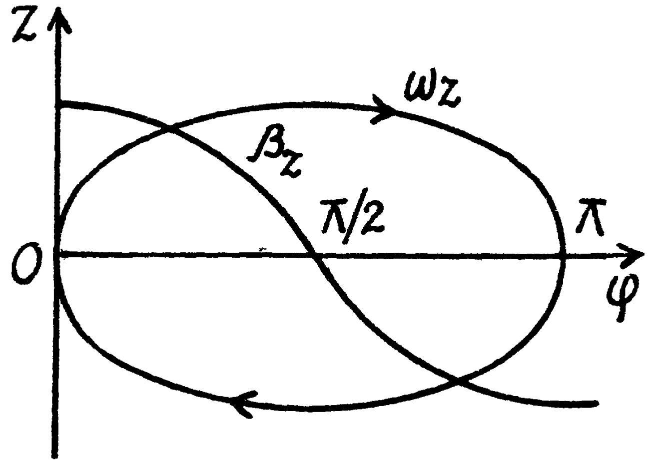
$$\omega = \omega_0\, e^{-kt}$$

1.50 Let us choose the positive direction of z-axis (stationary rotation axis) along the vector $\vec{\beta}_0$. In accordance with the equation

$$\frac{d\omega_z}{dt} = \beta_z \quad \text{or} \quad \omega_z \frac{d\omega_z}{d\varphi} = \beta_z$$

or, $\omega_z\, d\omega_z = \beta_z\, d\varphi = \beta \cos\varphi\, d\varphi$,

Integrating this Eq. within its limit for $\omega_z(\varphi)$



$$\text{or,} \quad \int_0^{\omega_z} d\omega_z = \beta_0 \int_0^{\varphi} \cos\varphi\, d\varphi$$

$$\text{or,} \quad \frac{\omega_z^2}{2} = \beta_0 \sin\varphi$$

Hence $\omega_z = \pm \sqrt{2\beta_0 \sin\varphi}$

The plot $\omega_z(\varphi)$ is shown in the Fig. It can be seen that as the angle φ grows, the vector $\vec{\omega}$ first increases, coinciding with the direction of the vector $\vec{\beta}_0$ $(\omega_z > 0)$, reaches the maximum at $\varphi = \varphi/2$, then starts decreasing and finally turns into zero at $\varphi = \pi$. After that the body starts rotating in the opposite direction in a similar fashion $(\omega_z < 0)$. As a result, the body will oscillate about the position $\varphi = \varphi/2$ with an amplitude equal to $\pi/2$.

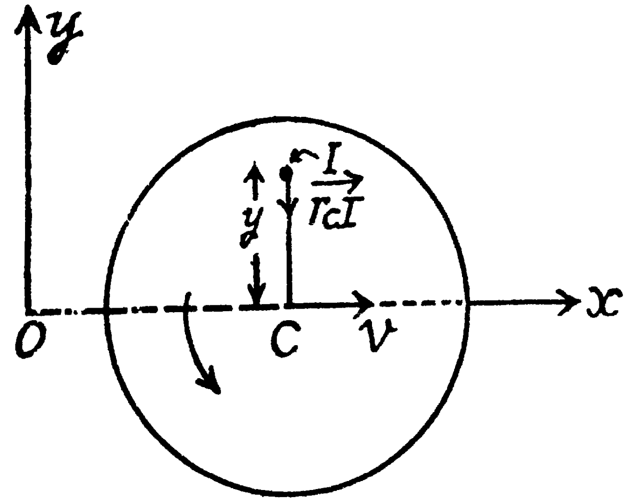
1.51 Rotating disc moves along the x-axis, in plane motion in $x-y$ plane. Plane motion of a solid can be imagined to be in pure rotation about a point (say I) at a certain instant known as instantaneous centre of rotation. The instantaneous axis whose positive sense is directed along $\vec{\omega}$ of the solid and which passes through the point I, is known as instantaneous axis of rotation.

Therefore the velocity vector of an arbitrary point (P) of the solid can be represented as :

$$\vec{v}_p = \vec{\omega} \times \vec{r}_{PI} \qquad (1)$$

On the basis of Eq. (1) for the C. M. (C) of the disc

$$\vec{v}_c = \vec{\omega} \times \vec{r}_{cI} \qquad (2)$$



According to the problem $\vec{v}_c \uparrow\uparrow \vec{i}$ and $\vec{\omega} \uparrow\uparrow \vec{k}$ i.e. $\vec{\omega} \perp x-y$ plane, so to satisy the Eqn. (2) $\vec{r}_{CI}$ is directed along $(-\vec{j})$. Hence point I is at a distance $r_{CI} = y$, above the centre of the disc along $y-$axis. Using all these facts in Eq. (2), we get

$$v_C = \omega y \text{ or } y = \frac{v_c}{\omega} \qquad (3)$$

(a) From the angular kinematical equation

$$\omega_z = \omega_{0z} + \beta_z t \qquad (4)$$

$$\omega = \beta t.$$

On the other hand $x = vt$, (where x is the x coordinate of the C.M.)

or, $$t = \frac{x}{v} \qquad (5)$$

From Eqs. (4) and (5), $\omega = \dfrac{\beta x}{v}$

Using this value of ω in Eq. (3) we get $y = \dfrac{v_c}{\omega} = \dfrac{v}{\beta x/v} = \dfrac{v^2}{\beta x}$ (hyperbola)

(b) As centre C moves with constant acceleration w, with zero initial velocity

So, $$x = \frac{1}{2} w t^2 \text{ and } v_c = wt$$

Therefore, $$v_c = w\sqrt{\frac{2x}{w}} = \sqrt{2xw}$$

Hence $$y = \frac{v_c}{\omega} = \frac{\sqrt{2wx}}{\omega} \text{ (parabola)}$$

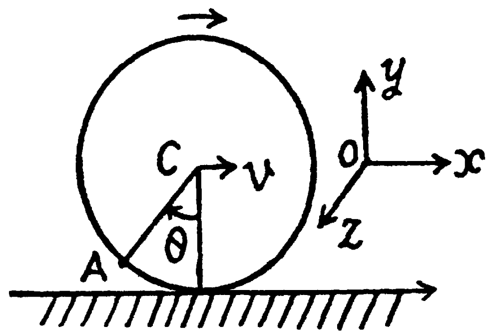
1.52 The plane motion of a solid can be imagined as the combination of translation of the C.M. and rotation about C.M.

So, we may write $\vec{v}_A = \vec{v}_C + \vec{v}_{AC}$

$$= \vec{v}_C + \vec{\omega} \times \vec{r}_{AC} \qquad (1) \text{ and}$$

$$\vec{w}_A = \vec{w}_C + \vec{w}_{AC}$$

$$= \vec{w}_C + \omega^2(-\vec{r}_{AC}) + (\vec{\beta} \times \vec{r}_{AC}) \qquad (2)$$



$\vec{r}_{AC}$ is the position of vector of A with respect to C.

In the problem $v_C = v =$ constant, and the rolling is without slipping i.e., $v_C = v = \omega R$, So, $w_C = 0$ and $\beta = 0$. Using these conditions in Eq. (2)

$$\vec{w}_A = \omega^2(-\vec{r}_{AC}) = \omega^2 R(-\hat{u}_{AC}) = \frac{v^2}{R}(-\hat{u}_{AC})$$

Here, $\hat{u}_{AC}$ is the unit vector directed along $\vec{r}_{AC}$.

Hence $w_A = \dfrac{v^2}{R}$ and $\vec{w}_A$ is directed along $(-\hat{u}_{AC})$ or directed toward the centre of the wheel.

(b) Let the centre of the wheel move toward right (positive x-axis) then for pure tolling on the rigid horizontal surface, wheel will have to rotate in clockwise sense. If ω be the angular velocity of the wheel then $\omega = \dfrac{v_C}{R} = \dfrac{v}{R}$.

Let the point A touches the horizontal surface at $t = 0$, further let us locate the point A at $t = t$,

When it makes $\theta = \omega t$ at the centre of the wheel.

From Eqn. (1) $\qquad \vec{v}_A = \vec{v}_C + \vec{\omega} \times \vec{r}_{AC}$

$$= v\vec{i} + \omega(-\vec{k}) \times [R\cos\theta(-\vec{j}) + R\sin\theta(-\vec{i})]$$

or, $\qquad \vec{v}_A = v\vec{i} + \omega R[\cos\omega t(-\vec{i}) + \sin\omega t\vec{j}]$

$$= (v - \cos\omega t)\vec{i} + v\sin\omega t\vec{j} \quad (\text{as } v = \omega R)$$

So, $\quad v_A = \sqrt{(v - v\cos\omega t)^2 + (v\sin\omega t)^2}$

$$= v\sqrt{2(1-\cos\omega t)} = 2v\sin(\omega t/2)$$

Hence distance covered by the point A during $T = 2\pi/\omega$

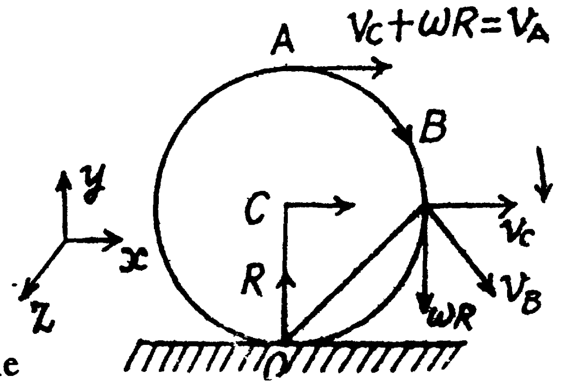
$$s = \int v_A\, dt = \int_0^{2\pi/\omega} 2v\sin(\omega t/2)\, dt = \frac{8v}{\omega} = 8R.$$

1.53 Let us fix the co-ordinate axis xyz as shown in the fig. As the ball rolls without slipping along the rigid surface so, on the basis of the solution of problem 1.52 :

Thus
$$\left.\begin{aligned} \vec{v}_0 = \vec{v}_c + \vec{\omega} \times \vec{r}_{oc} = 0 \\ v_c = \omega R \text{ and } \vec{\omega} \uparrow\uparrow (-\vec{k}) \text{ as } \vec{v}_c \uparrow\uparrow \vec{i} \end{aligned}\right\} \qquad (1)$$

and
$$\vec{\omega}_c + \vec{\beta} \times \vec{r}_{oc} = 0$$
$$w_c = \beta R \text{ and } \vec{\beta} \uparrow\uparrow (-\vec{k}) \text{ as } \vec{w}_c \uparrow\uparrow \vec{i}$$

At the position corresponding to that of Fig., in accordance with the problem,



$$w_c = w, \text{ so } v_c = wt$$

and
$$\omega = \frac{v_c}{R} = \frac{wt}{R} \text{ and } \beta = \frac{w}{R} \text{ (using 1)}$$

(a) Let us fix the co-ordinate system with the frame attached with the rigid surface as shown in the Fig.

As point O is the instantaneous centre of rotation of the ball at the moment shown in Fig.

so,
$$\vec{v}_0 = 0,$$

Now,
$$\vec{v}_A = \vec{v}_C + \vec{\omega} \times \vec{r}_{AC}$$
$$= v_C \vec{i} + \omega(-\vec{k}) \times R(\vec{j}) = (v_C + \omega R)\vec{i}$$

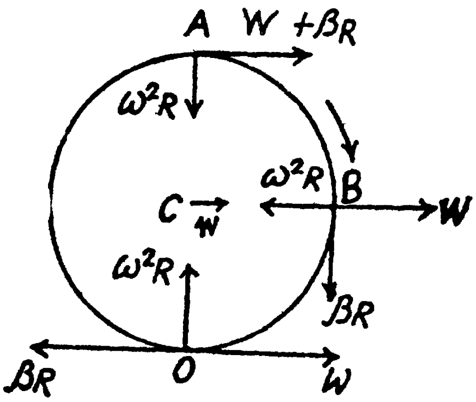
So,
$$\vec{v}_A = 2 v_C \vec{i} = 2 wt \vec{i} \text{ (using 1)}$$

Similalry
$$\vec{v}_B = \vec{v}_C + \vec{\omega} \times \vec{r}_{BC} = v_C \vec{i} + \omega(-\vec{k}) \times R(\vec{i})$$
$$= v_C \vec{i} + \omega R(-\vec{j}) = v_C \vec{i} + v_C(-\vec{j})$$

So, $v_B = \sqrt{2}\, v_c = \sqrt{2}\, wt$ and $\vec{v}_B$ is at an angle 45° from both $\vec{i}$ and $\vec{j}$ (Fig.)

(b)
$$\vec{w}_0 = \vec{w}_C + \omega^2(-\vec{r}_{oc}) + \vec{\beta} \times \vec{r}_{OC}$$
$$= \omega^2(-\vec{r}_{OC}) = \frac{v_C^2}{R}(-\hat{u}_{OC}) \text{ (using 1)}$$

where $\hat{u}_{OC}$ is the unit vector along $\vec{r}_{OC}$



so, $w_0 = \dfrac{v_0^2}{R} = \dfrac{w^2 t^2}{R}$ (using 2) and $\vec{w}_0$ is directed towards the centre of the ball

Now $\vec{w}_A = \vec{w}_C + \omega^2(-\vec{r}_{AC}) + \vec{\beta} \times \vec{r}_{AC}$

$= w\vec{i} + \omega^2 R(-\vec{j}) + \beta(-\vec{k}) \times R\vec{j}$

$= (w + \beta R)\vec{i} + \dfrac{v_c^2}{R}(-\vec{j})$ (using 1) $= 2w\vec{i} + \dfrac{w^2 t^2}{R}(-\vec{j})$

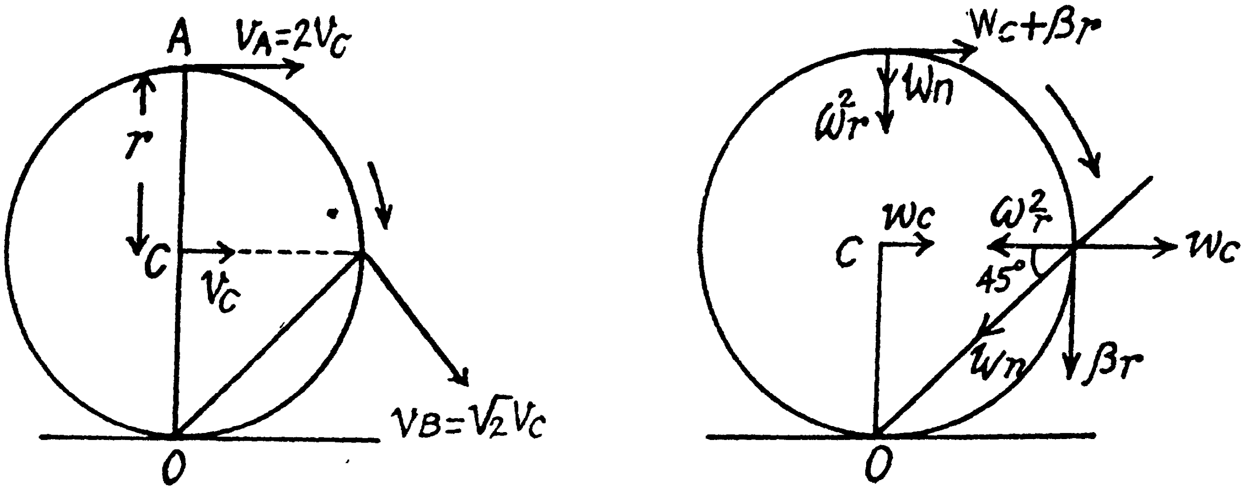
So,
$$w_A = \sqrt{4w^2 + \frac{w^4 t^4}{R^2}} = 2w\sqrt{1 + \left(\frac{wt^2}{2R}\right)^2}$$

Similarly
$$\vec{w}_B = \vec{w}_c + \omega^2(-\vec{r}_{BC}) + \vec{\beta} \times \vec{r}_{BC}$$
$$= w\vec{i} + \omega^2 R(-\vec{i}) + \beta(-\vec{k}) \times R(\vec{i})$$
$$= \left(v - \frac{v_C^2}{R}\right)\vec{i} + \beta R(-\vec{j}) \text{ (using 1)}$$

$$= \left(w - \frac{w^2 t^2}{R}\right)\vec{i} + w(-\vec{j}) \text{ (using 2)}$$

So, $$w_B = \sqrt{\left(w - \frac{w^2 t^2}{R}\right)^2 + w^2}$$

1.54 Let us draw the kinematical diagram of the rolling cylinder on the basis of the solution of problem 1.53.



As, an arbitrary point of the cylinder follows a curve, its normal acceleration and radius of curvature are related by the well known equation

$$w_n = \frac{v^2}{R}$$

so, for point A, $$w_{A(n)} = \frac{v_A^2}{R_A}$$

or, $$R_A = \frac{4 v_c^2}{\omega_r^2} = 4r \text{ (because } v_c = \omega r\text{, for pure rolling)}$$

Similarly for point B,

$$w_{B(n)} = \frac{v_B^2}{R_B}$$

$$\omega^2 r \cos 45° = \frac{(\sqrt{2}\, v_c)^2}{R_B},$$

or, $$R_B = 2\sqrt{2}\,\frac{v_C^2}{\omega^2 r} = 2\sqrt{2}\, r$$

1.55 The angular velocity is a vector as infinitesimal rotation commute. Then the relative angular velocity of the body 1 with respect to the body 2 is clearly.

$$\vec{\omega}_{12} = \vec{\omega}_1 - \vec{\omega}_2$$

as for relative linear velocity. The relative acceleration of 1 w.r.t. 2 is

$$\left(\frac{d\vec{\omega}_1}{dt}\right)_{S'}$$

where S' is a frame corotating with the second body and S is a space fixed frame with origin coinciding with the point of intersection of the two axes,

but $$\left(\frac{d\vec{\omega}_1}{dt}\right)_S = \left(\frac{d\vec{\omega}_1}{dt}\right)_{S'} + \vec{\omega}_2 \times \vec{\omega}_1$$

Since S' rotates with angular velocity $\vec{w}_2$. However $\left(\frac{d\vec{\omega}_1}{dt}\right)_S = 0$ as the first body rotates with constant angular velocity in space, thus

$$\vec{\beta}_{12} = \vec{\omega}_1 \times \vec{\omega}_2.$$

Note that for any vector $\vec{b}$, the relation in space forced frame (k) and a frame (k') rotating with angular velocity $\vec{\omega}$ is

$$\left.\frac{d\vec{b}}{dt}\right|_K = \left.\frac{d\vec{b}}{dt}\right|_{K'} + \vec{\omega} \times \vec{b}$$

1.56 We have $\vec{\omega} = at\,\vec{i} + bt^2\,\vec{j}$ (1)

So, $\omega = \sqrt{(at)^2 + (bt^2)^2}$, thus, $\omega|_{t=10s} = 7.81\ \text{rad/s}$

Differentiating Eq. (1) with respect to time

$$\vec{\beta} = \frac{d\vec{\omega}}{dt} = a\,\vec{i} + 2\,bt\,\vec{j} \tag{2}$$

So, $$\beta = \sqrt{a^2 + (2\,bt)^2}$$

and $$\beta|_{t=10s} = 1{\cdot}3\ \text{rad/s}^2$$

(b) $$\cos\alpha = \frac{\vec{\omega}\cdot\vec{\beta}}{\omega\beta} = \frac{(at\,\vec{i} + bt^2\,\vec{j})\cdot(a\vec{i} + 2\,bt\,\vec{j})}{\sqrt{(at)^2 + (bt^2)^2}\,\sqrt{a^2 + (2bt)^2}}$$

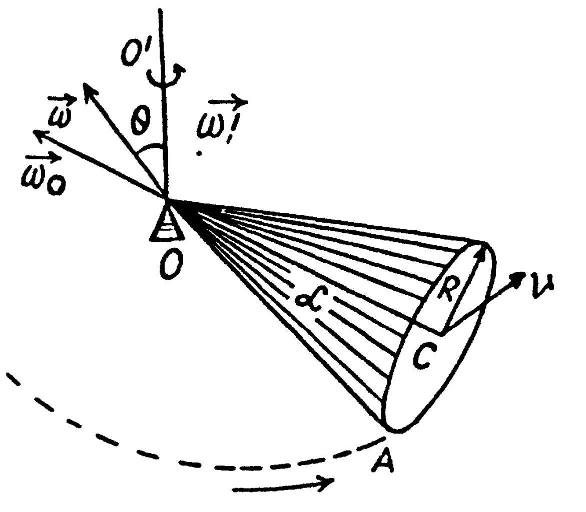
Putting the values of (a) and (b) and taking $t = 10\text{s}$, we get

$$\alpha = 17°$$

1.57 (a) Let the axis of the cone (OC) rotates in anticlockwise sense with constant angular velocity $\vec{\omega}'$ and the cone itself about it's own axis (OC) in clockwise sense with angular velocity $\vec{\omega}_0$ (Fig.). Then the resultant angular velocity of the cone.

$$\vec{\omega} = \vec{\omega}' + \vec{\omega}_0 \tag{1}$$

As the rolling is pure the magnitudes of the vectors $\vec{\omega}'$ and $\vec{\omega}_0$ can be easily found from Fig.



$$\omega' = \frac{v}{R\cot\alpha},\quad \omega_0 = v/R \tag{2}$$

As $\vec{\omega}' \perp \vec{\omega}_0$, from Eq. (1) and (2)

$$\omega = \sqrt{\omega'^2 + \omega_0^2}$$

$$\sqrt{\left(\frac{v}{R\cot\alpha}\right)^2 + \left(\frac{v}{R}\right)^2} = \frac{v}{R\cos\alpha} = 2{\cdot}3 \text{ rad/s}$$

(b) Vector of angular acceleration

$$\vec{\beta} = \frac{d\vec{\omega}}{dt} = \frac{d(\vec{\omega}' + \vec{\omega}_0)}{dt} \quad (\text{as } \vec{\omega}' = \text{constant.})$$

The vector $\vec{\omega}_0$ which rotates about the OO' axis with the angular velocity $\vec{\omega}'$, retains i magnitude. This increment in the time interval dt is equal to

$$|d\vec{\omega}_0| = \omega_0 \cdot \omega' \, dt \text{ or in vector form } d\vec{\omega}_0 = (\vec{\omega}' \times \vec{\omega}_0)\, dt.$$

Thus $\quad \vec{\beta} = \vec{\omega}' \times \vec{\omega}_0$ (3

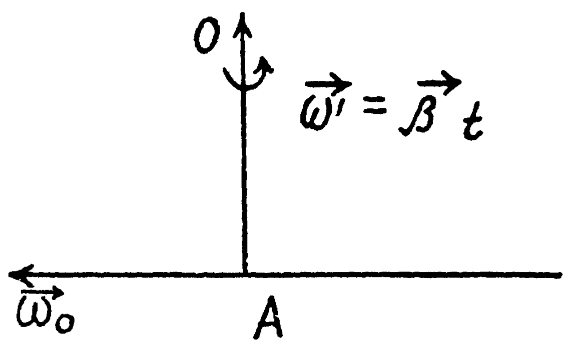
The magnitude of the vector $\vec{\beta}$ is equal to

$$\beta = \omega' \omega_0 \, (\text{as } \vec{\omega}' \perp \vec{\omega}_0)$$

So, $\quad \beta = \frac{v}{R\cot\alpha}\frac{v}{R} = \frac{v^2}{R^2}\tan\alpha = 2{\cdot}3 \text{ rad/s}$

1.58 The axis AB acquired the angular velocity

$$\vec{\omega}' = \vec{\beta}_o t \qquad (1)$$



Using the facts of the solution of 1.57, the angular velocity of the body

$$\omega = \sqrt{\omega_o^2 + \omega'^2}$$

$$= \sqrt{\omega_0^2 + \beta_0^2 t^2} = 0{\cdot}6 \text{ rad/s}$$

And the angular acceleration.

$$\vec{\beta} = \frac{d\vec{\omega}}{dt} = \frac{d(\vec{\omega}' + \vec{\omega}_0)}{dt} = \frac{d\vec{\omega}'}{dt} + \frac{d\vec{\omega}_0}{dt}$$

But $\quad \frac{d\vec{\omega}_0}{dt} = \vec{\omega}' \times \vec{\omega}_0$, and $\frac{d\vec{\omega}'}{dt} = \vec{\beta}_0 t$

So, $\quad \vec{\beta} = (\vec{\beta}_0 t \times \vec{\omega}_0) + \vec{\beta}_0$

As, $\vec{\beta}_0 \perp \vec{\omega}_0$ so, $\beta = \sqrt{(\omega_0 \beta_0 t)^2 + \beta_0^2} = \beta_0\sqrt{1 + (\omega_0 t)^2} = 0{\cdot}2 \text{ rad/s}^2$

1.2 THE FUNDAMENTAL EQUATION OF DYNAMICS

1.59 Let R be the constant upward thurst on the aerostat of mass m, coming down with a constant acceleration w. Applying Newton's second law of motion for the aerostat in projection form

$$F_y = mw_y$$

$$mg - R = mw \tag{1}$$

Now, if Δm be the mass, to be dumped, then using the Eq. $F_y = mw_y$

$$R - (m - \Delta m)\, g = (m - \Delta m)\, w, \tag{2}$$

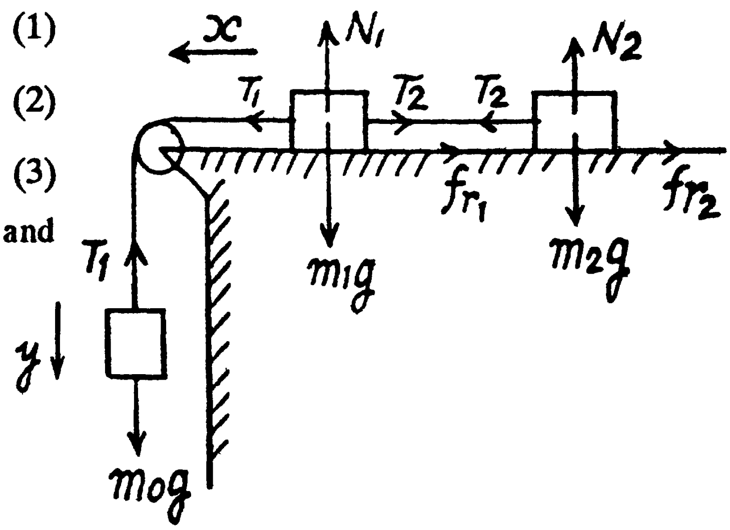
From Eqs. (1) and (2), we get, $\quad \Delta m = \dfrac{2\,mw}{g + w}$

1.60 Let us write the fundamental equation of dynamics for all the three blocks in terms of projections, having taken the positive direction of x and y axes as shown in Fig; and using the fact that kinematical relation between the accelerations is such that the blocks move with same value of acceleration (say w)

$$m_0\, g - T_1 = m_0\, w \tag{1}$$

$$T_1 - T_2 - km_1\, g = m_1\, w \tag{2}$$

and $$T_2 - km_2\, g = m_2\, w \tag{3}$$



The simultaneous solution of Eqs. (1), (2) and (3) yields,

$$w = g\,\frac{[\, m_0 - k\,(m_1 + m_2)\,]}{m_0 + m_1 + m_2}$$

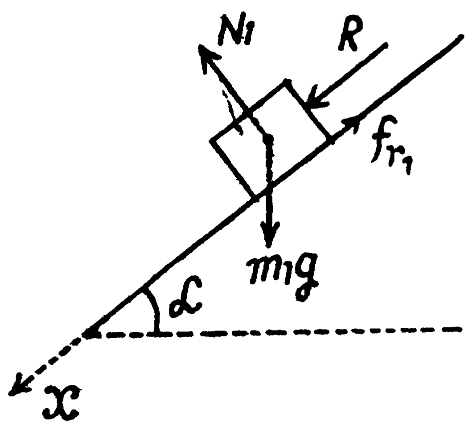
and $$T_2 = \frac{(1 + k)\, m_0}{m_0 + m_1 + m_2}\, m_2\, g$$

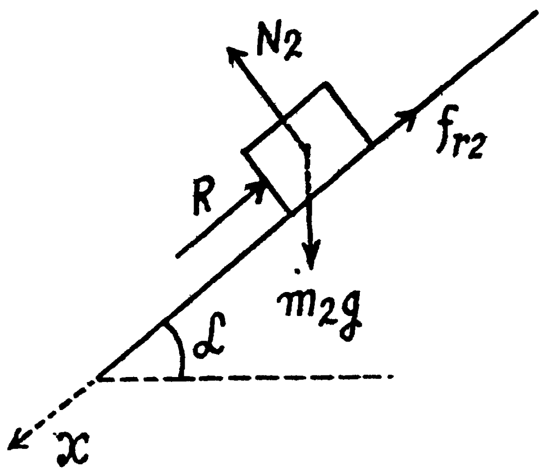
As the block m_0 moves down with acceleration w, so in vector form

$$\vec{w} = \frac{[m_0 - k\,(m_1 + m_2)\,]\,\vec{g}}{m_0 + m_1 + m_2}$$

1.61 Let us indicate the positive direction of x-axis along the incline (Fig.). Figures show the force diagram for the blocks.

Let, R be the force of interaction between the bars and they are obviously sliding down with the same constant acceleration w.





Newton's second law of motion in projection form along x-axis for the blocks gives :

$$m_1 g \sin\alpha - k_1 m_1 g \cos\alpha + R = m_1 w \tag{1}$$

$$m_2 g \sin\alpha - R - k_2 m_2 g \cos\alpha = m_2 w \tag{2}$$

Solving Eqs. (1) and (2) simultaneously, we get

$$w = g\sin\alpha - g\cos\alpha \frac{k_1 m_1 + k_2 m_2}{m_1 + m_2} \text{ and}$$

$$R = \frac{m_1 m_2 (k_1 - k_2) g \cos\alpha}{m_1 + m_2} \tag{3}$$

(b) when the blocks just slide down the plane, $w = 0$, so from Eqn. (3)

$$g\sin\alpha - g\cos\alpha \frac{k_1 m_1 + k_2 m_2}{m_1 + m_2} = 0$$

or, $(m_1 + m_2)\sin\alpha = (k_1 m_1 + k_2 m_2)\cos\alpha$

Hence $$\tan\alpha = \frac{(k_1 m_1 + k_2 m_2)}{m_1 + m_2}$$

1.62 Case 1. When the body is launched up :

Let k be the coefficeint of friction, u the velocity of projection and l the distance traversed along the incline. Retarding force on the block $= mg\sin\alpha + k\,mg\cos\alpha$ and hence the retardation $= g\sin\alpha + k\,g\cos\alpha$.

Using the equation of particle kinematics along the incline,

$$0 = u^2 - 2(g\sin\alpha + k\,g\cos\alpha)\,l$$

or, $$l = \frac{u^2}{2(g\sin\alpha + k\,g\cos\alpha)} \tag{1}$$

and $$0 = u - (g\sin\alpha + k\,g\cos\alpha)\,t$$

or, $$u = (g\sin\alpha + k\,g\cos\alpha)\,t \tag{2}$$

Using (2) in (1) $l = \frac{1}{2}(g\sin\alpha + k\,g\cos\alpha)\,t^2$ (3)

Case (2). When the block comes downward, the net force on the body $= mg\sin\alpha - km\,g\cos\alpha$ and hence its acceleration $= g\sin\alpha - k\,g\cos\alpha$

Let, t be the time required then,

$$l = \frac{1}{2}(g\sin\alpha - k\,g\cos\alpha)\,t'^2 \tag{4}$$

From Eqs. (3) and (4)

$$\frac{t^2}{t'^2} = \frac{\sin\alpha - k\cos\alpha}{\sin\alpha + k\cos\alpha}$$

But $\frac{t}{t'} = \frac{1}{\eta}$ (according to the question),

Hence on solving we get

$$k = \frac{(\eta^2 - 1)}{(\eta^2 + 1)}\tan\alpha = 0{\cdot}16$$

1.63 At the initial moment, obviously the tension in the thread connecting m_1 and m_2 equals the weight of m_2.

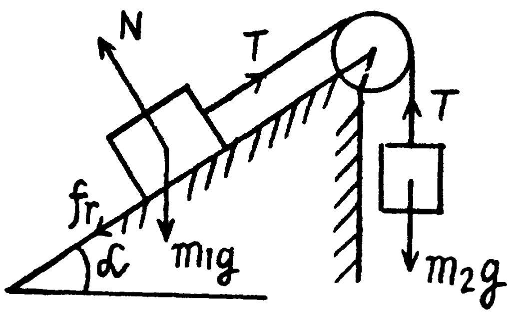
(a) For the block m_2 to come down or the block m_1 to go up, the conditions is

$$m_2 g - T \geq 0 \quad \text{and} \quad T - m_1 g \sin\alpha - fr \geq 0$$

where T is tension and f_r is friction which in the limiting case equals $km_1 g \cos\alpha$. Then

or $$m_2 g - m_1 \sin\alpha > k m_1 g \cos\alpha$$

or $$\frac{m_2}{m_1} > (k \cos\alpha + \sin\alpha)$$



(b) Similarly in the case

$$m_1 g \sin\alpha - m_2 g > fr_{\lim}$$

or, $m_1 g \sin\alpha - m_2 g > km_1 g \cos\alpha$

or, $$\frac{m_2}{m_1} < (\sin\alpha - k\cos\alpha)$$

(c) For this case, neither kind of motion is possible, and fr need not be limiting.

Hence, $$(k\cos\alpha + \sin\alpha) > \frac{m_2}{m_1} > (\sin\alpha - k\cos\alpha)$$

1.64 From the conditions, obtained in the previous problem, first we will check whether the mass m_2 goes up or down.

Here, $m_2/m_1 = \eta > \sin\alpha + k\cos\alpha$, (substituting the values). Hence the mass m_2 will come down with an acceleration (say w). From the free body diagram of previous problem,

$$m_2 - g - T = m_2 w \tag{1}$$

and $$T - m_1 g \sin\alpha - k m_1 g \cos\alpha = m_1 w \tag{2}$$

Adding (1) and (2), we get,

$$m_2 g - m_1 g \sin\alpha - k m_1 g \cos\alpha = (m_1 + m_2) w$$

$$w = \frac{(m_2/m_1 - \sin\alpha - k\cos\alpha) g}{(1 + m_2/m_1)} = \frac{(\eta - \sin\alpha - k\cos\alpha) g}{1+\eta}$$

Substituting all the values, $w = 0{\cdot}048\,g \approx 0{\cdot}05\,g$

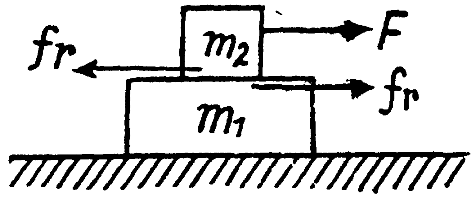
As m_2 moves down with acceleration of magnitude $w = 0.05\,g > 0$, thus in vector form acceleration of m_2:

$$\vec{w}_2 = \frac{(\eta - \sin\alpha - k\cos\alpha)\vec{g}}{1+\eta} = 0.05\,\vec{g}$$

1.65 Let us write the Newton's second law in projection form along positive x-axis for the plank and the bar

$$fr = m_1 w_1, \; fr = m_2 w_2 \tag{1}$$

At the initial moment, fr represents the static friction, and as the force F grows so does the friction force fr, but up to it's limiting value i.e. $fr = fr_{s(max)} = kN = k m_2 g$.



Unless this value is reached, both bodies moves as a single body with equal acceleration. But as soon as the force fr reaches the limit, the bar starts sliding over the plank i.e. $w_2 \geq w_1$.

Substituting here the values of w_1 and w_2 taken from Eq. (1) and taking into account that $f_r = k m_2 g$, we obtain, $(at - km_2 g)/m_2 \geq \frac{km_2}{m_1} g$, were the sign "=" corresponds to the moment $t = t_0$ (say)

Hence, $$t_0 = \frac{k g m_2 (m_1 + m_2)}{a m_1}$$

If $$t \leq t_0, \text{ then } w_1 = \frac{km_2 g}{m_1} \text{ (constant). and}$$

$$w_2 = (at - km_2 g)/m_2$$

On this basis $w_1(t)$ and $w_2(t)$, plots are as shown in the figure of answersheet.

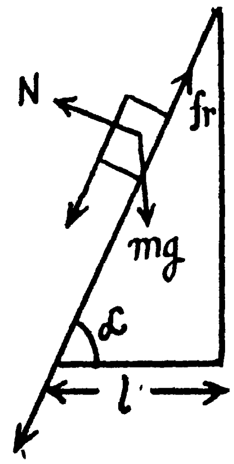
1.66 Let us designate the x-axis (Fig.) and apply $F_x = m w_x$ for body A :

$$mg \sin \alpha - k m g \cos \alpha = m w$$

or, $$w = g \sin \alpha - k g \cos \alpha$$

Now, from kinematical equation :

$$l \sec \alpha = 0 + (1/2) w t^2$$



or, $$t = \sqrt{2 l \sec \alpha / (\sin \alpha - k \cos \alpha) g}$$

$$= \sqrt{2 l / (\sin 2\alpha/2 - k \cos^2 \alpha) g}$$

(using Eq. (1)).

for t_{min}, $$\frac{d\left(\frac{\sin 2\alpha}{2} - k \cos^2 \alpha\right)}{d\alpha} = 0$$

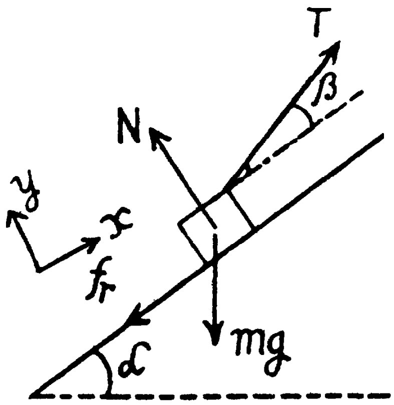
i.e. $$\frac{2 \cos 2\alpha}{2} + 2k \cos \alpha \sin \alpha = 0$$

or, $$\tan 2\alpha = -\frac{1}{k} \Rightarrow \alpha = 49°$$

and putting the values of α, k and l in Eq. (2) we get $t_{min} = 1$s.

1.67 Let us fix the $x - y$ co-ordinate system to the wedge, taking the x - axis up, along the incline and the y - axis perpendicular to it (Fig.).

Now, we draw the free body diagram for the bar.



Let us apply Newton's second law in projection form along x and y axis for the bar :

$T\cos\beta - m\,g\sin\alpha - fr = 0$ (1)

$T\sin\beta + N - m\,g\cos\alpha = 0$

or, $N = m\,g\cos\alpha - T\sin\beta$ (2)

But $f_r = kN$ and using (2) in (1), we get

$$T = m\,g\sin\alpha + k\,mg\cos\alpha/(\cos\beta + k\sin\beta) \quad (3)$$

For T_{min} the value of $(\cos\beta + k\sin\beta)$ should be maximum

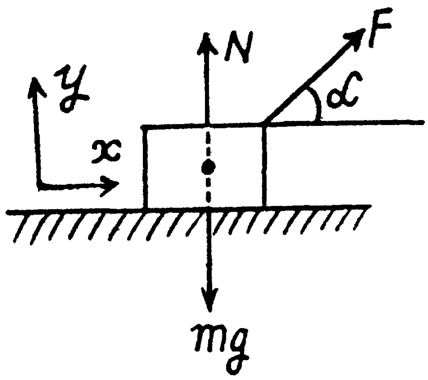
So, $$\frac{d(\cos\beta + k\sin\beta)}{d\beta} = 0 \quad \text{or} \quad \tan\beta = k$$

Putting this value of β in Eq. (3) we get,

$$T_{min} = \frac{m\,g(\sin\alpha + k\cos\alpha)}{1/\sqrt{1+k^2} + k^2/\sqrt{1+k^2}} = \frac{m\,g(\sin\alpha + k\cos\alpha)}{\sqrt{1+k^2}}$$

1.68 First of all let us draw the free body diagram for the small body of mass m and indicate x – axis along the horizontal plane and y – axis, perpendicular to it, as shown in the figure. Let the block breaks off the plane at $t = t_0$ i.e. $N = 0$

So, $N = m\,g - at_0\sin\alpha = 0$



or, $t_0 = \dfrac{mg}{a\sin\alpha}$ (1)

From $F_x = m\,w_x$, for the body under investigation :

$m\,d\,v_x/dt = at\cos\alpha$; Integrating within the limits for $v(t)$

$$m\int_0^v dv_x = a\cos\alpha\int_0 t\,dt \quad \text{(using Eq. 1)}$$

So, $$v = \frac{ds}{dt} = \frac{a\cos\alpha}{2m}t^2 \quad (2)$$

Integrating, Eqn. (2) for $s(t)$

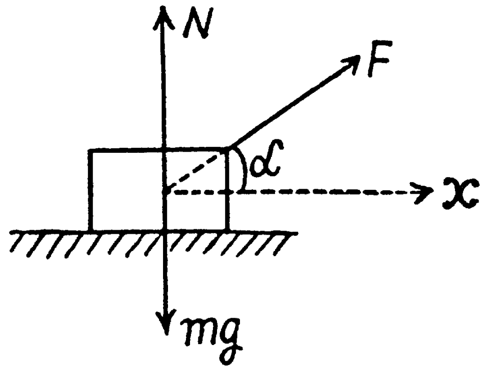
$$s = \frac{a\cos\alpha}{2m}\frac{t^3}{3} \quad (3)$$

Using the value of $t = t_0$ from Eq. (1), into Eqs. (2) and (3)

$$v = \frac{m\,g^2\cos\alpha}{2\,a\sin^2\alpha} \quad \text{and} \quad s = \frac{m^2\,g^3\cos\alpha}{6\,a^2\sin^3\alpha}$$

1.69 Newton's second law of motion in projection form, along horizontal or x-axis i.e. $F_x = m\, w_x$ gives.

$$F \cos(as) = m\, v \frac{dv}{ds} \quad (as\ \alpha = as)$$



or, $\quad F \cos(as)\, ds = m\, v\, dv$

Integrating, over the limits for $v(s)$

$$\frac{F}{m}\int_0^{\infty} \cos(as)\, ds = \frac{v^2}{2}$$

or $$v = \sqrt{\frac{2F \sin\alpha}{m\, a}}$$

$$= \sqrt{2g \sin\alpha / 3a} \ \left(\text{using } F = \frac{mg}{3}\right)$$

which is the sought relationship.

1.70 From the Newton's second law in projection from :

For the bar,

$$T - 2kmg = (2m)\, w \qquad (1)$$

For the motor,

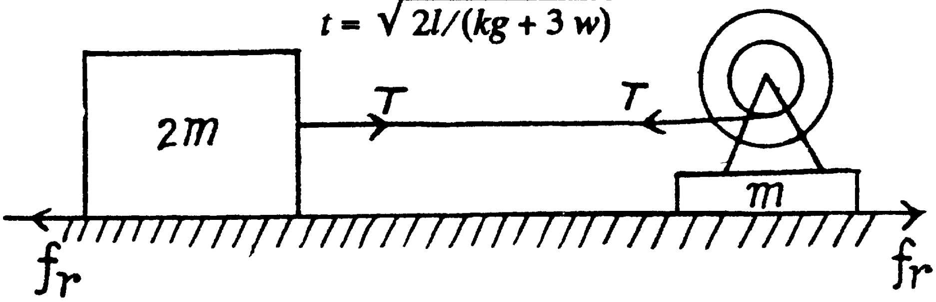
$$T - kmg = m\, w' \qquad (2)$$

Now, from the equation of kinematics in the frame of bar or motor :

$$l = \frac{1}{2}(w + w')\, t^2 \qquad (3)$$

From (1), (2) and (3) we get on eliminating T and w'

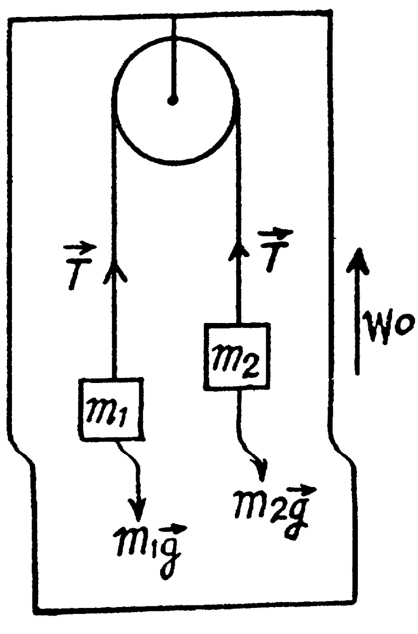
$$t = \sqrt{2l/(kg + 3w)}$$



1.71 Let us write Newton's second law in vector from $\vec{F} = m\vec{w}$, for both the blocks (in the frame of ground).

$$\vec{T} + m_1\vec{g} = m_1\vec{w}_1 \qquad (1)$$

$$\vec{T} + m_2\vec{g} = m_2\vec{w}_2 \qquad (2)$$



These two equations contain three unknown quantities $\vec{w}_1$, $\vec{w}_2$ and $\vec{T}$. The third equation is provided by the kinematic relationship between the accelerations :

$$\vec{w}_1 = \vec{w}_0 + \vec{w}', \quad \vec{w}_2 = \vec{w}_0 - \vec{w}' \qquad (3)$$

where $\vec{w}'$ is th acceleration of the mass m_1 with respect to the pulley or elevator car.

Summing up termwise the left hand and the right-hand sides of these kinematical equations, we get

$$\vec{w}_1 + \vec{w}_2 = 2\,\vec{w}_0 \tag{4}$$

The simultaneous solution of Eqs.(1), (2) and (4) yields

$$\vec{w}_1 = \frac{(m_1 - m_2)\,\vec{g} + 2\,m_2\,\vec{w}_0}{m_{1+}\,m_2}$$

Using this result in Eq. (3), we get,

$$\vec{w}' = \frac{m_1 - m_2}{m_1 + m_2}(\vec{g} - \vec{w}_0) \quad \text{and} \quad \vec{T} = \frac{2\,m_1\,m_2}{m_1 + m_2}(\vec{w}_0 - \vec{g})$$

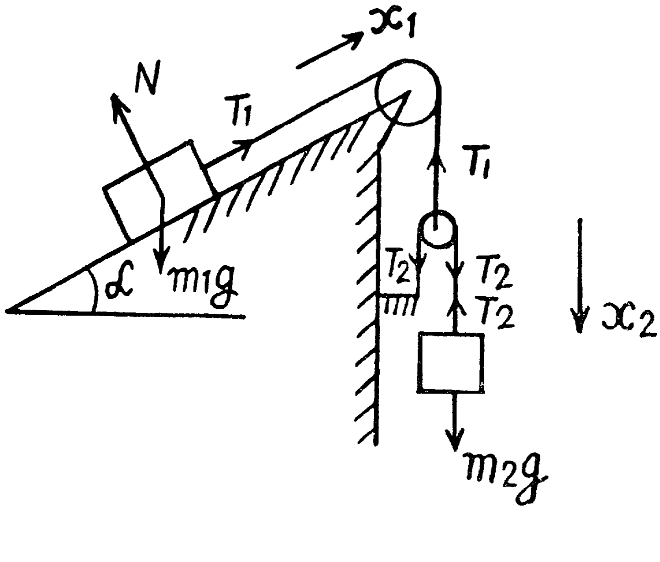
Using the results in Eq. (3) we get $\vec{w}' = \dfrac{m_1 - m_2}{m_1 + m_2}(\vec{g} - \vec{w}_0)$

(b) obviously the force exerted by the pulley on the celing of the car

$$\vec{F} = -2\,\vec{T} = \frac{4\,m_1\,m_2}{m_1 + m_2}(\vec{g} - \vec{w}_0)$$

Note : *one could also solve this problem in the frame of elevator car.*

1.72 Let us write Newton's second law for both, bar 1 and body 2 in terms of projection having taken the positive direction of x_1 and x_2 as shown in the figure and assuming that body 2 starts sliding, say, upward along the incline



$$T_1 - m_1\,g\sin\alpha = m_1\,w_1 \tag{1}$$

$$m_2\,g - T_2 = m_2\,w \tag{2}$$

For the pulley, moving in vertical direction from the equation $F_x = m\,w_x$

$$2\,T_2 - T_1 = (m_p)\,w_1 = 0$$

(as mass of the pulley $m_p = 0$)

or
$$T_1 = 2\,T_2 \tag{3}$$

As the length of the threads are constant, the kinematical relationship of accelerations becomes

$$w = 2\,w_1 \tag{4}$$

Simultaneous solutions of all these equations yields :

$$w = \frac{2\,g\left(2\dfrac{m_2}{m_1} - \sin\alpha\right)}{\left(4\dfrac{m_2}{m_1} + 1\right)} = \frac{2\,g\,(2\,\eta - \sin\alpha)}{(4\,\eta + 1)}$$

As $\eta > 1$, w is directed vertically downward, and hence in vector form

$$\vec{w} = \frac{2\,\vec{g}\,(2\,\eta - \sin\alpha)}{4\,\eta + 1}$$

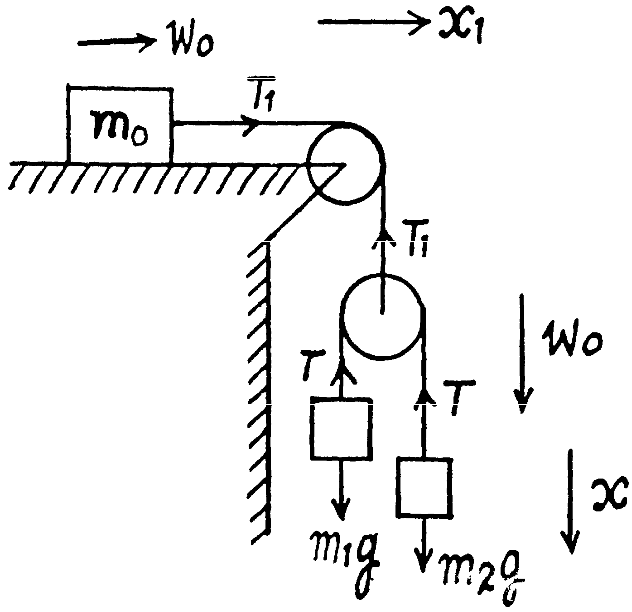
1.73 Let us write Newton's second law for masses m_1 and m_2 and moving pully in vertical direction along positive x – axis (Fig.) :

$$m_1 g - T = m_1 w_{1x} \qquad (1)$$

$$m_2 g - T = m_2 w_{2x} \qquad (2)$$

$$T_1 - 2T = 0 \text{ (as } m = 0) $$

or $$T_1 = 2T \qquad (3)$$



Again using Newton's second law in projection form for mass m_0 along positive x_1 direction (Fig.), we get

$$T_1 = m_0 w_0 \qquad (4)$$

The kinematical relationship between the accelerations of masses gives in terms of projection on the x – axis

$$w_{1x} + w_{2x} = 2 w_0 \qquad (5)$$

Simultaneous solution of the obtained five equations yields :

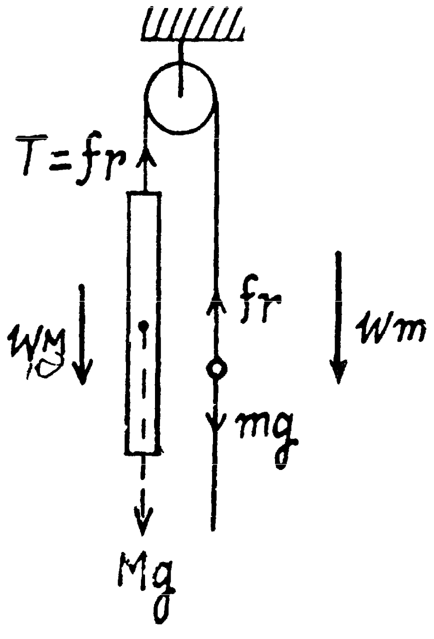
$$w_1 = \frac{[4 m_1 m_2 + m_0 (m_1 - m_2)] g}{4 m_1 m_2 + m_0 (m_1 + m_2)}$$

In vector form

$$\vec{w}_1 = \frac{[4 m_1 m_2 + m_0 (m_1 - m_2)] \vec{g}}{4 m_1 m_2 + m_0 (m_1 + m_2)}$$

1.74 As the thread is not tied with m, so if there were no friction between the thread and the ball m, the tension in the thread would be zero and as a result both bodies will have free fall motion. Obviously in the given problem it is the friction force exerted by the ball on the thread, which becomes the tension in the thread. From the condition or language of the problem $w_M > w_m$ and as both are directed downward so, relative acceleration of $M = w_M - w_m$ and is directed downward. Kinematical equation for the ball in the frame of rod in projection form along upward direction gives :

$$l = \frac{1}{2} (w_M - w_m) t^2 \qquad (1)$$



Newton's second law in projection form along vertically down direction for both, rod and ball gives,

$$Mg - fr = M w_M \qquad (2)$$

$$mg - fr = m w_m \qquad (3)$$

Multiplying Eq. (2) by m and Eq. (3) by M and then subtracting Eq. (3) from (2) and after using Eq. (1) we get

$$fr = \frac{2 l M m}{(M - m) t^2}$$

1.75 Suppose, the ball goes up with accleration w_1 and the rod comes down with the acceleration w_2.

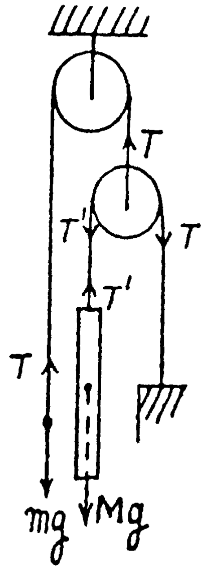
As the length of the thread is constant, $2\,w_1 = w_2$ (1)

From Newton's second law in projection form along vertically upward for the ball and vertically downward for the rod respectively gives,

$$T - m\,g = m\,w_1 \qquad (2)$$

and $$M\,g - T' = M\,w_2 \qquad (3)$$

but $T = 2\,T$ (because pulley is massless) (4)



From Eqs. (1), (2), (3) and (4)

$$w_1 = \frac{(2M - m)\,g}{m + 4M} = \frac{(2 - \eta)\,g}{\eta + 4} \text{ (in upward direction)}$$

and $$w_2 = \frac{2\,(2 - \eta)\,g}{(\eta + 4)} \text{ (downwards)}$$

From kinematical equation in projection form, we get

$$l = \frac{1}{2}(w_1 + w_2)\,t^2$$

as, w_1 and w_2 are in the opposite direction.

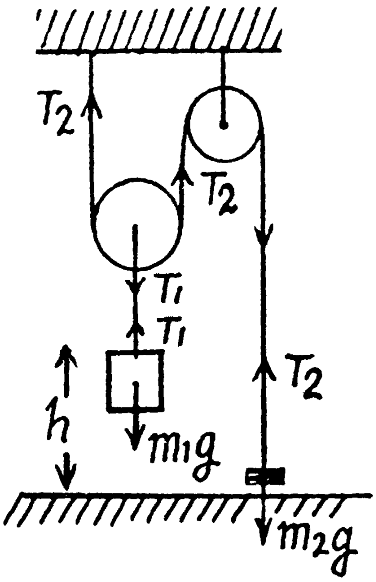
Putting the values of w_1 and w_2, the sought time becomes

$$t = \sqrt{2\,l\,(\eta + 4)/3\,(2 - \eta)\,g} = 1{\cdot}4\text{ s}$$

1.76 Using Newton's second law in projection form along x – axis for the body 1 and along negative x – axis for the body 2 respectively, we get

$$m_1\,g - T_1 = m_1\,w_1 \qquad (1)$$

$$T_2 - m_2\,g = m_2\,w_2 \qquad (2)$$



For the pulley lowering in downward direction from Newton's law along x axis,

$$T_1 - 2\,T_2 = 0 \text{ (as pulley is mass less)}$$

or, $$T_1 = 2\,T_2 \qquad (3)$$

As the length of the thread is constant so,

$$w_2 = 2\,w_1 \qquad (4)$$

The simultaneous solution of above equations yields,

$$w_2 = \frac{2\,(m_1 - 2m_2)\,g}{4\,m_2 + m_1} = \frac{2\,(\eta - 2)}{\eta + 4} \quad \left(\text{as } \frac{m_1}{m_2} = \eta\right) \qquad (5)$$

Obviously during the time interval in which the body 1 comes to the horizontal floor covering the distance h, the body 2 moves upward the distance $2h$. At the moment when the body 2 is at the height $2h$ from the floor its velocity is given by the expression :

$$v_2^2 = 2\,w_2\,(2h) = 2\left[\frac{2\,(\eta - 2)\,g}{\eta + 4}\right]2h = \frac{8\,h\,(\eta - 2)\,g}{\eta + 4}$$

After the body m_1 touches the floor the thread becomes slack or the tension in the thread zero, thus as a result body 2 is only under gravity for it's subsequent motion.

Owing to the velocity v_2 at that moment or at the height $2h$ from the floor, the body 2 further goes up under gravity by the distance,

$$h' = \frac{v_2^2}{2g} = \frac{4h(\eta - 2)}{\eta + 4}$$

Thus the sought maximum height attained by the body 2 :

$$H = 2h + h' = 2h + \frac{4h(\eta - 2)}{(\eta + 4)} = \frac{6\eta h}{\eta + 4}$$

1.77 Let us draw free body diagram of each body, i.e. of rod A and of wedge B and also draw the kinemetical diagram for accelerations, after analysing the directions of motion of A and B. Kinematical relationship of accelarations is :

$$\tan\alpha = \frac{w_A}{w_B} \tag{1}$$

Let us write Newton's second law for both bodies in terms of projections having taken positive directions of y and x axes as shown in the figure.

$$m_A g - N\cos\alpha = m_A w_A \tag{2}$$

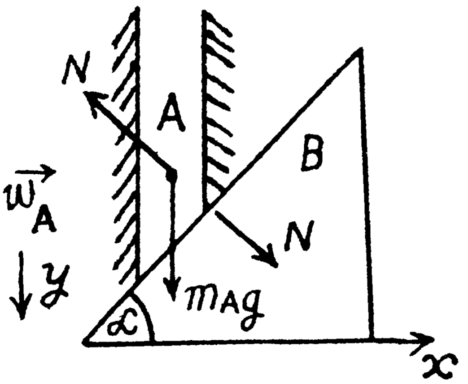
and

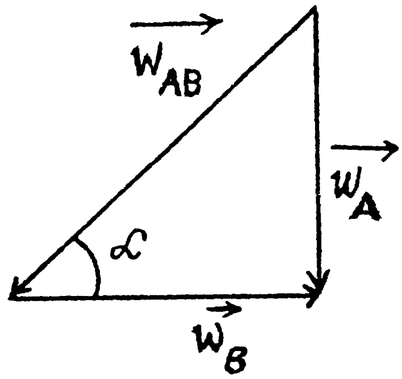
$$N\sin\alpha = m_B w_B \tag{3}$$

Simultaneous solution of (1), (2) and (3) yields :

$$w_A = \frac{m_A g\sin\alpha}{m_A\sin\alpha + m_B\cot\alpha\cos\alpha} = \frac{g}{(1 + \eta\cot^2\alpha)} \quad \text{and}$$

$$w_B = \frac{w_A}{\tan\alpha} = \frac{g}{(\tan\alpha + \eta\cot\alpha)}$$





Note : *We may also solve this problem using conservation of mechanical energy instead of Newton's second law.*

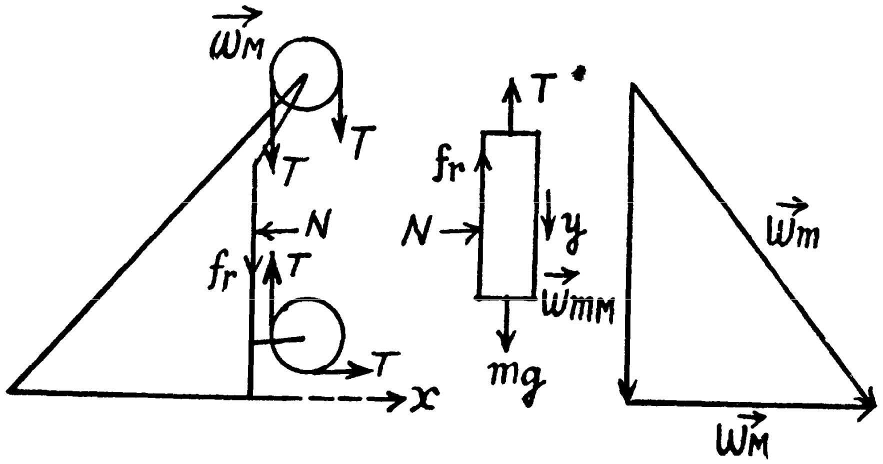
1.78 Let us draw free body diagram of each body and fix the coordinate system, as shown in the figure. After analysing the motion of M and m on the basis of force diagrams, let us draw the kinematical diagram for accelerations (Fig.).

As the length of threads are constant so,

$ds_{mM} = ds_M$ and as $\vec{v}_{mM}$ and $\vec{v}_M$ do not change their directions that why

$$\left|\vec{w}_{mM}\right| = \left|\vec{w}_M\right| = w \text{ (say) and}$$

$$\vec{w}_{mM} \uparrow\uparrow \vec{v}_M \quad \text{and} \quad \vec{w}_M \uparrow\uparrow \vec{v}_M$$



As $\vec{w}_m = \vec{w}_{mM} + \vec{w}_M$

so, from the triangle law of vector addition

$$w_m = \sqrt{2}\, w \qquad (1)$$

From the Eq. $F_x = m\, w_x$, for the wedge and block :

$$T - N = Mw, \qquad (2)$$

and

$$N = mw \qquad (3)$$

Now, from the Eq. $F_y = mw_y$, for the block

$$mg - T - kN = mw \qquad (4)$$

Simultaneous solution of Eqs. (2), (3) and (4) yields :

$$w = \frac{mg}{(km + 2m + M)} = \frac{g}{(k + 2 + M/m)}$$

Hence using Eq. (1)

$$w_m = \frac{g\sqrt{2}}{(2 + k + M/m)}$$

1.79 Bodies 1 and 2 will remain at rest with repect to bar A for $w_{min} \le w \le w_{max}$, where $w_{min}$is the sought minimum acceleration of the bar. Beyond these limits there will be a relative motion between bar and the bodies. For $0 \le w \le w_{min}$, the tendency of body 1 in relation to the bar A is to move towards right and is in the opposite sense for $w \ge w_{max}$. On the basis of above argument the static friction on 2 by A is directed upward and on 1 by A is directed towards left for the purpose of calculating w_{min}.

Let us write Newton's second law for bodies 1 and 2 in terms of projection along positive x – axis (Fig.).

$$T - fr_1 = m\, w \quad \text{or,} \quad fr_1 = T - mw \qquad (1)$$

$$N_2 = mw \qquad (2)$$

As body 2 has no acceleration in vertical direction, so

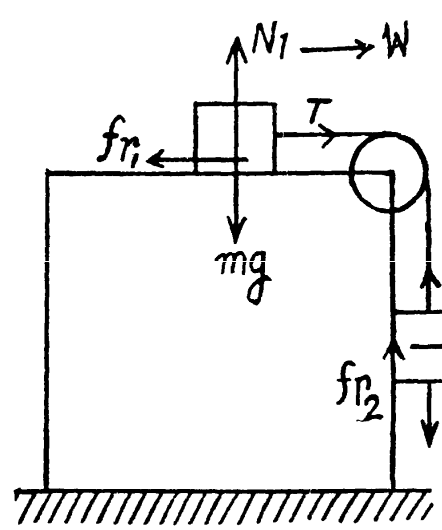
$$fr_2 = mg - T \qquad (3)$$

From (1) and (3)

$$(fr_1 + fr_2) = m\,(g - w) \qquad (4)$$

But $\quad fr_1 + fr_2 \le k\,(N_1 + N_2)$

or $\quad fr_1 + fr_2 \le k\,(mg + mw) \qquad (5)$



From (4) and (5)

$$m\,(g-w) \le mk\,(g+w)\,,\ \text{ or }\ w \ge \frac{g\,(1-k)}{(1+k)}$$

Hence $$w_{\min} = \frac{g\,(1-k)}{(1+k)}$$

1.80 On the basis of the initial argument of the solution of 1.79, the tendency of bar 2 with respect to 1 will be to move up along the plane.

Let us fix $(x-y)$ coordinate system in the frame of ground as shown in the figure.

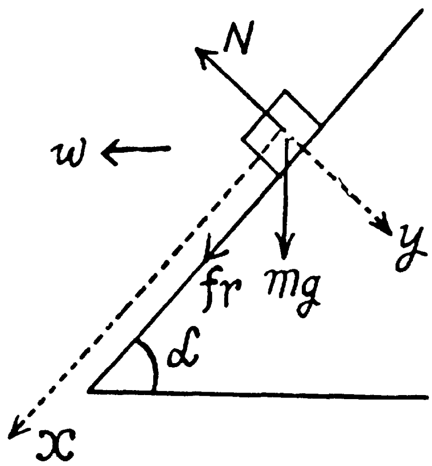
From second law of motion in projection form along y and x axes :

$m\,g\cos\alpha - N = m\,w\sin\alpha$

or, $\quad N = m\,(\,g\cos\alpha - w\sin\alpha\,)$ (1)

$m\,g\sin\alpha + fr = m\,w\cos\alpha$

or, $\quad fr = m\,(\,w\cos\alpha - g\sin\alpha\,)$ (2)



but $fr \le kN$, so from (1) and (2)

$(\,w\cos\alpha - g\sin\alpha\,) \le k\,(\,g\cos\alpha + w\sin\alpha\,)$

or, $w\,(\,\cos\alpha - k\sin\alpha\,) \le g\,(\,k\cos\alpha + \sin\alpha\,)$

or, $$w \le g\,\frac{(\,\cos\alpha + \sin\alpha\,)}{\cos\alpha - k\sin\alpha}\,,$$

So, the sought maximum acceleration of the wedge :

$$w_{\max} = \frac{(\,k\cos\alpha + \sin\alpha\,)\,g}{\cos\alpha - k\sin\alpha} = \frac{(\,k\cot\alpha + 1\,)\,g}{\cot\alpha - k}\quad \text{where } \cot\alpha > k$$

1.81 Let us draw the force diagram of each body, and on this basis we observe that the prism moves towards right say with an acceleration w_1 and the bar 2 of mass m_2 moves down the plane with respect to 1, say with acceleration $\vec{w}_{21}$, then, $\vec{w}_2 = \vec{w}_{21} + \vec{w}_1$ (Fig.)

Let us write Newton's second law for both bodies in projection form along positive y_2 and x_1 axes as shown in the Fig.

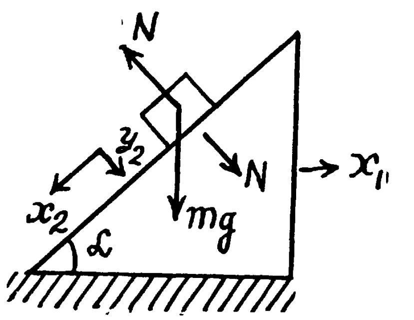
$$m_2\,g\cos\alpha - N = m_2\,w_{2(y_2)} = m_2\left[\,w_{21\,(y_2)} + w_{1\,(y_2)}\right] = m_2\left[\,0 + w_1\sin\alpha\,\right]$$

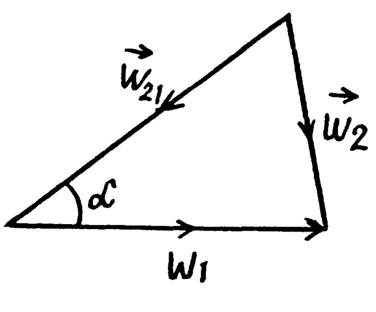
or, $$m_2\,g\cos\alpha - N = m_2\,w_1\sin\alpha \tag{1}$$

and $$N\sin\alpha = m_1 w_1 \tag{2}$$

Solving (1) and (2), we get

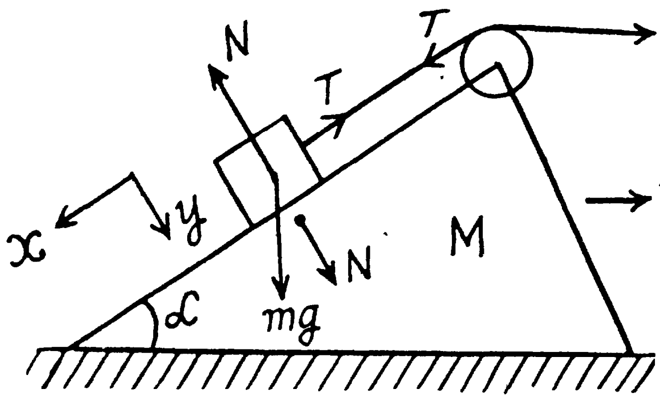
$$w_1 = \frac{m_2\,g\sin\alpha\cos\alpha}{m_1 + m_2\sin^2\alpha} = \frac{g\sin\alpha\cos\alpha}{(\,m_1/m_2\,) + \sin^2\alpha}$$

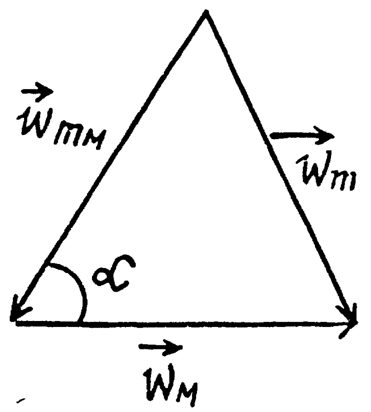




1.82 To analyse the kinematic relations between the bodies, sketch the force diagram of each body as shown in the figure.

On the basis of force diagram, it is obvious that the wedge M will move towards right and the block will move down along the wedge. As the length of the thread is constant, the distance travelled by the block on the wedge must be equal to the distance travelled by the wedge on the floor. Hence $d s_{mM} = d s_M$. As $\vec{v}_{mM}$ and $\vec{v}_M$ do not change their directions and acceleration that's why $\vec{w}_{mM} \uparrow\uparrow \vec{v}_{mM}$ and $\vec{w}_M \uparrow\uparrow \vec{v}_M$ and $w_{mM} = w_M = w$ (say) and accordingly the diagram of kinematical dependence is shown in figure.





As $\vec{w}_m = \vec{w}_{mM} + \vec{w}_M$, so from triangle law of vector addition.

$$w_m = \sqrt{w_M^2 + w_{mM}^2 - 2\, w_{mM}\, w_M \cos\alpha} = w\sqrt{2(1-\cos\alpha)} \tag{1}$$

From $F_x = m\, w_x$, (for the wedge),

$$T - T\cos\alpha + N\sin\alpha = M w \tag{2}$$

For the bar m let us fix $(x - y)$ coordinate system in the frame of ground Newton's law in projection form along x and y axes (Fig.) gives

$$mg\sin\alpha - T = m\, w_{m(x)} = m\left[w_{mM(x)} + w_{M(x)}\right]$$

$$= m\left[w_{mM} + w_M \cos(\pi - \alpha)\right] = m\, w\,(1 - \cos\alpha) \tag{3}$$

$$m\, g\cos\alpha - N = m\, w_{m(y)} = m\left[w_{mM(y)} + w_{M(y)}\right] = m\,[\,0 + w\sin\alpha\,] \tag{4}$$

Solving the above Eqs. simultaneously, we get

$$w = \frac{m\, g\sin\alpha}{M + 2m\,(1 - \cos\alpha)}$$

Note : *We can study the motion of the block m in the frame of wedge also, alternately we may solve this problem using conservation of mechanical energy.*

1.83 Let us sketch the diagram for the motion of the particle of mass m along the circle of radius R and indicate x and y axis, as shown in the figure.

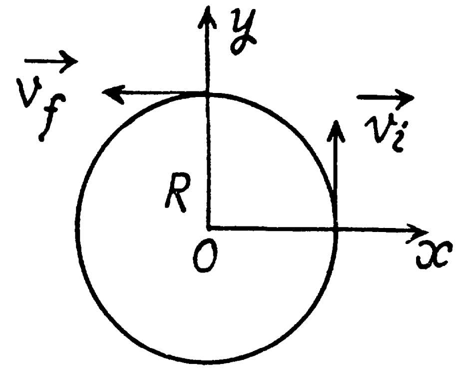
(a) For the particle, change in momentum $\Delta\vec{p} = mv(-\vec{i}) - m\, v(\vec{j})$

so, $\quad |\Delta\vec{p}| = \sqrt{2}\, m\, v$

and time taken in describing quarter of the circle,

$$\Delta t = \frac{\pi R}{2 v}$$

Hence, $< \vec{F} > = \frac{|\Delta \vec{p}|}{\Delta t} = \frac{\sqrt{2}\, m v}{\pi R / 2v} = \frac{2\sqrt{2}\, m v^2}{\pi R}$



(b) In this case

$\vec{p_i} = 0$ and $\vec{p_f} = m w_t t(-\vec{i})$,

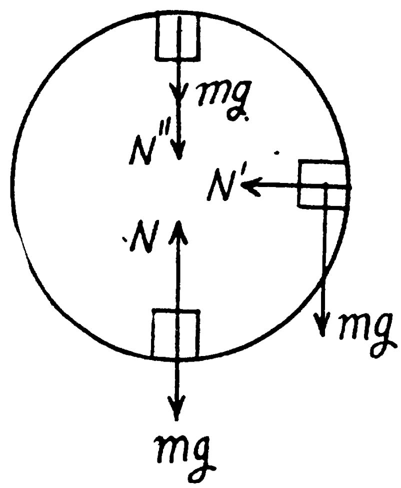
so $\quad |\Delta \vec{p}| = m w_t t$

Hence, $|< \vec{F} >| = \frac{|\Delta \vec{p}|}{t} = m w_t$

1.84 While moving in a loop, normal reaction exerted by the flyer on the loop at different points and uncompensated weight if any contribute to the weight of flyer at those points.

(a) When the aircraft is at the lowermost point, Newton's second law of motion in projection form $F_n = m w_n$ gives

$$N - mg = \frac{m v^2}{R}$$



or, $N = mg + \frac{m v^2}{R} = 2{\cdot}09$ kN

(b) When it is at the upper most point, again from $F_n = m w_n$ we get

$$N'' + mg = \frac{mv^2}{R}$$

$$N'' = \frac{m v^2}{R} - m g = 0{\cdot}7 \text{ kN}$$

(c) When the aircraft is at the middle point of the loop, again from $F_n = m w_n$

$$N' = \frac{m v^2}{R} = 1{\cdot}4 \text{ kN}$$

The uncompensated weight is mg. Thus effective weight $= \sqrt{N'^2 + m^2 g^2} = 1{\cdot}56$ kN acts obliquely.

1.85 Let us depict the forces acting on the small sphere m, (at an arbitrary position when the thread makes an angle θ from the vertical) and write equation $\vec{F} = m \vec{w}$ via projection on the unit vectors $\hat{u}_t$ and $\hat{u}_n$. From $F_t = m w_t$, we have

$$mg \sin \theta = m \frac{dv}{dt}$$

$$= m \frac{v\, dv}{ds} = m \frac{v\, dv}{l(-d\theta)}$$

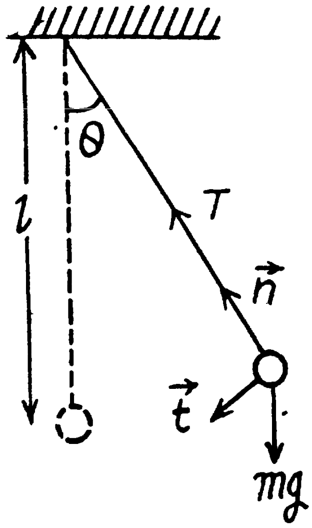
(as vertical is refrence line of angular position)

or $$v\,dv = -gl\sin\theta\,d\theta$$

Integrating both the sides :

$$\int_0^v v\,dv = -g\,l\int_{\pi/2}^{\theta}\sin\theta\,d\theta$$

or, $$\frac{v^2}{2} = g\,l\cos\theta$$



Hence $\frac{v^2}{l} = 2\,g\cos\theta = w_n$.(1)

(Eq. (1) can be easily obtained by the conservation of mechanical energy).

From $F_n = m\,w_n$

$$T - m\,g\cos\theta = \frac{m\,v^2}{l}.$$

Using (1) we have

$$T = 3\,mg\cos\theta \qquad (2)$$

Again from the Eq. $F_t = m\,w_t$:

$$mg\sin\theta = m\,w_t \text{ or } w_t = g\sin\theta \qquad (3)$$

Hence $w = \sqrt{w_t^2 + w_n^2} = \sqrt{(g\sin\theta)^2 + (2g\cos\theta)^2}$ (using 1 and 3)

$$= g\sqrt{1 + 3\cos^2\theta}$$

(b) Vertical component of velocity, $v_y = v\sin\theta$

So, $$v_y^2 = v^2\sin^2\theta = 2g\,l\cos\theta\sin^2\theta \text{ (using 1)}$$

For maximum v_y or v_y^2, $\frac{d(\cos\theta\sin^2\theta)}{d\theta} = 0$

which yields $$\cos\theta = \frac{1}{\sqrt{3}}$$

Therefore from (2) $T = 3mg\frac{1}{\sqrt{3}} = \sqrt{3}\,mg$

(c) We have $\vec{w} = w_t\,\hat{u}_t + w_n\,\hat{u}_n$ thus $w_y = w_{t(y)} + w_{n(y)}$

But in accordance with the problem $w_y = 0$

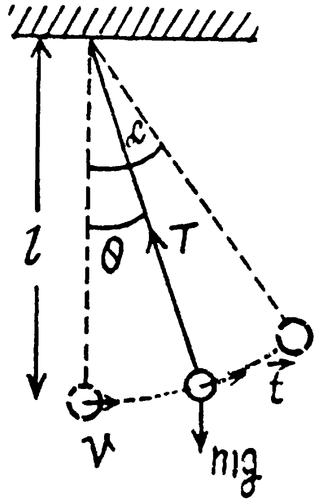
So, $$w_{t(y)} + w_{n(y)} = 0$$

or, $$g\sin\theta\sin\theta + 2g\cos^2\theta(-\cos\theta) = 0$$

or, $$\cos\theta = \frac{1}{\sqrt{3}} \text{ or, } \theta = 54{\cdot}7°$$

1.86 The ball has only normal acceleration at the lowest position and only tangential acceleration at any of the extreme position. Let v be the speed of the ball at its lowest position and l be the length of the thread, then according to the problem

$$\frac{v^2}{l} = g \sin \alpha \qquad (1)$$



where α is the maximum deflection angle

From Newton's law in projection form : $F_t = m w_t$

$$-mg \sin \theta = mv \frac{dv}{l \, d\theta}$$

or, $\quad -g\,l \sin \theta \, d\theta = v \, dv$

On integrating both the sides within their limits.

$$-gl \int_0^{\alpha} \sin \theta \, d\theta = \int_v^0 v \, dv$$

or, $\quad v^2 = 2gl\,(1 - \cos \alpha) \qquad (2)$

Note : Eq. (2) can easily be obtained by the conservation of mechanical energy of the ball in the uniform field of gravity.

From Eqs. (1) and (2) with $\theta = \alpha$

$$2gl\,(1 - \cos \alpha) = lg \cos \alpha$$

or, $\quad \cos \alpha = \frac{2}{3} \;$ so, $\; \alpha = 53°$

1.87 Let us depict the forces acting on the body A (which are the force of gravity $m\vec{g}$ and the normal reaction $\vec{N}$) and write equation $\vec{F} = m\vec{w}$ via projection on the unit vectors $\hat{u}_t$ and $\hat{u}_n$ (Fig.)

From $F_t = m w_t$

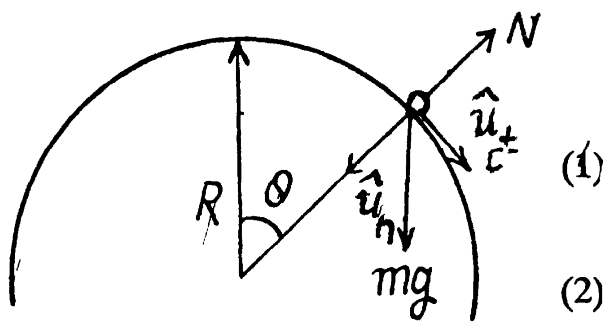
$$mg \sin \theta = m \frac{dv}{dt}$$

$$= m \frac{v \, dv}{ds} = m \frac{v \, dv}{R \, d\theta}$$

or, $\quad gR \sin \theta \, d\theta = v \, dv$

Integrating both side for obtaining $v\,(\theta)$

$$\int_0^{\theta} gR \sin \theta \, d\theta = \int_0^{v} v \, dv$$



or, $\quad v^2 = 2\,gR\,(1 - \cos \theta) \qquad (1)$

From $\; F_n = m w_n$

$$mg \cos \theta - N = m \frac{v^2}{R} \qquad (2)$$

At the moment the body loses contact with the surface, $N = 0$ and therefore the Eq. (2) becomes

$$v^2 = gR \cos \theta \qquad (3)$$

where v and θ correspond to the moment when the body loses contact with the surface. Solving Eqs. (1) and (3) we obtain $\cos\theta = \frac{2}{3}$ or, $\theta = \cos^{-1}(2/3)$ and $v = \sqrt{2gR/3}$.

1.88 At first draw the free body diagram of the device as, shown. The forces, acting on the sleeve are it's weight, acting vertically downward, spring force, along the length of the spring and normal reaction by the rod, perpendicular to its length.

Let F be the spring force, and Δl be the elongation.

From, $F_n = mw_n$:

$$N\sin\theta + F\cos\theta = m\,\omega^2 r \qquad (1)$$

where $r\cos\theta = (l_0 + \Delta l)$.

Similarly from $F_t = mw_t$

$$N\cos\theta - F\sin\theta = 0 \quad \text{or,} \quad N = F\sin\theta/\cos\theta \qquad (2)$$

From (1) and (2)

$$F(\sin\theta/\cos\theta)\cdot\sin\theta + F\cos\theta = m\,\omega^2 r$$

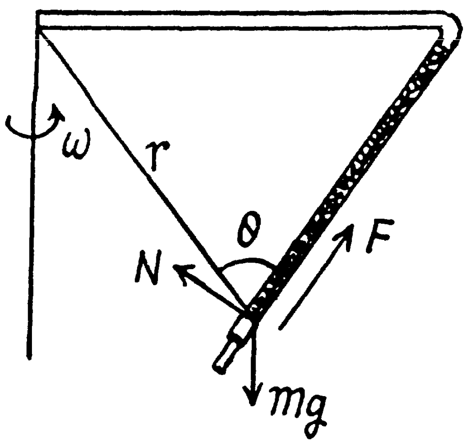
$$= m\,\omega^2(l_0 + \Delta l)/\cos\theta$$

On putting $F = \kappa\,\Delta l$,

$$\kappa\,\Delta l\sin^2\theta + \kappa\,\Delta l\cos^2\theta = m\,\omega^2(l_0 + \Delta l)$$

on solving, we get,

$$\Delta l = m\,\omega^2\frac{l_0}{\kappa - m\,\omega^2} = \frac{l_0}{(\kappa/m\,\omega^2 - 1)}$$



and it is independent of the direction of rotation.

1.89 According to the question, the cyclist moves along the circular path and the centripetal force is provided by the frictional force. Thus from the equation $F_n = m\,w_n$

$$fr = \frac{m v^2}{r} \quad \text{or} \quad kmg = \frac{m v^2}{r}$$

or
$$k_0\left(1 - \frac{r}{R}\right)g = \frac{v^2}{r} \quad \text{or} \quad v^2 = k_0(r - r^2/R)\,g \qquad (1)$$

For v_{max}, we should have $\dfrac{d\left(r - \dfrac{r^2}{R}\right)}{dr} = 0$

or,
$$1 - \frac{2r}{R} = 0, \quad \text{so } r = R/2$$

Hence, $v_{max} = \frac{1}{2}\sqrt{k_0 gR}$

1.90 As initial velocity is zero thus

$$v^2 = 2\,w_t\,s \qquad (1)$$

As $w_t > 0$ the speed of the car increases with time or distance. Till the moment, sliding starts, the static friction provides the required centripetal acceleration to the car.

Thus
$$fr = mw, \quad \text{but} \quad fr \leq kmg$$

So, $$w^2 \le k^2 g^2 \text{ or, } w_t^2 + \frac{v^2}{R} \le k^2 g^2$$

or, $$v^2 \le (k^2 g^2 - w_t^2) R$$

Hence $$v_{max} = \sqrt{(k^2 g^2 - w_t^2) R}$$

so, from Eqn. (1), the sought distance $s = \dfrac{v_{max}^2}{2 w_t} = \dfrac{1}{2} \sqrt{\left(\dfrac{kg}{w_t}\right)^2 - 1} = 60$ m.

1.91 Since the car follows a curve, so the maximum velocity at which it can ride without sliding at the point of minimum radius of curvature is the sought velocity and obviously in this case the static friction between the car and the road is limiting.

Hence from the equation $F_n = mw$

$$kmg \ge \frac{m v^2}{R} \quad \text{or} \quad v \le \sqrt{kRg}$$

so $$v_{max} = \sqrt{k R_{min} g}. \tag{1}$$

We know that, radius of curvature for a curve at any point (x, y) is given as,

$$R = \left| \frac{[1 + (dy/dx)^2]^{3/2}}{(d^2 y) / dx^2} \right| \tag{2}$$

For the given curve,

$$\frac{dy}{dx} = \frac{a}{\alpha} \cos\left(\frac{x}{\alpha}\right) \text{ and } \frac{d^2 y}{dx^2} = \frac{-a}{\alpha^2} \sin \frac{x}{\alpha}$$

Substituting this value in (2) we get,

$$R = \frac{[1 + (a^2/\alpha^2) \cos^2 (x/\alpha)]^{3/2}}{(a/\alpha^2) \sin (x/\alpha)}$$

For the minimum R, $\dfrac{x}{\alpha} = \dfrac{\pi}{2}$

and therefore, corresponding radius of curvature

$$R_{min} = \frac{\alpha^2}{a} \tag{3}$$

Hence from (1) and (2)

$$v_{max} = \alpha \sqrt{kg/a}$$

1.92 The sought tensile stress acts on each element of the chain. Hence divide the chain into small, similar elements so that each element may be assumed as a particle. We consider one such element of mass dm, which subtends angle $d\alpha$ at the centre. The chain moves along a circle of known radius R with a known angular speed ω and certain forces act on it. We have to find one of these forces.

From Newton's second law in projection form, $F_x = mw_x$ we get

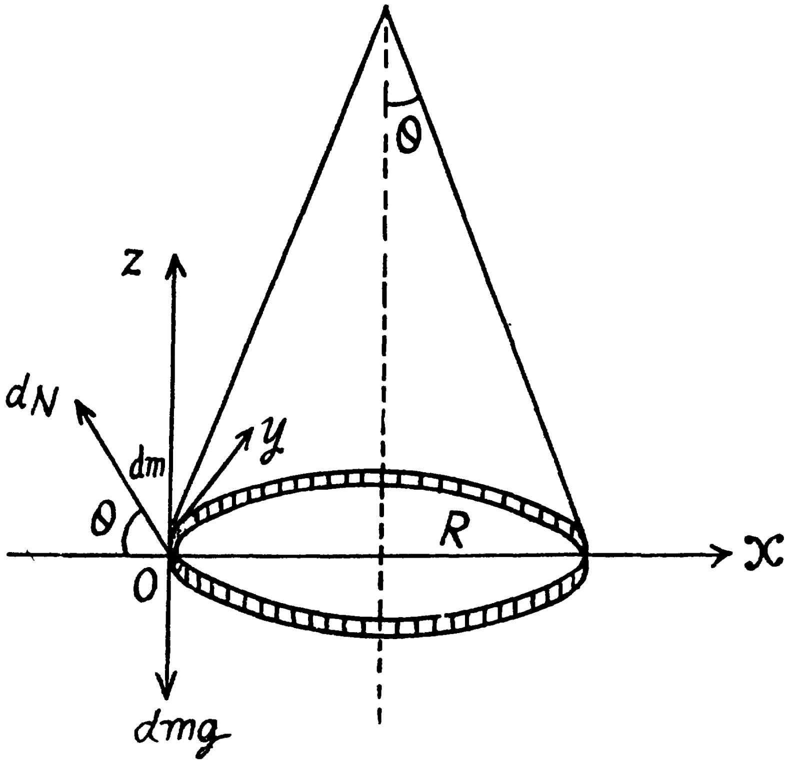
$$2T \sin (d\alpha/2) - dN \cos \theta = dm\, \omega^2 R$$

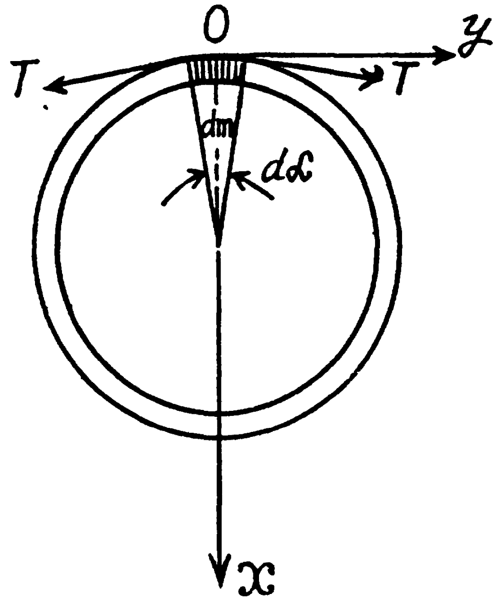
and from $F_Z = mw_Z$ we get

$$dN \sin\theta = g dm$$

Then putting $dm = m d\alpha/2\pi$ and $\sin (d\alpha/2) = d\alpha/2$ and solving, we get,

$$T = \frac{m (\omega^2 R + g \cot \theta)}{2\pi}$$





1.93 Let, us consider a small element of the thread and draw free body diagram for this element.

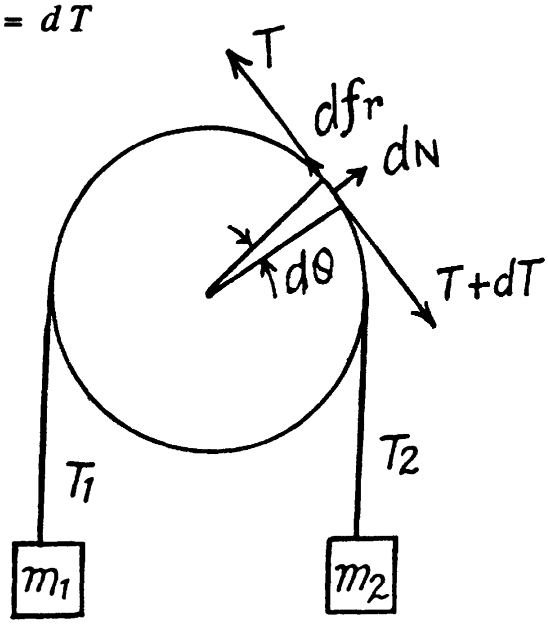
(a) Applying Newton's second law of motion in projection form, $F_n = mw_n$ for this element,

$$(T + dT)\sin(d\theta/2) + T\sin(d\theta/2) - dN = dm\,\omega^2 R = 0$$

or, $$2T\sin(d\theta/2) = dN, \text{ [negelecting the term}(dT\sin d\theta/2)\text{]}$$

or, $$T\,d\theta = dN, \text{ as } \sin\frac{d\theta}{2} = \frac{d\theta}{2} \tag{1}$$

Also, $$d\,fr = k\,dN = (T + dT) - T = dT \tag{2}$$



From Eqs. (1) and (2),

$$k\,T\,d\theta = dT \text{ or } \frac{dT}{T} = k\,d\theta$$

In this case $Q = \pi$ so,

or, or, $$\ln\frac{T_2}{T_1} = k\pi \tag{3}$$

So, $$k = \frac{1}{\pi}\ln\frac{T_2}{T_1} = \frac{1}{\pi}\ln\eta_0$$

as $$\frac{T_2}{T_1} = \frac{m_2 g}{m_1 g} = \frac{m_2}{m_1} = \eta_0$$

(b) When $\frac{m_2}{m_1} = \eta$, which is greater than η_0, the blocks will move with same value of acceleration. (say w) and clearly m_2 moves downward. From Newton's second law in projection form (downward for m_2 and upward for m_1) we get :

$$m_2 g - T_2 = m_2 w \tag{4}$$

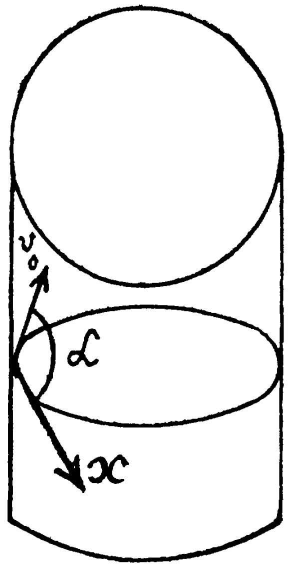
and $$T_1 - m_1 g = m_1 w \tag{5}$$

Also $$\frac{T_2}{T_1} = \eta_0 \qquad (6)$$

Simultaneous solution of Eqs. (4), (5) and (6) yields :

$$w = \frac{(m_2 - \eta_0 m_1)\, g}{(m_2 + \eta_0 m_1)} = \frac{(\eta - \eta_0)}{(\eta + \eta_0)}\, g \left(\text{as } \frac{m_2}{m_1} = \eta\right)$$

1.94 The force with which the cylinder wall acts on the particle will provide centripetal force necessary for the motion of the particle, and since there is no acceleration acting in the horizontal direction, horizontal component of the velocity will remain constant througout the motion.



So $$v_x = v_0 \cos\alpha$$

Using, $F_n = m\, w_n$, for the particle of mass m,

$$N = \frac{m\, v_x^2}{R} = \frac{m\, v_0^2 \cos^2\alpha}{R},$$

which is the required normal force.

1.95 Obviously the radius vector describing the position of the particle relative to the origin of coordinate is

$$\vec{r} = x\vec{i} + y\vec{j} = a \sin\omega t\, \vec{i} + b \cos\omega t\, \vec{j}$$

Differentiating twice with respect the time :

$$\vec{w} = \frac{d^2\vec{r}}{dt^2} = -\omega^2 (a \sin\omega t\, \vec{i} + b \cos\omega t\, \vec{j}) = -\omega^2 \vec{r} \qquad (1)$$

Thus $$\vec{F} = m\vec{w} = -m\omega^2 \vec{r}$$

1.96 (a) We have $$\Delta\vec{p} = \int \vec{F}\, dt$$

$$= \int_0^t m\vec{g}\, dt = m\vec{g}\, t \qquad (1)$$

(b) Using the solution of problem 1.28 (b), the total time of motion, $\tau = -\frac{2(\vec{v}_0 \cdot \vec{g})}{g^2}$

Hence using $$t = \tau \text{ in } (1)$$

$$|\Delta\vec{p}| = mg\tau$$

$$= -2m(\vec{v}_0 \cdot \vec{g})/g \quad (\vec{v}_0 \cdot \vec{g} \text{ is } -\text{ve})$$

1.97 From the equation of the given time dependence force $\vec{F} = \vec{a}\, t(\tau - t)$ at $t = \tau$, the force vanishes,

(a) Thus $$\Delta\vec{p} = \vec{p} = \int_0^\tau \vec{F}\, dt$$

or, $$\vec{p} = \int_0^{\tau} \vec{a}\, t\,(\tau - t)\, dt \; \frac{\vec{a}\,\tau^3}{6}$$

but $$\vec{p} = m\,\vec{v} \text{ so } \vec{v} = \frac{\vec{a}\,\tau^3}{6m}$$

(b) Again from the equation $\vec{F} = m\,\vec{w}$

$$\vec{a}t\,(\tau - t) = m\,\frac{d\,\vec{v}}{dt}$$

or, $$\vec{a}(t\tau - t^2)\, dt = md\,\vec{v}$$

Integrating within the limits for $\vec{v}(t)$,

$$\int_0^{t} \vec{a}(t\tau - t^2)\, dt = m\int_0^{\vec{v}} d\,\vec{v}$$

or, $$\vec{v} = \frac{\vec{a}}{m}\left(\frac{\tau\, t^2}{2} - \frac{t^3}{3}\right) = \frac{\vec{a}\,t^2}{m}\left(\frac{\tau}{2} - \frac{t}{3}\right)$$

Thus $$v = \frac{a\,t^2}{m}\left(\frac{\tau}{2} - \frac{t}{3}\right) \text{ for } t \le \tau$$

Hence distance covered during the time interval $t = \tau$,

$$s = \int_0^{\tau} v\, dt$$

$$= \int_0^{\tau} \frac{a\,t^2}{m}\left(\frac{\tau}{2} - \frac{t}{3}\right) dt = \frac{a}{m}\,\frac{\tau^4}{12}$$

1.98 We have $F = F_0 \sin \omega t$

or $$m\,\frac{d\,\vec{v}}{dt} = \vec{F}_0 \sin \omega t \text{ or } m\,d\,\vec{v} = \vec{F}_0 \sin \omega t\, dt$$

On integrating,

$$m\vec{v} = \frac{-\vec{F}_0}{\omega} \cos \omega t + C, \text{ (where } C \text{ is integration constant)}$$

When $$t = 0,\; v = 0, \text{ so } C = \frac{\vec{F}_0}{m\omega}$$

Hence, $\vec{v} = \dfrac{-\vec{F}_0}{m\,\omega} \cos \omega t + \dfrac{\vec{F}_0}{m\,\omega}$

As $|\cos \omega t \le 1$ so, $v = \dfrac{F_0}{m\,\omega}(1 - \cos \omega t)$

Thus $$s = \int_0^t v\,dt$$

$$= \frac{F_0 t}{m\omega} - \frac{F_0 \sin \omega t}{m\omega^2} = \frac{F_0}{m\omega^2}(\omega t - \sin \omega t)$$

(Figure in the answer sheet).

1.99 According to the problem, the force acting on the particle of mass m is, $\vec{F} = \vec{F}_0 \cos \omega t$

So, $$m\frac{d\vec{v}}{dt} = \vec{F}_0 \cos \omega t \quad \text{or} \quad d\vec{v} = \frac{\vec{F}_0}{m} \cos \omega t\, dt$$

Integrating, within the limits.

$$\int_0^{\vec{v}} d\vec{v} = \frac{\vec{F}_0}{m} \int_0^t \cos \omega t\, dt \quad \text{or} \quad \vec{v} = \frac{\vec{F}_0}{m\omega} \sin \omega t$$

It is clear from equation (1), that after starting at $t = 0$, the particle comes to rest fro the first time at $t = \frac{\pi}{\omega}$.

From Eq. (1), $v = |\vec{v}| = \frac{F_0}{m\omega} \sin \omega t$ for $t \leq \frac{\pi}{\omega}$ (2)

Thus during the time interval $t = \pi/\omega$, the sought distance

$$s = \frac{F_0}{mw} \int_0^{\pi/\omega} \sin \omega t\, dt = \frac{2F}{m\omega^2}$$

From Eq. (1)

$$v_{max} = \frac{F_0}{m\omega} \text{ as } |\sin \omega t| \leq 1$$

1.100 (a) From the problem $\vec{F} = -r\vec{v}$ so $m\frac{d\vec{v}}{dt} = -r\vec{v}$

Thus $$m\frac{dv}{dt} = -rv \; [\text{as } d\vec{v} \uparrow\downarrow \vec{v}]$$

or, $$\frac{dv}{v} = -\frac{r}{m}dt$$

On integrating $$\ln v = -\frac{r}{m}t + C$$

But at $t = 0$, $v = v_0$, so, $C = \ln v_0$

or, $$\ln \frac{v}{v_0} = -\frac{r}{m}t \quad \text{or,} \quad v = v_0 e^{-\frac{r}{m}t}$$

Thus for $t \to \infty$, $v = 0$

(b) We have $m\frac{dv}{dt} = -rv$ so $dv = \frac{-r}{m}ds$

Integrating within the given limits to obtain $v(s)$:

or, $$\int_{v_0}^{v} dv = -\frac{r}{m}\int_0^s ds \text{ or } v = v_0 - \frac{rs}{m} \tag{1}$$

Thus for $$v = 0,\ s = s_{total} = \frac{m v_0}{r}$$

(c) Let we have $$\frac{m\,dv}{v} = -rv \text{ or } \frac{dv}{v} = \frac{-r}{m}dt$$

or, $$\int_0^{v_0/\eta} \frac{dv}{v} = \frac{-r}{m}\int_0^t dt, \text{ or, } \ln\frac{v_0}{\eta v_0} = -\frac{r}{m}t$$

So $$t = \frac{-m \ln(1/\eta)}{r} = \frac{m \ln \eta}{r}$$

Now, average velocity over this time interval,

$$\langle v \rangle = \frac{\int v\,dt}{\int dt} = \frac{\int_0^{\frac{m}{r}\ln\eta} v_0 e^{-\frac{rt}{m}}\,dt}{\frac{m}{r}\ln\eta} = \frac{v_0(\eta - 1)}{\eta \ln \eta}$$

1.101 According to the problem

$$m\frac{dv}{dt} = -kv^2 \text{ or, } m\frac{dv}{v^2} = -k\,dt$$

Integrating, withing the limits,

$$\int_{v_0}^{v} \frac{dv}{v^2} = -\frac{k}{m}\int_0^t dt \text{ or, } t = \frac{m}{k}\frac{(v_0 - v)}{v_0 v} \tag{1}$$

To finc the valuè of k, rewrite

$$mv\frac{dv}{ds} = -kv^2 \text{ or, } \frac{dv}{v} = -\frac{k}{m}ds$$

On integrating

$$\int_{v_0}^{v} \frac{dv}{v} = -\frac{k}{m}\int_0^h ds$$

So, $$k = \frac{m}{h}\ln\frac{v_0}{v} \tag{2}$$

Putting the value of k from (2) in (1), we get

$$t = \frac{h(v_0 - v)}{v_0 v \ln\frac{v_0}{v}}$$

1.102 From Newton's second law for the bar in projection from, $F_x = m\,w_x$ along x direction we get

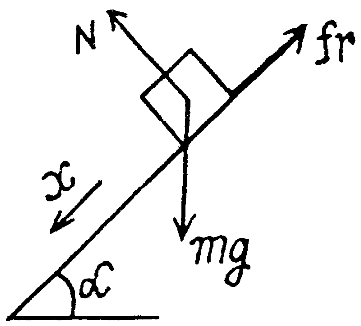
$$mg \sin\alpha - kmg\cos\alpha = mw$$

or, $$v\frac{dv}{dx} = g\sin\alpha - ax\,g\cos\alpha, \text{ (as } k = ax),$$

or, $$v\,dv = (g\sin\alpha - ax\,g\cos\alpha)\,dx$$

or, $$\int_0^v v\,dv = g\int_0^x (\sin\alpha - x\cos\alpha)\,dx$$

So, $$\frac{v^2}{2} = g\left(\sin\alpha\, x - \frac{x^2}{2} a\cos\alpha\right) \qquad (1)$$



From (1) $v = 0$ at either

$$x = 0, \text{ or } x = \frac{2}{a}\tan\alpha$$

As the motion of the bar is unidirectional it stops after going through a distance of $\frac{2}{a}\tan\alpha$.

From (1), for v_{max},

$$\frac{d}{dx}\left(\sin\alpha x - \frac{x^2}{2} a\cos\alpha\right) = 0, \text{ which yields } x = \frac{1}{a}\tan\alpha$$

Hence, the maximum velocity will be at the distance, $x = \tan\alpha/a$

Putting this value of x in (1) the maximum velocity,

$$v_{max} = \sqrt{\frac{g\sin\alpha\tan\alpha}{a}}$$

1.103 Since, the applied force is proportional to the time and the frictional force also exists, the motion does not start just after applying the force. The body starts its motion when F equals the limiting friction.

Let the motion start after time t_0, then

$$F = at_0 = kmg \text{ or, } t_0 = \frac{kmg}{a}$$

So, for $t = \leq t_0$, the body remains at rest and for $t > t_0$ obviously

$$\frac{mdv}{dt} = a(t - t_0) \quad \text{or,} \quad m\,dv = a(t - t_0)\,dt$$

Integrating, and noting $v = 0$ at $t = t_0$, we have for $t > t_0$

$$\int_0^v m\,dv = a\int_{t_0}^t (t - t_0)\,dt \quad \text{or} \quad v = \frac{a}{2m}(t - t_0)^2$$

Thus $$s = \int v\,dt = \frac{a}{2m}\int_{t_0}^t (t - t_0)^2\,dt = \frac{a}{6m}(t - t_0)^3$$

1.104 While going upward, from Newton's second law in vertical direction :

$$m\frac{v\,dv}{ds} = -(mg + kv^2) \quad \text{or} \quad \frac{v\,dv}{\left(g + \frac{kv^2}{m}\right)} = -ds$$

At the maximum height h, the speed $v = 0$, so

$$\int_{v_0}^{0} \frac{v\,dv}{g + (kv^2/m)} = -\int_{0}^{h} ds$$

Integrating and solving, we get,

$$h = \frac{m}{2k}\ln\left(1 + \frac{kv_o^2}{mg}\right) \tag{1}$$

When the body falls downward, the net force acting on the body in downward direction equals $(mg - kv^2)$,

Hence net acceleration, in downward direction, according to second law of motion

$$\frac{v\,dv}{ds} = g - \frac{kv^2}{m} \quad \text{or,} \quad \frac{v\,dv}{g - \frac{kv^2}{m}} = ds$$

Thus
$$\int_{0}^{v'} \frac{v\,dv}{g - kv^2/m} = \int_{0}^{h} ds$$

Integrating and putting the value of h from (1), we get,

$$v' = v_0 / \sqrt{1 + k v_0^2/mg}.$$

1.105 Let us fix $x - y$ co-ordinate system to the given plane, taking x-axis in the direction along which the force vector was oriented at the moment $t = 0$, then the fundamental equation of dynamics expressed via the projection on x and y-axes gives,

$$F\cos\omega t = m\frac{dv_x}{dt} \tag{1}$$

and
$$F\sin\omega t = m\frac{dv_y}{dt} \tag{2}$$

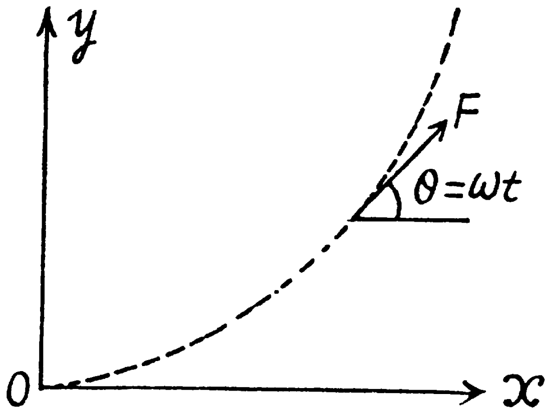
(a) Using the condition $v(0) = 0$, we obtain $v_x = \frac{F}{m\omega}\sin\omega t$ (3)

and

$$v_y = \frac{F}{m\omega}(1 - \cos\omega t) \tag{4}$$

Hence,
$$v = \sqrt{v_x^2 + v_y^2} = \left(\frac{2F}{m\omega}\right)\left|\sin\left(\frac{\omega t}{2}\right)\right|$$

(b) It is seen from this that the velocity v turns into zero after the time interval Δt, which can be found from the relation, $\omega \frac{\Delta t}{2} = \pi$. Consequentely, the sought distance, is



$$s = \int_0^{\Delta t} v\, dt = \frac{8F}{m\omega^2}$$

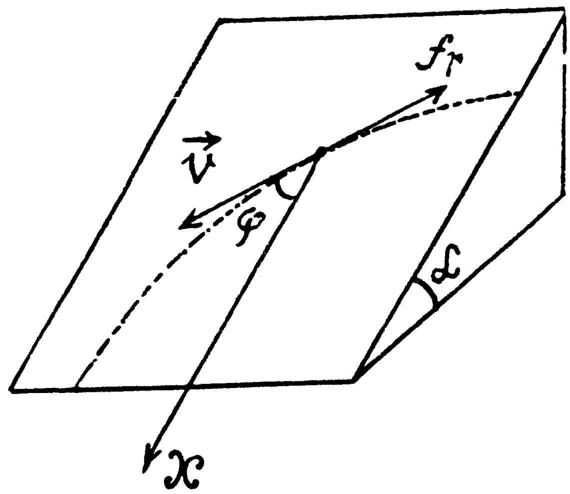
Average velocity, $\langle v \rangle = \frac{\int v\, dt}{\int dt}$

So, $\langle v \rangle = \int_0^{2\pi/\omega} \frac{2F}{m\omega} \sin\left(\frac{\omega t}{2}\right) dt \,/\, (2\pi\omega) = \frac{4F}{\pi m \omega}$

1.106 The acceleration of the disc along the plane is determined by the projection of the force of gravity on this plane $F_x = mg \sin\alpha$ and the friction force $fr = kmg\cos\alpha$. In our case $k = \tan\alpha$ and therefore

$$fr = F_x = mg \sin\alpha$$

Let us find the projection of the acceleration on the derection of the tangent to the trajectory and on the *x-axis* :



$m w_t = F_x \cos\varphi - fr = mg\sin\alpha\,(\cos\varphi - 1)$

$m w_x = F_x - fr\cos\varphi = mg\sin\alpha\,(1 - \cos\varphi)$

It is seen fromthis that $w_t = -w_x$, which means that the velocity v and its projection v_x differ only by a constant value C which does not change with time, i.e.

$$v = v_x + C,$$

where $v_x = v\cos\varphi$. The constant C is found from the initial condition $v = v_0$, whence $C = v_0$ since $\varphi = \frac{\pi}{2}$ initially. Finally we obtain

$$v = v_0 / (1 + \cos\varphi).$$

In the cource of time $\varphi \to 0$ and $v \to v_0/2$. (Motion then is unaccelerated.)

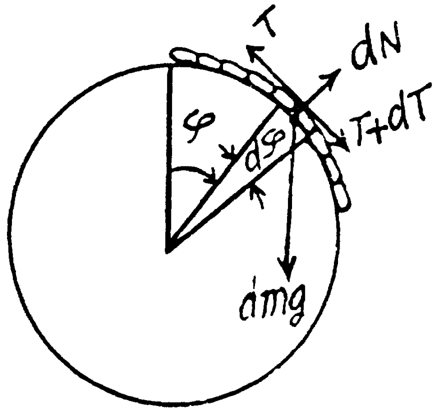
1.107 Let us consider an element of length ds at an angle φ from the vertical diameter. As the speed of this element is zero at initial instant of time, it's centripetal acceleration is zero, and hence, $dN - \lambda\, ds \cos\varphi = 0$, where λ is the linear mass density of the chain Let T and $T + dT$ be the tension at the upper and the lower ends of ds. we have from, $F_t = m w_t$

$$(T + dT) + \lambda\, ds\, g \sin\varphi - T = \lambda ds\, w_t$$

or, $$dT + \lambda R d\varphi\, g \sin\varphi = \lambda ds\, w_t$$

If we sum the above equation for all elements, the term $\int dT = 0$ because there is no tension at the free ends, so

$$\lambda gR \int_0^{l/R} \sin\varphi \, d\varphi = \lambda w_t \int ds = \lambda l w_t$$



Hence $\quad w_t = \frac{gR}{l}\left(1 - \cos\frac{l}{R}\right)$

As $w_n = a$ at initial moment

So, $w = |w_t| = \frac{gR}{l}\left(1 - \cos\frac{l}{R}\right)$

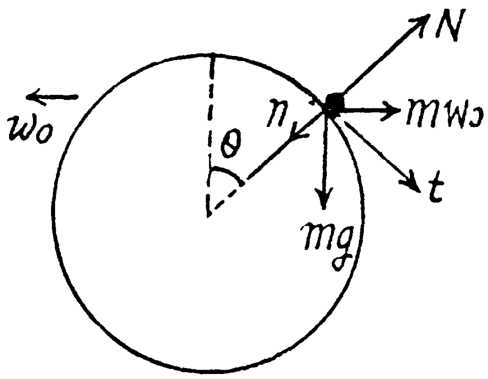
1.108 In the problem, we require the velocity of the body, realtive to the sphere, which itself moves with an acceleration w_0 in horizontal direction (say towards left). Hence it is advisible to solve the problem in the frame of sphere (non-inertial frame).

At an arbitary moment, when the body is at an angle θ with the vertical, we sketch the force diagram for the body and write the second law of motion in projection form $F_n = mw_n$

or, $mg\cos\theta - N - mw_0\sin\theta = \frac{mv^2}{R} \qquad (1)$

At the break off point, $N = 0$, $\theta = \theta_0$ and let $v = v_0$, so the Eq. (1) becomes,

$$\frac{v_0^2}{R} = g\cos\theta_0 - w_0\sin\theta_0 \qquad (2)$$



From, $F_t = mw_t$

$$mg\sin\theta - mw_0\cos\theta = m\frac{v\,dv}{ds} = m\frac{v\,dv}{R\,d\theta}$$

or, $v\,dv = R\,(g\sin\theta + w_0\cos\theta)\,d\theta$

Integrating, $\int_0^{v_0} v\,dv = \int_0^{\theta_0} R\,(g\sin\theta + w_0\cos\theta)\,d\theta$

$$\frac{v_0^2}{2R} = g(1 - \cos\theta_0) + w_0\sin\theta_0 \qquad (3)$$

Note that the Eq. (3) can also be obtained by the work-energy theorem $A = \Delta T$ (in the frame of sphere)

therefore, $\qquad mgR\,(1 - \cos\theta_0) + mw_0R\sin\theta_0 = \frac{1}{2}mv_0^2$

[here $mw_0R\sin\theta_0$ is the work done by the pseudoforce $(-m\vec{w}_0)$]

or, $\qquad \frac{v_0^2}{2R} = g\,(1 - \cos\theta_0) + w_0\sin\theta_0$

Solving Eqs. (2) and (3) we get,

$$v_0 = \sqrt{2gR/3} \text{ and } \theta_0 = \cos^{-1}\left[\frac{2+k\sqrt{5+9k^2}}{3(1+k^2)}\right], \text{ where } k = \frac{w_0}{g}$$

Hence $$\theta_0 \Big|_{w_0 = g} = 17°$$

1.109 This is not central force problem unless the path is a circle about the said point. Rather here F_t (tangential force) vanishes. Thus equation of motion becomes,

$$v_t = v_0 = \text{constant}$$

and, $$\frac{mv_0^2}{r} = \frac{A}{r^2} \text{ for } r = r_0$$

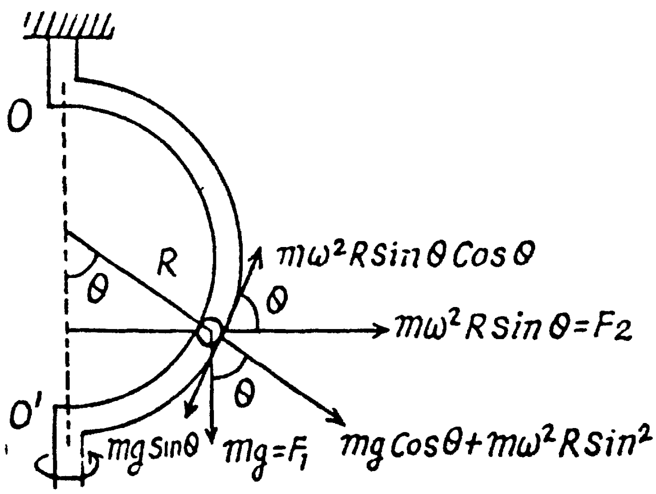
We can consider the latter equation as the equilibrium under two forces. When the motion is perturbed, we write $r = r_0 + x$ and the net force acting on the particle is,

$$-\frac{A}{(r_0+x)^n} + \frac{mv_0^2}{r_0+x} = \frac{-A}{r_0^n}\left(1-\frac{nx}{r_0}\right) + \frac{m\,v_0^2}{r_0}\left(1-\frac{x}{r_0}\right) = -\frac{m\,v_0^2}{r_0^2}(1-n)x$$

This is opposite to the displacement x, if $n < 1 \cdot \left(\frac{mv_0^2}{r}\right.$ is an outward directed centrifugul force while $\frac{-A}{r^n}$ is the inward directed external force).

1.110 There are two forces on the sleeve, the weight F_1 and the centrifugal force F_2. We resolve both forces into tangential and normal component then the net downward tangential force on the sleeve is,

$$mg\sin\theta\left(1-\frac{\omega^2 R}{g}\cos\theta\right)$$



This vanishes for $\theta = 0$ and for $\theta = \theta_0 = \cos^{-1}\left(\frac{g}{\omega^2 R}\right)$, which is real if $\omega^2 R > g$. If $\omega^2 R < g$, then $1 - \frac{\omega^2 R}{g}\cos\theta$ is always positive for small values of θ and hence the net tangential force near $\theta = 0$ opposes any displacement away from it. $\theta = 0$ is then stable.

If $\omega^2 R > g$, $1 - \frac{\omega^2 R}{g}\cos\theta$ is negative for small θ near $\theta = 0$ and $\theta = 0$ is then unstable.

However $\theta = \theta_0$ is stable because the force tends to bring the sleeve near the equilibrium position $\theta = \theta_0$.

If $\omega^2 R = g$, the two positions coincide and becomes a stable equilibrium point.

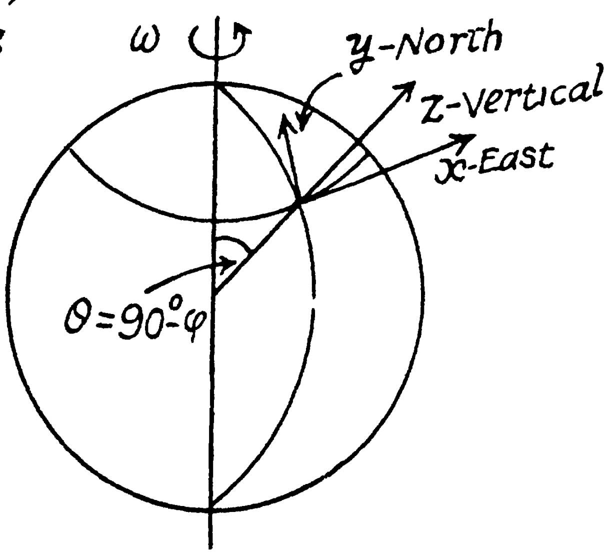
1.111 Define the axes as shown with z along the local vertical, x due east and y due north. (We assume we are in the northern hemisphere). Then the Coriolis force has the components.

$$\vec{F}_{cor} = -2m(\vec{\omega} \times \vec{v})$$

$= 2m\omega\left[v_y \cos\theta - v_z \sin\theta)\vec{i} - v_x \cos\theta \vec{j} + v_x \cos\theta \vec{k}\right] = 2m\omega(v_y \cos\theta - v_z \sin\theta)\vec{i}$

since v_x is small when the direction in which the gun is fired is due north. Thus the equation of motion (neglecting centrifugal forces) are

$\ddot{x} = 2m\omega(v_y \sin\varphi - v_z \cos\varphi), \ddot{y} = 0$ and $\ddot{z} = -g$



Integrating we get $\dot{y} = v$ (constant), $\dot{z} = -gt$

and $\dot{x} = 2\omega v \sin\varphi\, t + \omega g t^2 \cos\varphi$

Finally,

$$x = \omega v t^2 \sin\varphi + \frac{1}{3}\omega g t^3 \cos\varphi$$

Now $v >> gt$ in the present case. so,

$$x \approx \omega v \sin\varphi\left(\frac{s}{v}\right)^2 = \omega \sin\varphi \frac{s^2}{v}$$

$$\approx 7 \text{ cm (to the east).}$$

1.112 The disc exerts three forces which are mutually perpendicular. They are the reaction of the weight, mg, vertically upward, the Coriolis force $2mv'\omega$ perpendicular to the plane of the vertical and along the diameter, and $m\omega^2 r$ outward along the diameter. The resultant force is,

$$F = m\sqrt{g^2 + \omega^4 r^2 + (2v'\omega)^2}$$

1.113 The sleeve is free to slide along the rod AB. Thus only the centrifugal force acts on it. The equation is,

$$m\dot{v} = m\omega^2 r \quad \text{where } v = \frac{dr}{dt}.$$

$$\text{But } \dot{v} = v\frac{dv}{dr} = \frac{d}{dr}\left(\frac{1}{2}v^2\right)$$

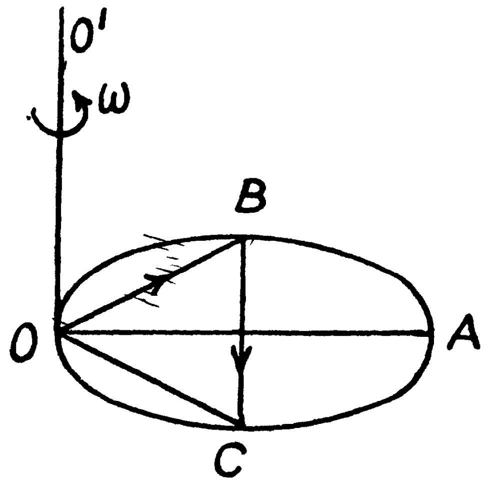
so,
$$\frac{1}{2}v^2 = \frac{1}{2}\omega^2 r^2 + \text{constant}$$

or,
$$v^2 = v_0^2 + \omega^2 r^2$$

v_0 being the initial velocity when $r = 0$. The Coriolis force is then,

$$2m\omega\sqrt{v_0^2 + \omega^2 r^2} = 2m\omega^2 r\sqrt{1 + v_0^2/\omega^2 r^2}$$

$$= 2{\cdot}83 \text{ N on putting the values.}$$



1.114 The disc $OBAC$ is rotating with angular velocity ω about the axis OO' passing through the edge point O. The equation of motion in rotating frame is,

$$m\vec{w}' = \vec{F} + m\omega^2 \vec{R} + 2m\vec{v}' \times \vec{\omega} = \vec{F} + \vec{F}_{in}$$

where $\vec{F}_{in}$ is the resultant inertial force (pseudo force) which is the vector sum of centrifugal and Coriolis forces.

(a) At A, F_{in} vanishes. Thus $0 = -2m\omega^2 R\,\hat{n} + 2mv'\,\omega\,\hat{n}$

where $\hat{n}$ is the inward drawn unit vector to the centre from the point in question, here A.

Thus, $$v' = \omega R$$

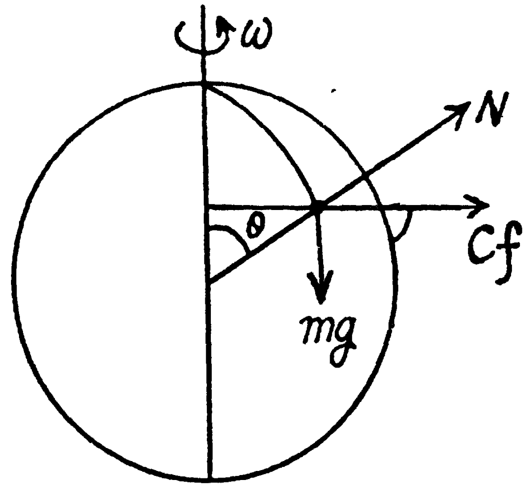
so, $$w = \frac{v'^2}{\rho} = \frac{v'^2}{R} = \omega^2 R.$$

(b) At B $$\vec{F}_{in} = m\omega^2 \vec{OC} + m\omega^2 \vec{BC}$$

its magnitude is $m\omega^2\sqrt{4R^2 - r^2}$, where $r = OB$.

1.115 The equation of motion in the rotating coordinate system is,

$$m\,\vec{w}' = \vec{F} + m\omega^2 \vec{R} + 2m\,(\vec{v}' \times \vec{\omega})$$



Now, $\vec{v}' = R\,\dot\theta\,\vec{e}_\theta + R\sin\theta\,\dot\varphi\,\vec{e}_\varphi$

and $\vec{w}' = w'\cos\theta\,\vec{e}_r - w'\sin\theta\,\vec{e}_\theta$

$$\frac{1}{2m}\vec{F}_{cor} = \begin{vmatrix} \vec{e}_r & \vec{e}_\theta & \vec{e}_\varphi \\ 0 & R\,\dot\theta & R\sin\theta\,\dot\varphi \\ \omega\cos\theta & -\omega\sin\theta & 0 \end{vmatrix}$$

$$= \vec{e}_r(\omega R\sin^2\theta\,\dot\varphi) + \omega R\sin\theta\cos\theta\,\dot\varphi\,\vec{e}_\theta - \omega R\,\dot\theta\cos\theta\,\vec{e}_\theta$$

Now on the sphere,

$$\vec{v}' = (-R\,\dot\theta^2 - R\sin^2\theta\,\dot\varphi^2)\,\vec{e}_r$$
$$+ (R\,\ddot\theta - R\sin\theta\cos\theta\,\dot\varphi^2)\,\vec{e}_\theta$$
$$+ (R\sin\theta\,\ddot\varphi + 2R\cos\theta\,\dot\theta\,\dot\varphi)\,\vec{e}_\varphi$$

Thus the equation of motion are,

$$m(-R\,\dot\theta^2 - R\sin^2\theta\,\dot\varphi^2) = N - mg\cos\theta + m\omega^2 R\sin^2\theta + 2m\omega R\sin^2\theta\,\dot\varphi$$

$$m(R\,\ddot\theta - R\sin\theta\cos\theta\,\dot\varphi^2) = mg\sin\theta + m\omega^2 R\sin\theta\cos\theta + 2m\omega R\sin\theta\cos\theta\,\dot\varphi$$

$$m(R\sin\theta\,\ddot\varphi + 2R\cos\theta\,\dot\theta\,\dot\varphi) = -2m\omega R\,\dot\theta\cos\theta$$

From the third equation, we get, $\dot\varphi = -\omega$

A result that is easy to understant by considering the motion in non-rotating frame. The eliminating $\dot\varphi$ we get,

$$mR\,\dot\theta^2 = mg\cos\theta - N$$
$$mR\,\ddot\theta = mg\sin\theta$$

Integrating the last equation,

$$\frac{1}{2}mR\,\dot\theta^2 = mg(1 - \cos\theta)$$

Hence $$N = (3 - 2\cos\theta)\, mg$$

So the body must fly off for $\theta = \theta_0 = \cos^{-1}\frac{2}{3}$, exactly as if the sphere were nonrotating.

Now, at this point F_{cf} = centrifugal force $= m\omega^2 R \sin\theta_0 = \sqrt{\frac{5}{9}}\, m\omega^2 R$

$$F_{cor} = \sqrt{\omega^2 R^2 \dot\theta^2 \cos^2\theta + (\omega^2 R^2)^2 \sin^2\theta} \times 2m$$

$$= \sqrt{\frac{5}{9}(\omega^2 R)^2 + \omega^2 R^2 \times \frac{4}{9} \times \frac{2g}{3R}} \times 2m = \frac{2}{3} m\omega^2 R \sqrt{5 + \frac{8g}{3\,\omega^2 R}}$$

1.116 (a) When the train is moving along a meridian only the Coriolis force has a lateral component and its magnitude (see the previous problem) is,

$$2m\,\omega\, v \cos\theta = 2m\,\omega \sin\lambda$$

(Here we have put $R\,\dot\theta \to v$)

So, $$F_{lateral} = 2 \times 2000 \times 10^3 \times \frac{2\pi}{86400} \times \frac{54000}{3600} \times \frac{\sqrt{3}}{2}$$

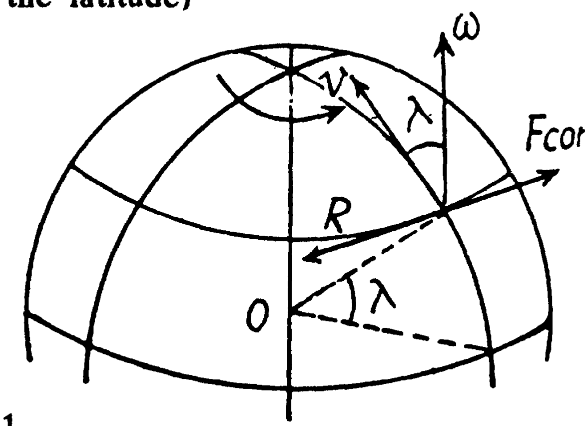
$$= 3{\cdot}77 \text{ kN}, \text{ (we write } \lambda \text{ for the latitude)}$$

(b) The resultant of the inertial forces acting on the train is,

$$\vec{F}_{in} = -2m\omega R\, \dot\theta \cos\theta\, \vec{e}_\varphi$$

$$+ (m\omega^2 R \sin\theta\cos\theta + 2m\,\omega R \sin\theta\cos\theta\,\dot\varphi)\,\vec{e}_\theta$$

$$+ (m\omega^2 R \sin^2\theta + 2m\,\omega R \sin^2\theta\,\dot\varphi)\,\vec{e}_r$$



This vanishes if $\dot\theta = 0,\ \dot\varphi = -\frac{1}{2}\omega$

Thus $$\vec{v} = v_\varphi \vec{e}_\varphi,\ v_\varphi = -\frac{1}{2}\omega R \sin\theta = -\frac{1}{2}\omega R \cos\lambda$$

(We write λ for the latitude here)

Thus the train must move from the east to west along the 60th parallel with a speed,

$$\frac{1}{2}\omega R \cos\lambda = \frac{1}{4} \times \frac{2\pi}{8{\cdot}64} \times 10^{-4} \times 6{\cdot}37 \times 10^6 = 115{\cdot}8 \text{ m/s} \approx 417 \text{ km/hr}$$

1.117 We go to the equation given in 1.111. Here $v_y = 0$ so we can take $y = 0$, thus we get for the motion in the $x\,z$ plane.

$$\ddot{x} = -2\omega\, v_z \cos\theta$$

and $$\ddot{z} = -g$$

Integrating, $$z = -\frac{1}{2} g t^2$$

$$\dot{x} = \omega\, g \cos\varphi\, t^2$$

So $$x = \frac{1}{3}\omega\, g\cos\varphi\, t^3 = \frac{1}{3}\omega\, g \cos\varphi \left(\frac{2h}{g}\right)^{3/2}$$

$$= \frac{2\,\omega h}{3}\cos\varphi \sqrt{\frac{2h}{g}}$$

There is thus a displacement to the east of

$$\frac{2}{3} \times \frac{2\pi}{8} 64 \times 500 \times 1 \times \sqrt{\frac{400}{9{\cdot}8}} \approx 26 \text{ cm.}$$

1.3 Laws of Conservation of Energy, Momentum and Angular Momentum.

1.118 As $\vec{F}$ is constant so the sought work done

$$A = \vec{F}\cdot\Delta\vec{r} = \vec{F}\cdot(\vec{r_2}-\vec{r_1})$$

or, $$A = (3\vec{i}+4\vec{j})\cdot[(2\vec{i}-3\vec{j})-(\vec{i}+2\vec{j})] = (3\vec{i}+4\vec{j})\cdot(\vec{i}-5\vec{j}) = 17\,\text{J}$$

1.119 Differentating $v(s)$ with respect to time

$$\frac{dv}{dt} = \frac{a}{2\sqrt{s}}\frac{ds}{dt} = \frac{a}{2\sqrt{s}}a\sqrt{s} = \frac{a^2}{2} = w$$

(As locomotive is in unidrectional motion)

Hence force acting on the locomotive $F = m\,w = \dfrac{ma^2}{2}$

Let, at $v = 0$ at $t = 0$ then the distance covered during the first t seconds

$$s = \frac{1}{2}wt^2 = \frac{1}{2}\frac{a^2}{2}t^2 = \frac{a^2}{4}t^2$$

Hence the sought work, $A = Fs = \dfrac{ma^2}{2}\dfrac{(a^2t^2)}{4} = \dfrac{m\,a^4t^2}{8}$

1.120 We have

$$T = \frac{1}{2}mv^2 = as^2 \quad \text{or,} \quad v^2 = \frac{2\,as^2}{m} \tag{1}$$

Differentating Eq. (1) with respect to time

$$2\,vw_t = \frac{4\,as}{m}v \quad \text{or,} \quad w_t = \frac{2\,as}{m} \tag{2}$$

Hence net acceleration of the particle

$$w = \sqrt{w_t^2 + w_n^2} = \sqrt{\left(\frac{2\,as}{m}\right)^2 + \left(\frac{2\,as^2}{mR}\right)^2} = \frac{2\,as}{m}\sqrt{1+(s/R)^2}$$

Hence the sought force, $F = mw = 2as\sqrt{1+(s/R)^2}$

1.121 Let $\vec{F}$ makes an angle θ with the horizontal at any instant of time (Fig.). Newton's second law in projection form along the direction of the force, gives :

$F = kmg\cos\theta + mg\sin\theta$ (because there is no acceleration of the body.)

As $\vec{F}\uparrow\uparrow d\vec{r}$ the differential work done by the force $\vec{F}$,

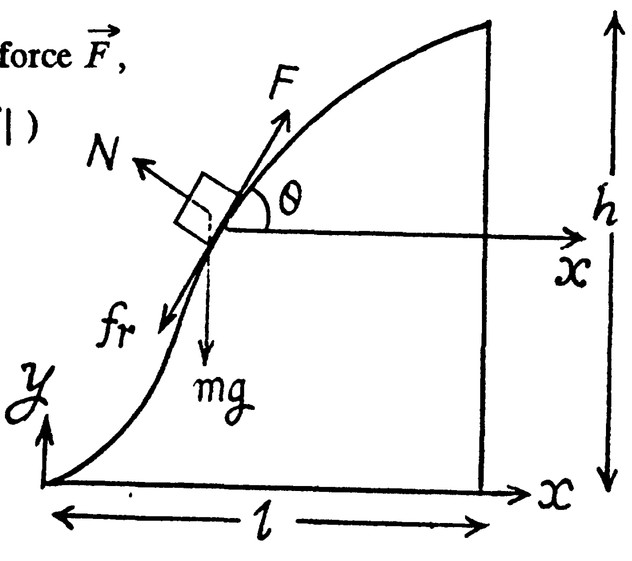
$$dA = \vec{F}\cdot d\vec{r} = F\,ds, \quad (\text{where } ds = |d\vec{r}|)$$

$$= kmg\,ds\,(\cos\theta) + mg\,ds\sin\theta$$

$$= kmg\,dx + mg\,dy.$$

Hence, $$A = kmg\int_0^l dx + mg\int_0^h dy$$

$$= kmg\,l + mgh = mg\,(k\,l + h).$$



1.122 Let s be the distance covered by the disc along the incline, from the Eq. of increment of M.E. of the disc in the field of gravity : $\Delta T + \Delta U = A_{fr}$

$$0 + (-mgs \sin \alpha) = -kmg \cos \alpha\, s - kmg\, l$$

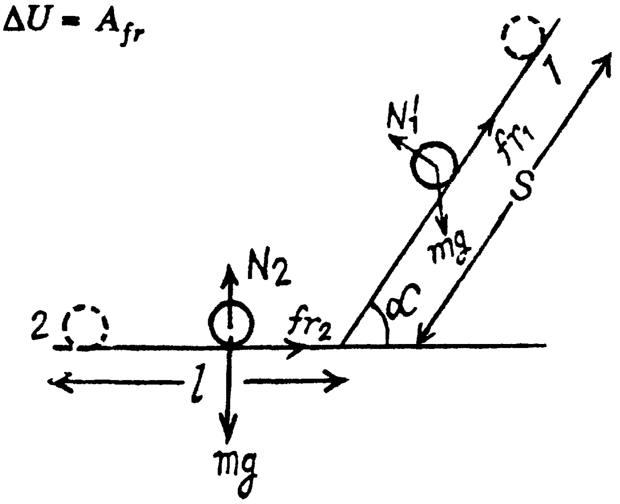
or, $$s = \frac{k\,l}{\sin \alpha - k \cos \alpha} \qquad (1)$$

Hence the sought work

$$A_{fr} = -kmg\,[s \cos \alpha + l\,]$$

$$A_{fr} = -\frac{k\,lmg}{1 - k \cot \alpha} \quad \text{[Using the Eqn. (1)]}$$

On puting the values $A_{fr} = -0.05$ J



1.123 Let x be the compression in the spring when the bar m_2 is about to shift. Therefore at this moment spring force on m_2 is equal to the limiting friction between the bar m_2 and horizontal floor. Hence

$$\kappa\, x = k\, m_2\, g \quad \text{[where } \kappa \text{ is the spring constant (say)]} \qquad (1)$$

For the block m_1 from work-energy theorem : $A = \Delta T = 0$ for minimum force. (A here indudes the work done in stretching the spring.)

so, $$Fx - \frac{1}{2}\kappa x^2 - kmg\,x = 0 \quad \text{or} \quad \kappa \frac{x}{2} = F - km_1\, g \qquad (2)$$

From (1) and (2),

$$F = kg\left(m_1 + \frac{m_2}{2}\right).$$

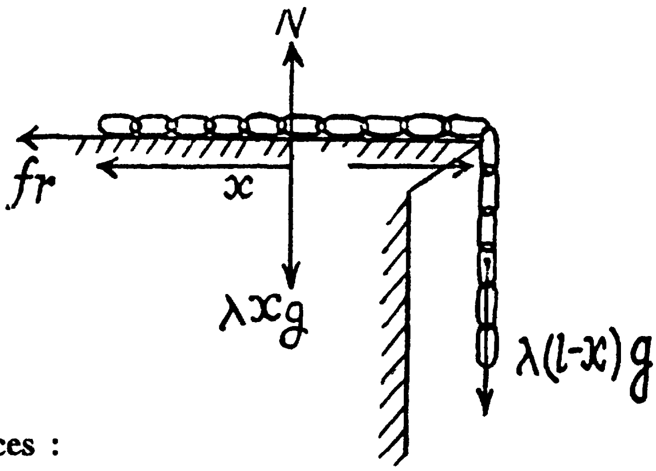
1.124 From the initial condition of the problem the limiting fricition between the chain lying on the horizontal table equals the weight of the over hanging part of the chain, i.e.

$\lambda\, \eta\, lg = k\, \lambda\, (1 - \eta)\, lg$ (where λ is the linear mass density of the chain)

So, $$k = \frac{\eta}{1 - \eta} \qquad (1)$$

Let (at an arbitrary moment of time) the length of the chain on the table is x. So the net friction force between the chain and the table, at this moment :

$$f_r = kN = k\lambda\, x\, g \qquad (2)$$



The differential work done by the friction forces :

$$dA = \vec{f_r} \cdot d\vec{r} = -f_r\, ds = -k\lambda\, x\, g\,(-dx) = \lambda g\left(\frac{\eta}{1 - \eta}\right) x\, dx \qquad (3)$$

(Note that here we have written $ds = -dx$., because ds is essentially a positive term and as the length of the chain decreases with time, dx is negative)

Hence, the sought work done

$$A = \int_{(1-\eta)l}^{0} \lambda\, g \frac{\eta}{1 - \eta}\, x\, dx = -(1 - \eta)\, \eta\, \frac{mgl}{2} = -1{\cdot}3 \text{ J}$$

1.125 The velocity of the body, t seconds after the begining of the motion becomes $\vec{v} = \vec{v}_0 + \vec{g}t$. The power developed by the gravity ($m\vec{g}$) at that moment, is

$$P = m\vec{g}\cdot\vec{v} = m(\vec{g}\cdot\vec{v}_0 + g^2 t) = mg(gt - v_0 \sin\alpha) \qquad (1)$$

As $m\vec{g}$ is a constant force, so the average power

$$<P> = \frac{A}{\tau} = \frac{m\vec{g}\cdot\Delta\vec{r}}{\tau}$$

where $\Delta\vec{r}$ is the net displacement of the body during time of flight.

As, $\qquad m\vec{g}\perp\Delta\vec{r}$ so $<P> = 0$

1.126 We have $\qquad w_n = \frac{v^2}{R} = at^2, \quad \text{or,} \quad v = \sqrt{aR}\,t,$

t is defined to start from the begining of motion from rest.

So, $\qquad w_t = \frac{dv}{dt} = \sqrt{aR}$

Instantaneous power, $P = \vec{F}\cdot\vec{v} = m(w_t\hat{u}_t + w_n\hat{u}_t)\cdot(\sqrt{aR}\,t\,\hat{u}_t)$,

(where $\hat{u}_t$ and $\hat{u}_t$ are unit vectors along the direction of tangent (velocity) and normal respectively)

So, $\qquad P = mw_t\sqrt{aR}\,t = maRt$

Hence the sought average power

$$<P> = \frac{\int_0^t P\,dt}{\int_0^t dt} = \frac{\int_0^t maRt\,dt}{t}$$

Hence $\qquad <P> = \frac{maRt^2}{2t} = \frac{maRt}{2}$

1.127 Let the body m acquire the horizontal velocity v_0 along positive x – axis at the point O.

(a) Velocity of the body t seconds after the begining of the motion,

$$\vec{v} = \vec{v}_0 + \vec{w}t = (v_0 - kgt)\vec{i} \qquad (1)$$

Instantaneous power $P = \vec{F}\cdot\vec{v} = (-kmg\,\vec{i})\cdot(v_0 - kgt)\vec{i} = -kmg(v_0 - kgt)$

From Eq. (1), the time of motion $\tau = v_0/kg$

Hence sought average power during the time of motion

$$<P> = \frac{\int_0^\tau -kmg(v_0 - kgt)\,dt}{\tau} = -\frac{kmg\,v_0}{2} = -2\text{ W (On substitution)}$$

From $F_x = mw_x$

$$-kmg = mw_x = mv_x\frac{dv_x}{dx}$$

or, $\qquad v_x\,dv_x = -kg\,dx \doteq -\alpha g x\,dx$

To find $v(x)$, let us integrate the above equation

$$\int_{v_0}^{v} v_x\, dv_x = -\alpha g \int_0^x x\, dx \quad \text{or,} \quad v^2 = v_0^2 - \alpha g x^2 \tag{1}$$

Now, $$\vec{P} = \vec{F} \cdot \vec{v} = -m\alpha x g \sqrt{v_0^2 - \alpha g x^2} \tag{2}$$

For maximum power, $\frac{d}{dt}\left(\sqrt{v^2{}_0 x^2 - \lambda g x^4}\right) = 0$ which yields $x = \frac{v_0}{\sqrt{2\,\alpha g}}$

Putting this value of x, in Eq. (2) we get,

$$P_{max} = -\frac{1}{2} m\, v_0^2 \sqrt{\alpha\, g}$$

1.128 Centrifugal force of inertia is directed outward along radial line, thus the sought work

$$A = \int_{r_1}^{r_2} m\omega^2\, r\, dr = \frac{1}{2} m\omega^2 \left(r_2^2 - r_1^2\right) = 0{\cdot}20\text{ T} \quad \text{(On substitution)}$$

1.129 Since the springs are connected in series, the combination may be treated as a single spring of spring constant.

$$\kappa = \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}$$

From the equation of increment of M.E., $\Delta T + \Delta U = A_{ext}$

$$0 + \frac{1}{2}\kappa\, \Delta l^2 = A, \quad \text{or,} \quad A = \frac{1}{2}\left(\frac{\kappa\, \kappa_2}{\kappa_1 + \kappa_2}\right) \Delta\, l^2$$

1.130 First, let us find the total height of ascent. At the beginning and the end of the path of velocity of the body is equal to zero, and therefore the increment of the kinetic energy of the body is also equal to zero. On the other hand, in according with work-energy theorem ΔT is equal to the algebraic sum of the works A performed by all the forces, i.e. by the force F and gravity, over this path. However, since $\Delta T = 0$ then $A = 0$. Taking into account that the upward direction is assumed to coincide with the positive direction of the y – axis, we can write

$$A = \int_0^h (\vec{F} + m\vec{g}) \cdot d\,\vec{r} = \int_0^h (F_y - mg)\, dy$$

$$= mg \int_0^h (1 - 2\,ay)\, dy = mgh\,(1 - ah) = 0.$$

whence $h = 1/a$.

The work performed by the force F over the first half of the ascent is

$$A_F = \int_0^{h/2} F_y\, dy = 2mg \int_0^{h/2} (1 - ay)\, dy = 3\, mg/4a.$$

The corresponding increment of the potential energy is

$$\Delta\, U = mgh/2 = mg/2a.$$

1.131 From the equation $F_r = -\frac{dU}{dr}$ we get $F_r = \left[-\frac{2a}{r^3} + \frac{b}{r^2}\right]$

(a) we have at $r = r_0$, the particle is in equilibrium position. i.e. $F_r = 0$ so, $r_0 = \frac{2a}{b}$

To check, whether the position is steady (the position of stable equilibrium), we have to satisfy

$$\frac{d^2 U}{dr^2} > 0$$

We have
$$\frac{d^2 U}{dr^2} = \left[\frac{6a}{r^4} - \frac{2b}{r^3}\right]$$

Putting the value of $r = r_0 = \frac{2a}{b}$, we get

$\frac{d^2 U}{dr^2} = \frac{b^4}{8a^3}$, (as a and b are positive constant)

So,
$$\frac{d^2 U}{dr^2} = \frac{b^2}{8a^3} > 0,$$

which indicates that the potential energy of the system is minimum, hence this position is steady.

(b) We have
$$F_r = -\frac{dU}{dr} = \left[-\frac{2a}{r^3} + \frac{b}{r^2}\right]$$

For F_r to be maximum, $\frac{dF_r}{dr} = 0$

So, $r = \frac{3a}{b}$ and then $F_{r(\max)} = \frac{-b^3}{27a^2}$,

As F_r is negative, the force is attractive.

1.132 (a) We have

$$F_x = -\frac{\partial U}{dx} = -2\alpha x \text{ and } F_y = \frac{-\partial U}{dy} = -2\beta y$$

So,
$$\vec{F} = 2\alpha x\vec{i} - 2\beta y\vec{i} \text{ and, } F = 2\sqrt{\alpha^2 x^2 + \beta^2 y^2} \qquad (1)$$

For a central force, $\vec{r}\times\vec{F} = 0$

Here,
$$\vec{r}\times\vec{F} = (x\vec{i} + y\vec{j}) \times (-2\alpha x\vec{i} - 2\beta y\vec{j})$$
$$= -2\beta xy\vec{k} - 2\alpha xy(\vec{k}) \neq 0$$

Hence the force is not a central force.

(b) As $U = \alpha x^2 + \beta y^2$

So,
$$F_x = \frac{\partial U}{\partial x} = -2\alpha x \text{ and } F_y = \frac{-\partial U}{\partial y} = -2\beta y.$$

So,
$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{4\alpha^2 x^2 + 4\beta^2 y^2}$$

According to the problem

$$F = 2\sqrt{\alpha^2 x^2 + \beta^2 y^2} = C \text{ (constant)}$$

or, $$\alpha^2 x^2 + \beta^2 y^2 = \frac{C^2}{2}$$

or, $$\frac{x^2}{\beta^2} + \frac{y^2}{\alpha^2} = \frac{C^2}{2\,\alpha^2\beta^2} = k\,(\text{say}) \tag{2}$$

Therefore the surfaces for which F is constant is an ellipse.

For an equipotential surface U is constant.

So, $$\alpha x^2 + \beta y^2 = C_0\ (\text{constant})$$

or, $$\frac{x^2}{\sqrt{\beta^2}} + \frac{y^2}{\sqrt{\alpha^2}} = \frac{C_0}{\alpha\,\beta} = K_0\ (\text{constant})$$

Hence the equipotential surface is also an ellipse.

1.133 Let us calculate the work performed by the forces of each field over the path from a certain point 1 (x_1, y_1) to another certain point 2 (x_2, y_2)

(i) $dA = \vec{F}\cdot d\vec{r} = ay\,\vec{i}\cdot d\vec{r} = aydx$ or, $A = a\int\limits_{x_1}^{x_2} y\,dx$

(ii) $dA = \vec{F}\cdot d\vec{r} = (a\vec{xi} + by\,\vec{i}\,)\cdot d\vec{r} = axdx + bydy$

Hence $$A = \int\limits_{x_1}^{x_2} a\,xdx + \int\limits_{y_1}^{y_2} bydy$$

In the first case, the integral depends on the function of type $y(x)$, i.e. on the shape of the path. Consequently, the first field of force is not potential. In the second case, both the integrals do not depend on the shape of the path. They are defined only by the coordinate of the initial and final points of the path, therefore the second field of force is potential.

1.134 Let s be the sought distance, then from the equation of increment of M.E. $\Delta T + \Delta U = A_{fr}$

$$\left(0 - \frac{1}{2} m v_0^2\right) + (+\,mg\,s\sin\alpha) = -\,kmg\cos\alpha\,s$$

or, $$s = \frac{v_0^2}{2g} \Big/ (\sin\alpha + k\cos\alpha)$$

Hence $$A_{fr} = -\,kmg\cos\alpha\,s = \frac{-\,kmv_0^2}{2\,(k + \tan\alpha)}$$

1.135 Velocity of the body at height h, $v_h = \sqrt{2g\,(H-h)}$, horizontally (from the figure given in the problem). Time taken in falling through the distance h.

$t = \sqrt{\frac{2h}{g}}$ (as initial vertical component of the velocity is zero.)

Now $$s = v_h\,t = \sqrt{2g\,(H+h)} \times \sqrt{\frac{2h}{g}} = \sqrt{4\,(Hh - h^2)}$$

For s_{max}, $\frac{d}{ds}(Hh - h^2) = 0$, which yields $h = \frac{H}{2}$

Putting this value of h in the expression obtained for s, we get,

$$s_{max} = H$$

1.136 To complete a smooth vertical track of radius R, the minimum height at which a particle starts, must be equal to $\frac{5}{2}R$ (one can proved it from energy conservation). Thus in our problem body could not reach the upper most point of the vertical track of radius $R/2$. Let the particle A leave the track at some point O with speed v (Fig.). Now from energy conservation for the body A in the field of gravity :

$$mg\left[h - \frac{h}{2}(1 + \sin\theta)\right] = \frac{1}{2}mv^2$$

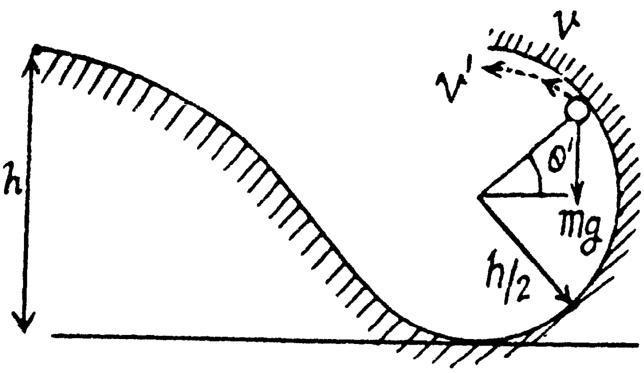
or, $\quad v^2 = gh(1 - \sin\theta) \qquad (1)$

From Newton's second law for the particle at the point O; $F_n = mw_n$,

$$N + mg\sin\theta = \frac{mv^2}{(h/2)}$$

But, at the point O the normal reaction $N = 0$

So, $\quad v^2 = \frac{gh}{2}\sin\theta \qquad (2)$



From (3) and (4), $\sin\theta = \frac{2}{3}$ and $v = \sqrt{\frac{gh}{3}}$

After leaving the track at O, the particle A comes in air and further goes up and at maximum height of it's trajectory in air, it's velocity (say v') becomes horizontal (Fig.). Hence, the sought velocity of A at this point.

$$v' = v\cos(90 - \theta) = v\sin\theta = \frac{2}{3}\sqrt{\frac{gh}{3}}$$

1.137 Let, the point of suspension be shifted with velocity v_A in the horizontal direction towards left then in the rest frame of point of suspension the ball starts with same velocity horizontally towards right. Let us work in this, frame. From Newton's second law in projection form towards the point of suspension at the upper most point (say B) :

$$mg + T = \frac{mv_B^2}{l} \quad \text{or,} \quad T = \frac{mv_B^2}{l} - mg \qquad (1)$$

Condition required, to complete the vertical circle is that $T \geq 0$. But $\qquad (2)$

$$\frac{1}{2}mv_A^2 = mg(2l) + \frac{1}{2}mv_B^2 \quad \text{So,} \quad v_B^2 = v_A^2 - 4gl \qquad (3)$$

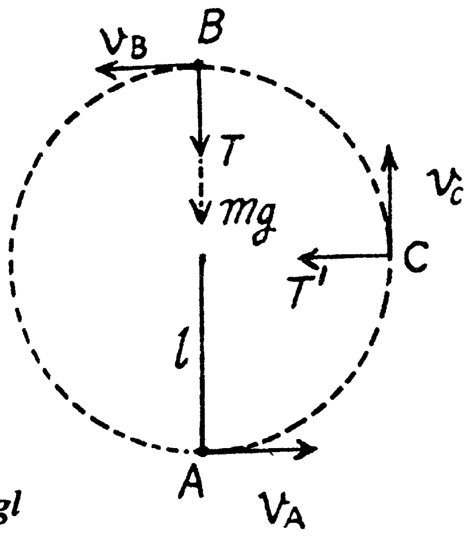
From (1), (2) and (3)

$$T = \frac{m(v_A^2 - 4gl)}{l} - mg \geq 0 \quad \text{or,} \quad v_A \geq \sqrt{5gl}$$

Thus $\quad v_{A\,(\min)} = \sqrt{5gl}$

From the equation $F_n = mw_n$ at point C

$$T' = \frac{mv_c^2}{l} \qquad (4)$$



Again from energy conservation

$$\frac{1}{2}mv_A^2 = \frac{1}{2}mv_c^2 + mgl \qquad (5)$$

From (4) and (5)

$$T = 3\,mg$$

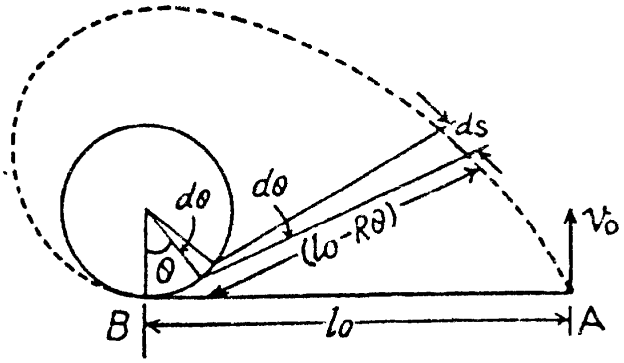
1.138 Since the tension is always perpendicular to the velocity vector, the work done by the tension force will be zero. Hence, according to the work energy theorem, the kinetic energy or velocity of the disc will remain constant during it's motion. Hence, the sought time $t = \frac{s}{v_0}$, where s is the total distance traversed by the small disc during it's motion.

Now, at an arbitary position (Fig.)

$ds = (l_0 - R\,\theta)\,d\,\theta,$

$$\text{so,} \quad s = \int_0^{l/R} (l_0 - R\,\theta)\,d\,\theta$$

$$\text{or,} \qquad s = \frac{l_0^2}{R} - \frac{R\,l_0^2}{2R^2} = \frac{l_0^2}{2R}$$



Hence, the required time, $\quad t = \dfrac{l_0^2}{2R\,v_0}$

It should be clearly understood that the only uncompensated force acting on the disc A in this case is the tension T, of the thread. It is easy to see that there is no point here, relative to which the moment of force T is invarible in the process of motion. Hence conservation of angular momentum is not applicable here.

1.139 Suppose that Δl is the elongation of the rubbler cord. Then from energy conservation,

$$\Delta U_{gr} + \Delta U_{el} = 0 \text{ (as } \Delta T = 0)$$

$$\text{or,} \qquad -mg(l + \Delta l) + \frac{1}{2}\kappa\,\Delta l^2 = 0$$

$$\text{or,} \qquad \frac{1}{2}\kappa\,\Delta l^2 - mg\,\Delta l - mg\,l = 0$$

or, $$\Delta l = \frac{mg \pm \sqrt{(mg)^2 + 4 \times \frac{\kappa}{2} mgl}}{2 \times \frac{\kappa}{2}} \times \frac{\kappa}{2} = \frac{mg}{\kappa}\left[1 + \sqrt{1 \pm \frac{2\kappa l}{mg}}\right]$$

Since the value of $\sqrt{1 + \frac{2\kappa l}{mg}}$ is certainly greater than 1, hence negative sign is avoided.

So, $$\Delta l = \frac{mg}{\kappa}\left(1 + \sqrt{1 + \frac{2\kappa l}{mg}}\right)$$

1.140 When the thread *PA* is burnt, obviously the speed of the bars will be equal at any instant of time until it breaks off. Let v be the speed of each block and θ be the angle, which the elongated spring makes with the vertical at the moment, when the bar *A* breaks off the plane. At this stage the elongation in the spring.

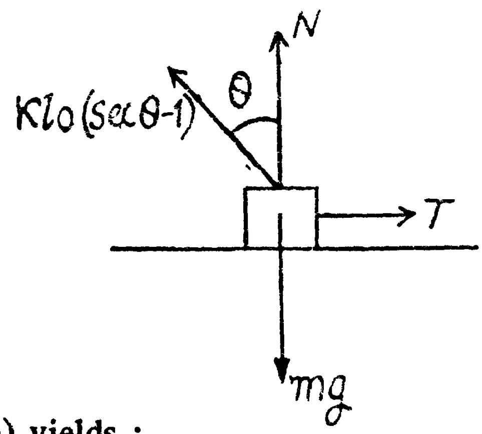
$$\Delta l = l_0 \sec\theta - l_0 = l_0(\sec\theta - 1) \qquad (1)$$

Since the problem is concerned with position and there are no forces other than conservative forces, the mechanical energy of the system (both bars + spring) in the field of gravity is conserved, i.e. $\Delta T + \Delta U = 0$

So, $$2\left(\frac{1}{2} mv^2\right) + \frac{1}{2}\kappa l_0^2(\sec\theta - 1)^2 - mgl_0 \tan\theta = 0 \qquad (2)$$

From Newton's second law in projection form along vertical direction :

$mg = N + \kappa\, l_0(\sec\theta - 1)\cos\theta$



But, at the moment of break off, $N = 0$.

Hence, $\kappa\, l_0(\sec\theta - 1)\cos\theta = mg$

or, $$\cos\theta = \frac{\kappa\, l_0 - mg}{\kappa\, l_0} \qquad (3)$$

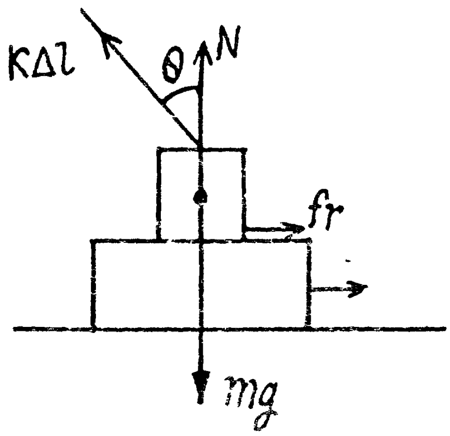
Taking $\kappa = \frac{5\,mg}{l_0}$, simultaneous solution of (2) and (3) yields :

$$v = \sqrt{\frac{19\, g\, l_0}{32}} = 1{\cdot}7\ \text{m/s}.$$

1.141 Obviously the elongation in the cord, $\Delta l = l_0(\sec\theta - 1)$, at the moment the sliding first starts and at the moment horizontal projection of spring force equals the limiting friction.

So, $$\kappa_1\, \Delta l \sin\theta = kN \qquad (1)$$

(where κ_1 is the elastic constant).



From Newton's law in projection form along vertical direction :

$$\kappa_1\, \Delta l \cos\theta + N = mg.$$

or, $$N = mg - \kappa_1\, \Delta l \cos\theta \qquad (2)$$

From (1) and (2),

$$\kappa_1\, \Delta l \sin\theta = k(mg - \kappa_1\, \Delta l \cos\theta)$$

or, $\quad \kappa_1 = \dfrac{kmg}{\Delta l \sin\theta + k\,\Delta l \cos\theta}$

From the equation of the increment of mechanical energy : $\Delta U + \Delta T = A_{fr}$

or, $$\left(\frac{1}{2}\kappa_1\,\Delta l^2\right) = A_{fr}$$

or, $$\frac{kmg\,\Delta l^2}{2\,\Delta l\,(\sin\theta + k\cos\theta)} = A_{fr}$$

Thus $$A_{fr} = \frac{kmg\,l_0\,(\sec\theta - 1)}{2\,(\sin\theta - k\cos\theta)} = 0{\cdot}09\text{ J} \quad \text{(on substitution)}$$

1.142 Let the deformation in the spring be Δl, when the rod AB has attained the angular velocity ω. From the second law of motion in projection form $F_n = mw_n$.

$$\kappa\,\Delta l = m\,\omega^2\,(l_0 + \Delta l) \quad \text{or,} \quad \Delta l = \frac{m\omega^2\,l_0}{\kappa - m\omega^2}$$

From the energy equation, $A_{ext} = \dfrac{1}{2}mv^2 + \dfrac{1}{2}\kappa\,\Delta l^2$

$$= \frac{1}{2}m\omega^2\,(l_0 + \Delta l)^2 + \frac{1}{2}\kappa\,\Delta l^2$$

$$= \frac{1}{2}m\omega^2\left(l_0 + \frac{m\omega^2\,l_0}{\kappa - m\omega^2}\right)^2 + \frac{1}{2}\kappa\left(\frac{m\omega^2\,l_0^2}{\kappa - m\omega^2}\right)^2$$

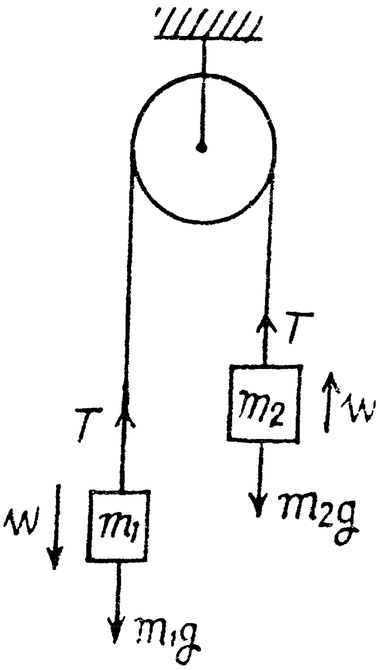
On solving $$A_{ext} = \frac{\kappa}{2}\,\frac{l_0^2\,\eta\,(1+\eta)}{(1-\eta)^2}, \quad \text{where } \eta = \frac{m\omega^2}{\kappa}$$

1.143 We know that acceleration of centre of mass of the system is given by the expression.

$$\vec{w}_C = \frac{m_1\,\vec{w}_1 + m_2\,\vec{w}_2}{m_1 + m_2}$$

Since $$\vec{w}_1 = -\vec{w}_2$$

$$\vec{w}_C = \frac{(m_1 - m_2)\,\vec{w}_1}{m_1 + m_2} \qquad (1)$$



Now from Newton's second law $\vec{F} = m\vec{w}$, for the bodies m_1 and m_2 respectively.

$$\vec{T} + m_1\,\vec{g} = m_1\,\vec{w}_1 \qquad (2)$$

and $$\vec{T} + m_2\,\vec{g} = m_2\,\vec{w}_2 = -m_2\,\vec{w}_1 \qquad (3)$$

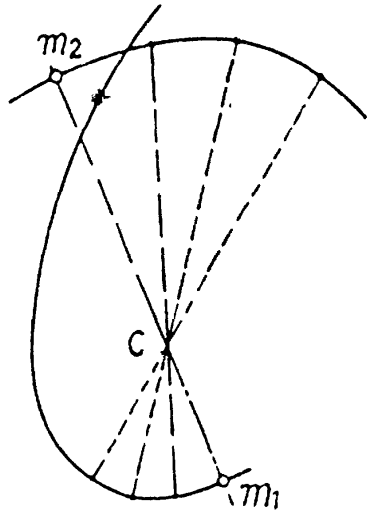
Solving (2) and (3)

$$\vec{w}_1 = \frac{(m_1 - m_2)\,\vec{g}}{m_1 + m_2} \qquad (4)$$

Thus from (1), (2) and (4),

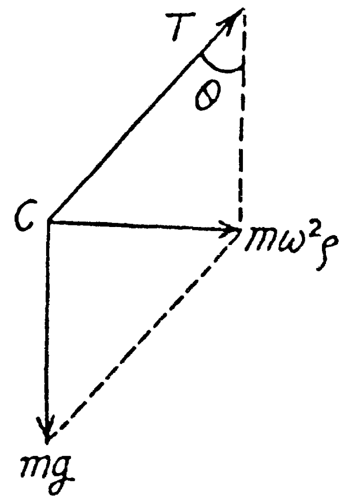
$$\vec{w}_c = \frac{(m_1 - m_2)^2 \vec{g}}{(m_1 + m_2)^2}$$

1.144 As the closed system consisting two particles m_1 and of m_2 is initially at rest the C.M. of the system will remain at rest. Further as $m_2 = m_1/2$, the C.M. of the system divides the line joining m_1 and m_2 at all the moments of time in the ratio 1 : 2. In addition to it the total linear momentum of the system at all the times is zero. So, $\vec{p_1} = -\vec{p_2}$ and therefore the velocities of m_1 and m_2 are also directed in opposite sense. Bearing in mind all these thing, the sought trajectory is as shown in the figure.



1.145 First of all, it is clear that the chain does not move in the vertical direction during the uniform rotation. This means that the vertical component of the tension T balances gravity. As for the horizontal component of the tension T, it is constant in magnitude and permanently directed toward the rotation axis. It follows from this that the C.M. of the chain, the point C, travels along horizontal circle of radius ρ (say). Therefore we have,

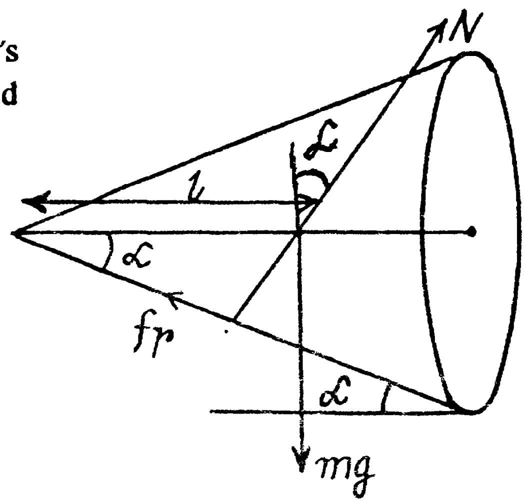
$$T \cos\theta = mg \quad \text{and} \quad T \sin\theta = m\omega^2 \rho$$



Thus $\quad \rho = \dfrac{g \tan\theta}{\omega^2} = 0{\cdot}8$ cm

and $\quad T = \dfrac{mg}{\cos\theta} = 5$ N

1.146 (a) Let us draw free body diagram and write Newton's second law in terms of projection along vertical and horizontal direction respectively.



$$N \cos\alpha - mg + fr \sin\alpha = 0 \qquad (1)$$

$$fr \cos\alpha - N \sin\alpha = m\omega^2 l \qquad (2)$$

From (1) and (2)

$$fr \cos\alpha - \frac{\sin\alpha}{\cos\alpha}(-fr \sin\alpha + mg) = m\omega^2 l$$

So, $$fr = mg\left(\sin\alpha + \frac{\omega^2 l}{g}\cos\alpha\right) = 6\text{N} \tag{3}$$

(b) For rolling, without sliding,

$$fr \le k\text{N}$$

but, $N = mg\cos\alpha - m\,\omega^2 l\sin\alpha$

$$mg\left(\sin\alpha + \frac{\omega^2 l}{g}\cos\alpha\right) \le k(mg\cos\alpha - m\,\omega^2 l\sin\alpha) \text{ [Using (3)]}$$

Rearranging, we get,

$$m\,\omega^2 l(\cos\alpha + k\sin\alpha) \le (k\,mg\cos\alpha - mg\sin\alpha)$$

Thus $$\omega \le \sqrt{g(k - \tan\alpha)/(1 + k\tan\alpha)\,l} = 2 \text{ rad/s}$$

1.147 (a) Total kinetic energy in frame K' is

$$T = \frac{1}{2}m_1(\vec{v}_1 - \vec{V})^2 + \frac{1}{2}m_2(\vec{v}_2 - \vec{V})^2$$

This is minimum with respect to variation in $\vec{V}$, when

$$\frac{\delta T'}{\delta\vec{V}} = 0, \text{ i.e. } m_1(\vec{v}_1 - \vec{V})^2 + m_2(\vec{v}_2 - \vec{V}) = 0$$

or $$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{V}_2}{m_1 + m_2} = \vec{v}_c$$

Hence, it is the frame of C.M. in which kinetic energy of a system is minimum.

(b) Linear momentum of the particle 1 in the K' or C frame

$$\tilde{\vec{p}}_1 = m_1(\vec{v}_1 - \vec{v}_c) = \frac{m_1 m_2}{m_1 + m_2}(\vec{v}_1 - \vec{v}_2)$$

or, $$\tilde{\vec{p}}_1 = \mu(\vec{v}_1 - \vec{v}_2), \text{ where, } \mu = \frac{m_1 m_2}{m_1 + m_2} = \text{reduced mass}$$

Similarly, $$\tilde{\vec{p}}_2 = \mu(\vec{v}_2 - \vec{v}_1)$$

So, $$|\tilde{\vec{p}}_1| = |\tilde{\vec{p}}_2| = \tilde{p} = \mu\,v_{rel} \text{ where, } v_{rel} = |\vec{v}_1 - \vec{v}_2| \tag{3}$$

Now the total kinetic energy of the system in the C frame is

$$\tilde{T} = \tilde{T}_1 + \tilde{T}_2 = \frac{\tilde{p}^2}{2m_1} + \frac{\tilde{p}^2}{2m_2} = \frac{\tilde{p}^2}{2\mu}$$

Hence $$\tilde{T} = \frac{1}{2}\mu\,v_{rel}^2 = \frac{1}{2}\mu\,|\vec{v}_1 - \vec{v}_2|^2$$

1.148 To find the relationship between the values of the mechanical energy of a system in the K and C reference frames, let us begin with the kinetic energy T of the system. The velocity of the i-th particle in the K frame may be represented as $\vec{v}_i = \vec{\tilde{v}}_i + \vec{v}_C$. Now we can write

$$T = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i (\vec{\tilde{v}}_1 + \vec{v}_C) \cdot (\vec{\tilde{v}}_i + \vec{v}_C)$$

$$= \sum \frac{1}{2} m_i \tilde{v}_i^2 + \vec{v}_C \sum m_i \vec{\tilde{v}}_1 + \sum \frac{1}{2} m_i v_C^2$$

Since in the C frame $\sum m_i \vec{\tilde{v}}_i = 0$, the previous expression takes the form

$$T = \tilde{T} + \frac{1}{2} m\, v_C^2 = \tilde{T} + \frac{1}{2} m\, V^2 \text{ (since according to the problem } v_C = V) \qquad (1)$$

Since the internal potential energy U of a system depends only on its configuration, the magnitude U is the same in all refrence frames. Adding U to the left and right hand sides of Eq. (1), we obtain the sought relationship

$$E = \tilde{E} + \frac{1}{2} m\, V^2$$

1.149 As initially $U = \tilde{U} = 0$, so, $\tilde{E} = \tilde{T}$

From the solution of 1.147 (b)

$$\tilde{T} = \frac{1}{2} \mu \, | \vec{v}_1 - \vec{v}_2 |,$$

As $\vec{v}_1 \perp \vec{v}_2$

Thus $\tilde{T} = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (v_1^2 + v_2^2)$

1.150 Velocity of masses m_1 and m_2, after t seconds are respectively.

$$\vec{v}_1{}' = \vec{v}_1 + \vec{g}\,t \text{ and } \vec{v}_2{}' = \vec{v}_2 + \vec{g}\,t$$

Hence the final momentum of the system,

$$\vec{p} = m_1 \vec{v}_1{}' + m_2 \vec{v}_2{}' = m_1 \vec{v}_1 + m_2 \vec{v}_2 + (m_1 + m_2)\,\vec{g}\,t$$

$$= \vec{p}_0 + m\,\vec{g}\,t, \text{ (where, } \vec{p}_0 = m_2 \vec{v}_1 + m_2 \vec{v}_2 \text{ and } m = m_1 + m_2)$$

And radius vector, $\vec{r}_C = \vec{v}_C\, t + \frac{1}{2} \vec{w}_C\, t^2$

$$\frac{(m_1 \vec{v}_1 + m_2 \vec{v}_2)\, t}{(m_1 + m_2)} + \frac{1}{2} \vec{g}\, t^2$$

$$= \vec{v}_0\, t + \frac{1}{2} \vec{g}\, t^2, \text{ where } \vec{v}_0 = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

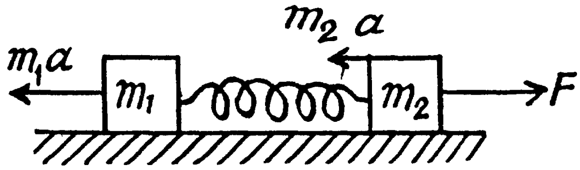
1.151 After releasing the bar 2 acquires the velocity v_2, obtained by the energy, conservation :

$$\frac{1}{2} m_2 v_2^2 = \frac{1}{2} \kappa x^2 \text{ or, } v_2 = x\sqrt{\frac{\kappa}{m_2}} \tag{1}$$

Thus the sought velocity of C.M.

$$v_{cm} = \frac{0 + m_2 x \sqrt{\frac{\kappa}{m_2}}}{m_1 + m_2} = \frac{x\sqrt{m_2 \kappa}}{(m_1 + m_2)}$$

1.152 Let us consider both blocks and spring as the physical system. The centre of mass of the system moves with acceleration $a = \dfrac{F}{m_1 + m_2}$ towards right. Let us work in the frame of centre of mass. As this frame is a non-inertial frame (accelerated with respect to the ground) we have to apply a pseudo force $m_1 a$ towards left on the block m_1 and $m_2\, a$ towards left on the block m_2



As the center of mass is at rest in this frame, the blocks move in opposite directions and come to instantaneous rest at some instant. The elongation of the spring will be maximum or minimum at this instant. Assume that the block m_1 is displaced by the distance x_1 and the block m_2 through a distance x_2 from the initial positions.

From the energy equation in the frame of C.M.

$$\Delta \tilde{T} + U = A_{ext},$$

(where A_{ext} also includes the work done by the pseudo forces)

Here,

$$\Delta \tilde{T} = 0, \; U = \frac{1}{2} k (x_1 + x_2)^2 \text{ and}$$

$$W_{ext} = \left(\frac{F - m_2 F}{m_1 + m_2}\right) x_2 + \frac{m_1 F}{m_1 + m_2} x_1 = \frac{m_1 F (x_1 + x_2)}{m_1 + m_2},$$

or, $$\frac{1}{2} k (x_1 + x_2)^2 = \frac{m_1 (x_1 + x_2) F}{m_1 + m_2}$$

So, $$x_1 + x_2 = 0 \text{ or, } x_1 + x_2 = \frac{2 m_1 F}{k (m_1 + m_2)}$$

Hence the maximum separation between the blocks equals : $l_0 + \dfrac{2 m_1 F}{k (m_1 + m_2)}$

Obviously the minimum sepation corresponds to zero elongation and is equal to l_0

1.153 (a) The initial compression in the spring Δl must be such that after burning of the thread, the upper cube rises to a height that produces a tension in the spring that is atleast equal to the weight of the lower cube. Actually, the spring will first go from its compressed

state to its natural length and then get elongated beyond this natural length. Let l be the maximum elongation produced under these circumstances.
Then

$$\kappa l = mg \tag{1}$$

Now, from energy conservation,

$$\frac{1}{2}\kappa \Delta l^2 = mg(\Delta l + l) + \frac{1}{2}\kappa l^2 \tag{2}$$

(Because at maximum elongation of the spring, the speed of upper cube becomes zero)
From (1) and (2),

$$\Delta l^2 - \frac{2mg\,\Delta l}{\kappa} - \frac{3m^2g^2}{\kappa^2} = 0 \quad \text{or,} \quad \Delta l = \frac{3mg}{\kappa}, \frac{-mg}{\kappa}$$

Therefore, acceptable solution of Δl equals $\frac{3mg}{\kappa}$

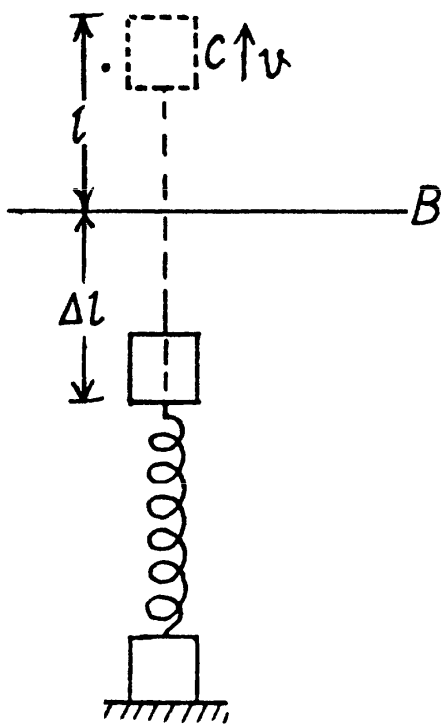
(b) Let v the velocity of upper cube at the position (say, at C) when the lower block breaks off the floor, then from energy conservation.

$$\frac{1}{2}mv^2 = \frac{1}{2}\kappa(\Delta l^2 - l^2) - mg(l + \Delta l)$$

$$\left(\text{where } l = mg/\kappa \text{ and } \Delta l = 7\frac{mg}{\kappa}\right)$$

or,
$$v^2 = 32\frac{mg^2}{\kappa} \tag{2}$$

At the position C, the velocity of C.M; $v_C = \frac{mv + 0}{2m} = \frac{v}{2}$ –Let, the C.M. of the system (spring+ two cubes) further rises up to Δy_{C2}.



Now, from energy conservation,

$$\frac{1}{2}(2m)v_C^2 = (2m)g\,\Delta r_{C2}$$

or, $$\Delta y_{C2} = \frac{v_C^2}{2g} = \frac{v^2}{8g} = \frac{4mg}{\kappa}$$

But, uptil position C, the C.M. of the system has already elevated by,

$$\Delta y_{C1} = \frac{(\Delta l + l)m + 0}{2m} = \frac{4mg}{\kappa}$$

Hence, the net displacement of the C.M. of the system, in upward direction

$$\Delta y_C = \Delta y_{C1} + \Delta y_{C2} = \frac{8mg}{\kappa}$$

1.154 Due to ejection of mass from a moving system (which moves due to inertia) in a direction perpendicular to it, the velocity of moving system does not change. The momentum change being adjusted by the forces on the rails. Hence in our problem velocities of buggies change only due to the entrance of the man coming from the other buggy. From the

Solving (1) and (2), we get

$$v_1 = \frac{mv}{M-m} \text{ and } v_2 = \frac{Mv}{M-m}$$

As $\qquad \vec{v_1} \uparrow\downarrow \vec{v} \text{ and } \vec{v_2} \uparrow\uparrow \vec{v}$

So, $$\vec{v_1} = \frac{-m\vec{v}}{(M-m)} \text{ and } \vec{v_2} = \frac{M\vec{v}}{(M-m)}$$

1.155 From momentum conservation, for the system "rear buggy with man"

$$(M+m)\vec{v_0} = m(\vec{u}+\vec{v_R}) + M\vec{v_R} \tag{1}$$

From momentum conservation, for the system (front buggy + man coming from rear buggy)

$$M\vec{v_0} + m(\vec{u}+\vec{v_R}) = (M+m)\vec{v_F}$$

So, $$\vec{v_F} = \frac{M\vec{v_0}}{M+m} + \frac{m}{M+m}(\vec{u}+\vec{v_R})$$

Putting the value of $\vec{v_R}$ from (1), we get

$$\vec{v_F} = \vec{v_0} + \frac{mM}{(M+m)^2}\vec{u}$$

1.156 (i) Let $\vec{v_1}$ be the velocity of the buggy after both man jump off simultaneously. For the closed system (two men + buggy), from the conservation of linear momentum,

$$M\vec{v_1} + 2m(\vec{u}+\vec{v_1}) = 0$$

or, $$\vec{v_1} = \frac{-2m\vec{u}}{M+2m} \tag{1}$$

(ii) Let $\vec{v}'$ be the velocity of buggy with man, when one man jump off the buggy. For the closed system (buggy with one man + other man) from the conservation of linear momentum :

$$0 = (M+m)\vec{v}' + m(\vec{u}+\vec{v}') \tag{2}$$

Let $\vec{v_2}$ be the sought velocity of the buggy when the second man jump off the buggy; then from conservation of linear momentum of the system (buggy + one man) :

$$(M+m)\vec{v}' = M\vec{v_2} + m(\vec{u}+\vec{v_2}) \tag{3}$$

Solving equations (2) and (3) we get

$$\vec{v_2} = \frac{m(2M+3m)\vec{u}}{(M+m)(M+2m)} \tag{4}$$

From (1) and (4)

$$\frac{v_2}{v_1} = 1 + \frac{m}{2(M+m)} > 1$$

Hence $v_2 > v_1$

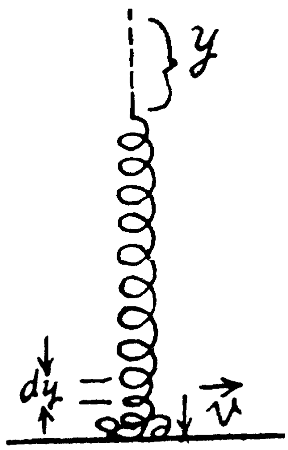
1.157 The descending part of the chain is in free fall, it has speed $v = \sqrt{2gh}$ at the instant, all its points have descended a distance y. The length of the chain which lands on the floor during the differential time interval dt following this instant is vdt.

For the incoming chain element on the floor :

From $dp_y = F_y\,dt$ (where y – axis is directed down)

$$0 - (\lambda\, v dt)\, v = F_y\, dt$$

or $$F_y = -\lambda\, v^2 = -2\,\lambda\, g\, y$$



Hence, the force exerted on the falling chain equals $\lambda\, v^2$ and is directed upward. Therefore from third law the force exerted by the falling chain on the table at the same instant of time becomes $\lambda\, v^2$ and is directed downward.

Since a length of chain of weight $(\lambda y g)$ already lies on the table the total force on the floor is $(2\lambda y g) + (\lambda y g) = (3\lambda y g)$ or the weight of a length $3y$ of chain.

1.158 Velocity of the ball, with which it hits the slab, $v = \sqrt{2\,gh}$

After first impact, $v' = ev$ (upward) but according to the problem $v' = \frac{v}{\eta}$, so $e = \frac{1}{\eta}$ (1)

and momentum, imparted to the slab,

$$= mv - (-mv') = mv\,(1+e)$$

Similarly, velocity of the ball after second impact,

$$v'' = ev' = e^2\, v$$

And momentum imparted $= m\,(v' + v'') = m\,(1+e)\, ev$

Again, momentum imparted during third impact,

$$= m\,(1+e)\, e^2\, v, \text{ and so on,}$$

Hence, net momentum, imparted $= mv\,(1+e) + mve\,(1+e) + mve^2\,(1+e) + \ldots$

$$= mv\,(1+e)\,(1+e+e^2+\ldots)$$

$$= mv\,\frac{(1+e)}{(1-e)}, \text{ (from summation of G.P.)}$$

$$= \sqrt{2\,gh}\;\frac{\left(1+\frac{1}{\eta}\right)}{\left(1-\frac{1}{\eta}\right)} = m\,\sqrt{2\,gh}\,/\,(\eta+1)\,/\,(\eta-1) \text{ (Using Eq. 1)}$$

$$= 0{\cdot}2 \text{ kg m/s. (On substitution)}$$

1.159 (a) Since the resistance of water is negligibly small, the resultant of all external forces acting on the system "a man and a raft" is equal to zero. This means that the position of the C.M. of the given system does not change in the process of motion.

i.e. $\vec{r}_C =$ constant or, $\Delta\vec{r}_C = 0$ i.e. $\sum m_i\,\Delta\vec{r}_i = 0$

or, $$m\left(\Delta\vec{r}_{mM} + \Delta\vec{r}_M\right) + M\,\Delta\vec{r}_M = 0$$

Thus, $$m\,(\vec{l}\,' + \vec{l}) + M\,\vec{l} = 0, \text{ or, } \vec{l} = -\frac{m\vec{l}\,'}{m+M}$$

(b) As net external force on "man-raft" system is equal to zero, therefore the momentum of this system does not change,

So, $$0 = m\,[\,\vec{v}\,'(t) + \vec{v}_2(t)\,] + M\,\vec{v}_2(t)$$

1.159 (a) Since the resistance of water is negligibly small, the resultant of all external forces acting on the system "a man and a raft" is equal to zero. This means that the position of the C.M. of the given system does not change in the process of motion.

i.e. $\vec{r}_C =$ constant or, $\Delta\vec{r}_C = 0$ i.e. $\sum m_i \Delta\vec{r}_i = 0$

or, $$m\left(\Delta\vec{r}_{mM} + \Delta\vec{r}_M\right) + M\,\Delta\vec{r}_M = 0$$

Thus, $$m(\vec{l}' + \vec{l}) + M\vec{l} = 0, \text{ or, } \vec{l} = -\frac{m\vec{l}'}{m+M}$$

(b) As net external force on "man-raft" system is equal to zero, therefore the momentum of this system does not change,

So, $$0 = m[\vec{v}'(t) + \vec{v}_2(t)] + M\vec{v}_2(t)$$

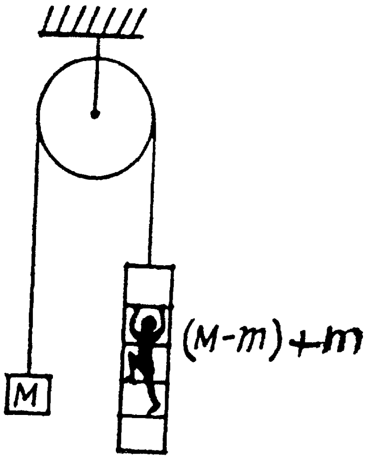
or, $$\vec{v}_2(t) = -\frac{m\,\vec{v}'(t)}{m+M} \tag{1}$$

As $\vec{v}'(t)$ or $\vec{v}_2(t)$ is along horizontal direction, thus the sought force on the raft

$$= \frac{M\,d\vec{v}_2(t)}{dt} = -\frac{Mm}{m+M}\frac{d\vec{v}'(t)}{dt}$$

Note : *we may get the result of part* (a), *if we integrate Eq.* (1) *over the time of motion of man or raft.*

1.160 In the refrence frame fixed to the pulley axis the location of C.M. of the given system is described by the radius vector



$$\Delta\vec{r}_C = \frac{M\,\Delta\vec{r}_M + (M-m)\,\Delta\vec{r}_{(M-m)} + m\,\Delta\vec{r}_m}{2M}$$

But $\Delta\vec{r}_M = -\Delta\vec{r}_{(M-m)}$

and $\Delta\vec{r}_m = \Delta\vec{r}_{m(M-m)} + \Delta\vec{r}_{(M-m)}$

Thus $\Delta\vec{r}_C = \dfrac{m\,\vec{l}'}{2M}$

Note : *one may also solve this problem using momentum conservation.*

1.161 Velocity of cannon as well as that of shell equals $\sqrt{2gl\sin\alpha}$ down the inclined plane taken as the positive x – axis. From the linear impulse momentum theorem in projection form along x – axis for the system (connon + shell) i.e. $\Delta p_x = F_x\,\Delta t$:

$p\cos\alpha - M\sqrt{2gl\sin\alpha} = Mg\sin\alpha\,\Delta t$ (as mass of the shell is neligible)

or, $$\Delta t = \frac{p\cos\alpha - M\sqrt{2gl\sin\alpha}}{Mg\sin\alpha}$$

1.162 From conservation of momentum, for the system (bullet + body) along the initial direction of bullet

$$mv_0 = (m+M)\,v, \text{ or, } v = \frac{mv_0}{m+M}$$

1.163 When the disc breaks off the body M, its velocity towards right (along x-axis) equals the velocity of the body M, and let the disc's velocity in upward direction (along y-axis) at that moment be v'_y

From conservation of momentum, along x-axis for the system (disc + body)

$$mv = (m+M)\,v'_x \quad \text{or} \quad v'_x = \frac{mv}{m+M} \tag{1}$$

And from energy conservation, for the same system in the field of gravity :

$$\frac{1}{2}mv^2 = \frac{1}{2}(m+M)\,v'^2_x + \frac{1}{2}m\,v'^2_y + mgh',$$

where h' is the height of break off point from initial level. So,

$$\frac{1}{2}mv^2 = \frac{1}{2}(m+M)\frac{m^2v^2}{(M+m)} + \frac{1}{2}m\,v'^2_y + mgh', \qquad \text{using (1)}$$

or, $$v'^2_y = v^2 - \frac{mv^2}{(m+M)} - 2\,gh'$$

Also, if h'' is the height of the disc, from the break-off point,

then, $$v'^2_y = 2\,gh''$$

So, $$2g\,(h''+h') = v^2 - \frac{m\,v^2}{(M+m)}$$

Hence, the total height, raised from the initial level

$$= h' + h'' = \frac{M\,v^2}{2g\,(M+m)}$$

1.164 (a) When the disc slides and comes to a plank, it has a velocity equal to $v = \sqrt{2\,gh}$. Due to friction between the disc and the plank the disc slows down and after some time the disc moves in one piece with the plank with velocity v' (say).

From the momentum conservation for the system (disc + plank) along horizontal towards right :

$$mv = (m+M)\,v' \quad \text{or} \quad v' = \frac{mv}{m+M}$$

Now from the equation of the increment of total mechanical energy of a system :

$$\frac{1}{2}(M+m)\,v'^2 - \frac{1}{2}mv^2 = A_{fr}$$

or, $$\frac{1}{2}(M+m)\frac{m^2v^2}{(m+M)^2} - \frac{1}{2}mv^2 = A_{fr}$$

so, $$\frac{1}{2}v^2\left[\frac{m^2}{M+m} - m\right] = A_{fr}$$

Hence, $$A_{fr} = -\left(\frac{mM}{m+M}\right)gh = -\,\mu gh$$

$$\left(\text{where } \mu = \frac{mM}{m+M} = \text{reduced mass}\right)$$

(b) We look at the problem from a frame in which the hill is moving (together with the disc on it) to the right with speed u. Then in this frame the speed of the disc when it just gets onto the plank is, by the law of addition of velocities, $\overline{v} = u + \sqrt{2gh}$. Similarly the common speed of the plank and the disc when they move together is

$$\overline{v}' = u + \frac{m}{m+M}\sqrt{2gh}.$$

Then as above $\overline{A}_{fr} = \frac{1}{2}(m+M)\overline{v}'^2 - \frac{1}{2}m\overline{v}^2 - \frac{1}{2}Mu^2$

$$= \frac{1}{2}(m+M)\left\{u^2 + \frac{2m}{m+M}u\sqrt{2gh} + \frac{m^2}{(m+M)^2}2gh\right\} - \frac{1}{2}(m+M)u^2 - \frac{1}{2}m\,2u\sqrt{2gh} - mgh$$

We see that $\overline{A}_{fr}$ is independent of u and is in fact just $-\mu g h$ as in (a). Thus the result obtained does not depend on the choice of reference frame.

Do note however that it will be in correct to apply "conservation of enegy" formula in the frame in which the hill is moving. The energy carried by the hill is not negligible in this frame. See alsó the next problem.

1.165 In a frame moving relative to the earth, one has to include the kinetic energy of the earth as well as earth's acceleration to be able to apply conservation of energy to the problem. In a reference frame falling to the earth with velocity v_0, the stone is initially going up with velocity v_0 and so is the earth. The final velocity of the stone is $0 = v_o - \text{gt}$ and that of the earth is $v_o + \frac{m}{M}gt$ (M is the mass of the earth), from Newton's third law, where $t =$ time of fall. From conservation of energy

$$\frac{1}{2}mv_0^2 + \frac{1}{2}Mv_0^2 + mgh = \frac{1}{2}M\left(v_0 + \frac{m}{M}v_0\right)^2$$

Hence
$$\frac{1}{2}v_0^2\left(m + \frac{m^2}{M}\right) = mgh$$

Negecting $\frac{m}{M}$ in comparison with 1, we get

$$v_0^2 = 2gh \text{ or } v_o = \sqrt{2gh}$$

The point is this in earth's rest frame the effect of earth's accleration is of order $\frac{m}{M}$ ànd can be neglected but in a frame moving with respect to the earth the effect of earth's acceleration must be kept because it is of order one (i.e. large).

1.166 From conservation of momentum, for the closed system "both colliding particles"

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{v}$$

or,
$$\vec{v} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \frac{1(3\vec{i} - 2\vec{j}) + 2(4\vec{j} - 6\vec{k})}{3} = \vec{i} + 2\vec{j} - 4\vec{k}$$

Hence
$$|\vec{v}| = \sqrt{1 + 4 + 16}\text{ m/s} = 4{\cdot}6\text{ m/s}$$

1.167 For perfectly inelastic collision, in the C.M. frame, final kinetic energy of the colliding system (both spheres) becomes zero. Hence initial kinetic energy of the system in C.M. frame completely turns into the internal energy (Q) of the formed body. Hence

$$Q = \tilde{T}_i = \frac{1}{2}\mu \left|\vec{v}_1 - \vec{v}_2\right|^2$$

Now from energy conservation $\Delta T = -Q = -\frac{1}{2}\mu \left|\vec{v}_1 - \vec{v}_2\right|^2$,

In lab frame the same result is obtained as

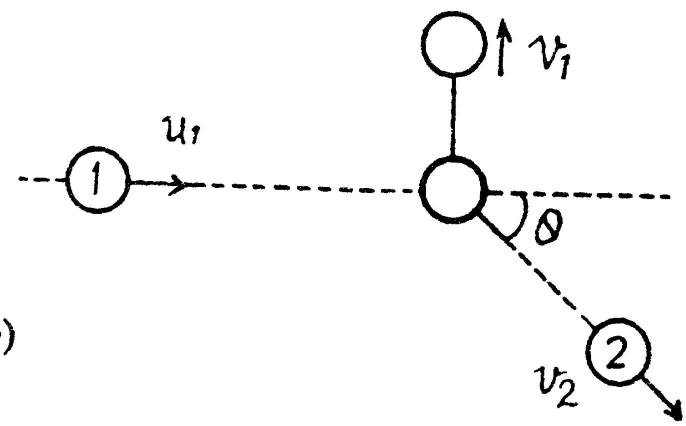
$$\Delta T = \frac{1}{2}\frac{(m_1\vec{v}_1 + m_2\vec{v}_2)^2}{m_1 + m_2} - \frac{1}{2} m_1 \left|\vec{v}_1\right|^2 + m_2 \left|\vec{v}_2\right|^2$$

$$= -\frac{1}{2}\mu \left|\vec{v}_1 - \vec{v}_2\right|^2$$

1.168 (a) Let the initial and final velocities of m_1 and m_2 are $\vec{u}_1$, $\vec{u}_2$ and $\vec{v}$, $\vec{v}_2$ respectively. Then from conservation of momentum along horizontal and vertical directions, we get :

$$m_1 u_1 = m_2 v_2 \cos\theta \qquad (1)$$

and $\quad m_1 v_1 = m_2 v_2 \sin\theta \qquad (2)$



Squaring (1) and (2) and then adding them,

$$m_2^2 v_2^2 = m_1^2 (u_1^2 + v_1^2)$$

Now, from kinetic energy conservation,

$$\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_1 v_1^2 \qquad (3)$$

or, $\quad m\,(u_1^2 - v_1^2) = m_2 v_2^2 = m_2 \frac{m_1^2}{m_2^2}(u_1^2 + v_1^2)$ [Using (3)]

or, $$u_1^2\left(1 - \frac{m_1}{m_2}\right) = v_1^2\left(1 + \frac{m_1}{m_2}\right)$$

or, $$\left(\frac{v_1}{u_1}\right)^2 = \frac{m_2 - m_1}{m_1 + m_2} \qquad (4)$$

So, fraction of kinetic energy lost by the particle 1,

$$= \frac{\frac{1}{2} m_1 u_1^2 - \frac{1}{2} m_1 v_1^2}{\frac{1}{2} m_1 u_1^2} = 1 - \frac{v_1^2}{u_1^2}$$

$$= 1 - \frac{m_2 - m_1}{m_1 + m_2} = \frac{2\,m_1}{m_1 + m_2} \quad \text{[Using (4)]} \qquad (5)$$

(b) When the collision occurs head on,

$$m_1 u_1 = m_1 v_1 + m_2 v_2 \qquad (1)$$

and from conservation of kinetic energy,

$$\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 \left[\frac{m_1 (u_1 - v_1)^2}{m_2} \right]^2 \text{ [Using (5)]}$$

or, $$v_1 \left(1 + \frac{m_1}{m_2}\right) = u_1 \left(\frac{m_1}{m_2} - 1\right)$$

or, $$\frac{v_1}{u_1} = \frac{(m_1 / m_2 - 1)}{(1 + m_1 / m_2)} \quad (6)$$

Fraction of kinetic energy, lost

$$= 1 - \frac{v_1^2}{u_1^2} = 1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 = \frac{4 m_1 m_2}{(m_1 + m_2)^2} \text{ [Using (6)]}$$

1.169 (a) When the particles fly apart in opposite direction with equal velocities (say v), then from conservatin of momentum,

$$m_1 u + 0 = (m_2 - m_1) v \quad (1)$$

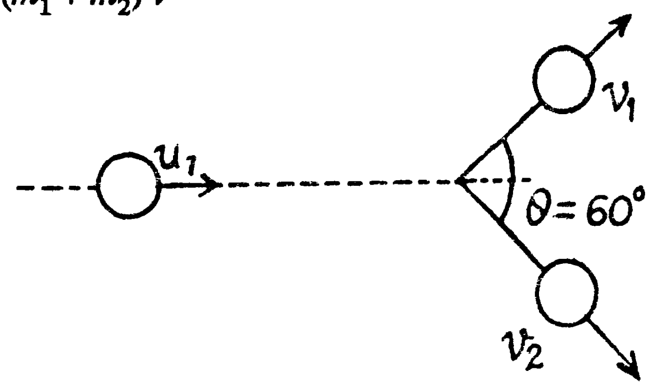
and from conservation of kinetic energy,

$$\frac{1}{2} m_1 u^2 = \frac{1}{2} m_1 v^2 + \frac{1}{2} m_2 v^2$$

or, $$m_1 u^2 = (m_1 + m_2) v^2 \quad (2)$$

From Eq. (1) and (2),

$$m_1 u^2 = (m_1 + m_2) \frac{m_1^2 u^2}{(m_2 - m_1)^2}$$



or, $$m_2^2 - 3 m_1 m_2 = 0$$

Hence $$\frac{m_1}{m_2} = \frac{1}{3} \text{ as } m_2 \neq 0$$

(b) When they fly apart symmetrically relative to the initial motion direction with the angle of divergence $\theta = 60°$,

From conservation of momentum, along horizontal and vertical direction,

$$m_1 u_1 = m_1 v_1 \cos(\theta/2) + m_2 v_2 \cos(\theta/2) \quad (1)$$

and $$m_1 v_1 \sin(\theta/2) = m_2 v_2 \sin(\theta/2)$$

or, $$m_1 v_1 = m_2 v_2 \quad (2)$$

Now, from conservation of kinetic energy,

$$\frac{1}{2} m_1 u_1^2 + 0 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \quad (3)$$

From (1) and (2),

$$m_1 u_1 = \cos(\theta/2) \left(m_1 v_1 + \frac{m_1 v_1}{m_2} m_2 \right) = 2 m_1 v_1 \cos(\theta/2)$$

So, $$u_1 = 2\, v_1 \cos(\theta/2) \tag{4}$$

From (2), (3), and (4)

$$4\, m_1 \cos^2(\theta/2)\, v_1^2 = m_1\, v_1^2 + \frac{m_2\, m_1^2\, v_1^2}{m_2^2}$$

$$\text{or,} \quad 4\cos^2(\theta/2) = 1 + \frac{m_1}{m_2}$$

or, $$\frac{m_1}{m_2} = 4\cos^2\frac{\theta}{2} - 1$$

and putting the value of θ, we get, $\dfrac{m_1}{m_2} = 2$

1.170 If (v_{1x}, v_{1y}) are the instantaneous velocity components of the incident ball and (v_{2x}, v_{2y}) are the velocity components of the struck ball at the same moment, then since there are no external impulsive forces (i.e. other than the mutual interaction of the balls)

We have $$u \sin\alpha = v_{1y} \;,\quad v_{2y} = 0$$

$$m\, u \cos\alpha = m\, v_{1x} + m\, v_{2x}$$

The impulsive force of mutual interaction satisfies

$$\frac{d}{dt}(v_{1x}) = \frac{F}{m} = -\frac{d}{dt}(v_{2x})$$

(F is along the x axis as the balls are smooth. Thus Y component of momentum is not transferred.) Since loss of K.E. is stored as deformation energy D, we have

$$D = \frac{1}{2} m\, u^2 - \frac{1}{2} m\, v_1^{\,2} - \frac{1}{2} m v_2^{\,2}$$

$$= \frac{1}{2} m u^2 \cos^2\alpha - \frac{1}{2} m v_{1x}^{\,2} - \frac{1}{2} m v_{2x}^{\,2}$$

$$= \frac{1}{2m}\left[\, m^2 u^2 \cos^2\alpha - m^2 v_{1x}^{\,2} - (m u \cos\alpha - m v_{1x})^2 \,\right]$$

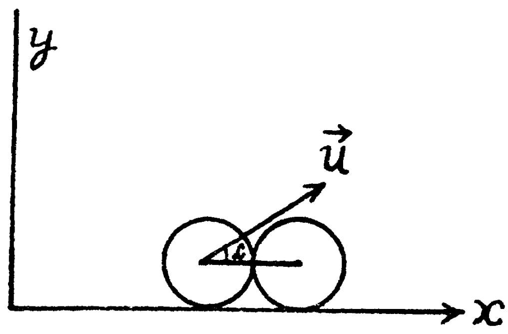
$$= \frac{1}{2m}\left[\, 2 m^2 u \cos\alpha\, v_{1x} - 2 m^2 v_{1x}^{\,2} \,\right] = m\,(v_{1x} u \cos\alpha - v_{1x}^{\,2})$$

$$= m\left[\frac{u^2\cos^2\alpha}{4} - \left(\frac{u\cos\alpha}{2} - v_{1x}\right)^2\right]$$

We see that D is maximum when

$$\frac{u\cos\alpha}{2} = v_{1x}$$

and $$D_{\max} = \frac{m u^2 \cos^2\alpha}{4}$$



Then $$\eta = \frac{D_{\max}}{\frac{1}{2} m u^2} = \frac{1}{2}\cos^2\alpha = \frac{1}{4}$$

On substiuting $\alpha = 45°$

1.171 From the conservation of linear momentum of the shell just before and after its fragmentation

$$3\vec{v} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \quad (1)$$

where $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are the velocities of its fragments.

From the energy conservation $\quad 3\eta v^2 = v_1^2 + v_2^2 + v_3^2 \quad (2)$

Now $$\vec{\tilde{v}}_i \text{ or } \vec{v}_{iC} = \vec{v}_i - \vec{v}_C = \vec{v}_i - \vec{v} \quad (3)$$

where $\vec{v}_C = \vec{v}$ = velocity of the C.M. of the fragments the velocity of the shell. Obviously in the C.M. frame the linear momentum of a system is equal to zero, so

$$\vec{\tilde{v}}_1 + \vec{\tilde{v}}_2 + \vec{\tilde{v}}_3 = 0 \quad (4)$$

Using (3) and (4) in (2), we get

$$3\eta v^2 = (\vec{v} + \vec{\tilde{v}}_1)^2 + (\vec{v} + \vec{\tilde{v}}_2)^2 + (\vec{v} - \vec{\tilde{v}}_1 - \vec{\tilde{v}}_2)^2 \quad = 3v^2 + 2\tilde{v}_1^2 + 2\tilde{v}_2^2 + 2\,\vec{\tilde{v}}_1 \cdot \vec{\tilde{v}}_2$$

or, $$2\tilde{v}_1^2 + 2\tilde{v}_1\,\tilde{v}_2\cos\theta + 2\tilde{v}_2^2 + 3\,(1-\eta)v^2 = 0 \quad (5)$$

If we have had used $\vec{\tilde{v}}_2 = -\vec{\tilde{v}}_1 - \vec{\tilde{v}}_3$, then Eq. 5 were contain $\tilde{v}_3$ instead of $\tilde{v}_2$ and so on.

The problem being symmetrical we can look for the maximum of any one. Obviously it will be the same for each.

For $\tilde{v}_1$to be real in Eq. (5)

$$4\,\tilde{v}_2^2\cos^2\theta \geq 8(2\tilde{v}_2^2 + 3\,(1-\eta)\,v^2) \text{ or } 6(\eta - 1)v^2 \geq (4 - \cos^2\theta)\tilde{v}_2^2$$

So, $$\tilde{v}_2 \leq v\sqrt{\frac{6\,(\eta-1)}{4-\cos^2\theta}} \quad \text{or} \quad \tilde{v}_{2(\max)} = \sqrt{2\,(\eta-1)}\;\; v$$

Hence $$v_{2\,(\max)} = |\vec{v} + \vec{\tilde{v}}_2|_{\max} = v + \sqrt{2\,(\eta-1)}\;v = v\left(1 + \sqrt{2\,(\eta-1)}\right) = 1 \text{ km/s}$$

Thus owing to the symmetry

$$v_{1(\max)} = v_{2\,(\max)} = v_{3(\max)} = v\left(1 + \sqrt{2(\eta-1)}\right) = 1 \text{ km/s}$$

1.172 Since, the collision is head on, the particle 1 will continue moving along the same line as before the collision, but there will be a change in the magnitude of it's velocity vector. Let it starts moving with velocity v_1 and particle 2 with v_2 after collision, then from the conservation of momentum

$$mu = mv_1 + mv_2 \quad \text{or,} \quad u = v_1 + v_2 \quad (1)$$

And from the condition, given,

$$\eta = \frac{\frac{1}{2}mu^2 - \left(\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2\right)}{\frac{1}{2}mu^2} = 1 - \frac{v_1^2 + v_2^2}{u^2}$$

or, $$v_1^2 + v_2^2 = (1-\eta)\,u^2 \quad (2)$$

From (1) and (2),

$$v_1^2 + (u - v_1)^2 = (1-\eta)\,u^2$$

or, $$v_1^2 + u^2 - 2\,uv_1 + v_1^2 = (1-\eta)\,u^2$$

or, $$2v_1^2 - 2v_1 u + \eta u^2 = 0$$

So, $$v_1 = 2u \pm \frac{\sqrt{4u^2 - 8\eta u^2}}{4}$$

$$= \frac{1}{2}\left[u \pm \sqrt{u^2 - 2\eta u^2}\right] = \frac{1}{2}u\,(1 \pm \sqrt{1 - 2\eta}\,)$$

Positive sign gives the velocity of the 2nd particle which lies ahead. The negative sign is correct for v_1 .

So, $v_1 = \frac{1}{2}u\,(1 - \sqrt{1-2\eta}\,) = 5$ m/s will continue moving in the same direction.

Note that $v_1 = 0$ if $\eta = 0$ as it must.

1.173 Since, no external impulsive force is effective on the system "$M + m$", its total momentum along any direction will remain conserved.

So from $p_x =$ const.

$$mu = Mv_1 \cos\theta \quad \text{or,} \quad v_1 = \frac{m}{M}\frac{u}{\cos\theta} \tag{1}$$

and from $p_y =$ const

$$mv_2 = Mv_1 \sin\theta \quad \text{or,} \quad v_2 = \frac{M}{m}v_1 \sin\theta = u\tan\theta, \quad \text{[using (1)]}$$

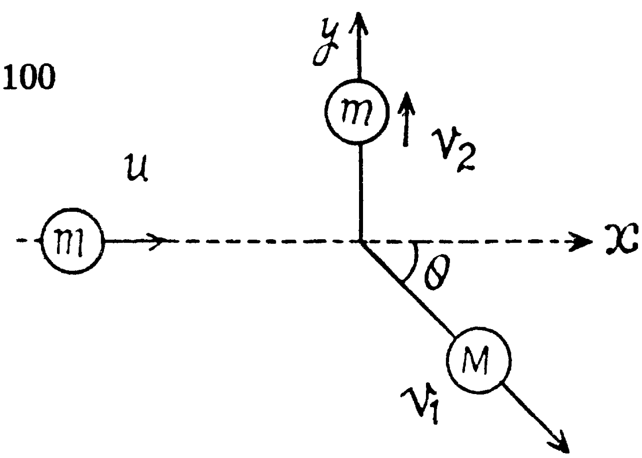
Final kinetic energy of the system

$$T_f = \frac{1}{2}mv_2^2 + \frac{1}{2}Mv_1^2$$

And initial kinetic energy of the system $= \frac{1}{2}mu^2$

So, $$\%\ \text{change} = \frac{T_f - T_i}{T_i} \times 100$$

$$= \frac{\frac{1}{2}m\,u^2\tan^2\theta + \frac{1}{2}M\frac{m^2}{M^2}\frac{u^2}{\cos^2\theta} - \frac{1}{2}mu^2}{\frac{1}{12}mu^2} \times 100$$



$$= \frac{\frac{1}{2}u^2\tan^2\theta + \frac{1}{2}\frac{m}{M}u^2\sec^2\theta - \frac{1}{2}u^2}{\frac{1}{2}u^2} \times 100$$

$$= \left(\tan^2\theta + \frac{m}{M}\sec^2\theta - 1\right) \times 100$$

and putting the values of θ and $\frac{m}{M}$, we get % of change in kinetic energy $= -40\ \%$

1.174 (a) Let the particles m_1 and m_2 move with velocities $\vec{v_1}$ and $\vec{v_2}$ respectively. On the basis of solution of problem 1.147 (b)

$$\tilde{p} = \mu\, v_{rel} = \mu\,\left|\vec{v_1} - \vec{v_2}\right|$$

As $\qquad \vec{v_1} \perp \vec{v_2}$

So, $\qquad \tilde{p} = \mu \sqrt{v_1^2 + v_2^2}$ where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

(b) Again from 1.147 (b)

$$\tilde{T} = \frac{1}{2}\mu v_{rel}^2 = \frac{1}{2}\mu \left|\vec{v_1} - \vec{v_2}\right|^2$$

So, $\qquad \tilde{T} = \dfrac{1}{2}\mu (v_1^2 + v_2^2)$

1.175 From conservation of momentum

$$\vec{p_1} = \vec{p_1}' + \vec{p_2}'$$

so $\qquad \left(\vec{p_1} - \vec{p_1}'\right)^2 = p_1^2 - 2p_1 p_1' \cos\theta_1 + p_1'^2 = p_2'^2$

From conservation of energy

$$\frac{p_1^2}{2m_1} = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2}$$

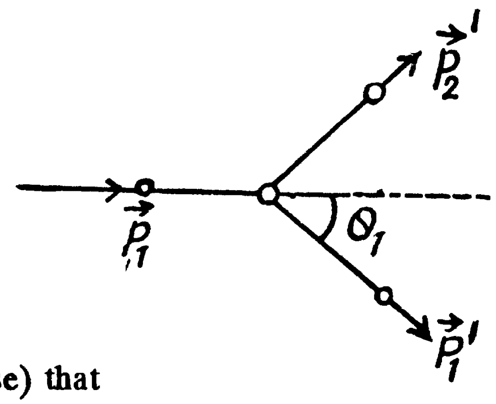
Eliminating p_2' we get

$$0 = p_1'^2\left(1 + \frac{m_2}{m_1}\right) - 2p_1' p_1 \cos\theta_1 + p_1^2\left(1 - \frac{m_2}{m_1}\right)$$

This quadratic equation for p_1' has a real solution in terms of p_1 and $\cos\theta_1$ only if

$$4\cos^2\theta_1 \geq 4\left(1 - \frac{m_2^2}{m_1^2}\right)$$

or $\qquad \sin^2\theta_1 \leq \dfrac{m_2^2}{m_1^2}$



or $\quad \sin\theta_1 \leq +\dfrac{m_2}{m_1}$ or $\sin\theta_1 \geq -\dfrac{m_2}{m_1}$

This clearly implies (since only + sign makes sense) that

$$\sin\theta_{1\,max} = \frac{m_2}{m_1}.$$

1.176 From the symmetry of the problem, the velocity of the disc A will be directed either in the initial direction or opposite to it just after the impact. Let the velocity of the disc A after the collision be v' and be directed towards right after the collision. It is also clear from the symmetry of problem that the discs B and C have equal speed (say v'') in the directions, shown. From the condition of the problem,

$$\cos\theta = \frac{\eta \frac{d}{2}}{d} = \frac{\eta}{2} \text{ so, } \sin\theta = \sqrt{4 - \eta^2}\,/\,2 \qquad (1)$$

For the three discs, system, from the conservation of linear momentum in the symmetry direction (towards right)

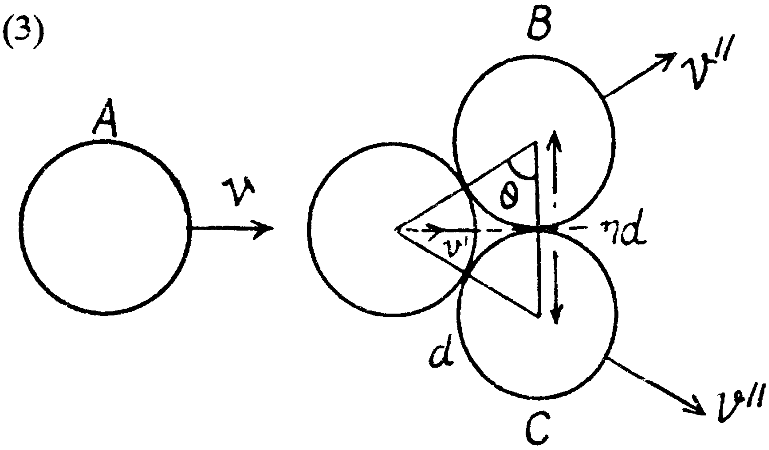
$$mv = 2m\,v''\sin\theta + m\,v' \text{ or, } v = 2\,v''\sin\theta + v' \qquad (2)$$

From the definition of the coefficeint of restitution, we have for the discs A and B (or C)

$$e = \frac{v'' - v' \sin \theta}{v \sin \theta - 0}$$

But $e = 1$, for perfectly elastic collision,

So, $v \sin \theta = v'' - v' \sin \theta$ (3)



From (2) and (3),

$$v' = \frac{v\,(1 - 2 \sin^2 \theta)}{(1 + 2 \sin^2 \theta)}$$

$$= \frac{v\,(\eta^2 - 2)}{6 - \eta^2} \quad \{\text{using (1)}\}$$

Hence we have,

$$v' = \frac{v\,(\eta^2 - 2)}{6 - \eta^2}$$

Therefore, the disc A will recoil if $\eta < \sqrt{2}$ and stop if $\eta = \sqrt{2}$.

Note : *One can write the equations of momentum conservation along the direction perpendicular to the initial direction of disc A and the consevation of kinetic energy instead of the equation of restitution.*

1.177 (a) Let a molecule comes with velocity $\vec{v_1}$ to strike another stationary molecule and just after collision their velocities become $\vec{v}'_1$ and $\vec{v}'_2$ respectively. As the mass of the each molecule is same, conservation of linear momentum and conservation of kinetic energy for the system (both molecules) respectively gives :

$$\vec{v_1} = \vec{v}'_1 + \vec{v}'_2$$

and

$$v_1^2 = v_1'^2 + v_2'^2$$

From the property of vector addition it is obvious from the obtained Eqs. that

$$\vec{v}'_1 \perp \vec{v}'_2 \quad \text{or} \quad \vec{v}'_1 \cdot \vec{v}'_2 = 0$$

(b) Due to the loss of kinetic energy in inelastic collision $v_1^2 > v_1'^2 + v_2'^2$

so, $\vec{v}'_1 \cdot \vec{v}'_2 > 0$ and therefore angle of divergence $< 90°$.

1.178 Suppose that at time t, the rocket has the mass m and the velocity $\vec{v}$, relative to the reference frame, employed. Now consider the inertial frame moving with the velocity that the rocket has at the given moment. In this reference frame, the momentum increament that the rocket & ejected gas system acquires during time dt is,

$$d\vec{p} = m\,d\vec{v} + \mu\,dt\,\vec{u} = \vec{F}\,dt$$

or,

$$m\frac{d\vec{v}}{dt} = \vec{F} - \mu\,\vec{u}$$

or,

$$m\,\vec{w} = \vec{F} - \mu\,\vec{u}$$

1.179 According to the question, $\vec{F} = 0$ and $\mu = -dm/dt$ so the equation for this system becomes,

$$m\frac{d\vec{v}}{dt} = \frac{dm}{dt}\vec{u}$$

As $d\vec{v} \uparrow\downarrow \vec{u}$ so, $m\,dv = -u\,dm.$

Integrating within the limits :

$$\frac{1}{u}\int_0^v dv = -\int_{m_0}^{m}\frac{dm}{m} \quad \text{or} \quad \frac{v}{u} = \ln\frac{m_0}{m}$$

Thus, $v = u\ln\dfrac{m_0}{m}$

As $d\vec{v} \uparrow\downarrow \vec{u}$, so in vector form $\vec{v} = -\vec{u}\ln\dfrac{m_0}{m}$

1.180 According to the question, $\vec{F}$ (external force) = 0

So, $$m\frac{d\vec{v}}{dt} = \frac{dm}{dt}\vec{u}$$

As $$d\vec{v} \uparrow\downarrow \vec{u},$$

so, in scalar form, $$m\,dv = -u\,dm$$

or, $$\frac{wdt}{u} = -\frac{dm}{m}$$

Integrating within the limits for $m(t)$

$$\frac{wt}{u} = -\int_{m_0}^{m}\frac{dm}{m} \quad \text{or,} \quad \frac{v}{u} = -\ln\frac{m}{m_0}$$

Hence, $$m = m_0\,e^{-(wt/u)}$$

1.181 As $\vec{F} = 0$, from the equation of dynamics of a body with variable mass;

$$m\frac{d\vec{v}}{dt} = \vec{u}\frac{dm}{dt} \quad \text{or,} \quad d\vec{v} = \vec{u}\frac{dm}{m} \tag{1}$$

Now $d\vec{v} \uparrow\downarrow \vec{u}$ and since $\vec{u} \perp \vec{v}$, we must have $|d\vec{v}| = v_0\,d\alpha$ (because v_0 is constant) where $d\alpha$ is the angle by which the spaceship turns in time dt.

So, $$-u\frac{dm}{m} = v_0\,d\alpha \quad \text{or,} \quad d\alpha = -\frac{u}{v_0}\frac{dm}{m}$$

or, $$\alpha = -\frac{u}{v_0}\int_{m_0}^{m}\frac{dm}{m} = \frac{u}{v_0}\ln\left(\frac{m_0}{m}\right)$$

1.182 We have $\frac{dm}{dt} = -\mu$ or, $dm = -\mu\, dt$

Integrating $$\int_{m_0}^{m} dm = -\mu \int_0^t dt \text{ or, } m = m_0 - \mu t$$

As $\vec{u} = 0$ so, from the equation of variable mass system :

$$(m_0 - \mu t)\frac{d\vec{v}}{dt} = \vec{F} \text{ or, } \frac{d\vec{v}}{dt} = \vec{w} = \vec{F}/(m_0 - \mu t)$$

or, $$\int_0^{\vec{v}} d\vec{v} = \vec{F}\int_0^t \frac{dt}{(m_0 - \mu t)}$$

Hence $$\vec{v} = \frac{\vec{F}}{\mu} \ln\left(\frac{m_0}{m_0 - \mu t}\right)$$

1.183 Let the car be moving in a reference frame to which the hopper is fixed and at any instant of time, let its mass be m and velocity $\vec{v}$.

Then from the general equation, for variable mass system.

$$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{u}\frac{dm}{dt}$$

We write the equation, for our system as,

$$m\frac{d\vec{v}}{dt} = \vec{F} - \vec{v}\frac{dm}{dt} \text{ as, } \vec{u} = -\vec{v} \qquad (1)$$

So $$\frac{d}{dt}(m\vec{v}) = \vec{F}$$

and $$\vec{v} = \frac{\vec{F}t}{m} \text{ on integration.}$$

But $$m = m_0 + \mu t$$

so, $$\vec{v} = \frac{\vec{F}t}{m_0\left(1 + \frac{\mu t}{m_0}\right)}$$

Thus the sought acceleration, $$\vec{w} = \frac{d\vec{v}}{dt} = \frac{\vec{F}}{m_0\left(1 + \frac{\mu t}{m_0}\right)^2}$$

1.184 Let the length of the chain inside the smooth horizontal tube at an arbitrary instant is x. From the equation,

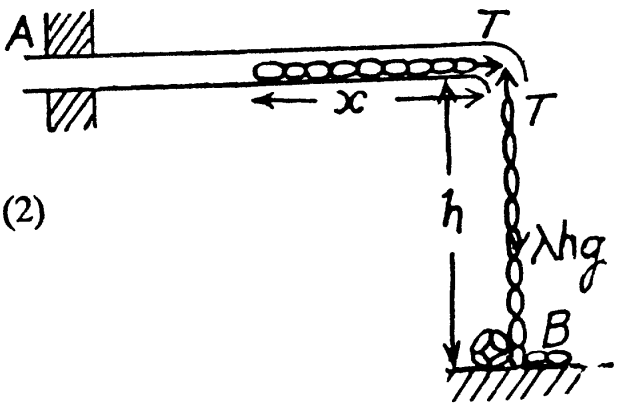
$$m\vec{w} = \vec{F} + \vec{u}\frac{dm}{dt}$$

as $\vec{u} = 0$, $\vec{F} \uparrow\uparrow \vec{w}$, for the chain inside the tube

$$\lambda x w = T \text{ where } \lambda = \frac{m}{l} \qquad (1)$$

Similarly for the overhanging part,

$$\vec{u} = 0$$



Thus $\qquad mw = F$

or $\qquad \lambda h w = \lambda h g - T \qquad (2)$

From (1) and (2),

$$\lambda (x + h) w = \lambda h g \quad \text{or,} \quad (x + h) v \frac{dv}{ds} = hg$$

or,
$$(x + h) v \frac{dv}{(-dx)} = gh,$$

[As the length of the chain inside the tube decreases with time, $ds = -dx$.]

or,
$$v\,dv = -g h \frac{dx}{x + h}$$

Integrating,
$$\int_0^v v\,dv = -g h \int_{(l-h)}^{0} \frac{dx}{x + h}$$

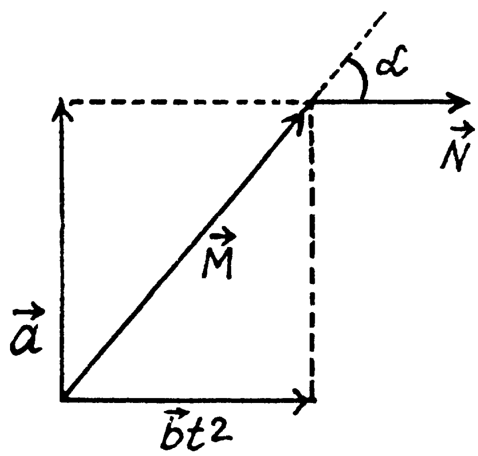
or,
$$\frac{v^2}{2} = gh \ln\left(\frac{l}{h}\right) \quad \text{or} \quad v = \sqrt{2gh \ln\left(\frac{l}{h}\right)}$$

1.185 Force moment relative to point O ;

$$\vec{N} = \frac{d\vec{M}}{dt} = 2\vec{bt}$$

Let the angle between $\vec{M}$ and $\vec{N}$,

$\alpha = 45°$ at $t = t_0$,



$$\text{Then } \frac{1}{\sqrt{2}} = \frac{\vec{M} \cdot \vec{N}}{|\vec{M}||\vec{N}|} = \frac{(\vec{a} + \vec{b} t_0^2) \cdot (2\vec{b} t_0)}{\sqrt{a^2 + b^2 t_0^4}\; 2bt_0}$$

$$= \frac{2b^2 t_0^3}{\sqrt{a^2 + b^2 t_0^4}\; 2bt_0} = \frac{b t_0^2}{\sqrt{a^2 + b^2 t_0^4}}$$

So, $\quad 2 b^2 t_0^4 = a^2 + b^2 t_0^4 \quad$ or, $\quad t_0 = \sqrt{\frac{a}{b}}$ (as t_0 cannot be negative)

It is also obvious from the figure that the angle α is equal to 45° at the moment t_0, when $a = b t_0^2$, i.e. $t_0 = \sqrt{a/b}$ and $\vec{N} = 2\sqrt{\frac{a}{b}}\,\vec{b}$.

1.186 $\vec{M}(t) = \vec{r} \times \vec{p} = \left(\vec{v}_0 t + \frac{1}{2}\vec{g}t^2\right) \times m\left(\vec{v}_0 + \vec{g}t\right)$

$$= mv_0 gt^2 \sin\left(\frac{\pi}{2} + \alpha\right)(-\vec{k}) + \frac{1}{2}mv_0 gt^2 \sin\left(\frac{\pi}{2} + \alpha\right)(\vec{k})$$

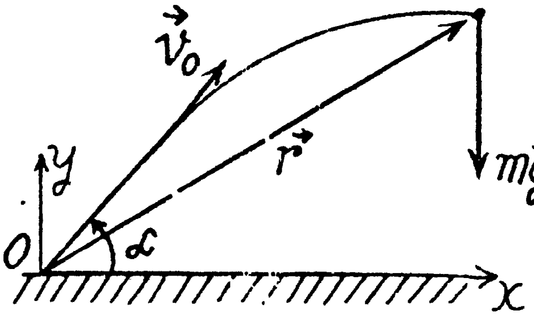
$$= \frac{1}{2}mv_0 gt^2 \cos\alpha\,(-\vec{k}):$$

Thus $M(t) = \dfrac{mv_0 g t^2 \cos\alpha}{2}$

Thus angular momentum at maximum height

i.e. at $t = \dfrac{\tau}{2} = \dfrac{v_0 \sin\alpha}{g}$,

$M\left(\dfrac{\tau}{2}\right) = \left(\dfrac{m v_0^3}{2g}\right) \sin^2\alpha \cos\alpha = 37 \text{ kg} - \text{m}^2/\text{s}$

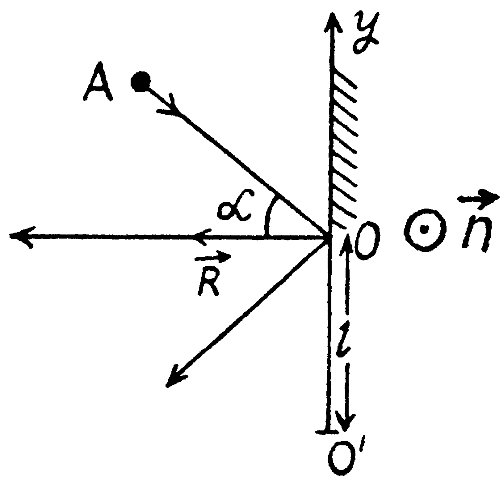


Alternate :

$$\vec{M}(0) = 0 \text{ so, } \vec{M}(t) = \int_0^t \vec{N}\,dt = \int_0^t (\vec{r} \times m\vec{g})$$

$$= \int_0^t \left[\left(\vec{v}_0 t + \frac{1}{2}\vec{g}t^2\right) \times m\vec{g}\right] dt = \left(\vec{v}_0 \times m\vec{g}\right)\frac{t^2}{2}$$

1.187 (a) The disc experiences gravity, the force of reaction of the horizontal surface, and the force $\vec{R}$ of reaction of the wall at the moment of the impact against it. The first two forces counter-balance each other, leaving only the force $\vec{R}$. It's moment relative to any point of the line along which the vector $\vec{R}$ acts or along normal to the wall is equal to zero and therefore the angular momentum of the disc relative to any of these points does not change in the given process.



(b) During the course of collision with wall the position of disc is same and is equal to $\vec{r}_{oo'}$. Obviously the increment in linear momentum of the ball $\Delta\vec{p} = 2mv\cos\alpha\,\hat{n}$

Here, $\Delta\vec{M} = \vec{r}_{oo'} \times \Delta\vec{p} = 2\,mv\,l\cos\alpha\,\hat{n}$ and directed normally emerging from the plane of figure

Thus $|\Delta\vec{M}| = 2\,mv\,l\cos\alpha$

1.188 (a) The ball is under the influence of forces $\vec{T}$ and $m\vec{g}$ at all the moments of time, while moving along a horizontal circle. Obviously the vertical component of $\vec{T}$ balance $m\vec{g}$ and

so the net moment of these two about any point becoems zero. The horizontal component of $\vec{T}$, which provides the centripetal acceleration to ball is already directed toward the centre (C) of the horizontal circle, thus its moment about the point C equals zero at all the moments of time. Hence the net moment of the force acting on the ball about point C equals zero and that's why the angular mommetum of the ball is conserved about the horizontal circle.

(b) Let α be the angle which the thread forms with the vertical.

Now from equation of particle dynamics :

$T\cos\alpha = mg$ and $T\sin\alpha = m\omega^2 l\sin\alpha$

Hence on solving $\cos\alpha = \frac{g}{\omega^2 l}$ (1)

As $|\vec{M}|$ is constant in magnitude so from figure.

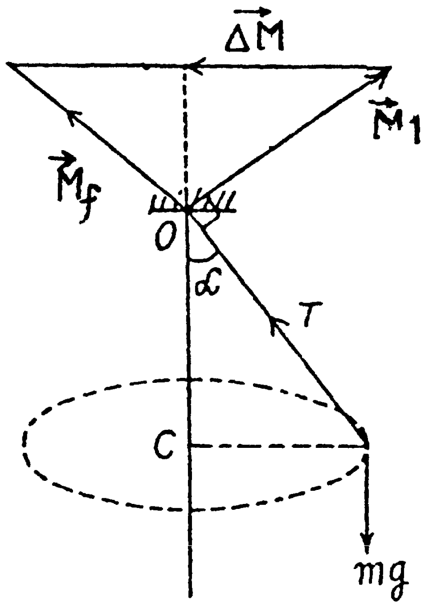
$|\Delta\vec{M}| = 2M\cos\alpha$ where

$M = |\vec{M}_i| = |\vec{M}_f|$

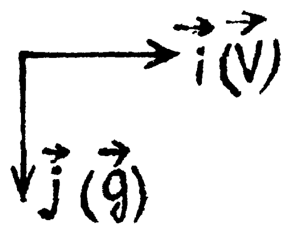
$= |\vec{r}_{bo}\times m\vec{v}| = mvl\left(\text{as } \vec{r}_{bo}\perp\vec{v}\right)$

Thus$|\Delta\vec{M}| = 2mvl\cos\alpha = 2m\omega l^2\sin\alpha\cos\alpha$

$$= \frac{2mgl}{\omega}\sqrt{1-\left(\frac{g}{\omega^2 t}\right)^2} \text{ (using 1).}$$



1.189 During the free fall time $t = \tau = \sqrt{\frac{2h}{g}}$, the reference point O moves in hoizontal direction (say towards right) by the distance $V\tau$. In the translating frame as $\vec{M}(O) = 0$, so



$$\Delta\vec{M} = \vec{M}_f = \vec{r}$$
$$= (-V\tau\vec{i} + h\vec{j})\times m[g\tau\vec{j} - V\vec{i}]$$
$$= -mVg\tau^2\vec{h} + mVh(+\vec{k})$$
$$= -mVg\left(\frac{2h}{g}\right)\vec{k} + mVh(+\vec{k}) = -mVh\vec{k}$$

Hence $|\Delta\vec{M}| = mVh$

1.190 The Coriolis force is $(2m\vec{v}' \times \vec{\omega})$.
Here $\vec{\omega}$ is along the z-axis (vertical). The moving disc is moving with velocity v_0 which is constant. The motion is along the x-axis say. Then the Coriolis force is along y-axis and has the magnitude $2m v_0\omega$. At time t, the distance of the centre of moving disc from O is $v_0 t$ (along x-axis). Thus the torque N due to the coriolis force is
$N = 2m v_0\omega\cdot v_0 t$ along the z-axis.

Hence equating this to $\frac{dM}{dt}$

$$\frac{dM}{dt} = 2m\, v_0^2\, \omega t \quad \text{or} \quad M = m v_0^2\, \omega t^2 + \text{constant.}$$

The constant is irrelevant and may be put equal to zero if the disc is originally set in motion from the point O.

This discussion is approximate. The Coriolis force will cause the disc to swerve from straight line motion and thus cause deviation from the above formula which will be substantial for large t.

1.191 If $\dot{r}$ = radial velocity of the particle then the total energy of the particle at any instant is

$$\frac{1}{2} m \dot{r}^2 + \frac{M^2}{2mr^2} + kr^2 = E \qquad (1)$$

where the second term is the kinetic energy of angular motion about the centre O. Then the extreme values of r are determined by $\dot{r} = 0$ and solving the resulting quadratic equation

$$k(r^2)^2 - Er^2 + \frac{M^2}{2m} = 0$$

we get

$$r^2 = \frac{E \pm \sqrt{E^2 - \frac{2M^2 k}{m}}}{2k}$$

From this we see that

$$E = k(r_1^2 + r_2^2) \qquad (2)$$

where r_1 is the minimum distance from O and r_2 is the maximum distance. Then

$$\frac{1}{2} m v_2^2 + 2kr_2^2 = k\,(\,r_1^2 + r_2^2\,)$$

Hence, $$m = \frac{2kr^2}{v_2^2}$$

Note : Eq. (1) can be derived from the standard expression for kinetic energy and angular momentum in plane poler coordinates :

$$T = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$$

M = angular momentum = $mr^2 \dot{\theta}$

1.192 The swinging sphere experiences two forces : The gravitational force and the tension of the thread. Now, it is clear from the condition, given in the problem, that the moment of these forces about the vertical axis, passing through the point of suspension $N_z = 0$. Consequently, the angular momentum M_z of the sphere relative to the given axis (z) is constant.

Thus $$m v_0\,(l \sin\theta) = m v\, l \qquad (1)$$

where m is the mass of the sphere and v is it s velocity in the position, when the thread forms an angle $\frac{\pi}{2}$ with the vertical. Mechanical energy is also conserved, as the sphere is

under the influence if only one other force, i.e. tension, which does not perform any work, as it is always perpendicular to the velocity.

So, $$\frac{1}{2} m v_0^2 + m g\, l \cos\theta = \frac{1}{2} m v^2 \qquad (2)$$

From (1) and (2), we get,

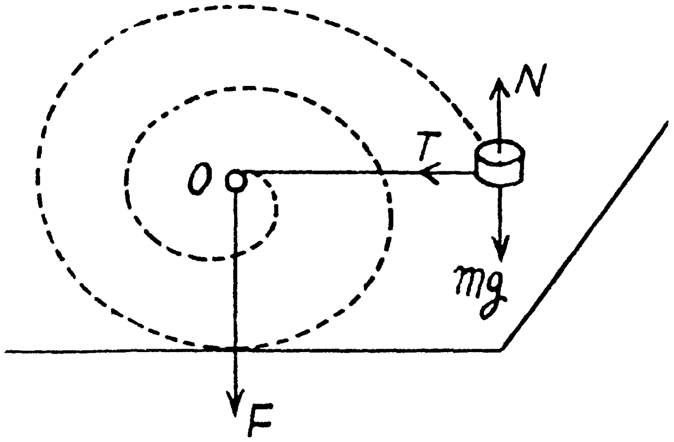
$$v_0 = \sqrt{2gl/\cos\theta}$$

1.193 Forces, acting on the mass m are shown in the figure. As $\vec{N} = m\vec{g}$, the net torque of these two forces about any fixed point must be equal to zero. Tension T, acting on the mass m is a central force, which is always directed towards the centre O. Hence the moment of force T is also zero about the point O and therefore the angular momentum of the particle m is conserved about O.

Let, the angular velocity of the particle be ω, when the separation between hole and particle m is r, then from the conservation of momentum about the point O, :

$$m(\omega_0 r_0) r_0 = m(\omega r) r,$$

or $$\omega = \frac{\omega_0 r_0^2}{r^2}$$



Now, from the second law of motion for m,

$$T = F = m\omega^2 r$$

Hence the sought tension;

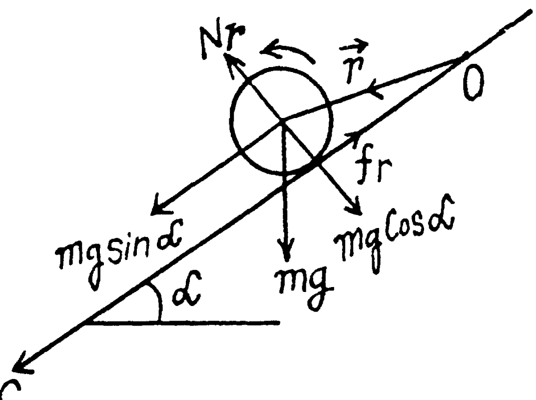
$$F = \frac{m\omega_0^2 r_0^4 r}{r^4} = \frac{m\omega_0^2 r_0^4}{r^3}$$

1.194 On the given system the weight of the body m is the only force whose moment is effective about the axis of pulley. Let us take the sense of $\vec{\omega}$ of the pulley at an arbitrary instant as the positive sense of axis of rotation (z-axis)

As $$M_z(0) = 0, \text{ so, } \Delta M_z = M_z(t) = \int N_z\, dt$$

So, $$M_z(t) = \int_0^t mgR\, dt = mgRt$$

1.195 Let the point of contact of sphere at initial moment ($t = 0$) be at O. At an arbitrary moment, the forces acting on the sphere are shown in the figure. We have normal reaction $N_r = mg\sin\alpha$ and both pass through same line and the force of static friction passes through the point O, thus the moment about point O becomes zero. Hence $mg\sin\alpha$ is the only force which has effective torque about point O, and is given by $|\vec{N}| = mgR\sin\alpha$ normally emerging from the plane of figure.



As $\vec{M}(t=0) = 0$, so, $\Delta\vec{M} = \vec{M}(t) = \int \vec{N}\, dt$

Hence, $\quad M(t) = Nt = mgR\sin\alpha t$

1.196 Let position vectors of the particles of the system be $\vec{r_i}$ and $\vec{r_i}'$ with respect to the points O and O' respectively. Then we have,

$$\vec{r_i} = \vec{r_i}' + \vec{r_0} \tag{1}$$

where $\vec{r_0}$ is the radius vector of O' with respect to O.

Now, the angular momentum of the system relative to the point O can be written as follows;

$$\vec{M} = \sum \left(\vec{r_i} \times \vec{p_i}\right) = \sum \left(\vec{r_i}' \times \vec{p_i}\right) + \sum \left(\vec{r_0} \times \vec{p_i}\right) \qquad \text{[using (1)]}$$

or, $$\vec{M} = \vec{M}' + \left(\vec{r_0} \times \vec{p}\right), \text{ where, } \vec{p} = \sum \vec{p_i} \tag{2}$$

From (2), if the total linear momentum of the system, $\vec{p} = 0$, then its angular momentum does not depend on the choice of the point O.

Note that in the C.M. frame, the system of particles, as a whole is at rest.

1.197 On the basis of solution of problem 1.196, we have concluded that; "in the C.M. frame, the angular momentum of system of particles is independent of the choice of the point, relative to which it is determined" and in accordance with the problem, this is denoted by $\vec{M}$.

We denote the angular momentum of the system of particles, relative to the point O, by $\vec{M}$. Since the internal and proper angular momentum $\tilde{\vec{M}}$, in the C.M. frame, does not depend on the choice of the point O', this point may be taken coincident with the point O of the K-frame, at a given moment of time. Then at that moment, the radius vectors of all the particles, in both reference frames, are equal $(\vec{r_i}' = \vec{r_i})$ and the velocities are related by the equation,

$$\vec{v_i} = \tilde{\vec{v}}_i + \vec{v_c}, \tag{1}$$

where $\vec{v_c}$ is the velocity of C.M. frame, relative to the K-frame. Consequently, we may write,

$$\vec{M} = \sum m_i \left(\vec{r_i} \times \vec{v_i}\right) = \sum m_i \left(\vec{r_i}' \times \tilde{\vec{v}}_i\right) + \sum m_i \left(\vec{r_i} \times \vec{v_c}\right)$$

or, $$\vec{M} = \tilde{\vec{M}} + m\left(\vec{r_c} \times \vec{v_c}\right), \text{ as } \sum m_i \vec{r_i} = m\vec{r_c}, \text{ where } m = \sum m_i.$$

or, $$\vec{M} = \tilde{\vec{M}} + (\vec{r_c} \times m\vec{v_c}) = \tilde{\vec{M}} + (\vec{r_c} \times \vec{p})$$

1.198 From conservation of linear momentum along the direction of incident ball for the system consists with colliding ball and phhere

$$mv_0 = mv' + \frac{m}{2}v_1 \tag{1}$$

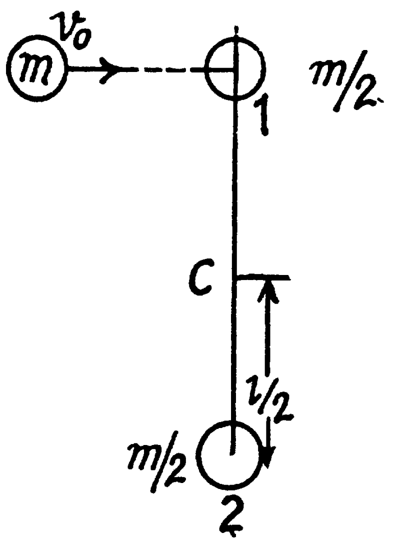
where v' and v_1 are the velocities of ball and sphere 1 respectively after collision. (Remember that the collision is head on).

As the collision is perfectly elastic, from the definition of co-efficeint of restitution,

$$1 = \frac{v' - v_1}{0 - v_0} \text{ or, } v' - v_1 = -v_0 \tag{2}$$

Solving (1) and (2), we get,

$v_1 = \frac{4\,v_0}{3}$, directed towards right.



In the C.M. frame of spheres 1 and 2 (Fig.)

$\vec{\tilde{p}}_1 = -\vec{\tilde{p}}_2$ and $|\vec{\tilde{p}}_1| = |\vec{\tilde{p}}_2| = \mu\,|\vec{v}_1 - \vec{v}_2|$

Also, $\vec{r}_{1C} = -\vec{r}_{2C}$, thus $\vec{\tilde{M}} = 2\,[\vec{r}_{1C} \times \vec{\tilde{p}}_1]$

As $\vec{r}_{1C} \perp \vec{\tilde{p}}_1$, so, $\vec{\tilde{M}} = 2\left[\frac{l}{2}\,\frac{m/2}{2}\,\frac{4\,v_0}{3}\,\hat{n}\right]$

(where $\hat{n}$ is the unit vector in the sense of $\vec{r}_{1C} \times \vec{\tilde{p}}_1$)

Hence $\tilde{M} = \frac{m\,v_0\,l}{3}$

1.199 In the C.M. frame of the system (both the discs + spring), the linear momentum of the discs are related by the relation, $\vec{\tilde{p}}_1 = -\vec{\tilde{p}}_2$, at all the moments of time.

where, $\tilde{p}_1 = \tilde{p}_2 = \tilde{p} = \mu\,v_{rel}$

And the total kinetic energy of the system,

$$T = \frac{1}{2}\mu\,v_{rel}^2 \text{ [See solution of 1.147 (b)]}$$

Bearing in mind that at the moment of maximum deformation of the spring, the projection of $\vec{v}_{rel}$ along the length of the spring becomes zero, i.e. $v_{rel\,(x)} = 0$.

The conservation of mechanical energy of the considered system in the C.M. frame gives.

$$\frac{1}{2}\left(\frac{m}{2}\right)v_0^2 = \frac{1}{2}\kappa x^2 + \frac{1}{2}\left(\frac{m}{2}\right)v_{rel\,(y)}^2 \qquad (1)$$

Now from the conservation of angular momentum of the system about the C.M.,

$$\frac{1}{2}\left(\frac{l_0}{2}\right)\left(\frac{m}{2}v_0\right) = 2\left(\frac{l_0 + x}{2}\right)\frac{m}{2}v_{rel\,(y)}$$

or, $$v_{rel\,(y)} = \frac{v_0\,l_0}{(l_0 + x)} = v_0\left(1 + \frac{x}{l_0}\right)^{-1} \approx v_0\left(1 - \frac{x}{l_0}\right), \text{ as } x \ll l_0 \qquad (2)$$

Using (2) in (1), $$\frac{1}{2}m\,v_0^2\left[1 - \left(1 - \frac{x}{l_0}\right)^2\right] = \kappa x^2$$

or, $$\frac{1}{2}m\,v_0^2\left[1 - \left(1 - \frac{2x}{l_0} + \frac{x^2}{l_0^2}\right)^2\right] = \kappa x^2$$

or, $$\frac{m\,v_0^2\,x}{l_0} \approx \kappa x^2, \text{ [neglecting } x^2/l_0^2]$$

As $$x \neq 0, \text{ thus } x = \frac{m\,v_0^2}{\kappa\,l_0}$$

1.4 UNIVERSAL GRAVITATION

1.200 We have

$$\frac{Mv^2}{r} = \frac{\gamma M m_s}{r^2} \text{ or } r = \frac{\gamma m_s}{v^2}$$

Thus
$$\omega = \frac{v}{r} = \frac{v}{\gamma m_s / v^2} = \frac{v^3}{\gamma m_s}$$

(Here m_s is the mass of the Sun.)

So $$T = \frac{2\pi \gamma m_s}{v^3} = \frac{2\pi \times 6{\cdot}67 \times 10^{-11} \times 1{\cdot}97 \times 10^{30}}{(34{\cdot}9 \times 10^3)^3} = 1{\cdot}94 \times 10^7 \text{ sec} = 225 \text{ days.}$$

(The answer is incorrectly written in terms of the planetary mass M)

1.201 For any planet

$$M R \omega^2 = \frac{\gamma M m_s}{R^2} \text{ or } \omega = \sqrt{\frac{\gamma m_s}{R^3}}$$

So,
$$T = \frac{2\pi}{\omega} = 2\pi R^{3/2} / \sqrt{\gamma m_s}$$

(a) Thus
$$\frac{T_J}{T_E} = \left(\frac{R_J}{R_E}\right)^{3/2}$$

So
$$\frac{R_J}{R_E} = (T_J / T_E)^{2/3} = (12)^{2/3} = 5 \cdot 24.$$

(b)
$$V_J^2 = \frac{\gamma m_s}{R_J}, \text{ and } R_J = \left(T \frac{\sqrt{\gamma m_s}}{2\pi}\right)^{2/3}$$

So
$$V_J^2 = \frac{(\gamma m_s)^{2/3} (2\pi)^{2/3}}{T^{2/3}} \quad \text{or, } V_J = \left(\frac{2\pi \gamma m_s}{T}\right)^{2/3}$$

where $T = 12$ years. $m_s =$ mass of ths Sun.

Putting the values we get $V_J = 12{\cdot}97$ km/s

$$\text{Acceleration} = \frac{v_J^2}{R_J} = \left(\frac{2\pi \gamma m_s}{T}\right)^{2/3} \times \left(\frac{2\pi}{T\sqrt{\gamma m_s}}\right)^{2/3}$$

$$= \left(\frac{2\pi}{T}\right)^{4/3} (\gamma m_s)^{1/3}$$

$$= 2{\cdot}15 \times 10^{-4} \text{ km/s}^2$$

1.202 Semi-major axis= $(r+R)/2$

It is sufficient to consider the motion be along a circle of semi-major axis $\frac{r+R}{2}$ for T does not depend on eccentricity.

Hence
$$T = \frac{2\pi\left(\frac{r+R}{2}\right)^{3/2}}{\sqrt{\gamma m_s}} = \pi\sqrt{(r+R)^3/2\gamma m_s}$$

(again m_s is the mass of the Sun)

1.203 We can think of the body as moving in a very elongated orbit of maximum distance R and minimum distance 0 so semi major axis $= R/2$. Hence if τ is the time of fall then

$$\left(\frac{2\tau}{T}\right)^2 = \left(\frac{R/2}{R}\right)^3 \quad \text{or} \quad \tau^2 = T^2/32$$

or
$$\tau = T/4\sqrt{2} = 365/4\sqrt{2} = 64{\cdot}5 \text{ days.}$$

1.204 $T = 2\pi R^{3/2}/\sqrt{\gamma m_s}$

If the distances are scaled down, $R^{3/2}$ decreases by a factor $\eta^{3/2}$ and so does m_s. Hence T does not change.

1.205 The double star can be replaced by a single star of mass $\frac{m_1 m_2}{m_1+m_2}$ moving about the centre of mass subjected to the force $\gamma m_1 m_2/r^2$. Then

$$T = \frac{2\pi r^{3/2}}{\sqrt{\gamma m_1 m_2 \Big/ \frac{m_1 m_2}{m_1+m_2}}} = \frac{2\pi r^{3/2}}{\sqrt{\gamma M}}$$

So
$$r^{3/2} = \frac{T}{2\pi}\sqrt{\gamma M}$$

or,
$$r = \left(\frac{T}{2\pi}\right)^{2/3}(\gamma M)^{1/3} = \sqrt[3]{\gamma M (T/2\pi)^2}$$

1.206 (a) The gravitational potential due to m_1 at the point of location of m_2 :

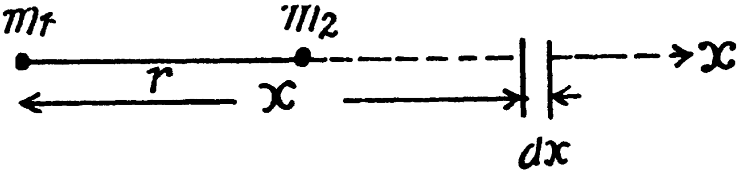
$$V_2 = \int_r^{\infty} \vec{G}\cdot d\vec{r} = \int_r^{\infty} -\frac{\gamma m_1}{x^2}dx = -\frac{\gamma m_1}{r}$$

So,
$$U_{21} = m_2 V_2 = -\frac{\gamma m_1 m_2}{r}$$

Similarly
$$U_{12} = -\frac{\gamma m_1 m_2}{r}$$

Hence

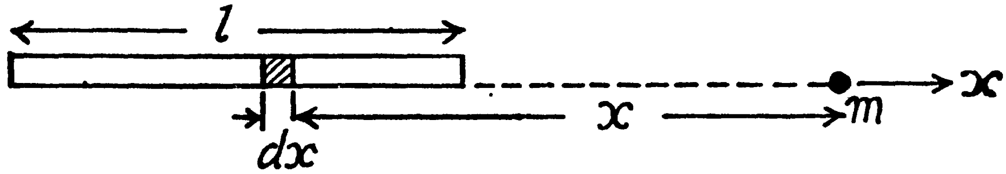
$$U_{12} = U_{21} = U = -\frac{\gamma m_1 m_2}{r}$$



(b) Choose the location of the point mass as the origin. Then the potential energy dU of an element of mass $dM = \frac{M}{l}dx$ of the rod in the field of the point mass is

$$dU = -\gamma m \frac{M}{l} dx \frac{1}{x}$$

where x is the distance between the element and the point. (Note that the rod and the point mass are on a straight line.) If then a is the distance of the nearer end of the rod from the point mass.



$$U = -\gamma \frac{mM}{l} \int_a^{a+l} \frac{dx}{x} = -\gamma m \frac{M}{l} \ln\left(1 + \frac{l}{a}\right)$$

The force of interaction is

$$F = -\frac{\partial U}{\partial a}$$

$$= \gamma \frac{mM}{l} \times \frac{1}{1 + \frac{l}{a}} \left(-\frac{l}{a^2}\right) = -\frac{\gamma mM}{a(a+l)}$$

Minus sign means attraction.

1.207 As the planet is under central force (gravitational interaction), its angular momentum is conserved about the Sun (which is situated at one of the focii of the ellipse)

So, $$m v_1 r_1 = m v_2 r_2 \quad \text{or,} \quad v_1^2 = \frac{v_2^2 r_2^2}{r_1^2} \tag{1}$$

From the conservation of mechanical energy of the system (Sun + planet),

$$-\frac{\gamma m_s m}{r_1} + \frac{1}{2} m v_1^2 = -\frac{\gamma m_s m}{r_2} + \frac{1}{2} m v_2^2$$

or, $$-\frac{\gamma m_s}{r_1} + \frac{1}{2} v_2^2 \frac{r_2^2}{r_1^2} = -\left(\frac{\gamma m_s}{r_2}\right) + \frac{1}{2} v_2^2 \quad \text{[Using (1)]}$$

Thus, $$v_2 = \sqrt{2 \gamma m_s r_1 / r_2 (r_1 + r_2)} \tag{2}$$

Hence $$M = m v_2 r_2 = m \sqrt{2 \gamma m_s r_1 r_2 / (r_1 + r_2)}$$

1.208 From the previous problem, if r_1 , r_2 are the maximum and minimum distances from the sun to the planet and v_1 , v_2 are the corresponding velocities, then, say,

$$E = \frac{1}{2} m v_2^2 - \frac{\gamma m m_s}{r_2}$$

$$= \frac{\gamma m m_s}{r_1 + r_2} \cdot \frac{r_1}{r_2} - \frac{\gamma m m_s}{r_2} = -\frac{\gamma m m_s}{r_1 + r_2} = -\frac{\gamma m m_s}{2a} \text{ [Using Eq. (2) of 1.207]}$$

where $2a$ = major axis $= r_1 + r_2$. The same result can also be obtained directly by writing an equation analogous to Eq (1) of problem 1.191.

$$E = \frac{1}{2} m \dot{r}^2 + \frac{M^2}{2mr^2} - \frac{\gamma m m_s}{r}$$

(Here M is angular momentum of the planet and m is its mass). For extreme position $\dot{r} = 0$ and we get the quadratic

$$Er^2 + \gamma m m_s r - \frac{M^2}{2m} = 0$$

The sum of the two roots of this equation are

$$r_1 + r_2 = -\frac{\gamma m m_s}{E} = 2a$$

Thus

$$E = -\frac{\gamma m m_s}{2a} = \text{ constant}$$

1.209 From the conservtion of angular momentum about the Sun.

$$m v_0 r_0 \sin \alpha = m v_1 r_1 = m v_2 r_2 \quad \text{or,} \quad v_1 r_1 = v_2 r_2 = v_0 r_2 \sin \alpha \tag{1}$$

From conservation of mechanical energy,

$$\frac{1}{2} m v_0^2 - \frac{\gamma m_s m}{r_0} = \frac{1}{2} m v_1^2 - \frac{\gamma m_s m}{r_1}$$

or,

$$\frac{v_0^2}{2} - \frac{\gamma m_s}{r_0} = \frac{v_0^2 r_0^2 \sin^2 \alpha}{2 r_1^2} - \frac{\gamma m_s}{r_1} \text{ (Using 1)}$$

or,

$$\left(v_0^2 - \frac{2 \gamma m_s}{r_0}\right) r_1^2 + 2 \gamma m_s r_1 - v_0^2 r_0^2 \sin \alpha = 0$$

So,

$$r_1 = \frac{-2 \gamma m_s \pm \sqrt{4 \gamma^2 m_s^2 + 4 \left(v_0^2 r_0^2 \sin^2 \alpha\right) \left(v_0^2 - \frac{2 \gamma m_s}{r_0}\right)}}{2 \left(v_0^2 - \frac{2 \gamma m_s}{r_0}\right)}$$

$$= \frac{1 \pm \sqrt{1 - \frac{v_0^2 r_0^2 \sin^2 \alpha}{\gamma m_s} \left(\frac{2}{r_0} - \frac{v_0^2}{r m_s}\right)}}{\left(\frac{2}{r_0} - \frac{v_0}{\gamma m_s}\right)} = \frac{r_0 \left[1 \pm \sqrt{1 - (2 - \eta) \eta \sin^2 \alpha} \right]}{(2 - \eta)}$$

where $\eta = v_0^2 r_0 / \gamma m_s$, (m_s is the mass of the Sun).

1.210 At the minimum separation with the Sun, the cosmic body's velocity is perpendicular to its position vector relative to the Sun. If r_{min} be the sought minimum distance, from conservation of angular momentum about the Sun (C).

$$m v_0 l = m v r_{min} \text{ or, } v = \frac{v_0 l}{r_{min}} \tag{1}$$

From conservation of mechanical energy of the system (sun + cosmic body),

$$\frac{1}{2} m v_0^2 = -\frac{\gamma m_s m}{r_{min}} + \frac{1}{2} m v^2$$

So, $$\frac{v_0^2}{2} = -\frac{\gamma m_s}{r_{min}} + \frac{v_0^2}{2 r_{min}^2} \quad \text{(using 1)}$$

or, $$v_0^2 r_{min}^2 + 2\gamma m_s r_{min} - v_0^2 l^2 = 0$$

So, $$r_{min} = \frac{-2\gamma m_s \pm \sqrt{4\gamma^2 m^2 + 4 v_0^2 v_0^2 l^2}}{2 v_0^2} = \frac{-\gamma m_s \pm \sqrt{\gamma^2 m_s^2 + v_0^4 l^2}}{v_0^2}$$

Hence, taking positive root

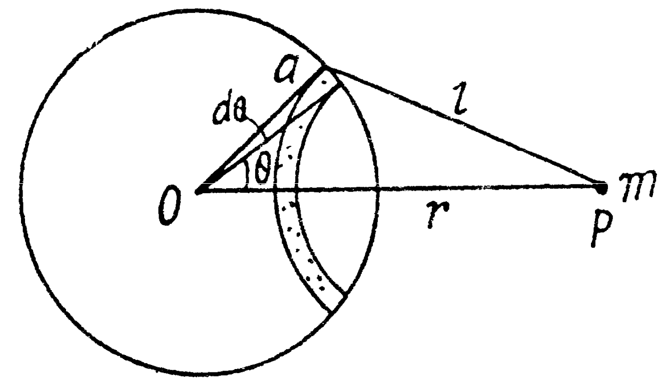
$$r_{min} = (\gamma m_s / v_0^2)\left[\sqrt{1 + (l v_0^2/\gamma m_s)^2} - 1\right]$$

1.211 Suppose that the sphere has a radius equal to a. We may imagine that the sphere is made up of concentric thin spherical shells (layers) with radii ranging from 0 to a, and each spherical layer is made up of elementry bands (rings). Let us first calculate potential due to an elementry band of a spherical layer at the point of location of the point mass m (say point P) (Fig.). As all the points of the band are located at the distance l from the point P, so,

$$\partial \varphi = -\frac{\gamma \, \partial M}{l} \quad \text{(where mass of the band)} \tag{1}$$

$$\partial M = \left(\frac{dM}{4\pi a^2}\right)(2\pi a \sin\theta)(a \, d\theta)$$

$$= \left(\frac{dM}{2}\right) \sin\theta \, d\theta \tag{2}$$



And $$l^2 = a^2 + r^2 - 2 ar\cos\theta \tag{3}$$

Differentiating Eq. (3), we get

$$l dl = ar \sin\theta \, d\theta \tag{4}$$

Hence using above equations

$$\partial \varphi = -\left(\frac{\gamma \, dM}{2ar}\right) dl \tag{5}$$

Now integrating this Eq. over the whole spherical layer

$$d\varphi = \int \partial\varphi = -\frac{\gamma\, dM}{2ar} \int_{r-a}^{r+a}$$

So $$d\varphi = -\frac{\gamma\, dM}{r} \tag{6}$$

Equation (6) demonstrates that the potential produced by a thin uniform spherical layer outside the layer is such as if the whole mass of the layer were concentrated at it's centre; Hence the potential due to the sphere at point P;

$$\varphi = \int d\varphi = -\frac{\gamma}{r}\int dM = -\frac{\gamma M}{r} \tag{7}$$

This expression is similar to that of Eq. (6)

Hence thte sought potential energy of gravitational interaction of the particle m and the sphere,

$$U = m\varphi = -\frac{\gamma Mm}{r}$$

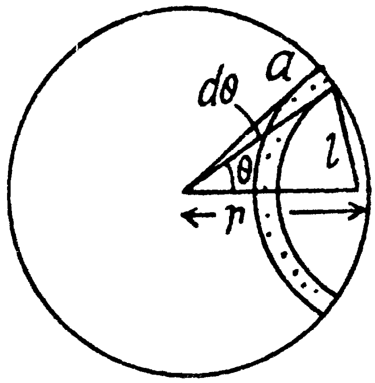
(b) Using the Eq., $$G_r = -\frac{\partial\varphi}{\partial r}$$

$$G_r = -\frac{\gamma M}{r^2} \qquad \text{(using Eq. 7)}$$

So $$\vec{G} = -\frac{\gamma M}{r^3}\vec{r} \text{ and } \vec{F} = m\vec{G} = -\frac{\gamma m M}{r^3}\vec{r} \tag{8}$$

1.212 (The problem has already a clear hint in the answer sheet of the problem book). Here we adopt a different method.

Let m be the mass of the spherical layer, wich is imagined to be made up of rings. At a point inside the spherical layer at distance r from the centre, the gravitational potential due to a ring element of radius a equals,



$$d\varphi = -\frac{\gamma m}{2ar}\, dl \quad \text{(see Eq. (5) of solution of 1.211)}$$

So, $$\varphi = \int d\varphi = -\frac{\gamma m}{2ar}\int_{a-r}^{a+r} dl = -\frac{\gamma m}{a} \tag{1}$$

Hence $$G_r = -\frac{\partial\varphi}{\partial r} = 0 \,.$$

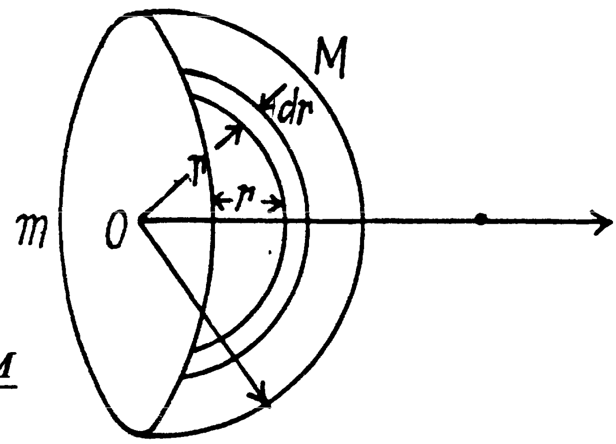
Hence gravitational field strength as well as field force becomes zero, inside a thin sphereical layer.

1.213 One can imagine that the uniform hemisphere is made up of thin hemispherical layers of radii ranging from 0 to R. Let us consider such a layer (Fig.). Potential at point O, due to this layer is,

$$d\varphi = -\frac{\gamma dm}{r} = -\frac{3\gamma M}{R^3} r dr, \text{ where } dm = \frac{M}{(2/3)\pi R^3}\left(\frac{4\pi r^2}{2}\right)dr$$

(This is because all points of each hemispherical shell are equidistant from O.)

Hence, $\varphi = \int d\varphi = -\frac{3\gamma M}{R^3}\int_0^R r dr = -\frac{3\gamma M}{2R}$



Hence, the work done by the gravitational field force on the particle of mass m, to remove it to infinity is given by the formula

$A = m\varphi$, since $\varphi = 0$ at infinity.

Hence the sought work,

$$A_{0\to\infty} = -\frac{3\gamma mM}{2R}$$

(The work done by the external agent is $-A$.)

1.214 In the solution of problem 1.211, we have obtained φ and G due to a uniform shpere, at a distance r from it's centre outside it. We have from Eqs. (7) and (8) of 1.211,

$$\varphi = -\frac{\gamma M}{r} \text{ and } \vec{G} = -\frac{\gamma M}{r^3}\vec{r} \qquad \text{(A)}$$

Accordance with the Eq. (1) of the solution of 1.212, potential due to a spherical shell of radius a, at any point, inside it becomes

$$\varphi = \frac{\gamma M}{a} = \text{Const. and } G_r = -\frac{\partial \varphi}{\partial r} = 0 \qquad \text{(B)}$$

For a point (say P) which lies inside the uniform solid sphere, the potential φ at that point may be represented as a sum.

$$\varphi_{inside} = \varphi_1 + \varphi_2$$

where φ_1 is the potential of a solid sphere having radius r and φ_2 is the potential of the layer of radii r and R. In accordance with equation (A)

$$\varphi_1 = -\frac{\gamma}{r}\left(\frac{M}{(4/3)\pi R^3}\frac{4}{3}\pi r^3\right) = -\frac{\gamma M}{R^3}r^2$$

The potential φ_2 produced by the layer (thick shell) is the same at all points inside it. The potential φ_2 is easiest to calculate, for the point positioned at the layer's centre. Using Eq. (B)

$$\varphi_2 = -\gamma\int_r^R \frac{dM}{r} = -\frac{3}{2}\frac{\gamma M}{R^3}(R^2 - r^2)$$

where $dM = \frac{M}{(4/3)\pi R^3} 4\pi r^2 dr = \left(\frac{3M}{R^3}\right) r^2 dr$

is the mass of a thin layer between the radii r and $r + dr$.

Thus $$\varphi_{inside} = \varphi_1 + \varphi_2 = \left(\frac{\gamma M}{2R}\right)\left(3 - \frac{r^2}{R^2}\right) \qquad \text{(C)}$$

From the Eq. $$G_r = \frac{-\partial \varphi}{\partial r}$$

$$G_r = \frac{\gamma M r}{R^3}$$

or $$\vec{G} = -\frac{\gamma M}{R^3}\vec{r} = -\gamma \frac{4}{3}\pi \rho \vec{r}$$

(where $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$, is the density of the sphere) (D)

The plots $\varphi(r)$ and $G(r)$ for a uniform sphere of radius R are shown in figure of answersheet.

Alternate : Like Gauss's theorem of electrostatics, one can derive Gauss's theorem for gravitation in the form $\oint \vec{G} \cdot d\vec{S} = -4\pi\gamma\, m_{inclosed}$. For calculation of $\vec{G}$ at a point inside the sphere at a distance r from its centre, let us consider a Gaussian surface of radius r, Then,

$$G_r 4\pi r^2 = -4\pi\gamma\left(\frac{M}{R^3}\right)r^3 \text{ or, } G_r = -\frac{\gamma M}{R^3}r$$

Hence, $$\vec{G} = -\frac{\gamma M}{R^3}\vec{r} = -\gamma\frac{4}{3}\pi\rho\vec{r}\left(\text{as } \rho = \frac{M}{(4/3)\pi R^3}\right)$$

So, $$\varphi = \int_r^\infty G_r\, dr = \int_r^R -\frac{\gamma M}{R^3}\, r\, dr + \int_R^\infty -\frac{\gamma M}{r^2}\, dr$$

Integrating and summing up, we get,

$$\varphi = -\frac{\gamma M}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

And from Gauss's theorem for outside it :

$$G_r 4\pi r^2 = -4\pi\gamma M \text{ or } G_r = -\frac{\gamma M}{r^2}$$

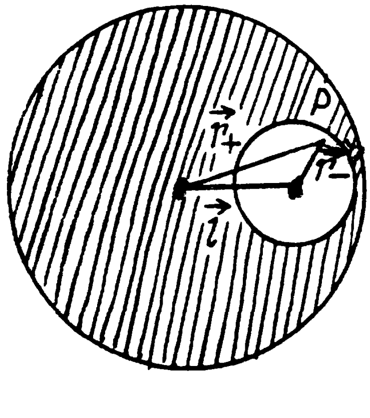
Thus $$\varphi(r) = \int_r^\infty G_r\, dr = -\frac{\gamma M}{r}$$

1.215 Treating the cavity as negative mass of density $-\rho$ in a uniform sphere density $+\rho$ and using the superposition principle, the sought field strength is :

$$\vec{G} = \vec{G}_1 + \vec{G}_2$$

or $$\vec{G} = -\frac{4}{3}\pi\gamma\rho\vec{r}_+ + -\frac{4}{3}\gamma\pi(-\rho)\,\vec{r}_-$$

(where $\vec{r}_+$ and $\vec{r}_-$ are the position vectors of an orbitrary point P inside the cavity with respect to centre of sphere and cavity respectively.)

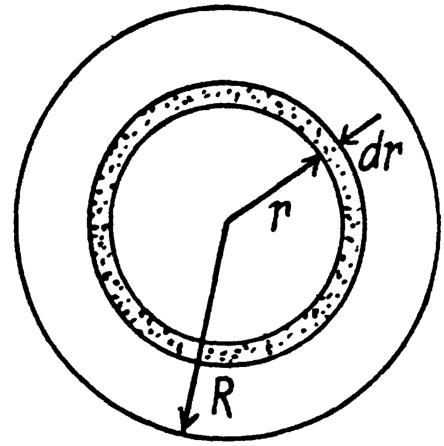


Thus $$\vec{G} = -\frac{4}{3}\pi\gamma\rho\left(\vec{r}_+ - \vec{r}_-\right) = -\frac{4}{3}\pi\gamma\rho\,\vec{l}$$

1.216 We partition the solid sphere into thin spherical layers and consider a layer of thickness dr lying at a distance r from the centre of the ball. Each spherical layer presses on the layers within it. The considered layer is attracted to the part of the sphere lying within it (the outer part does not act on the layer). Hence for the considered layer

$$dp\,4\pi r^2 = dF$$

or, $$dP\,4\pi r^2 = \frac{\gamma\left(\frac{4}{3}\pi r^3\rho\right)(4\pi r^2 dr\,\rho)}{r^2}$$



(where ρ is the mean density of sphere)

or, $$dp = \frac{4}{3}\pi\gamma\rho^2 r\,dr$$

Thus $$p = \int_r^R dp = \frac{2\pi}{3}\gamma\rho^2(R^2 - r^2)$$

(The pressure must vanish at $r = R$.)

or, $$p = \frac{3}{8}\left(1 - (r^2/R^2)\right)\gamma M^2/\pi R^4, \text{ Putting } \rho = M/(4/3)\pi R^3$$

Putting $r = 0$, we have the pressure at sphere's centre, and treating it as the Earth where mean density is equal to $\rho = 5{\cdot}5 \times 10^3\ \text{kg/m}^3$ and $R = 64 \times 10^2$ km

we have, $$p = 1{\cdot}73 \times 10^{11}\ \text{Pa} \quad \text{or} \quad 1{\cdot}72 \times 10^6\ \text{atms}.$$

1.217 (a) Since the potential at each point of a spherical surface (shell) is constant and is equal to $\varphi = -\frac{\gamma m}{R}$, [as we have in Eq. (1) of solution of problem 1.212]

We obtain in accordance with the equation

$$U = \frac{1}{2}\int dm\,\varphi = \frac{1}{2}\varphi\int dm$$

$$= \frac{1}{2}\left(-\frac{\gamma m}{R}\right)m = -\frac{\gamma m^2}{2R}$$

(The factor $\frac{1}{2}$ is needed otherwise contribution of different mass elements is counted twice.)

(b) In this case the potential inside the sphere depends only on r (see Eq. (C) of the solution of problem 1.214)

$$\varphi = -\frac{3\gamma m}{2R}\left(1 - \frac{r^2}{3R^2}\right)$$

Here dm is the mass of an elementry spherical layer confined between the radii r and $r + dr$:

$$dm = (4\pi r^2 dr\,\rho) = \left(\frac{3m}{R^3}\right)r^2 dr$$

$$U = \frac{1}{2}\int dm\,\varphi$$

$$= \frac{1}{2}\int_0^R \left(\frac{3\,m}{R^3}\right) r^2\,dr \left\{-\frac{3\,\gamma\,m}{2\,R}\left(1-\frac{r^2}{3\,R^2}\right)\right\}$$

After integrating, we get

$$U = -\frac{3}{5}\frac{\gamma\,m^2}{R}$$

1.218 Let $\omega = \sqrt{\frac{\gamma M_E}{r^3}}$ = circular frequency of the satellite in the outer orbit,

$\omega_0 = \sqrt{\frac{\gamma M_E}{(r-\Delta r)^3}}$ = circular frequency of the satellite in the inner orbit.

So, relative angular velocity = $\omega_0 \pm \omega$ where – sign is to be taken when the satellites are moving in the same sense and + sign if they are moving in opposite sense.

Hence, time between closest approaches

$$= \frac{2\,\pi}{\omega_0 \pm \omega} = \frac{2\,\pi}{\sqrt{\gamma M_E}\,/\,r^{3/2}}\,\frac{1}{\frac{3\,\Delta r}{2\,r}+\delta} = \begin{cases} 4{\cdot}5 \text{ days } (\delta = 0) \\ 0{\cdot}80 \text{ hour } (\delta = 2) \end{cases}$$

where δ is 0 in the first case and 2 in the second case.

1.219 $\omega_1 = \frac{\gamma M}{R^2} = \frac{6{\cdot}67 \times 10^{-11} \times 5{\cdot}96 \times 10^{24}}{(6{\cdot}37 \times 10^6)^2} = 9{\cdot}8 \text{ m/s}^2$

$$\omega_2 = \omega^2 R = \left(\frac{2\,\pi}{T}\right)^2 R = \left(\frac{2 \times 22}{24 \times 3600 \times 7}\right)^2 6{\cdot}37 \times 10^6 = 0{\cdot}034 \text{ m/s}^2$$

and $\quad \omega_3 = \frac{\gamma M_S}{R^2_{mean}} = \frac{6{\cdot}67 \times 10^{-11} \times 1{\cdot}97 \times 10^{30}}{(149{\cdot}50 \times 10^6 \times 10^3)^2} = 5{\cdot}9 \times 10^{-3} \text{ m/s}^2$

Then $\quad \omega_1 : \omega_2 : \omega_3 = 1 : 0{\cdot}0034 : 0{\cdot}0006$

1.220 Let h be the sought height in the first case. so

$$\frac{99}{100}\,g = \frac{\gamma M}{(R+h)^2}$$

$$= \frac{\gamma M}{R^2\left(1+\frac{h}{R}\right)^2} = \frac{g}{\left(1+\frac{h}{R}\right)^2}$$

or $$\frac{99}{100} = \left(1 + \frac{h}{R}\right)^{-2}$$

From the statement of the problem, it is obvious that in this case $h << R$

Thus $$\frac{99}{100} = \left(1 - \frac{2h}{R}\right) \text{ or } h = \frac{R}{200} = \left(\frac{6400}{200}\right) \text{km} = 32 \text{ km}$$

In the other case if h' be the sought height, than

$$\frac{g}{2} = g\left(1 + \frac{h'}{R}\right)^{-2} \text{ or } \frac{1}{2} = \left(1 + \frac{h'}{R}\right)^{-2}$$

From the language of the problem, in this case h' is not very small in comparision with R. Therefore in this case we cannot use the approximation adopted in the previous case.

Here, $\left(1 + \frac{h'}{R}\right)^2 = 2$ So, $\frac{h'}{R} = \pm \sqrt{2} - 1$

As $-ve$ sign is not acceptable

$$h' = (\sqrt{2} - 1) R = (\sqrt{2} - 1)\, 6400 \text{ km} = 2650 \text{ km}$$

1.221 Let the mass of the body be m and let it go upto a height h.
From conservation of mechanical energy of the system

$$-\frac{\gamma M m}{R} + \frac{1}{2} m v_0^2 = \frac{-\gamma M m}{(R+h)} + 0$$

Using $\frac{\gamma M}{R^2} = g$, in above equation and on solving we get,

$$h = \frac{R v_0^2}{2 g R - v_0^2}$$

1.222 Gravitational pull provides the required centripetal acceleration to the satelite. Thus if h be the sought distance, we have

so, $$\frac{m v^2}{(R+h)} = \frac{\gamma m M}{(R+h)^2} \text{ or, } (R+h) v^2 = \gamma M$$

or, $$R v^2 + h v^2 = g R^2, \text{ as } g = \frac{\gamma M}{R^2}$$

Hence $$h = \frac{g R^2 - R v^2}{v^2} = R\left[\frac{g R}{v^2} - 1\right]$$

1.223 A satellite that hovers above the earth's equator and corotates with it moving from the west to east with the diurnal angular velocity of the earth appears stationary to an observer on the earth. It is called geostationary. For this calculation we may neglect the annual motion of the earth as well as all other influences. Then, by Newton's law,

$$\frac{\gamma M m}{r^2} = m\left(\frac{2\pi}{T}\right)^2 r$$

where M = mass of the earth, T = 86400 seconds = period of daily rotation of the earth and r = distance of the satellite from the centre of the earth. Then

$$r = \sqrt[3]{\gamma M \left(\frac{T}{2\pi} \right)^2}$$

Substitution of $M = 5{\cdot}96 \times 10^{24}$ kg gives

$$r = 4{\cdot}220 \times 10^4 \text{ km}$$

The instantaneous velocity with respect to an inertial frame fixed to the centre of the earth at that moment will be

$$\left(\frac{2\pi}{T} \right) r = 3{\cdot}07 \text{ km/s}$$

and the acceleration will be the centripetal acceleration.

$$\left(\frac{2\pi}{T} \right)^2 r = 0{\cdot}223 \text{ m/s}^2$$

1.224 We know from the previous problem that a satellite moving west to east at a distance $R = 2{\cdot}00 \times 10^4$ km from the centre of the earth will be revolving round the earth with an angular velocity faster than the earth's diurnal angualr velocity. Let

$$\omega = \text{angular velocity of the satellite}$$

$$\omega_0 = \frac{2\pi}{T} = \text{anuglar velocity of the earth. Then}$$

$$\omega - \omega_0 = \frac{2\pi}{\tau}$$

as the relative angular velocity with respect to earth. Now by Newton's law

$$\frac{\gamma M}{R^2} = \omega^2 R$$

So,

$$M = \frac{R^3}{\gamma} \left(\frac{2\pi}{\tau} + \frac{2\pi}{T} \right)^2$$

$$= \frac{4\pi^2 R^3}{\gamma T^2} \left(1 + \frac{T}{\tau} \right)^2$$

Substitution gives

$$M = 6{\cdot}27 \times 10^{24} \text{ kg}$$

1.225 The velocity of the satellite in the inertial space fixed frame is $\sqrt{\frac{\gamma M}{R}}$ east to west. With respect to the Earth fixed frame, from the $\vec{v_1}' = \vec{v} - (\vec{w} \times \vec{r})$ the velocity is

$$v' = \frac{2\pi R}{T} + \sqrt{\frac{\gamma M}{R}} = 7{\cdot}03 \text{ km/s}$$

Here M is the mass of the earth and T is its period of rotation about its own axis.

It would be $-\frac{2\pi R}{T}+\sqrt{\frac{\gamma M}{R}}$, if the satellite were moving from west to east.

To find the acceleration we note the formula

$$m\vec{w}' = \vec{F}+2m(\vec{v}'\times\vec{\omega})+m\omega^2\vec{R}$$

Here $\vec{F} = -\frac{\gamma M m}{R^3}\vec{R}$ and $\vec{v}'\perp\vec{\omega}$ and $\vec{v}'\times\vec{\omega}$ is directed towards the centre of the Earth.

Thus $$w' = \frac{\gamma M}{R^2}+2\left(\frac{2\pi R}{T}+\sqrt{\frac{\gamma M}{R}}\right)\frac{2\pi}{T}-\left(\frac{2\pi}{T}\right)^2 R$$

toward the earth's rotation axis

$$= \frac{\gamma M}{R^2}+\frac{2\pi}{T}\left[\frac{2\pi R}{T}+2\sqrt{\frac{\gamma M}{R}}\right] = 4{\cdot}94 \text{ m/s}^2 \text{ on substitution.}$$

1.226 From the well known relationship between the velocities of a particle w.r.t a space fixed frame (K) rotating frame (K') $\vec{v} = \vec{v}' + (\vec{w}\times\vec{r})$

$$v'_1 = v-\left(\frac{2\pi}{T}\right)R$$

Thus kinetic energy of the satellite in the earth's frame

$$T_1' = \frac{1}{2}m v'^2_1 = \frac{1}{2}m\left(v-\frac{2\pi R}{T}\right)^2$$

Obviously when the satellite moves in opposite sense comared to the rotation of the Earth its velocity relative to the same frame would be

$$v_2' = v+\left(\frac{2\pi}{T}\right)R$$

And kinetic energy

$$T_2' = \frac{1}{2}m v'^2_2 = \frac{1}{2}m\left(v+\frac{2\pi R}{T}\right)^2 \qquad (2)$$

From (1) and (2)

$$T' = \frac{\left(v+\frac{2\pi R}{T}\right)^2}{\left(v-\frac{2\pi R}{T}\right)^2} \qquad (3)$$

Now from Newton's second law

$$\frac{\gamma M m}{R^2} = \frac{m v^2}{R} \quad \text{or} \quad v = \sqrt{\frac{\gamma M}{R}} = \sqrt{gR} \qquad (4)$$

Using (4) and (3)

$$\frac{T_2'}{T_1'} = \frac{\left(\sqrt{gR}+\frac{2\pi R}{T}\right)^2}{\left(\sqrt{gR}-\frac{2\pi R}{T}\right)^2} = 1{\cdot}27 \quad \text{nearly (Using Appendices)}$$

1.227 For a satellite in a circular orbit about any massive body, the following relation holds between kinetic, potential & total energy :

$$T = -E,\ U = 2E \qquad (1)$$

Thus since total mechanical energy must decrease due to resistance of the cosmic dust, the kintetic energy will increase and the satellite will 'fall', We see then, by work energy theorm

$$dT = -dE = -dA_{fr}$$

So, $$mvdv = \alpha v^2 vdt \quad \text{or,} \quad \frac{\alpha dt}{m} = \frac{dv}{v^2}$$

Now from Netow's law at an arbitray radius r from the moon's centre.

$$\frac{v^2}{r} = \frac{\gamma M}{r^2} \quad \text{or} \quad v = \sqrt{\frac{\gamma M}{r}}$$

(M is the mass of the moon.) Then

$$v_i = \sqrt{\frac{\gamma M}{\eta R}},\ v_f = \sqrt{\frac{\gamma M}{R}}$$

where R= moon's radius. So

$$\int_{v_1}^{v_f} \frac{dv}{v^2} = \frac{\alpha}{m} \int_0^{\tau} dt = \frac{\alpha \tau}{m}$$

or, $$\tau = \frac{m}{\alpha}\left(\frac{1}{v_i} - \frac{1}{v_f}\right) = \frac{m}{\alpha \sqrt{\frac{M}{\gamma R}}}(\sqrt{\eta} - 1) = \frac{m}{\alpha \sqrt{gR}}(\sqrt{\eta} - 1)$$

where g is moon's gravity. The averaging implied by Eq. (1) (for noncircular orbits) makes the result approximate.

1.228 From Newton's second law

$$\frac{\gamma Mm}{R^2} = \frac{mv_0^2}{R} \text{ or } v_0 = \sqrt{\frac{\gamma M}{R}} = 1{\cdot}67 \text{ km/s} \qquad (1)$$

From conservation of mechanical energy

$$\frac{1}{2}mv_e^2 - \frac{\gamma Mm}{R} = 0 \text{ or, } v_e = \sqrt{\frac{2\gamma M}{R}} = 2{\cdot}37 \text{ km/s}^2 \qquad (2)$$

In Eq. (1) and (2), M and R are the mass of the moon and its radius. In Eq. (1) if M and R represent the mass of the earth and its radius, then, using appendices, we can easily get

$$v_0 = 7{\cdot}9 \text{ km/s} \quad \text{and} \quad v_c = 11{\cdot}2 \text{ km/s}.$$

1.229 In a parabolic orbit, $E = 0$

So
$$\frac{1}{2}mv_i^2 - \frac{\gamma Mm}{R} = 0 \text{ or, } v_i = \sqrt{2}\sqrt{\frac{\gamma M}{R}}$$

where M = mass of the Moon, R = its radius. (This is just the escape velocity.)
On the other hand in orbit

$$mv_f^2 R = \frac{\gamma Mm}{R^2} \text{ or } v_f = \sqrt{\frac{\gamma M}{R}}$$

Thus
$$\Delta v = (1 - \sqrt{2})\sqrt{\frac{\gamma M}{R}} = -0{\cdot}70 \text{ km/s}.$$

1.230 From 1.228 for the Earth surface

$$v_0 = \sqrt{\frac{\gamma M}{R}} \text{ and } v_e = \sqrt{\frac{2\gamma M}{R}}$$

Thus the sought additional velocity

$$\Delta v = v_e - v_0 = \sqrt{\frac{\gamma M}{R}}(\sqrt{2} - 1) = gR(\sqrt{2} - 1)$$

This 'kick' in velocity must be given along the direction of motion of the satellite in its orbit.

1.231 Let r be the sought distance, then

$$\frac{\gamma \eta M}{(nR - r)^2} = \frac{\gamma M}{r^2} \quad \text{or } \eta r^2 = (nR - r)^2$$

or
$$\sqrt{\eta}\, r = (nR - r) \text{ or } r = \frac{nR}{\sqrt{\eta} + 1} = 3{\cdot}8 \times 10^4 \text{ km}.$$

1.232 Between the earth and the moon, the potential energy of the spaceship will have a maximum at the point where the attractions of the earth and the moon balance each other. This maximum P.E. is approximately zero. We can also neglect the contribution of either body to the p.E. of the spaceship sufficiently near the other body. Then the minimum energy that must be imparted to the spaceship to cross the maximum of the P.E. is clearly (using E to denote the earth)

$$\frac{\gamma M_E m}{R_E}$$

With this energy the spaceship will cross over the hump in the P.E. and coast down the hill of p.E. towards the moon and crashland on it. What the problem seeks is the minimum energy reguired for softlanding. That reguies the use of rockets to loving about the braking of the spaceship and since the kinetic energy of the gases ejected from the rocket will always be positive, the total energy required for softlanding is greater than that required for crashlanding. To calculate this energy we assume that the rockets are used fairly close to the moon when the spaceship has nealy attained its terminal velocity on the moon $\sqrt{\frac{2\gamma M_O}{R_0}}$ where M_0 is the mass of the moon and R_0 is its radius. In general $dE = vdp$ and since the speed of the ejected gases is not less than the speed of the rocket, and momentum transfered to the ejected gases must equal the momentum of the spaceship the energy E of the gass ejected is not less than the kinetic energy of spaceship

$$\frac{\gamma M_0 m}{R_0}$$

Addding the two we get the minimum work done on the ejected gases to bring about the softlanding.

$$A_{min} = \gamma m \left(\frac{M_E}{R_E} + \frac{M_0}{R_0} \right)$$

On substitution we get $1{\cdot}3 \times 10^8$ kJ.

1.233 Assume first that the attraction of the earth can be neglected. Then the minimum velocity, that must be imparted to the body to escape from the Sun's pull, is, as in 1·230, equal to

$$(\sqrt{2} - 1)\, v_1$$

where $v_1^2 = \gamma M_s / r$, r = radius of the earth's orbit, M_s = mass of the Sun.

In the actual case near the earth, the pull of the Sun is small and does not change much over distances, which are several times the radius of the Earth. The velocity v_3 in question is that which overcomes the earth's pull with sufficient velocity to escape the Sun's pull. Thus

$$\frac{1}{2} m v_3^2 - \frac{\gamma M_E}{R} = \frac{1}{2} m (\sqrt{2} - 1)^2 v_1^2$$

where R = radius of the earth, M_E = mass of the earth.

Writing $v_1^2 = \gamma M_E / R$, we get

$$v_3 = \sqrt{2 v_2^2 + (\sqrt{2} - 1)^2 v_1^2} = 16{\cdot}6 \text{ km/s}$$

1.5 DYNAMICS OF A SOLID BODY

1.234 Since, motion of the rod is purely translational, net torque about the C.M. of the rod should be equal to zero.

Thus $$F_1 \frac{l}{2} = F_2 \left(\frac{l}{2} - a \right) \text{ or, } \frac{F_1}{F_2} = 1 - \frac{a}{l/2} \tag{1}$$

For the translational motion of rod.

$$F_2 - F_1 = m w_c \text{ or } 1 - \frac{F_1}{F_2} = \frac{m w_c}{F_2} \tag{2}$$

From (1) and (2)

$$\frac{a}{l/2} = \frac{m w_c}{F_2} \text{ or, } l = \frac{2 a F_2}{m w_c} = 1 \text{ m}$$

1.235 Sought moment $\vec{N} = \vec{r} \times \vec{F} = (a\vec{i} + b\vec{j}) \times (A\vec{i} + B\vec{j})$

$$= aB\,\vec{k} + Ab(-\vec{k}) = (aB - Ab)\,\vec{k}$$

and arm of the force $$l = \frac{N}{F} = \frac{aB - Ab}{\sqrt{A^2 + B^2}}$$

1.236 Relative to point O, the net moment of force :

$$\vec{N} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 = (a\vec{i} \times A\vec{j}) + (B\vec{j} \times B\vec{i})$$

$$= ab\,\vec{k} + AB(-\vec{k}) = (ab - AB)\,\vec{k} \tag{1}$$

Resultant of the external force

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = A\vec{j} + B\vec{i} \tag{2}$$

As $\vec{N} \cdot \vec{F} = 0$ (as $\vec{N} \perp \vec{F}$) so the sought arm l of the force $\vec{F}$

$$l = N/F = \frac{ab - AB}{\sqrt{A^2 + B^2}}$$

1.237 For coplanar forces, about any point in the same plane, $\sum \vec{r}_i \times \vec{F}_i = \vec{r} \times \vec{F}_{net}$

(where $\vec{F}_{net} = \sum \vec{F}_i =$ resultant force) or, $\vec{N}_{net} = \vec{r} \times \vec{F}_{net}$

Thus length of the arm, $l = \dfrac{N_{net}}{F_{net}}$

Here obviously $|\vec{F}_{net}| = 2F$ and it is directed toward right along AC. Take the origin at C. Then about C,

$\vec{N} = \left(\sqrt{2}\, a F + \dfrac{a}{\sqrt{2}} F - \sqrt{2}\, a F \right)$ directed normally into the plane of figure.

(Here a = side of the square.)

Thus $\vec{N} = F \dfrac{a}{\sqrt{2}}$ directed into the plane of the figure.

Hence $$l = \frac{F(a/\sqrt{2})}{2F} = \frac{a}{2\sqrt{2}} = \frac{a}{2} \sin 45^\circ$$

Thus the point of application of force is at the mid point of the side BC.

1.238 (a) Consider a strip of length dx at a perpendicular distance x from the axis about which we have to find the moment of inertia of the rod. The elemental mass of the rod equals

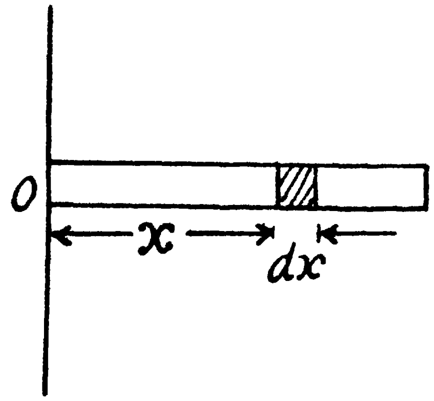
$$dm = \frac{m}{l}\,dx$$

Moment of inertia of this element about the axis

$$dI = dm\,x^2 = \frac{m}{l}\,dx \cdot x^2$$

Thus, moment of inertia of the rod, as a whole about the given axis

$$I = \int_0^l \frac{m}{l} x^2\,dx = \frac{m\,l^2}{3}$$



(b) Let us imagine the plane of plate as xy plane taking the origin at the intersection point of the sides of the plate (Fig.).

Obviously $$I_x = \int dm\,y^2$$

$$= \int_0^a \left(\frac{m}{ab}\,b\,dy\right) y^2$$

$$= \frac{m\,a^2}{3}$$

Similarly $$I_y = \frac{m\,b^2}{3}$$

Hence from perpendicular axis theorem

$$I_z = I_x + I_y = \frac{m}{3}\left(a^2 + b^2\right),$$

which is the sought ,moment of inertia.

1.239 (a) Consider an elementry disc of thickness dx. Moment of inertia of this element about the z-axis, passing through its C.M.

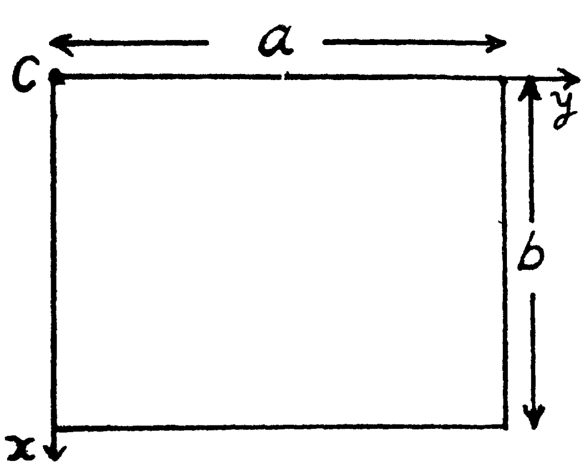
$$dI_z = \frac{(dm)\,R^2}{2} = \rho\,S\,dx\,\frac{R^2}{2}$$

where ρ = density of the material of the plate and S = area of cross section of the plate.

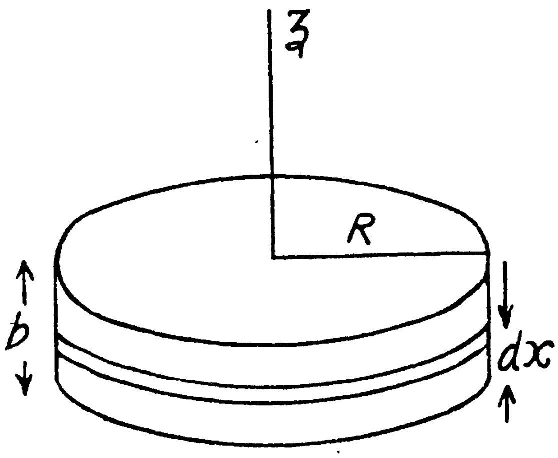
Thus the sought moment of inertia

$$I_z = \frac{\rho\,S R^2}{2}\int_0^b dx = \frac{R^2}{2}\rho\,S b$$

$$= \frac{\pi}{2}\rho\,b\,R^4\left(\text{as } S = \pi R^2\right)$$



putting all the vallues we get, $I_z = 2 \cdot \text{gm} \cdot \text{m}^2$



(b) Consider an element disc of radius r and thickness dx at a distance x from the point O. Then $r = x \tan\alpha$ and volume of the disc

$$= \pi x^2 \tan^2 \alpha \, dx$$

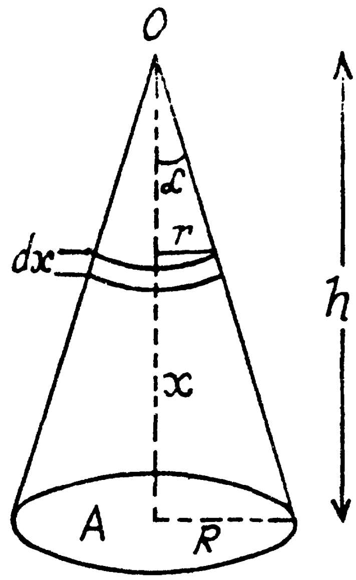
Hence, its mass $dm = \pi x^2 \tan\alpha \, dx \cdot \rho$ (where $\rho =$ density of the cone $= m / \frac{1}{3} \pi R^2 h$)

Moment of inertia of this element, about the axis OA,

$$dI = dm \frac{r^2}{2}$$

$$= (\pi x^2 \tan^2 \alpha \, dx) \frac{x^2 \tan^2 \alpha}{2}$$

$$= \frac{\pi \rho}{2} x^4 \tan^4 \alpha \, dx$$



Thus the sought moment of inertia $I = \frac{\pi \rho}{2} \tan^4 \alpha \int_0^h x^4 \, dx$

$$= \frac{\pi \rho R^4 \cdot h^5}{10 h^4} \left(\text{as } \tan\alpha = \frac{R}{h} \right)$$

Hence $\qquad I = \frac{3 m R^2}{10} \left(\text{putting } \rho = \frac{3m}{\pi R^2 h} \right)$

1.240 (a) Let us consider a lamina of an arbitrary shape and indicate by 1,2 and 3, three axes coinciding with x, y and $z-$ axes and the plane of lamina as $x-y$ plane.

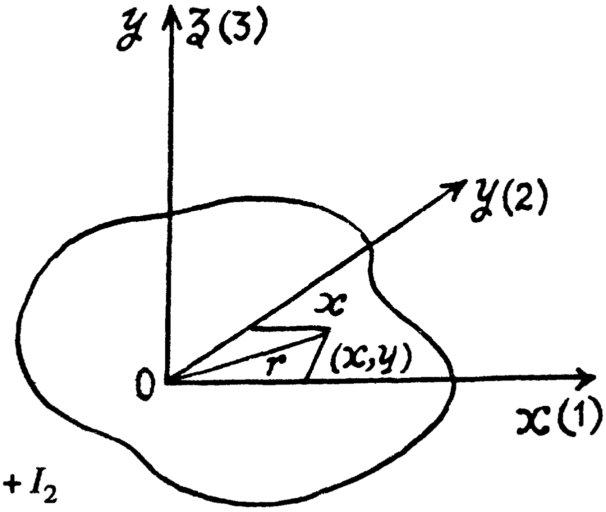
Now, moment of inertia of a point mass about $x-$ axis, $dI_x = dm \, y^2$

Thus moment of inertia of the lamina about this axis, $I_x = \int dm y^2$

Similarly, $I_y = \int dm x^2$

and $I_z = \int dm r^2$

$= \int dm \, (x^2 + y^2)$ as $r = \sqrt{x^2 + y^2}$



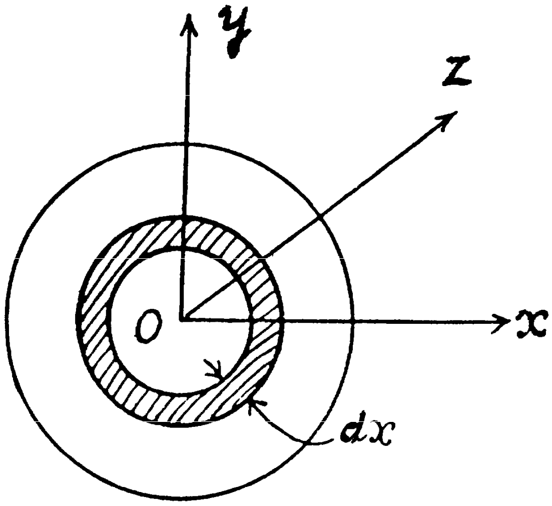
Thus, $\qquad I_z = I_x + I_y$ or, $I_3 = I_1 + I_2$

(b) Let us take the plane of the disc as $x-y$ plane and origin to the centre of the disc (Fig.) From the symmetry $I_x = I_y$. Let us consider a ring element of radius r and thickness dr, then the moment of inertia of the ring element about the $y-$ axis.

$$dI_z = dm\,r^2 = \frac{m}{\pi R^2}(2\pi r dr)\,r^2$$

Thus the moment of inertia of the disc about z – axis

$$I_z = \frac{2m}{R^2}\int_0^R r^3\,dr = \frac{mR^2}{2}$$



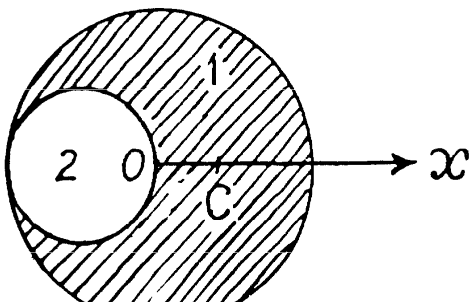
But we have $\qquad I_z = I_x + I_y = 2I_x$

Thus $\qquad I_x = \frac{I_z}{2} = \frac{mR^2}{4}$

1.241 For simplicity let us use a mathematical trick. We consider the portion of the given disc as the superposition of two complete discs (without holes), one of positive density and radius R and other of negative density but of same magnitude and radius $R/2$.

As (area) α (mass), the respective masses of the considered discs are $(4m/3)$ and $(-m/3)$ respectively, and these masses can be imagined to be situated at their respective centers (C.M). Let us take point O as origin and point x – axis towards right. Obviously the C.M. of the shaded position of given shape lies on the x – axis. Hence the C.M. (C) of the shaded portion is given by

$$x_c = \frac{(-m/3)(-R/2) + (4m/3)0}{(-m/3) + 4m/3} = \frac{R}{6}$$



Thus C.M. of the shape is at a distance $R/6$ from point O toward x – axis

Using parallel axis theorem and bearing in mind that the moment of inertia of a complete homogeneous disc of radius m_0 and radius r_0 equals $\frac{1}{2}m_0 r_0^2$. The moment of inetia of the small disc of mass $(-m/3)$ and radius $R/2$ about the axis passing through point C and perpendicular to the plane of the disc

$$I_{2C} = \frac{1}{2}\left(-\frac{m}{3}\right)\left(\frac{R}{2}\right)^2 + \left(-\frac{m}{3}\right)\left(\frac{R}{2}+\frac{R}{6}\right)^2$$

$$= -\frac{mR^2}{24} - \frac{4}{27}mR^2$$

Similarly $\qquad I_{1C} = \frac{1}{2}\left(\frac{4m}{3}\right)R^2 + \left(\frac{4m}{3}\right)\left(\frac{R}{6}\right)^2$

$$= \frac{2}{3}mR^2 + \frac{mR^2}{27}$$

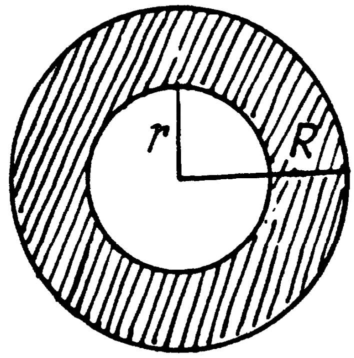
Thus the sought moment of inertia,

$$I_C = I_{1C} + I_{2C} = \frac{15}{24}mR^2 - \frac{3}{27}mR^2 = \frac{37}{72}mR^2$$

1.242 Moment of inertia of the shaded portion, about the axis passing through it's certre,

$$I = \frac{2}{5}\left(\frac{4}{3}\pi R^3 \rho\right) R^2 - \frac{2}{5}\left(\frac{4}{3}\pi r^3 \rho\right) r^3$$

$$= \frac{2}{5}\frac{4}{3}\pi \rho \left(R^5 - r^5\right)$$



Now, if $R = r + dr$, the shaded portion becomes a shell, which is the required shape to calculate the moment of inertia.

Now, $$I = \frac{2}{5} - \frac{4}{3}\pi\rho\left\{(r + dr)^5 - r^5\right\}$$

$$= \frac{2}{5}\frac{4}{3}\pi \rho \left(r^5 + 5r^4 dr + \ldots\ldots - r^5\right)$$

Neglecting higher terms.

$$= \frac{2}{3}\left(4\pi r^2 dr\rho\right) r^2 = \frac{2}{3} mr^2$$

1.243 (a) Net force which is effective on the system (cylinder M + body m) is the weight of the body m in a uniform gravitational field, which is a constant. Thus the initial acceleration of the body m is also constant.

From the conservation of mechanical energy of the said system in the uniform field of gravity at time $t = \Delta t$: $\Delta T + \Delta U = 0$

or $$\frac{1}{2} mv^2 + \frac{1}{2}\frac{MR^2}{2}\omega^2 - mg\,\Delta h = 0$$

or, $$\frac{1}{4}(2m + M)\, v^2 - mg\,\Delta h = 0 \text{ [as } v = \omega R \text{ at all times]} \quad (1)$$

But $$v^2 = 2w\,\Delta h$$

Hence using it in Eq. (1), we get

$$\frac{1}{4}(2m + M)\, 2w\,\Delta h - mg\,\Delta h = 0 \text{ or } w = \frac{2mg}{(2m + M)}$$

From the kinematical relationship, $\beta = \frac{w}{R} = \frac{2mg}{(2m + M)R}$

Thus the sought angular velocity of the cylinder

$$\omega(t) = \beta t = \frac{2mg}{(2m + M)R}\, t = \frac{gt}{(1 + M/2m)R}$$

(b) Sought kinetic energy.

$$T(t) = \frac{1}{2} mv^2 + \frac{1}{2}\frac{Ml^2}{2}\omega^2 = \frac{1}{4}(2m + M) R^2 \omega^2$$

1.244 For equilibrium of the disc and axle

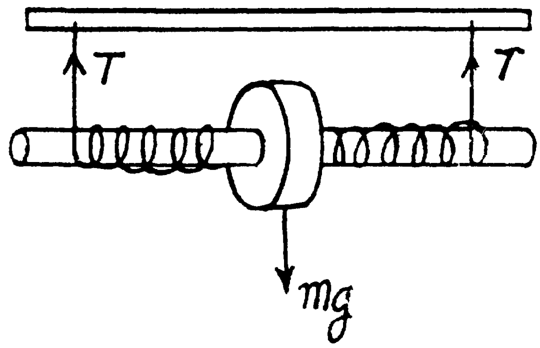
$$2T = mg \text{ or } T = mg/2$$

As the disc unwinds, it has an angular acceleration β given by

$$I\beta = 2Tr \text{ or } \beta = \frac{2Tr}{I} = \frac{mgr}{I}$$

The corresponding linear acceleration is

$$r\beta = w = \frac{mgr^2}{I}$$



Since the disc remains stationary under the combined action of this acceleration and the acceleration $(-w)$ of the bar which is transmitted to the axle, we must have

$$w = \frac{mgr^2}{I}$$

1.245 Let the rod be deviated through an angle φ from its initial position at an arbitrary instant of time, measured relative to the initial position in the positive direction. From the equation of the increment of the mechanical energy of the system.

$$\Delta T = A_{ext}$$

or, $$\frac{1}{2}I\omega^2 = \int N_z \, d\varphi$$

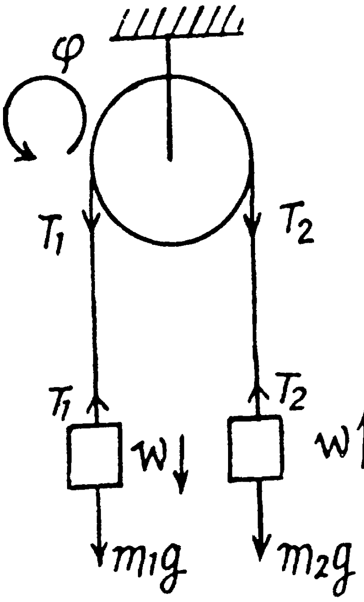
or, $$\frac{1}{2}\frac{Ml^2}{3}\omega^2 = \int_0^{\varphi} Fl\cos\varphi \, d\varphi = Fl\sin\varphi$$

Thus, $$\omega = \sqrt{\frac{6F\sin\varphi}{Ml}}$$

1.246 First of all, let us sketch free body diagram of each body. Since the cylinder is rotating and massive, the tension will be different in both the sections of threads. From Newton's law in projection form for the bodies m_1 and m_2 and noting that $w_1 = w_2 = w = \beta R$, (as no thread slipping), we have $(m_1 > m_2)$

$$m_1 g - T_1 = m_1 w = m_1 \beta R$$

and $$T_2 - m_2 g = m_2 w \qquad (1)$$



Now from the equation of rotational dynamics of a solid about stationary axis of rotation. i.e. $N_z = I\beta_z$, for the cylinder.

or, $$(T_1 - T_2)R = I\beta = mR^2\beta/2 \qquad (2)$$

Similtaneous solution of the above equations yields :

$$\beta = \frac{(m_1 - m_2)g}{R\left(m_1 + m_2 + \frac{m}{2}\right)} \text{ and } \frac{T_1}{T_2} = \frac{m_1(m + 4m_2)}{m_2(m + 4m_1)}$$

1.247 As the system $(m + m_1 + m_2)$ is under constant forces, the acceleration of body m_1 an m_2 is constant. In addition to it the velocities and accelerations of bodies m_1 and m_2 ai equal in magnitude (say v and w) because the length of the thread is constant.
From the equation of increament of mechanical energy i.e. $\Delta T + \Delta U = A_{fr}$, at time t whe block m_1 is distance h below from initial position corresponding to $t = 0$,

$$\frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}\left(\frac{mR^2}{2}\right)\frac{v^2}{R^2} - m_2 gh = -km_1 gh \qquad (1$$

(as angular velocity $\omega = v/R$ for no slipping of thread.)

But $$v^2 = 2wh$$

So using it in (1), we get

$$w = \frac{2(m_2 - km_1)g}{m + 2(m_1 + m_2)}. \qquad (2$$

Thus the work done by the friction force on m_1

$$A_{fr} = -km_1gh = -km_1g\left(\frac{1}{2}wt^2\right)$$

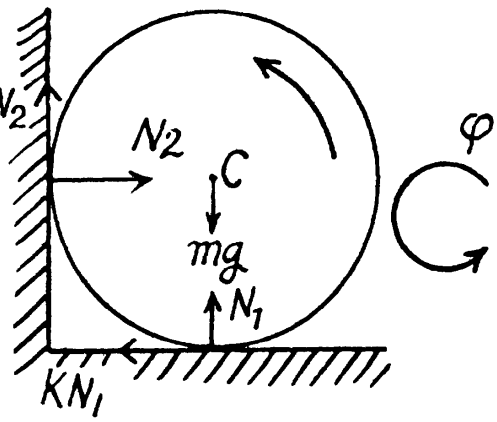
$$= -\frac{km_1(m_1 - km_1)g^2t^2}{m + 2(m_1 + m_2)} \qquad \text{(using 2).}$$

1.248 In the problem, the rigid body is in translation equlibrium but there is an angular retardation. We first sketch the free body diagram of the cylinder. Obviously the friction forces, acting on the cylinder, are kinetic. From the condition of translational equlibrium for the cylinder,

$$mg = N_1 + kN_2; \quad N_2 = kN_1$$

Hence, $$N_1 = \frac{mg}{1 + k^2}; \quad N_2 = k\frac{mg}{1 + k^2}$$

For pure rotation of the cylinder about its rotation axis, $N_z = I\beta_z$



or, $$-kN_1R - kN_2R = \frac{mR^2}{2}\beta_z$$

or, $$-\frac{kmgR(1 + k)}{1 + k^2} = \frac{mR^2}{2}\beta_z$$

or, $$\beta_z = -\frac{2k(1 + k)g}{(1 + k^2)R}$$

Now, from the kinematical equation,

$$\omega^2 = \omega_0^2 + 2\beta_z\Delta\varphi \text{ we have,}$$

$$\Delta\varphi = \frac{\omega_0^2(1 + k^2)R}{4k(1 + k)g}, \qquad \text{because } \omega = 0$$

Hence, the sought number of turns,

$$n = \frac{\Delta\varphi}{2\pi} = \frac{\omega_0^2(1+k^2)R}{8\pi k(1+k)g}$$

1.249 It is the moment of friction force which brings the disc to rest. The force of friction is applied to each section of the disc, and since these sections lie at different distances from the axis, the moments of the forces of friction differ from section to section.

To find N_z, where z is the axis of rotation of the disc let us partition the disc into thin rings (Fig.). The force of friction acting on the considered element

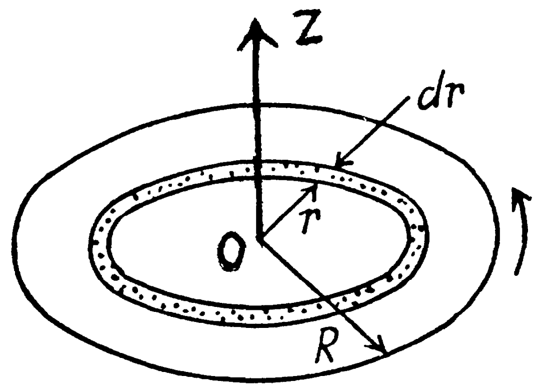
$dfr = k(2\pi r\, dr\, \sigma)g$, (where σ is the density of the disc)

The moment of this force of friction is

$$dN_z = -r\, dfr = -2\pi k \sigma g r^2 dr$$

Integrating with respect to r from zero to R, we get

$$N_z = -2\pi k \sigma g \int_0^R r^2 dr = -\frac{2}{3}\pi k \sigma g R^3.$$



For the rotation of the disc about the stationary axis z, from the equation $N_z = I\beta_z$

$$-\frac{2}{3}\pi k \sigma g R^3 = \frac{(\pi R^2 \sigma)R^2}{2}\beta_z \quad \text{or} \quad \beta_z = -\frac{4kg}{3R}$$

Thus from the angular kinematical equation $\omega_z = \omega_{0z} + \beta_z t$

$$0 = \omega_0 + \left(-\frac{4kg}{3R}\right)t \quad \text{or} \quad t = \frac{3R\,\omega_0}{4kg}$$

1.250 According to the question,

$$I\frac{d\omega}{dt} = -k\sqrt{\omega} \quad \text{or,} \quad I = \frac{d\omega}{\sqrt{\omega}} = -k\,dt$$

Integrating, $$\sqrt{\omega} = -\frac{kt}{2I} + \sqrt{\omega_0}$$

or, $$\omega = \frac{k^2t^2}{4I^2} - \frac{\sqrt{\omega_0}\,kt}{I} + \omega_0, \text{ (Noting that at } t = 0,\ \omega = \omega_0.)$$

Let the flywheel stops at $t = t_0$ then from Eq. (1), $t_0 = \dfrac{2I\sqrt{\omega_0}}{k}$

Hence sought average angular velocity

$$<\omega> = \frac{\displaystyle\int_0^{\frac{2I\sqrt{\omega_0}}{k}} \left(\frac{k^2t^2}{4I^2} - \frac{\sqrt{\omega_0}\,kt}{I} + \omega_0\right)dt}{\displaystyle\int_0^{\frac{2I\sqrt{\omega_0}}{k}} dt} = \frac{\omega_0}{3}$$

1.251 Let us use the equation $\frac{dM_z}{dt} = N_z$ relative to the axis through O (1)

For this purpose, let us find the angular momentum of the system M_z about the given rotation axis and the corresponding torque N_z. The angular momentum is

$$M_z = I\omega + mvR = \left(\frac{m_0}{2} + m\right)R^2\omega$$

[where $I = \frac{m_0}{2}R^2$ and $v = \omega R$ (no cord slipping)]

So,
$$\frac{dM_z}{dt} = \left(\frac{MR^2}{2} + mR^2\right)\beta_z \tag{2}$$

The downward pull of gravity on the overhanging part is the only external force, which exerts a torque about the z-axis, passing through O and is given by,

$$N_z = \left(\frac{m}{l}\right)xgR$$

Hence from the equation $\frac{dM_z}{dt} = N_z$

$$\left(\frac{MR^2}{2} + mR^2\right)\beta_z = \frac{m}{l}xgR$$

Thus,
$$\beta_z = \frac{2mgx}{lR\,(M + 2m)} > 0$$

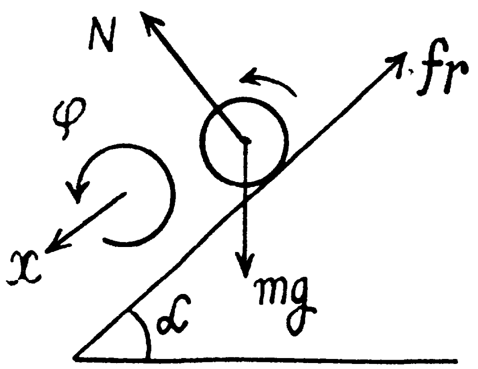
Note : *We may solve this problem using conservation of mechanical energy of the system (cylinder + thread) in the uniform field of gravity.*

1.252 (a) Let us indicate the forces acting on the sphere and their points of application. Choose positive direction of x and φ (rotation angle) along the incline in downward direction and in the sense of $\vec{\omega}$ (for undirectional rotation) respectively. Now from equations of dynamics of rigid body i.e. $F_x = mw_{cx}$ and $N_{cz} = I_c\,\beta_z$ we get :

$$mg\sin\alpha - f_r = mw \tag{1}$$

and
$$fr\,R = \frac{2}{5}mR^2\beta \tag{2}$$

But
$$fr \le kmg\cos\alpha \tag{3}$$



In addition, the absence of slipping provides the kinematical realtionship between the accelerations :

$$w = \beta R \tag{4}$$

The simultaneous solution of all the four equations yields :

$$k\cos\alpha \ge \frac{2}{7}\sin\alpha, \quad \text{or} \quad k \ge \frac{2}{7}\tan\alpha$$

(b) Solving Eqs. (1) and (2) [of part (a)], we get :

$$w_c = \frac{5}{7} g \sin \alpha.$$

As the sphere starts at $t = 0$ along positive x axis, for pure rolling

$$v_c(t) = w_c t = \frac{5}{7} g \sin \alpha\, t \qquad (5)$$

Hence the sought kinetic energy

$$T = \frac{1}{2} m v_c^2 + \frac{1}{2} \frac{2}{5} m R^2 \omega^2 = \frac{7}{10} m v_c^2 \text{ (as } \omega = v_c/R \text{)}$$

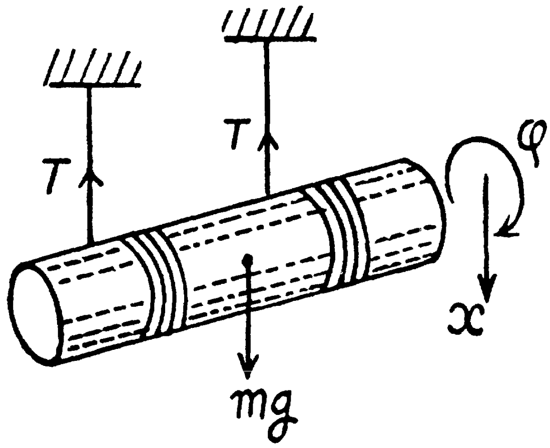
$$= \frac{7}{10} m \left(\frac{5}{7} g \sin \alpha\, t\right)^2 = \frac{5}{14} m g^2 \sin^2 \alpha\, t^2$$

1.253 (a) Let us indicate the forces and their points of application for the cylinder. Choosing the positive direction for x and φ as shown in the figure, we write the equation of motion of the cylinder axis and the equation of moments in the C.M. frame relative to that axís i.e. from equation $F_x = m w_c$ and $N_z = I_c \beta_z$,

$$mg - 2T = m w_c ; \; 2TR = \frac{mR^2}{2} \beta$$

As there is no slipping of thread on the cylinder

$$w_c = \beta R$$



From these three equations

$$T = \frac{mg}{6} = 13 \text{ N}, \; \beta = \frac{2}{5} \frac{g}{R} = 5 \times 10^2 \text{ rad/s}^2$$

(b) we have $\beta = \frac{2}{3} \frac{g}{R}$

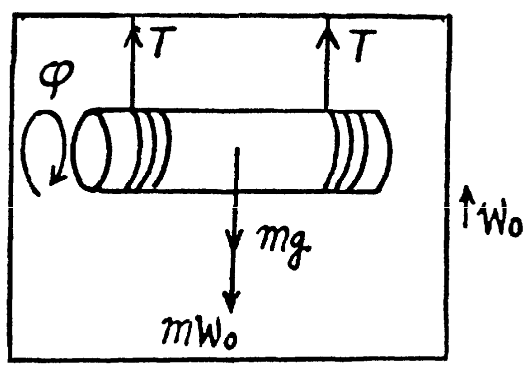
So, $w_c = \frac{2}{3} g > 0$ or, in vector form $\vec{w}_c = \frac{2}{3} \vec{g}$

$$P = \vec{F} \cdot \vec{v} = \vec{F} \cdot (\vec{w}_c t)$$

$$= m \vec{g} \cdot \left(\frac{2}{3} \vec{g} t\right) = \frac{2}{3} m g^2 t$$

1.254 Let us depict the forces and their points of application corresponding to the cylinder attached with the elevator. Newton's second law for solid in vector form in the frame of elevator, gives :

$$2\vec{T} + m\vec{g} + m(-\vec{w}_0) = m\vec{w}' \qquad (1)$$



The equation of moment in the C.M. frame relative to the cylinder axis i.e. from $N_z = I_c \beta_z$ -

$$2TR = \frac{mR^2}{2} \beta = \frac{mR^2}{2} \frac{w'}{R}$$

[as thread does not slip on the cylinder, $w' = \beta R$]

or, $$T = \frac{mw'}{4}$$

As (1) $\vec{T} \uparrow\downarrow \vec{w}$

so in vector form

$$\vec{T} = -\frac{m\vec{w}}{4} \tag{2}$$

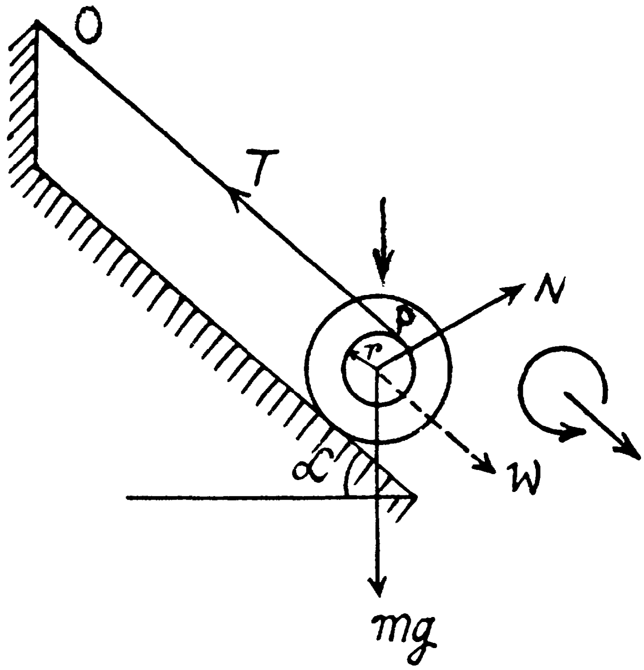
Solving Eqs. (1) and (2), $\vec{w}' = \frac{2}{3}(\vec{g} - \vec{w}_0)$ and sought force

$$\vec{F} = 2\vec{T} = \frac{1}{3}m(\vec{g} - \vec{w}_0).$$

1.255 Let us depict the forces and their points of application for the spool. Choosing the positive direction for x and φ as shown in the fig., we apply $F_x = mw_{cx}$ and $N_{cz} = I_c \beta_z$ and get

$$mg \sin\alpha - T = mw\,; \; Tr = I\beta$$

"Notice that if a point of a solid in plane motion is connected with a thread, the projection of velocity vector of the solid's point of contact along the length of the thread equals the velocity of the other end of the thread (if it is not slacked)"



Thus in our problem, $v_p = v_0$ but $v_0 = 0$, hence point P is the instantaneous centre of rotation of zero velocity for the spool. Therefore $v_c = \omega r$ and subsequently $w_c = \beta r$.

Solving the equations simultaneously, we get

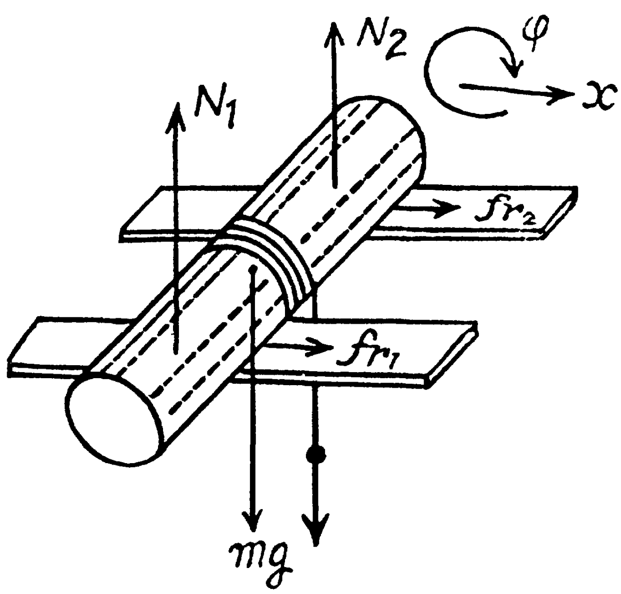
$$w = \frac{g \sin\alpha}{1 + \frac{I}{mr^2}} = 1{\cdot}6 \text{ m/s}^2$$

1.256 Let us sketch the force diagram for solid cylinder and apply Newton's second law in projection form along x and y axes (Fig.) :

$$fr_1 + fr_2 = mw_c \tag{1}$$

and $$N_1 + N_2 - mg - F = 0$$

or $$N_1 + N_2 = mg + F \tag{2}$$



Now choosing positive direction of φ as shown in the figure and using $N_{cz} = I_c \beta_z$,

we get

$$FR - (fr_1 + fr_2)R = \frac{mR^2}{2}\beta = \frac{mR^2}{2}\frac{w_c}{R} \tag{3}$$

[as for pure rolling $w_c = \beta R$]. In addition to,

$$fr_1 + fr_2 \leq k(N_1 + N_2) \tag{4}$$

Solving the Eqs., we get

$$F \le \frac{3\,kmg}{(2-3k)}, \quad \text{or} \quad F_{\max} = \frac{3\,kmg}{2-3\,k}$$

and $$w_{c\,(\max)} = \frac{k\,(N_1+N_2)}{m}$$

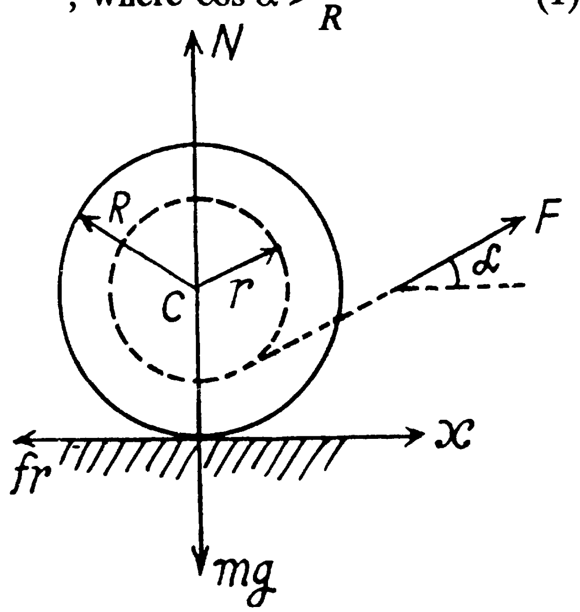
$$= \frac{k}{m}\,[mg + F_{\max}\,] = \frac{k}{m}\left[mg + \frac{3\,kmg}{2-3k}\right] = \frac{2\,kg}{2-3k}$$

1.257 (a) Let us choose the positive direction of the rotation angle φ, such that w_{cx} and β_z have identical signs (Fig.). Equation of motion, $F_x = mw_{cx}$ and $N_{cz} = I_c\,\beta_z$ gives :

$$F\cos\alpha - fr = mw_{cx} : fr\,R - Fr = I_c\,\beta_z = \gamma mR^2\,\beta_z$$

In the absence of the slipping of the spool $w_{cx} = \beta_z R$

From the three equations $w_{cx} = w_c = \dfrac{F\,[\cos\alpha - (r/R)\,]}{m\,(1+\gamma)}$, where $\cos\alpha > \dfrac{r}{R}$ (1)



(b) As static friction (fr) does not work on the spool, from the equation of the increment of mechanical energy $A_{ext} = \Delta T$.

$$A_{ext} = \frac{1}{2}mv_c^2 + \frac{1}{2}\gamma\,mR^2\,\frac{v_c^2}{R^2} = \frac{1}{2}m\,(1+\gamma)\,v_c^2$$

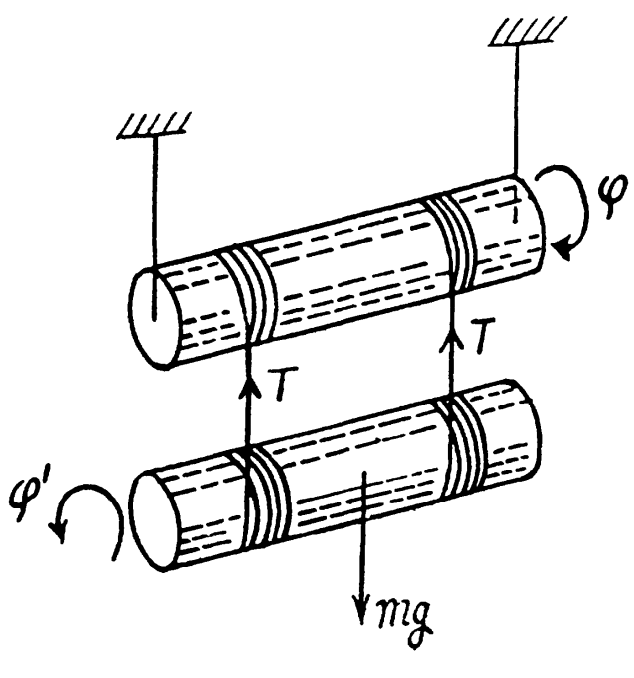
$$= \frac{1}{2}m\,(1+\gamma)\,2w_c\,x = \frac{1}{2}m\,(1+\gamma)\,2\,w_c\left(\frac{1}{2}w_c\,t^2\right)$$

$$= \frac{F^2\left(\cos\alpha - \dfrac{r}{R}\right)^2 t^2}{2\,m\,(1+\gamma)}$$

Note that at $\cos\alpha = r/R$, *there is no rolling and for* $\cos\alpha < r/R$, $w_{cx} < 0$, *i.e. the spool will move towards negative x-axis and rotate in anticlockwise sense.*

1.258 For the cylinder from the equation $N_z = I\,\beta_z$ about its stationary axis of rotation.

$$2\,Tr = \frac{mr^2}{2}\beta \quad \text{or} \quad \beta = \frac{4T}{mr} \qquad (1)$$



For the rotation of the lower cylinder from the equation $N_{cz} = I_c\,\beta_z$

$$2Tr = \frac{mr^2}{2}\beta' \quad \text{or,} \quad \beta' = \frac{4T}{mr} = \beta$$

Now for the translational motion of lower cylinder from the Eq. $F_x = mw_{cx}$:

$$mg - 2\,T = mw_c \qquad (2)$$

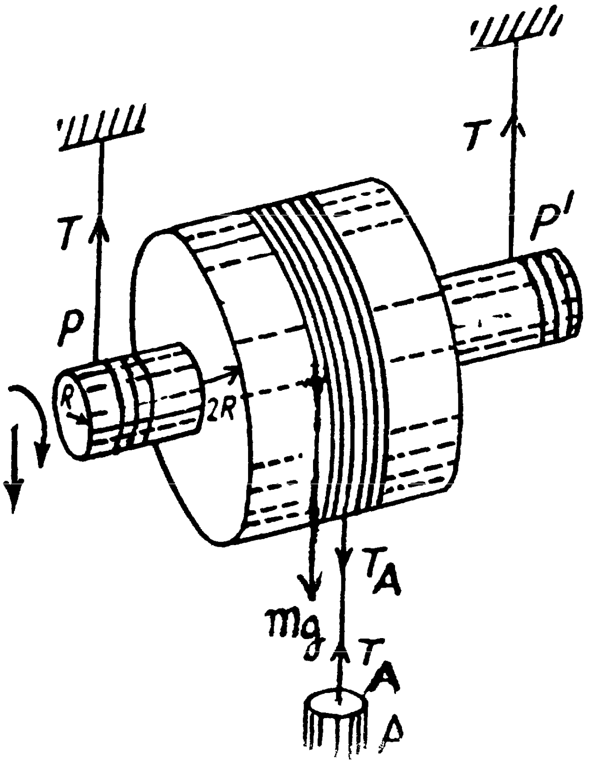
As there is no slipping of threads on the cylinders :

$$w_c = \beta'\,r + \beta r = 2\,\beta r \qquad (3)$$

Simultaneous solution of (1), (2) and (3) yields

$$T = \frac{mg}{10}.$$

1.259 Let us depict the forces acting on the pulley and weight A, and indicate positive direction for x and φ as shown in the figure. For the cylinder from the equation $F_x = m\, w_x$ and $N_{cz} = I_c \beta_z$, we get



$$Mg + T_A - 2T = M w_c \qquad (1)$$

and $$2TR + T_A (2R) = I\beta = \frac{I w_c}{R} \qquad (2)$$

For the weight A from the equation

$$F_x = m w_x$$

$$mg - T_A = m w_A \qquad (3)$$

As there is no slipping of the threads on the pulleys.

$$w_A = w_c + 2\beta R = w_c + 2 w_c = 3 w_c \qquad (4)$$

Simultaneous solutions of above four equations gives :

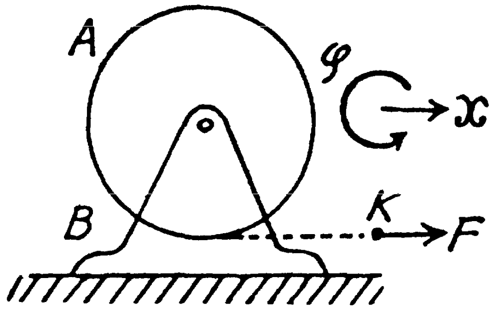
$$w_A = \frac{3(M + 3m) g}{\left(M + 9m + \frac{I}{R^2}\right)}$$

1.260 (a) For the translational motion of the system $(m_1 + m_2)$, from the equation : $F_x = m w_{cx}$

$$F = (m_1 + m_2) w_c \quad \text{or,} \quad w_c = F/(m_1 + m_2) \qquad (1)$$

Now for the rotational motion of cylinder from the equation : $N_{cx} = I_c \beta_z$

$$Fr = \frac{m_1 r^2}{2}\beta \quad \text{or} \quad \beta r = \frac{2F}{m_1} \qquad (2)$$



But $\quad w_K = w_c + \beta r$, So

$$w_K = \frac{F}{m_1 + m_2} + \frac{2F}{m_1} = \frac{F(3m_1 + 2m_2)}{m_1(m_1 + m_2)} \qquad (3)$$

(b) From the equation of increment of mechanical energy : $\Delta T = A_{ext}$

Here $\quad \Delta T = T(t)$, so, $\quad T(t) = A_{ext}$

As force F is constant and is directed along x-axis the sought work done.

$$A_{ext} = Fx$$

(where x is the displacement of the point of application of the force F during time interval t)

$$= F\left(\frac{1}{2}w_K t^2\right) = \frac{F^2 t^2 (3m_1 + 2m_2)}{2m_1(m_1+m_2)} = T(t)$$

(using Eq. (3)

Alternate : $T(t) = T_{translation}(t) + T_{rotation}(t)$

$$= \frac{1}{2}(m_1+m_2)\left(\frac{Ft}{(m_1+m_2)}\right)^2 + \frac{1}{2}\frac{m_1 r^2}{2}\left(\frac{2Ft}{m_1 r}\right)^2 = \frac{F^2 t^2(3m_1+2m_2)}{2m_1(m_{1+}m_2)}$$

1.261 Choosing the positive direction for x and φ as shown in Fig, let us we write the equation of motion for the sphere $F_x = mw_{cx}$ and $N_{cz} = I_c\,\beta_z$

$fr = m_2 w_2$; $fr\, r = \frac{2}{5} m_2 r^2 \beta$

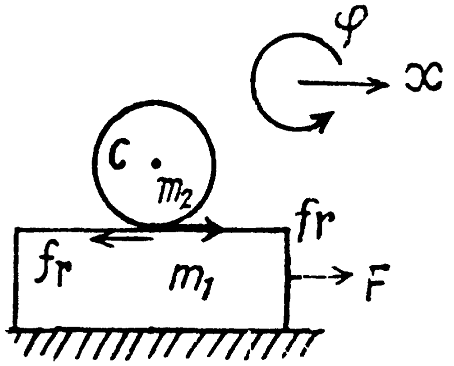
(w_2 is the acceleration of the C.M. of sphere.)

For the plank from the Eq. $F_x = mw_x$

$F - f_r = m_1 w_1$

In addition, the condition for the absence of slipping of the sphere yields the kinematical relation between the accelerations :

$$w_1 = w_2 + \beta r$$



Simultaneous solution of the four equations yields :

$$w_1 = \frac{F}{\left(m_1 + \frac{2}{7}m_2\right)} \quad \text{and} \quad w_2 = \frac{2}{7}w_1$$

1.262 (a) Let us depict the forces acting on the cylinder and their point of applications for the cylinder and indicate positive direction of x and φ as shown in the figure. From the equations for the plane motion of a solid $F_x = mw_{cx}$ and $N_{cz} = I_c\,\beta_z$:

$$k\,mg = m\,w_{cx} \quad \text{or} \quad w_{cx} = kg \tag{1}$$

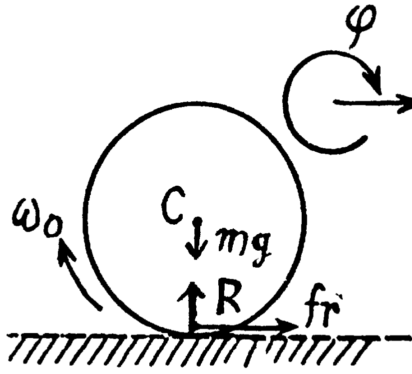
$$-kmg\,R = \frac{mR^2}{2}\beta^z \quad \text{or} \quad \beta_z = -2\frac{kg}{R} \tag{2}$$

Let the cylinder starts pure rolling at $t = t_0$ after releasing on the horizontal floor at $t = 0$.

From the angular kinematical equation

$$\omega_z = \omega_{oz} + \beta_z t,$$

or $$\omega = \omega_0 - 2\frac{kg}{R}t \tag{3}$$



From the equation of the linear kinematics,

$$v_{cx} = v_{ocx} + w_{cx}t$$

or $$v_c = 0 + kg\,t_0 \tag{4}$$

But at the moment $t = t_0$, when pure rolling starts $v_c = \omega R$

so, $$kg\, t_0 = \left(\omega_0 - 2\frac{kg}{R} t_0\right) R$$

Thus $$t_0 = \frac{\omega_0 R}{3\, kg}$$

(b) As the cylinder pick, up speed till it starts rolling, the point of contact has a purely translatory movement equal to $\frac{1}{2} w_c\, t_0^2$ in the forward directions but there is also a backward movement of the point of contact of magnitude $(\omega_0 \tau_0 - \frac{1}{2}\beta\, t_0^2) R$. Because of slipping the net displacement is backwards. The total work done is then,

$$A_{fr} = kmg\left[\frac{1}{2} w_c\, t_0^2 - (\omega_0 t_0 + \frac{1}{2}\beta t_0^2) R\right]$$

$$= kmg\left[\frac{1}{2} kg\, t_0^2 - \frac{1}{2}\left(-\frac{2kg}{R}\right) t_0^2\, R - \omega_0 t_0 R\right]$$

$$= kmg\,\frac{\omega_0 R}{3kg}\left[\frac{\omega_0 R}{6} + \frac{\omega_0 R}{3} - \omega_0 R\right] = -\frac{m\omega_0^2 R^2}{6}$$

The same result can also be obtained by the work-energy theorem, $A_{fr} = \Delta T$.

1.263 Let us write the equation of motion for the centre of the sphere at the moment of breaking-off:

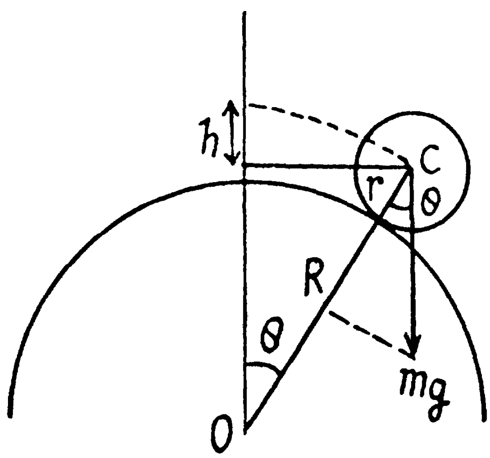
$$mv^2/(R + r) = mg\cos\theta,$$

where v is the velocity of the centre of the sphere at that moment, and θ is the corresponding angle (Fig.). The velocity v can be found from the energy conservation law :

$$mgh = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2,$$

where I is the moment of inertia of the sphere relative to the axis passing through the sphere's centre. i.e. $I = \frac{2}{5} mr^2$. In addition,

$$v = \omega r\,;\ h = (R + r)(1 - \cos\theta).$$



From these four equations we obtain

$$\omega = \sqrt{10\, g\,(R + r)\, 17\, r^2}.$$

1.264 Since the cylinder moves without sliding, the centre of the cylinder rotates about the point O, while passing through the common edge of the planes. In other words, the point O becomes the foot of the instantaneous axis of rotation of the cylinder.

It at any instant during this motion the velocity of the C.M. is v_1 when the angle (shown in the figure) is β, we have

$$\frac{m\, v_1^2}{R} = mg\cos\beta - N,$$

where N is the normal reaction of the edge

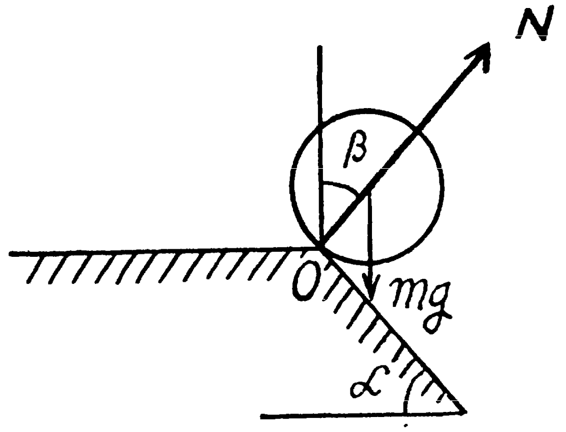
or, $\qquad v_1^2 = gR\cos\beta - \frac{NR}{m} \qquad (1)$

From the energy conservation law,

$$\frac{1}{2}I_0\frac{v_1^2}{R^2} - \frac{1}{2}I_0\frac{v_0^2}{R^2} = mgR(1-\cos\beta)$$

But $\qquad I_0 = \frac{mR^2}{2} + mR^2 = \frac{3}{2}mR^2,$

(from the parallel axis theorem)



Thus, $$v_1^2 = v_0^2 + \frac{4}{3}gR(1-\cos\beta) \qquad (2)$$

From (1) and (2)

$$v_0^2 = \frac{gR}{3}(7\cos\beta - 4) - \frac{NR}{m}$$

The angle β in this equation is clearly smaller than or equal to α so putting $\beta = \alpha$ we get

$$v_0^2 = \frac{gR}{3}(7\cos\alpha - 4) - \frac{N_0R}{M}$$

where N_0 is the corresponding reaction. Note that $N \geq N_0$. No jumping occurs during this turning if $N_0 > 0$. Hence, v_0 must be less than

$$v_{max} = \sqrt{\frac{gR}{3}(7\cos\alpha - 4)}$$

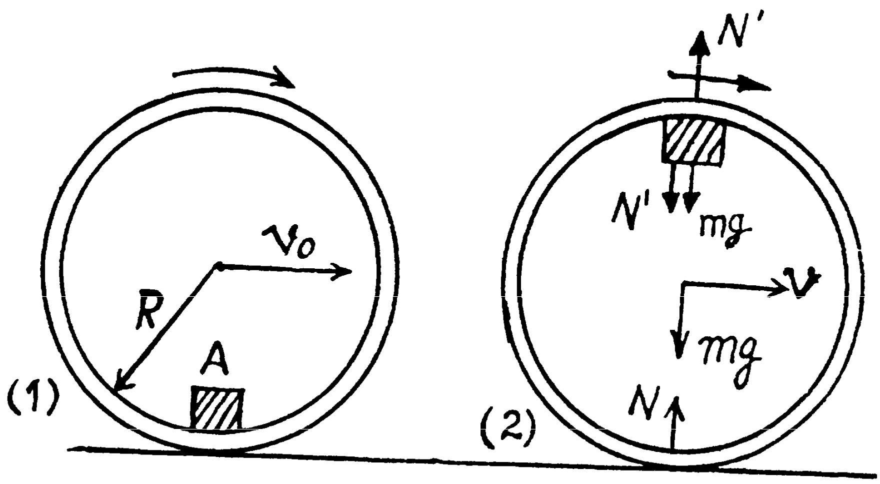
1.265 Clearly the tendency of bouncing of the hoop will be maximum when the small body A, will be at the highest point of the hoop during its rolling motion. Let the velocity of C.M. of the hoop equal v at this position. The static friction does no work on the hoop, so from conservation of mechanical energy; $E_1 = E_2$

or, $$0 + \frac{1}{2}mv_0^2 + \frac{1}{2}mR^2\left(\frac{v_0}{R}\right)^2 - mgR = \frac{1}{2}m(2v)^2 + \frac{1}{2}mv^2 + \frac{1}{2}mR^2\left(\frac{v}{R}\right)^2 + mgR$$

or, $$3v^2 = v_0^2 - 2gR \qquad (1)$$

From the equation $F_n = mw_n$ for body A at final position 2 :

$$mg + N' = m\omega^2 R = m\left(\frac{v}{R}\right)^2 R \qquad (2)$$



As the hoop has no acceleration in vertical direction, so for the hoop,

$$N + N' = mg \tag{3}$$

From Eqs. (2) and (3),

$$N = 2\,mg - \frac{m\,v^2}{R} \tag{4}$$

As the hoop does not bounce, $N \geq 0$ (5)

So from Eqs. (1), (4) and (5),

$$\frac{8\,gR - v_0^2}{3R} \geq 0 \quad \text{or} \quad 8\,gR \geq v_0^2$$

Hence $\qquad v_0 \leq \sqrt{8\,gR}$

1.266 Since the lower part of the belt is in contact with the rigid floor, velocity of this part becomes zero. The crawler moves with velocity v, hence the velocity of upper part of the belt becomes $2v$ by the rolling condition and kinetic energy of upper part $= \frac{1}{2}\left(\frac{m}{2}\right)(2v)^2 = mv^2$, which is also the sought kinetic energy, assuming that the length of the belt is much larger than the radius of the wheels.

1.267 The sphere has two types of motion, one is the rotation about its own axis and the other is motion in a circle of radius R. Hence the sought kinetic energy

$$T = \frac{1}{2} I_1\,\omega_1^2 + \frac{1}{2} I_2\,\omega_2^2 \tag{1}$$

where I_1 is the moment of inertia about its own axis, and I_2 is the moment of inertia about the vertical axis, passing through O,

But, $I_1 = \frac{2}{5} mr^2$ and $I_2 = \frac{2}{5} mr^2 + mR^2$ (using parallel axis theorem,) (2)

In addition to

$$\omega_1 = \frac{v}{r} \text{ and } \omega_2 = \frac{v}{R} \tag{3}$$

Using (2) and (3) in (1), we get $T' = \frac{7}{10}\,mv^2\left(1 + \frac{2r^2}{7R^2}\right)$

1.268 For a point mass of mass dm, looked at from C rotating frame, the equation is

$$dm\,\vec{w}' = \vec{f} + dm\,\omega^2\,\vec{r}' + 2\,dm\,(\vec{v}' \times \vec{\omega})$$

where $\vec{r}'$ = radius vector in the rotating frame with respect to rotation axis and $\vec{v}'$ = velocity in the same frame. The total centrifugal force is clearly

$$\vec{F}_{cf} = \sum dm\,\omega^2\,\vec{r}' = m\omega^2\,\vec{R}_c$$

$\vec{R}_c$ is the radius vector of the C.M. of the body with respect to rotation axis, also

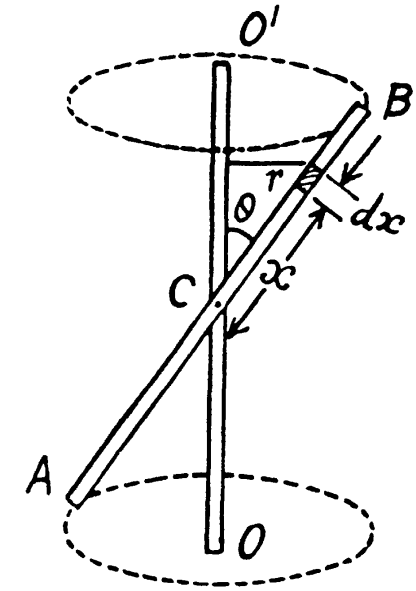
$$\vec{F}_{cor} = 2m\,\vec{v}_c' \times \vec{\omega}$$

where we have used the definitions

$$m\,\vec{R}_c = \sum dm\,\vec{r}' \text{ and } m\,\vec{v}_c' = \sum dm\,\vec{v}'$$

1.269 Consider a small element of length dx at a distance x from the point C, which is rotating in a circle of radius $r = x \sin\theta$

Now, mass of the element $= \left(\frac{m}{l}\right) dx$



So, centrifugal force acting on this element $= \left(\frac{m}{l}\right) dx\, \omega^2 x \sin\theta$ and moment of this force about C,

$$|dN| = \left(\frac{m}{l}\right) dx\, \omega^2 x \sin\theta \cdot x\cos\theta$$

$$= \frac{m\,\omega^2}{2l} \sin 2\theta\, x^2\, dx$$

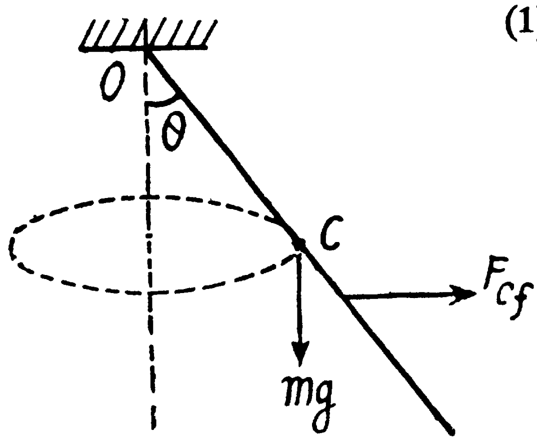
and hence, total moment

$$N = 2\int_0^{l/2} \frac{m\,\omega^2}{2l} \sin 2\theta\; x^2\, dx = \frac{1}{24} m\,\omega^2 l^2 \sin 2\theta,$$

1.270 Let us consider the system in a frame rotating with the rod. In this frame, the rod is at rest and experiences not only the gravitational force $m\vec{g}$ and the reaction force $\vec{R}$, but also the centrifugal force $\vec{F}_{cf}$.

In the considered frame, from the condition of equilibrium i.e. $N_{0z} = 0$

or, $$N_{cf} = mg\frac{l}{2}\sin\theta \qquad (1)$$



where N_{cf} is the moment of centrifugal force about O. To calculate N_{cf}, let us consider an element of length dx, situated at a distance x from the point O. This element is subjected to a horizontal pseudo force $\left(\frac{m}{l}\right) dx\, \omega^2 x \sin\theta$.

The moment of this pseudo force about the axis of rotation through the point O is

$$dN_{cf} = \left(\frac{m}{l}\right) dx\, \omega^2 x \sin\theta\, x\cos\theta$$

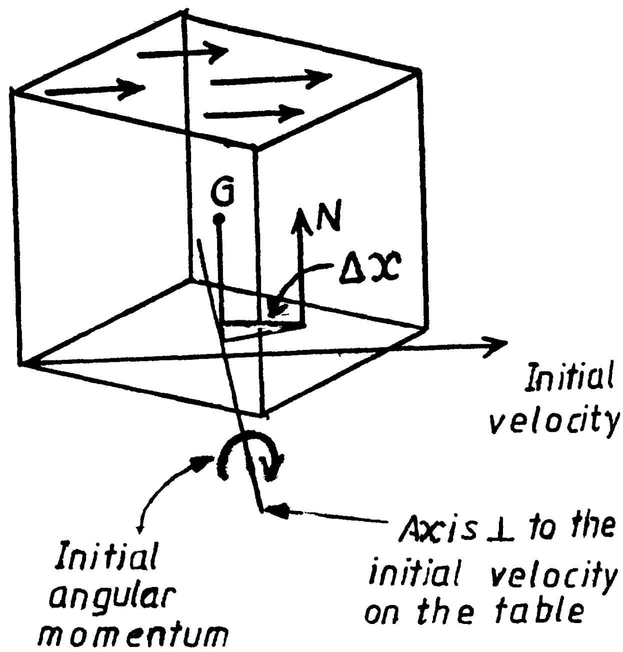
$$= \frac{m\,\omega^2}{l} \sin\theta\cos\theta\; x^2\, dx$$

So $$N_{cf} = \int_0^l \frac{m\,\omega^2}{l}\sin\theta\cos\theta\, x^2\, dx = \frac{m\,\omega^2 l^2}{3}\sin\theta\cos\theta \qquad (2)$$

It follows from Eqs. (1) and (2) that,

$$\cos\theta = \left(\frac{3g}{2\,\omega^2 l}\right) \text{ or } \theta = \cos^{-1}\left(\frac{3g}{2\,\omega^2 l}\right) \qquad (3)$$

1.271 When the cube is given an initial velocity on the table in some direction (as shown) it acquires an angular momentum about an axis on the table perpendicular to the initial velocity and (say) just below the C.G.. This angular momentum will disappear when the cube stops and this can only by due to a torque. Frictional forces cannot do this by themselves because they act in the plain containing the axis. But if the force of normal reaction act eccentrically (as shown), their torque can bring about the vanishing of the angular momentum. We can calculate the distance Δx between the point of application of the normal reaction and the C.G. of the cube as follows. Take the moment about C.G. of all the forces. This must vanish because the cube does not turn or turnble on the table. Then if the force of friction is fr



$$fr\,\frac{a}{2} = N\,\Delta x$$

But $N = mg$ and $fr = kmg$, so

$$\Delta x = ka/2$$

1.272 In the process of motion of the given system the kinetic energy and the angular momentum relative to rotation axis do not vary. Hence, it follows that

$$\frac{1}{2}\frac{Ml^2}{3}\omega_0^2 = \frac{1}{2}m(\omega^2 l^2 + v'^2) + \frac{1}{2}\frac{Ml^2}{3}\omega^2$$

(ω is the final angular velocity of the rod)

and $$\frac{Ml^2}{3}\omega_0 = \frac{Ml^2}{3}\omega + ml^2\omega$$

From these equations we obtain

$$\omega = \omega_0 \Big/ \left(1 + \frac{3M}{M}\right) \text{ and}$$

$$v' = \omega_0 l \Big/ \sqrt{1 + 3m/M}$$

1.273 Due to hitting of the ball, the angular impulse received by the rod about the C.M. is equal to $p\,\frac{1}{2}$. If ω is the angular velocity acquired by the rod, we have

$$\frac{ml^2}{12}\omega = \frac{pl}{2} \quad \text{or} \quad \omega = \frac{6p}{ml} \tag{1}$$

In the frame of C.M., the rod is rotating about an axis passing through its mid point with the angular velocity ω. Hence the force exerted by one half on the other = mass of one half × acceleration of C.M. of that part, in the frame of C.M.

$$= \frac{m}{2}\left(\omega^2\frac{l}{4}\right) = m\frac{\omega^2 l}{8} = \frac{9p^2}{2ml} = 9\,\text{N}$$

1.274 (a) In the process of motion of the given system the kinetic energy and the angular momentum relative to rotation axis do not vary. Hence it follows that

$$\frac{1}{2} m v^2 = \frac{1}{2} m v'^2 + \frac{1}{2}\left(\frac{Ml^2}{3}\right)\omega^2$$

and
$$mv\frac{l}{2} = mv'\frac{l}{2} + \frac{Ml^2}{3}\omega$$

From these equations we obtain

$$v' = \left(\frac{3m - 4M}{3m + 4M}\right)v. \quad \text{and} \quad \omega = \frac{4v}{l(1 + 4m/3M)}$$

As $\vec{v}' \uparrow\uparrow \vec{v}$, so in vector form $\vec{v}' = \left(\frac{3m - 4M}{3m + 4M}\right)\vec{v}$

(b) Obviously the sought force provides the centripetal acceleration to the C.M. of the rod and is

$$F_n = m w_{cn}$$

$$= M\omega^2 \frac{l}{2} = \frac{8Mv^2}{l(1 + 4M/3m)^2}$$

1.275 (a) About the axis of rotation of the rod, the angular momentum of the system is conserved. Thus if the velocity of the flying bullet is v.

$$mvl = \left(ml^2 + \frac{Ml^2}{3}\right)\omega$$

$$\omega = \frac{mv}{\left(m + \frac{M}{3}\right)l} \approx \frac{3mv}{Ml} \quad \text{as } m \ll M \tag{1}$$

Now from the conservation of mechanical energy of the system (rod with bullet) in the uniform field of gravity

$$\frac{1}{2}\left(ml^2 + \frac{Ml^2}{3}\right)\omega^2 = (M + m)g\frac{l}{2}(1 - \cos\alpha) \tag{2}$$

[because C.M. of rod raises by the height $\frac{l}{2}(1 - \cos\alpha)$]

Solving (1) and (2), we get

$$v = \left(\frac{M}{m}\right)\sqrt{\frac{2}{3}gl}\,\sin\frac{\alpha}{2} \quad \text{and} \quad \omega = \sqrt{\frac{6g}{l}}\,\sin\frac{\alpha}{2}$$

(b) Sought $\Delta p = \left[m(\omega l) + M\left(\omega\frac{l}{2}\right)\right] - mv$

where ωl is the veloccity of the bullet and $\omega\frac{l}{2}$ equals the velocity of C.M. of the rod after the impact. Putting the value of v and ω we get

$$\Delta p = \frac{1}{2}mv = M\sqrt{\frac{gl}{6}}\,\sin\frac{\alpha}{2}$$

This is caused by the reaction at the hinge on the upper end.

(c) Let the rod starts swinging with angular velocity ω', in this case. Then, like part (a)

$$mvx = \left(\frac{Ml^2}{3} + mx^2\right)\omega' \quad \text{or} \quad \omega' \approx \frac{3mvx}{Ml^2}$$

Final momentum is

$$p_f \approx m\,x\,\omega' + \int_0^l y\,\omega'\frac{M}{l}dy \approx \frac{M}{2}\omega'\,l \approx \frac{3}{2}m\,v\,\frac{x}{l}$$

So, $$\Delta p \approx p_f - p_i \approx m\,v\left(\frac{3\,x}{2l} - 1\right)$$

This vanishes for $$x \approx \frac{2}{3}l$$

1.276 (a) As force F on the body is radial so its angular momentum about the axis becomes zero and the angular momentum of the system about the given axis is conserved. Thus

$$\frac{MR^2}{2}\omega_0 + m\,\omega_0 R^2 = \frac{MR^2}{2}\omega \quad \text{or} \quad \omega = \omega_0\left(1 + \frac{2m}{M}\right)$$

(b) From the equation of the increment of the mechanical energy of the system :

$$\Delta T = A_{ext}$$

$$\frac{1}{2}\frac{MR^2}{2}\omega^2 - \frac{1}{2}\left(\frac{MR^2}{2} + mR^2\right)\omega_0^2 = A_{ext}$$

Putting the value of ω from part (a) and solving we get

$$A_{ext} = \frac{m\,\omega_0^2 R^2}{2}\left(1 + \frac{2m}{M}\right)$$

1.277 (a) Let z be the rotation axis of disc and φ be its rotation angle in accordance with right-hand screw rule (Fig.). (φ and φ' are to be measured in the same sense algebraically.) As M_z of the system (disc + man) is conserved and $M_{z(initial)} = 0$, we have at any instant,

$$0 = \frac{m_2 R^2}{2}\frac{d\varphi}{dt} + m_1\left[\left(\frac{d\varphi'}{dt}\right)R + \left(\frac{d\varphi}{dt}\right)R\right]R$$

or, $$d\varphi = \left[-\frac{m_1}{m_1 + (m_2/2)}\right]d\varphi'$$

On integrating $$\int_0^{\varphi} d\varphi = -\int_0^{\varphi'}\left(\frac{m_1}{m_1 + (m_2/2)}\right)d\varphi'$$

or, $$\varphi = -\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\varphi' \qquad (1)$$

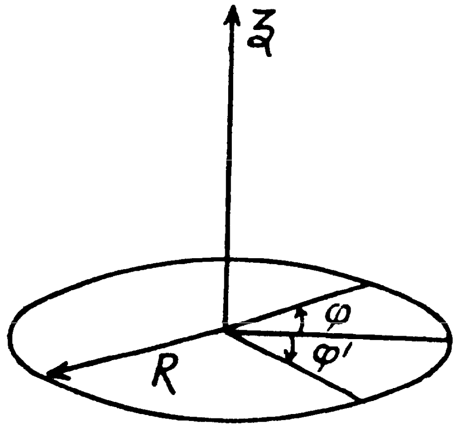
This gives the total angle of rotation of the disc.

(b) From Eq. (1)

$$\frac{d\varphi}{dt} = -\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\frac{d\varphi'}{dt} = -\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\frac{v'(t)}{R}$$

Differentiating with respect to time

$$\frac{d^2\varphi}{dt^2} = -\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\frac{1}{R}\frac{dv'(t)}{dt}$$



Thus the sought force moment from the Eq. $N_z = I\beta_z$

$$N_z = \frac{m_2 R^2}{2}\frac{d^2\varphi}{dt^2} = -\frac{m_2 R^2}{2}\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\frac{1}{R}\frac{dv'(t)}{dt}$$

Hence $$N_z = -\frac{m_1 m_2 R}{2m_1 + m_2}\frac{dv'(t)}{dt}$$

1.278 (a) Frome the law of conservation of angular momentum of the system relative to vertical axis z, it follows that:

$$I_1\omega_{1z} + I_2\omega_{2z} = (I_1 + I_2)\omega_z$$

Hence $$\omega_z = (I_1\omega_{1z} + I_2\omega_{2z}) / (I_1 + I_2) \quad (1)$$

Not that for $\omega_z > 0$, the corresponding vector $\vec{\omega}$ coincides with the poitive direction to the z axis, and vice versa. As both discs rotates about the same vertical axis z, thus in vector form.

$$\vec{\omega} = I_1\vec{\omega}_1 + I_2\vec{\omega}_2 / (I_1 + I_2)$$

However, the problem makes sense only if $\vec{\omega}_1 \uparrow\uparrow \vec{\omega}_2$ or $\vec{\omega}_1 \uparrow\downarrow \vec{\omega}_2$

(b) From the equation of increment of mechanical energy of a system: $A_{fr} = \Delta T$.

$$= \frac{1}{2}(I_1 + I_2)\omega_z^2 - \frac{1}{2}I_1\omega_{1z}^2 + \frac{1}{2}I_2\omega_{2z}^2$$

Using Eq. (1)

$$A_{fr} = -\frac{I_1 I_2}{2(I_1 + I_2)}(\omega_{1z} - \omega_{2z})^2$$

1.279 For the closed system (disc + rod), the angular momentum is conserved about any axis. Thus from the conservation of angular momentum of the system about the rotation axis of rod passing through its C.M. gives :

$$mv\frac{l}{2} = mv'\frac{l}{2} + \frac{\eta m l^2}{12}\omega \quad (1)$$

(v' is the final velocity of the disc and ω angular velocity of the rod)

For the closed system linear momentum is also conserved. Hence

$$mv = mv' + \eta\, mv_c \qquad (2)$$

(where v_c is the velocity of C.M. of the rod)

From Eqs (1) and (2) we get

$$v_c = \frac{l\,\omega}{3} \quad \text{and} \qquad v - v' = \eta v_c$$

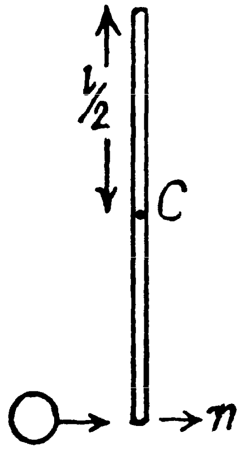
Applying conservation of kinetic energy, as the collision is elastic

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^{\,2} + \frac{1}{2}\eta m v_c^{\,2} + \frac{1}{2}\,\frac{\eta m l^2}{12}\,\omega^2 \qquad (3)$$

or $\qquad v^2 - v'^2 = 4\eta v_c^{\,2}$ and hence $v + v' = 4v_c$

Then

$$v' = \frac{4-\eta}{4+\eta}\,v \text{ and } \omega = \frac{12\,v}{(4+\eta)\,l}$$



Vectorially, noting that we have taken $\vec{v}\,'$ parallel to $\vec{v}$

$$\vec{u}\,' = \left(\frac{4-\eta}{4+\eta}\right)\vec{v}$$

So, $\vec{u}\,' = 0$ for $\eta = 4$ and $\vec{u}\,' \downarrow\uparrow \vec{v}$ for $\eta > 4$

1.280 See the diagram in the book (Fig. 1.72)

(a) When the shaft BB' is turned through 90° the platform must start turning with angular velocity Ω so that the angular momentum remains constant. Here

$$(I + I_0)\,\Omega = I_0\,\omega_0 \quad \text{or,} \quad \Omega = \frac{I_0\,\omega_0}{I + I_0}$$

The work performed by the motor is therefore

$$\frac{1}{2}(I + I_0)\,\Omega^2 = \frac{1}{2}\frac{I_0^2\,\omega_0^2}{I + I_0}$$

If the shaft is turned through 180°, angular velocity of the sphere changes sign. Thus from conservation of angular momentum,

$$I\,\Omega - I_0\,\omega_0 = I_0\,\omega_0$$

(Here $-I_0\,\omega_0$ is the complete angular momentum of the sphere i. e. we assume that the angular velocity of the sphere is just $-\omega_0$). Then

$$\Omega = 2I_0\,\frac{\omega_0}{I}$$

and the work done must be,

$$\frac{1}{2}I\,\Omega^2 + \frac{1}{2}I_0\,\omega_0^2 - \frac{1}{2}I_0\,\omega_0^2 = \frac{2I_0^2\,\omega_0^2}{I}$$

(b) In the case (a), first part, the angular momentum vector of the sphere is precessing with angular velocity Ω. Thus a torque,

$$I_0 \omega_0 \Omega = \frac{I_0^2 \omega_0^2}{I + I_0} \text{ is needed.}$$

1.281 The total centrifugal force can be calculated by,

$$\int_0^{l_0} \frac{m}{l_0} \omega^2 x\, dx = \frac{1}{2} m\, l_0\, \omega^2$$

Then for equilibrium,

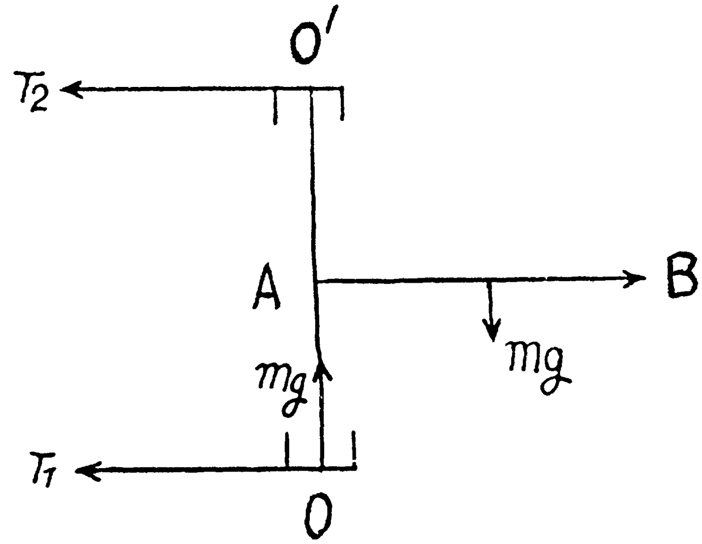
$$(T_2 - T_1)\frac{l}{2} = mg\frac{l_0}{2}$$

and, $T_2 + T_1 = \frac{1}{2} m\, l_0\, \omega^2$

Thus T_1 vanishes, when

$$\omega^2 = \frac{2g}{l}, \quad \omega = \sqrt{\frac{2g}{l}} = 6 \text{ rad/s}$$

Then $T_2 = mg\frac{l_0}{l} = 25$ N



1.282 See the diagram in the book (Fig. 1.71).

(a) The angular velocity $\vec{\omega}$ about OO' can be resolved into a component parallel to the rod and a component $\omega \sin\theta$ perpendicular to the rod through C. The component parallel to the rod does not contribute so the angular momentum

$$M = I\omega \sin\theta = \frac{1}{12} m\, l^2 \omega \sin\theta$$

Also, $$M_z = M \sin\theta = \frac{1}{12} m\, l^2 \omega \sin^2\theta$$

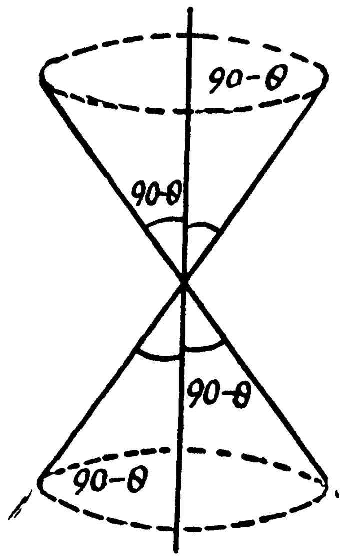
This can be obtained directly also,

(b) The modulus of $\vec{M}$ does not change but the modulus of the change of $\vec{M}$ is $|\Delta \vec{M}|$.

$$|\Delta \vec{M}| = 2M \sin(90 - \theta) = \frac{1}{12} m\, l^2 \omega \sin 2\theta$$

(c) Here $M_\perp = M\cos\theta = I\omega \sin\theta \cos\theta$

Now $$\left|\frac{d\vec{M}}{dt}\right| = I\omega \sin\theta \cos\theta\, \frac{\omega\, dt}{dt} = \frac{1}{24} m l^2 \omega^2 \sin^2\theta$$



as $\vec{M}$ precesses with angular velocity ω.

1.283 Here $M = I\omega$ is along the symmetry axis. It has two components, the part $I\omega\cos\theta$ is constant and the part $M_{\perp} = I\omega\sin\theta$ presesses, then

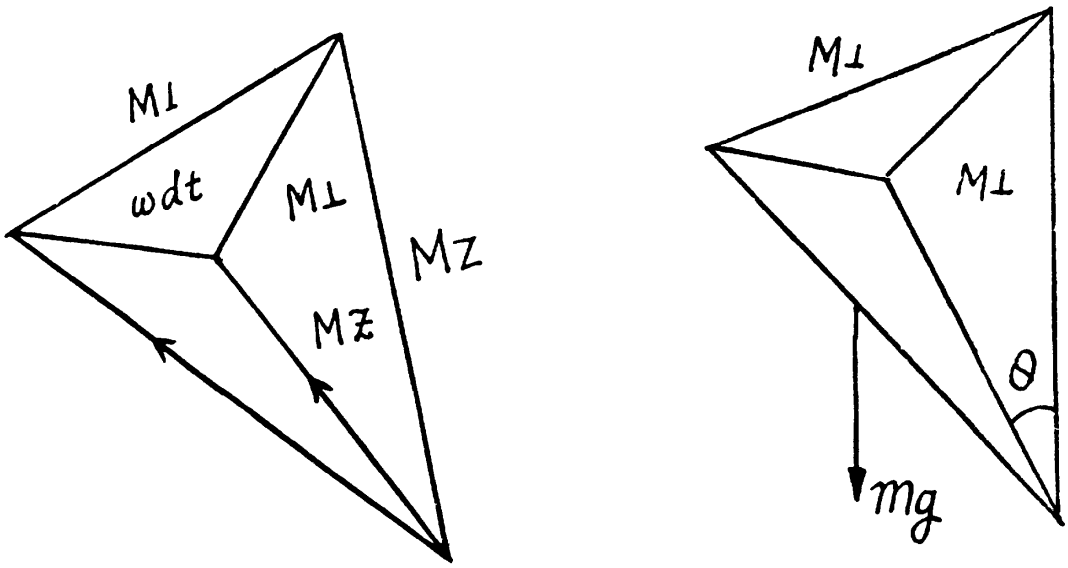
$$\left|\frac{d\vec{M}}{dt}\right| = I\omega\sin\theta\,\omega' = mgl\sin\theta$$

or, $$\omega' = \text{precession frequency} = \frac{mgl}{I\omega} = 0{\cdot}7\ \text{rad/s}$$

(b) This force is the centripetal force due to precession. It acts inward and has the magnitude

$$|\vec{F}| = \left|\sum m_i\,\omega'^2\,\vec{\rho_i}\right| = m\,\omega'^2\,l\sin\theta = 12\ \text{mN}.$$

$\vec{\rho_i}$ is the distance of the $i\,th$ element from the axis. This is the force that the table will exert on the top. See the diagram in the answer sheet



1.284 See the diagram in the book (Fig. 1.73).

The moment of inertia of the disc about its symmentry axis is $\frac{1}{2}mR^2$. If the angular velocity of the disc is ω then the angular momentum is $\frac{1}{2}mR^2\omega$. The precession frequency being $2\pi\,n$,

we have $$\left|\frac{d\vec{M}}{dt}\right| = \frac{1}{2}mR^2\,\omega\times 2\pi n$$

This must equal $m\,(g + w)\,l$, the effective gravitational torques (g being replaced by $g + w$ in the elevator). Thus,

$$\omega = \frac{(g+w)\,l}{\pi R^2\,n} = 300\ \text{rad/s}$$

1.285 The effective g is $\sqrt{g^2+w^2}$ inclined at angle $\tan^{-1}\frac{w}{g}$ with the vertical. Then with reference to the new " vertical" we proceed as in problem 1·283. Thus

$$\omega' = \frac{ml\sqrt{g^2+w^2}}{I\omega} = 0{\cdot}8 \text{ rad/s.}$$

The vector $\vec{\omega}'$ forms an angle $\theta = \tan^{-1}\frac{w}{g} = 6°$ with the normal vertical.

1.286 The moment of inertia of the sphere is $\frac{2}{5}mR^2$ and hence the value of angular momentum is $\frac{2}{5}mR^2\omega$. Since it precesses at speed ω' the torque required is

$$\frac{2}{5}mR^2\,\omega\,\omega' = F'\,l$$

So,
$$F' = \frac{2}{5}mR^2\,\omega\,\omega'/l = 300 \text{ N}$$

(The force F' must be vertical.)

1.287 The moment of inertia is $\frac{1}{2}mr^2$ and angular momentum is $\frac{1}{2}mr^2\omega$. The axle oscillates about a horizontal axis making an instantaneous angle.

$$\varphi = \varphi_m \sin\frac{2\pi t}{T}$$

This means that there is a variable precession with a rate of precession $\frac{d\varphi}{dt}$. The maximum value of this is $\frac{2\pi\varphi_m}{T}$. When the angle between the axle and the axis is at its maximum value, a torque $I\,\omega\,\Omega$

$$= \frac{1}{2}mr^2\,\omega\,\frac{2\pi\varphi_m}{T} = \frac{\pi\,mr^2\,\omega\varphi_m}{T} \quad \text{acts on it.}$$

The corresponding gyroscopic force will be $\frac{\pi\,mr^2\,\omega\varphi_m}{lT} = 90$ N

1.288 The revolutions per minute of the flywheel being n, the angular momentum of the flywheel is $I \times 2\pi n$. The rate of precession is $\frac{v}{R}$

Thus $N = 2\pi INV/R = 5{\cdot}97$ kN. m.

1.289 As in the previous problem a couple $2\pi Inv/R$ must come in play. This can be done if a force, $\frac{2\pi Inv}{Rl}$ acts on the rails in opposite directions in addition to the centrifugal and other forces. The force on the outer rail is increased and that on the inner rail decreased. The additional force in this case has the magnitude 1·4 kN. m.

1.6 ELASTIC DEFORMATIONS OF A SOLID BODY

1.290 Variation of length with temperature is given by

$$l_t = l_0(1+\alpha\Delta t) \text{ or } \frac{\Delta l}{l_0} = \alpha\,\Delta t = \varepsilon \tag{1}$$

But $$\varepsilon = \frac{\sigma}{E},$$

Thus $\sigma = \alpha \Delta t E$, which is the sought stress of pressure.

Putting the value of α and E from Appendix and taking $\Delta t = 100°\text{C}$, we get

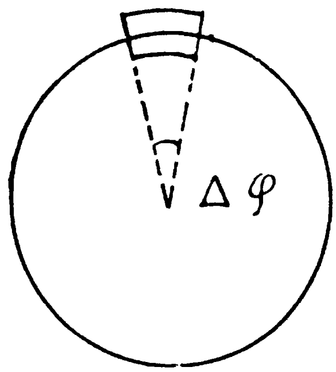
$$\sigma = 2{\cdot}2 \times 10^3 \text{ atm.}$$

1.291 (a) Consider a transverse section of the tube and concentrate on an element which subtends an angle $\Delta\varphi$ at the centre. The forces acting on a portion of length Δl on the element are

(1) tensile forces side ways of magnitude $\sigma \Delta r \Delta l$.

The resultant of these is

$$2\sigma\Delta r\Delta l \sin\frac{\Delta\varphi}{2} \approx \sigma\Delta r\Delta l\Delta\varphi$$



radially towards the cente.

(2) The force due to fluid pressure $= pr\Delta\varphi\Delta l$

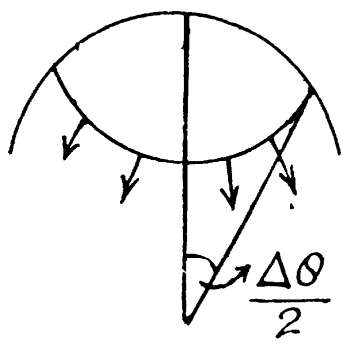
Since these balance, we get $\quad p_{\max} \approx \sigma_m \dfrac{\Delta r}{r}$

where σ_m is the maximum tensile force.

Putting the values we get $p_{\max} = 19{\cdot}7$ atmos.

(b) Consider an element of area $dS = \pi(r\,\Delta\theta/2)^2$ about z-axis chosen arbitrarily. There are tangential tensile forces all around the ring of the cap. Their resultant is

$$\sigma\left[2\pi\left(r\frac{\Delta\theta}{2}\right)\Delta r\right]\sin\frac{\Delta\theta}{2}$$



Hence in the limit

$$p_m \pi\left(\frac{r\Delta\theta}{2}\right)^2 = \sigma_m \pi\left(\frac{r\Delta\theta}{2}\right)\Delta r\,\Delta\theta$$

or $$p_m = \frac{2\sigma_m \Delta r}{r} = 39{\cdot}5 \text{ atmos.}$$

1.292 Let us consider an element of rod at a distance x from its rotation axis (Fig.). From Newton's second law in projection form directed towards the rotation axis

$$-dT = (dm)\,\omega^2 x = \frac{m}{l}\,\omega^2 x\,dx$$

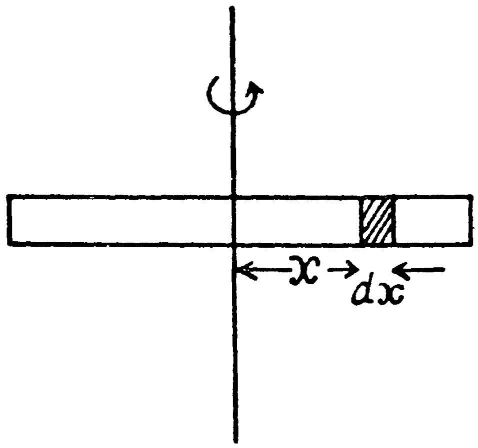
On integrating

$$-T = \frac{m\omega^2}{l}\frac{x^2}{2} + \text{C (constant)}$$

But at $\qquad x = \pm\frac{l}{2}$ or free end, $T = 0$

Thus $\qquad 0 = \frac{m\omega^2}{2}\frac{l^2}{4} + C$ or $C = -\frac{m\omega^2 l}{8}$

Hence $\qquad T = \frac{m\omega^2}{2}\left(\frac{l}{4} - \frac{x^2}{l}\right)$



Thus $\qquad T_{\max} = \frac{m\omega^2 l}{8}$ (at mid point)

Condition required for the problem is

$$T_{\max} = S\sigma_m$$

So, $\frac{m\omega^2 l}{8} = S\sigma_m$ or $\omega = \frac{2}{l}\sqrt{\frac{2\sigma_m}{\rho}}$

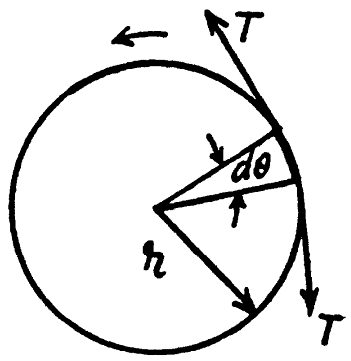
Hence the sought number of r p s

$$n = \frac{\omega}{2\pi} = \frac{1}{\pi l}\sqrt{\frac{2\sigma_m}{\rho}}$$ [using the table $n = 0{\cdot}8 \times 10^2$rps]

1.293 Let us consider an element of the ring (Fig.). From Newton's law $F_n = mw_n$ for this element, we get,

$$T d\theta = \left(\frac{m}{2\pi}d\theta\right)\omega^2 r$$ [see solution of 1.93or 1.92]

So, $\qquad T = \frac{m}{2\pi}\omega^2 r$



Condition for the problem is :

$\frac{T}{\pi r^2} \le \sigma_m$ or, $\frac{m\omega^2 r}{2\pi^2 r^2} \le \sigma_m$

or, $\qquad \omega^2_{\max} = \frac{2\pi^2\sigma_m r}{\pi r^2(2\pi r\rho)} = \frac{\sigma_m}{\rho r^2}$

Thus sought number of rps

$$n = \frac{\omega_{\max}}{2\pi} = \frac{1}{2\pi r}\sqrt{\frac{\sigma_m}{\rho}}$$

Using the table of appendices $n = 23$rps

1.294 Let the point O desend by the distance x (Fig.). From the condition of equilibrium of point O.

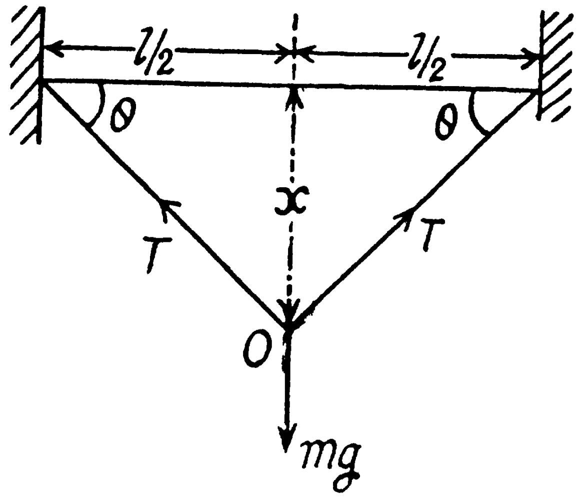
$$2T\sin\theta = mg \text{ or } T = \frac{mg}{2\sin\theta} = \frac{mg}{2x}\sqrt{(l/2)^2 + x^2} \qquad (1)$$

Now, $$\frac{T}{\pi(d/2)^2} = \sigma = \varepsilon E \text{ or } T = \varepsilon E\pi\frac{d^2}{4} \qquad (2)$$

(σ here is stress and ε is strain.)

In addition to it,

$$\varepsilon = \frac{\sqrt{(l/2)^2 + x^2} - \frac{l}{2}}{l/2} = \sqrt{1 + \left(\frac{2x}{l}\right)^2} - 1 \quad (3)$$



From Eqs. (1), (2) and (3)

$$x - \frac{x}{\sqrt{1 + \left(\frac{2x}{l}\right)^2}} = \frac{mgl}{\pi E d^2} \quad \text{as } x << l$$

So, $\frac{4x^3}{2l^2} \approx \frac{mgl}{\pi E d^2}$

or, $x = l\left(\frac{mg}{2\pi E d^2}\right)^{1/3} = 2{\cdot}5 \text{ cm}$

1.295 Let us consider an element of the rod at a distance x from the free end (Fig.). For the considered element '$T - T$' are internal restoring forces which produce elongation and dT provides the acceleration to the element. For the element from Newton's law :

$$dT = (dm)\, w = \left(\frac{m}{l} dx\right) \frac{F_o}{m} = \frac{F_o}{l} dx$$

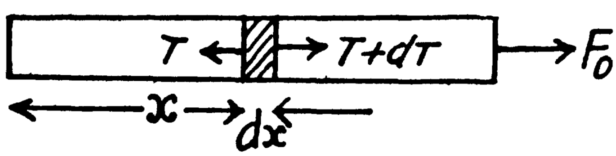
As free end has zero tension, on integrating the above expression,

$$\int_o^T dT = \frac{F_o}{l} \int_o^x dx \quad \text{or} \quad T = \frac{F_o}{l} x$$

Elongation in the considered element of lenght dx :

$$\partial \xi = \frac{\sigma}{E}(x)\ dx = \frac{T}{SE} dx = \frac{F_{xo} x dx}{SEL}$$

Thus total elengation $\xi = \frac{F_o}{SEl} \int_x^l x\, dx = \frac{F_o l}{2SE}$



Hence the sought strain

$$\sigma = \frac{\xi}{l} = \frac{F_o}{2SE}$$

1.296 Let us consider an element of the rod at a distance r from it's rotation axis. As the element rotates in a horizontal circle of radius r, we have from Newton's second law in projection form directed toward the axis of rotation :

$$T - (T + dT) = (dm)\, \omega^2 r$$

or,

$$-dT = \left(\frac{m}{l} dr\right) \omega^2 r = \frac{m}{l} \omega^2 r\, dr$$

At the free end tension becomes zero. Integrating the above experession we get, thus

$$-\int_T^0 dT = \frac{m}{l}\omega^2 \int_r^l r\,dr$$

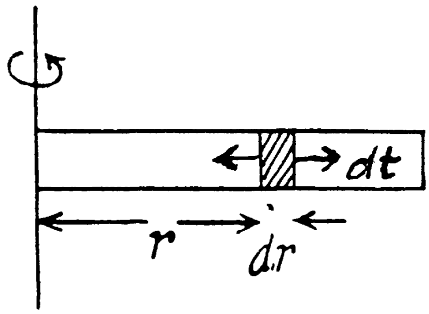
Thus $$T = \frac{m\omega^2}{l}\left(\frac{l^2 - r^2}{2}\right) = \frac{m\,\omega^2\,l}{2}\left(1 - \frac{r^2}{l^2}\right)$$

Elongation in elemental length dr is given by :

$$\partial\,\xi = \frac{\sigma\,(r)}{E}\,dr = \frac{T}{SE}\,dr$$

(where S is the cross sectional area of the rod and T is the tension in the rod at the considered element)

or, $$\partial\,\xi = \frac{m\,\omega^2\,l}{2\,SE}\left(1 - \frac{r^2}{l^2}\right)dr$$



Thus the sought elongation

$$\xi = \int d\,\xi = \frac{m\,\omega^2\,l}{2\,SE}\int_0^l \left(1 - \frac{r^2}{l^2}\right)dr$$

or, $$\xi = \frac{m\,\omega^2\,l}{2\,SE}\,\frac{2l}{3} = \frac{(Sl\,\rho)}{3\,SE}\,\omega^2\,l^3$$

$$= \frac{1}{3}\,\frac{\rho\,\omega^2\,l^3}{E}$$ (where ρ is the density of the copper.)

1.297 Volume of a solid cylinder

$$V = \pi\,r^2\,l$$

So, $$\frac{\Delta V}{V} = \frac{\pi\,2\,r\,\Delta\,r\,l}{\pi\,r^2\,l} + \frac{\pi\,r^2\,\Delta\,l}{\pi\,r^2\,l} = \frac{2\,\Delta\,r}{r} + \frac{\Delta\,l}{l} \qquad (1)$$

But longitudinal strain $\Delta l/l$ and accompanying lateral strain $\Delta\,r/r$ are related as

$$\frac{\Delta\,r}{r} = -\,\mu\,\frac{\Delta\,l}{l} \qquad (2)$$

Using (2) in (1), we get :

$$\frac{\Delta V}{V} = \frac{\Delta l}{l}\,(1 - 2\,\mu) \qquad (3)$$

But $$\frac{\Delta l}{l} = \frac{-\,F/\pi\,r^2}{E}$$

(Because the increment in the length of cylinder Δl is negative)

So, $$\frac{\Delta V}{V} = \frac{-\,F}{\pi\,r^2\,E}\,(1 - 2\,\mu)$$

Thus, $$\Delta V = \frac{-Fl}{E}(1-2\mu)$$

Negative sign means that the volume of the cylinder has decreased.

1.298 (a) As free end has zero tension, thus the tension in the rod at a vestical distance y from its lower end

$$T = \frac{m}{l} g y \qquad (1)$$

Let ∂l be the elongation of the element of length dy, then

$$\partial l = \frac{\sigma(y)}{E} dy$$

$$= \frac{T}{SE} dy = \frac{mgydy}{SlE} = \rho\, gydy/E \quad \text{(where } \rho \text{ is the density of the copper)}$$

Thus the sought elongation

$$\Delta l = \int \partial l = \rho g \int_0^l \frac{y\,dy}{E} = \frac{1}{2} \rho g l^2/E \qquad (2)$$

(b) If the longitudinal (tensile) strain is $\varepsilon = \frac{\Delta l}{l}$, the accompanying lateral (compressive) strain is given by

$$\varepsilon' = \frac{\Delta r}{r} = -\mu\varepsilon \qquad (3)$$

Then since $V = \pi r^2 l$ we have

$$\frac{\Delta V}{V} = \frac{2\Delta r}{r} + \frac{\Delta l}{l}$$

$$= (1-2\mu)\frac{\Delta l}{l} \quad \text{[Using (3)]}$$

where $\frac{\Delta l}{l}$ is given in part (a), μ is the Poisson ratio for copper.

1.299 Consider a cube of unit length before pressure is applied. The pressure acts on each face. The pressures on the opposite faces constitute a tensile stress producing longitudianl compression and lateral extension. The compressions is $\frac{p}{E}$ and the lateral extension is $\mu\frac{p}{E}$ The net result is a compression

$$\frac{p}{E}(1-2\mu) \text{ in each side.}$$

Hence $\frac{\Delta V}{V} = -\frac{3p}{E}(1-2\mu)$ because from symmetry $\frac{\Delta V}{V} = 3\frac{\Delta l}{l}$

(b) Let us consider a cube under an equal compressive stress σ, acting on all its faces.

Then,
$$\text{volume strain} = -\frac{\Delta V}{V} = \frac{\sigma}{k}, \tag{1}$$

where k is the bulk modulus of elasticity.

So
$$\frac{\sigma}{k} = \frac{3\,\sigma}{E}(1-2\,\mu)$$

or,
$$E = 3\,k\,(1-2\,\mu) = \frac{3}{\beta}(1-2\,\mu)\left(\text{as } k = \frac{1}{\beta}\right)$$

$\mu \le \frac{1}{2}$ if E and β are both to remain positive.

1.300 A beam clamped at one end and supporting an applied load at the free end is called a cantilever. The theory of cantilevers is discussed in advanced text book on mechanics. The key result is that elastic forces in the beam generate a couple, whose moment, called the moment of resistances, balances the external bending moment due to weight of the beam, load etc. The moment of resistance, also called internal bending moment (I.B.M) is given by

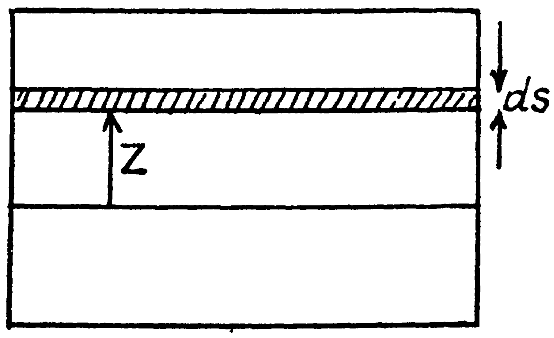
$$\text{I.B.M.} = \text{EI/R}$$

Here R is the radius of curvature of the beam at the representative point (x, y). I is called the geometrical moment of inertia

$$I = \int z^2\, dS$$

of the cross section relative to the axis passing through the netural layer which remains unstretched. (Fig.1.). The section of the beam beyond P exerts the bending moment $N(x)$ and we have,

$$\frac{EI}{R} = N(x)$$



If there is no load other than that due to the weight of the beam, then

$$N(x) = \frac{1}{2}\rho\, g\,(l-x)^2\, bh$$

where ρ = density of steel.

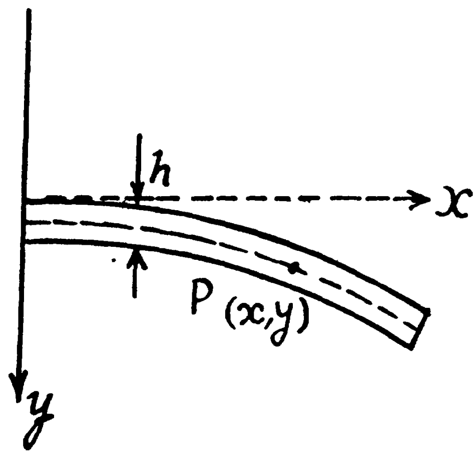
Hence, at $x = 0$

$$\left(\frac{I}{R}\right)_0 = \frac{\rho\, g\, l^2\, b\, h}{2\,E\,I}$$

Here b = width of the beam perpendicular to paper.

Also,
$$I = \int_{-h/2}^{h/2} z^2\, b dz = \frac{bh^3}{12}.$$

Hence,
$$\left(\frac{1}{R}\right)_0 = \frac{6\,\rho\, gl^2}{Eh^2} = (0.121\ \text{km})^{-1}$$



1.301 We use the equation given above and use the result that when y is small

$$\frac{1}{R} \approx \frac{d^2y}{dx^2}. \text{ Thus, } \frac{d^2y}{dx^2} = \frac{N(x)}{EI}$$

(a) Here $N(x) = N_0$ is a constant. Then integration gives,

$$\frac{dy}{dx} = \frac{N_0 x}{EI} + C_1$$

But $\left(\frac{dy}{dx}\right) = 0$ for $x = 0$, so $C_1 = 0$. Integrating again,

$$y = \frac{N_0 x^2}{2EI}$$

where we have used $y = 0$ for $x = 0$ to set the constant of integration at zero. This is the equation of a parabola. The sag of the free end is

$$\lambda = y(x = l) = \frac{N_0 l^2}{2EI}$$

(b) In this case $N(x) = F(l - x)$ because the load F at the extremity is balanced by a similar force at F directed upward and they constitute a couple. Then

$$\frac{d^2y}{dx^2} = \frac{F(l-x)}{EI}.$$

Integrating, $$\frac{dy}{dx} = \frac{F(lx - x^2/2)}{EI} + C_1$$

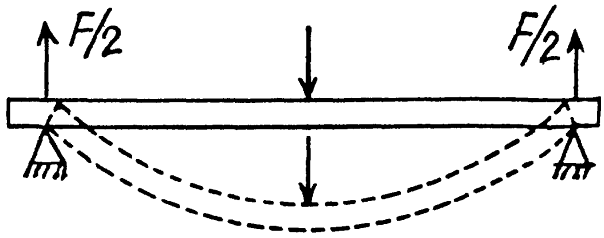
As before $C_1 = 0$. Integrating again, using $y = 0$ for $x = 0$

$$y = \frac{F\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)}{EI} \quad \text{here } \lambda = \frac{Fl^3}{3EI}$$

Here for a square cross section

$$I = \int_{-a/2}^{a/2} z^2 a\, dz = a^4/12.$$

1.302 One can think of it as analogous to the previous case but with a beam of length $l/2$ loaded upward by a force $F/2$.



Thus $$\lambda = \frac{Fl^3}{48EI},$$

On using the last result of the previous problem.

1.303 (a) In this case $N(x) = \frac{1}{2}\rho g b h (l - x)^2$ where $b =$ width of the girder.

Also $I = bh^3/12$. Then,

$$\frac{E b h^2}{12}\frac{d^2 y}{dx^2} = \frac{\rho g b h}{2}(l^2 - 2lx + x^2).$$

Integrating, $$\frac{dy}{dx} = \frac{6\rho g}{E h^2}\left(l^2 x - lx^2 + \frac{x^3}{3}\right)$$

using $\frac{dy}{dx} = 0$ for $x = 0$. Again integrating

$$y = \frac{6\rho g}{Eh^2}\left(\frac{l^2 x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right)$$

Thus $$\lambda = \frac{6\rho g l^4}{Eh^2}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{12}\right)$$

$$= \frac{6\rho g l^4}{Eh^2}\frac{3}{12} = \frac{3\rho g l^4}{2Eh^2}$$

(b) As before, $EI\frac{d^2 y}{dx^2} = N(x)$ where $N(x)$ is the bending moment due to section PB.

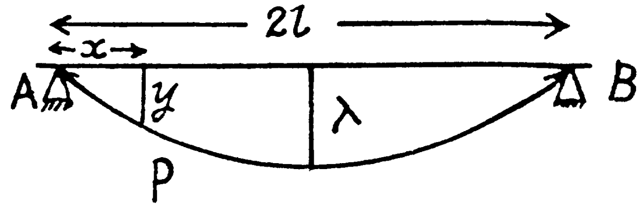
This bending moment is clearly

$$N = \int_x^{2l} w\, d\xi\,(\xi - x) - wl\,(2l - x)$$

$$= w\left(2l^2 - 2xl + \frac{x^2}{2}\right) - wl\,(2l - x) = w\left(\frac{x^2}{2} - xl\right)$$

(Here $w = \rho g b h$ is weight of the beam per unit length)

Now integrating, $EI\frac{dy}{dx} = w\left(\frac{x^3}{6} - \frac{x^2 l}{2}\right) + c_0$



or since $\frac{dy}{dx} = 0$ for $x = l$, $c_0 = w l^3/3$

Integrating again, $EIy = w\left(\frac{x^4}{24} - \frac{x^3 l}{6}\right) + \frac{wl^3 x}{3} + c_1$

As $y = 0$ for $x = 0$, $c_1 = 0$. From this we find

$$\lambda = y\,(x = l) = \frac{5 w l^4}{24}\Big/ EI = \frac{5\rho g l^4}{2 E h^2}$$

1.304 The deflection of the plate can be noticed by going to a co- rotating frame. In this frame each element of the plate experiences a pseudo force proportional to its mass. These forces have a moment which constitutes the bending moment of the problem. To calculate this moment we note that the acceleration of an element at a distance ξ from the axis is $a = \xi\beta$ and the moment of the forces exerted by the section between x and l is

$$N = \rho l h \beta \int_x^l \xi^2 d\xi = \frac{1}{3}\rho l h \beta (l^3 - x^3).$$

From the fundamental equation

$$EI\frac{d^2 y}{dx^2} = \frac{1}{3}\rho l h \beta (l^3 - x^3).$$

The moment of inertia $I = \int_{-h/2}^{+h/2} z^2 l \, dz = \frac{l h^3}{12.}$

Note that the neutral surface (i.e. the surface which contains lines which are neither stretched nor compressed) is a vertical plane here and z is perpendicular to it.

$$\frac{d^2 y}{dx^2} = \frac{4\rho\beta}{Eh^2}(l^3 - x^3). \text{ Integrating}$$

$$\frac{dy}{dx} = \frac{4\rho\beta}{Eh^2}\left(l^3 x - \frac{x^4}{4}\right) + c_1$$

Since $\frac{dy}{dx} = 0$, for $x = 0$, $c_1 = 0$. Integrating again,

$$y = \frac{4\rho\beta}{Eh^2}\left(\frac{l^3 x^2}{2} - \frac{x^5}{20}\right) + c_2$$

$c_2 = 0$ because $y = 0$ for $x = 0$

Thus $$\lambda = y(x = l) = \frac{9\rho\beta l^5}{5Eh^2}$$

1.305 (a) Consider a hollow cylinder of length l, outer radius $r + \Delta r$ inner radius r, fixed at one end and twisted at the other by means of a couple of moment N. The angular displacement φ, at a distance l from the fixed end, is proportional to both l and N. Consider an element of length dx at the twisted end. It is moved by an angle φ as shown. A vertical section is also shown and the twisting of the parallelopipe of length l and area $\Delta r\, dx$ under the action of the twisting couple can be discussed by elementary means. If f is the tangential force generated then shearing stress is $f/\Delta r\, dx$ and this must equal

$$G\theta = G\frac{r\varphi}{l}, \text{ since } \theta = \frac{r\varphi}{l}.$$

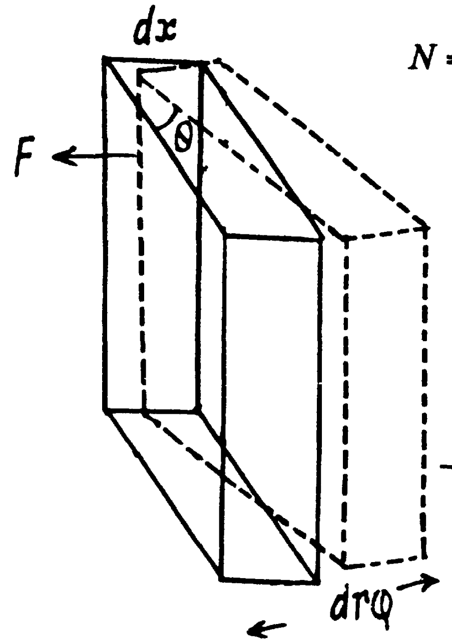
Hence, $$f = G\,\Delta r\, dx \frac{r\varphi}{l}.$$

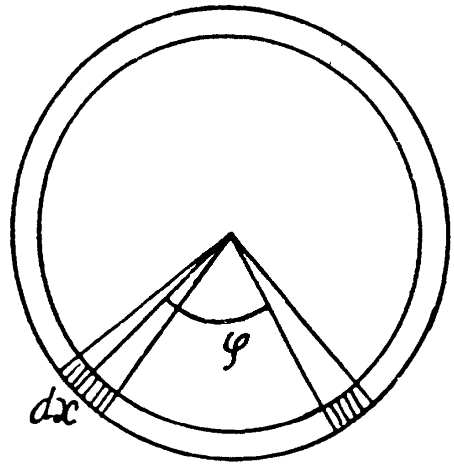
The force f has moment fr about the axis and so the total moment is

$$N = G\,\Delta r \frac{\varphi}{l} r^2 \int dx = \frac{2\pi r^3 \Delta r\, \varphi}{l} G$$

(b) For a solid cylinder we must integrate over r. Thus

$$N = \int_0^r \frac{2\pi r^3\, dr\, \varphi G}{l} = \frac{\pi r^4 G \varphi}{2l}$$





1.306 Clearly $N = \int_{d_1/2}^{d_2/2} \frac{2\pi r^3\, d r\, \varphi G}{l} = \frac{\pi}{32\, l} G \varphi \left(d_2^4 - d_1^4\right)$

using
$$G = 81 \text{ G Pa} = 8{\cdot}1 \times 10^{10} \frac{\text{N}}{\text{m}^2}$$

$$d_2 = 5 \times 10^{-2} \text{ m}, \; d_1 = 3 \times 10^{-2} \text{ m}$$

$$\varphi = 2{\cdot}0° = \frac{\pi}{90} \text{ radians}, \; l = 3 \text{ m}$$

$$N = \frac{\pi \times 8{\cdot}1 \times \pi}{32 \times 3 \times 90} (625 - 81) \times 10^2 \text{ N·m}$$

$$= 0{\cdot}5033 \times 10^3 \text{ N·m} \approx 0{\cdot}5 \text{ k N·m}$$

1.307 The maximum power that can be transmitted by means of a shaft rotating about its axis is clearly $N\omega$ where N is the moment of the couple producing the maximum permissible torsion, φ. Thus

$$P = \frac{\pi r^4 G \varphi}{2l} \cdot \omega = 16{\cdot}9 \text{ kw}$$

1.308 Consider an elementary ring of width dr at a distant r from the axis. The part outside exerts a couple $N + \frac{dN}{dr} dr$ on this ring while the part inside exerts a couple N in the opposite direction. We have for equilibrium

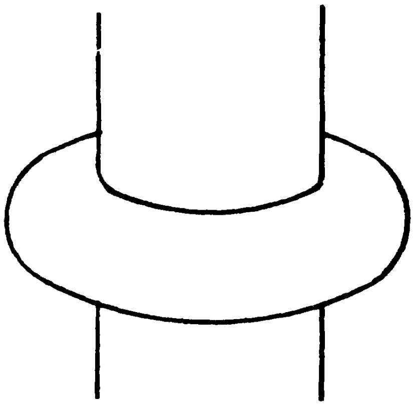
$$\frac{dN}{dr} dr = -dI\beta$$

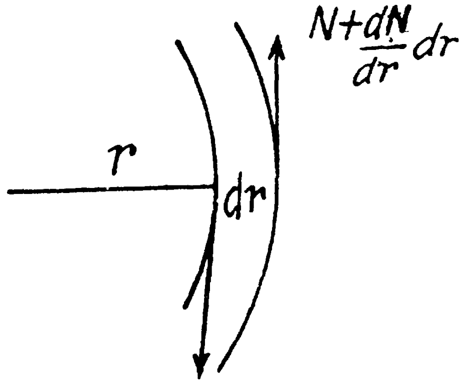
where dI is the moment of inertia of the elementary ring, β is the angular acceleration and minus sign is needed because the couple $N(r)$ decreases, with distance vanshing at the outer radius, $N(r_2) = 0$. Now

$$dI = \frac{m}{\pi(r_2^2 - r_1^2)} 2\pi r\, dr\, r^2$$

Thus
$$dN = \frac{2m\beta}{(r_1^2 - r_1^2)} r^3\, dr$$

or,
$$N = \frac{1}{2} \frac{m\beta}{(r_2^2 - r_1^2)} (r_2^4 - r^4), \text{ on integration}$$





1.309 We assume that the deformation is wholly due to external load, neglecting the effect of the weight of the rod (see next problem). Then a well known formula says,

elastic energy per unit volume

$$= \frac{1}{2} \text{stress} \times \text{strain} = \frac{1}{2} \sigma\, \varepsilon$$

This gives $\frac{1}{2} \frac{m}{\rho} E\, \varepsilon^2 \approx 0{\cdot}04$ kJ for the total deformation energy.

1.310 When a rod is deformed by its own weight the stress increases as one moves up, the stretching force being the weight of the portion below the element considered.

The stress on the element dx is

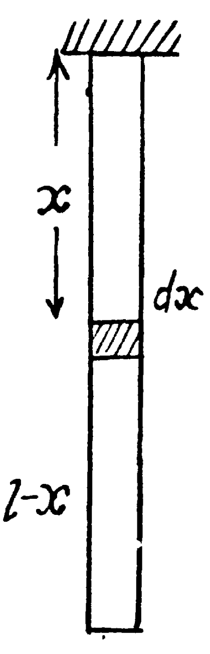
$$\rho\, \pi r^2 (l - x)\, g / \pi r^2 = \rho\, g\, (l - x)$$

The extension of the element is

$$\Delta\, dx = d\, \Delta x = \rho\, g\, (l - x)\, dx / E$$

Integrating $\Delta l = \frac{1}{2} \rho\, g\, l^2 / E$ is the extension of the whole rod. The elastic energy of the element is

$$\frac{1}{2} \rho\, g\, (l - x)\, \frac{\rho\, g\, (l - x)}{E}\, \pi r^2\, dx$$



Integrating

$$\Delta U = \frac{1}{6} \pi r^2 \rho^2 g^2 l^3 / E = \frac{2}{3} \pi r^2 l\, E \left(\frac{\Delta l}{l}\right)^2$$

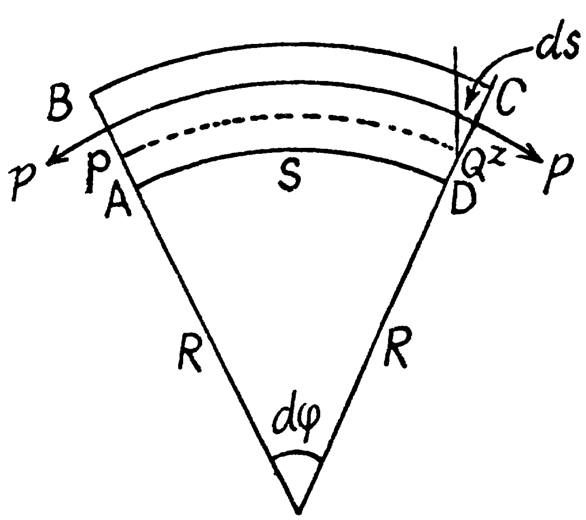
1.311 The work done to make a loop out of a steel band appears as the elastic energy of the loop and may be calculated from the same.

If the length of the band is l, the radius of the loop $R = \frac{l}{2\pi}$. Now consider an element $ABCD$ of the loop. The elastic energy of this element can be calculated by the same sort of arguments as used to derive the formula for internal bending moment. Consider a fibre at a distance z from the neutral surface PQ. This fibre experiences a force p and undergoes an extension ds where $ds = Z\,d\varphi$, while $PQ = s = R\,d\varphi$. Thus strain $\frac{ds}{s} = \frac{Z}{R}$. If α is the cross sectional area of the fibre, the elastic energy associated with it is

$$\frac{1}{2}E\left(\frac{Z}{R}\right)^2 R\,d\varphi\,\alpha$$

Summing over all the fibres we get

$$\frac{EI\varphi}{2R}\sum \alpha Z^2 = \frac{E\,l\,d\varphi}{2R}$$



For the whole loop this gives,

using $\int d\varphi = 2\pi$,

$$\frac{E\,l\,\pi}{R} = \frac{2EI\pi^2}{l}$$

Now $$I = \int_{-\delta/2}^{\delta/2} Z^2 h\,dZ = \frac{h\delta^3}{12}$$

So the energy is $$\frac{1}{6}\frac{\pi^2 E h \delta^3}{l} = 0{\cdot}08\text{ kJ}$$

1.312 When the rod is twisted through an angle θ, a couple

$N(\theta) = \frac{\pi r^4 G}{2l}\theta$ appears to resist this. Work done in twisting the rod by an angle φ is then

$$\int_0^{\varphi} N(\theta)\,d\theta = \frac{\pi r^4 G}{4l}\varphi^2 = 7\text{ J} \quad \text{on putting the values.}$$

1.313 The energy between radii r and $r + dr$ is, by differentiation, $\frac{\pi r^3\,dr}{l} G\varphi^2$

Its density is $$\frac{\pi r^3\,dr}{2\pi r\,dr\,l}\frac{G\varphi^2}{l} = \frac{1}{2}\frac{G\varphi^2 r^2}{l^2}$$

1.314 The energy density is as usual $1/2$ stress $\times$ strain. Stress is the pressure $\rho g h$. Strain is $\beta \times \rho g h$ by defination of β. Thus

$$u = \frac{1}{2}\beta(\rho g h)^2 = 23{\cdot}5\text{ kJ/m}^3 \quad \text{on putting the values.}$$

1.7 HYDRODYNAMICS

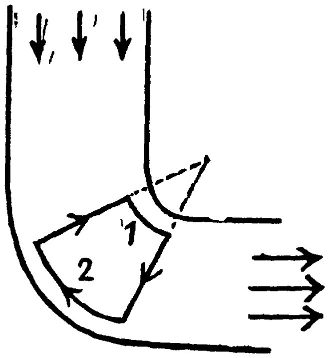
1.315 Between 1 and 2 fluid particles are in nearly circular motion and therefore have centripetal acceleration. The force for this acceleration, like for any other situation in an ideal fluid, can only come from the pressure variation along the line joining 1 and 2. This requires that pressure at 1 should be greater than the pressure at 2 i.e.

$$p_1 > p_2$$

so that the fluid particles can have required acceleration. If there is no turbulence, the motion can be taken as irrotational. Then by considering

$$\oint \vec{v} \cdot d\vec{l} = 0$$

along the circuit shown we infer that



$$v_2 > v_1$$

(The portion of the circuit near 1 and 2 are streamlines while the other two arms are at right angle to streamlines)

In an incompressible liquid we also have div $\vec{v} = 0$

By electrostatic analogy we then find that the density of streamlines is proportional to the velocity at that point.

1.316 From the conservation of mass

$$v_1 S_1 = v_2 S_2 \qquad (1)$$

But $S_1 < S_2$ as shown in the figure of the problem, therefore

$$v_1 > v_2$$

As every streamline is horizontal between 1 & 2, Bernoull's theorem becomes

$$p + \frac{1}{2} \rho v^2 = \text{ constant, which gives}$$

$$p_1 < p_2 \text{ as } v_1 > v_2$$

As the difference in height of the water column is Δh, therefore

$$p_2 - p_1 = \rho g \Delta h \qquad (2)$$

From Bernoull's theorem between points 1 and 2 of a streamline

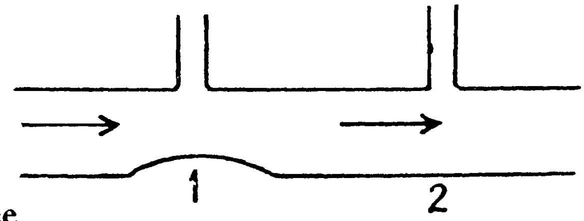
$$p_1 + \frac{1}{2} \rho v_1^2 = p_2 + \frac{1}{2} \rho v_2^2$$

or, $$p_2 - p_1 = \frac{1}{2} \rho (v_1^2 - v_2^2)$$

or $$\rho g \Delta h = \frac{1}{2} \rho (v_1^2 - v_2^2) \qquad (3) \text{ (using Eq. 2)}$$

using (1) in (3), we get

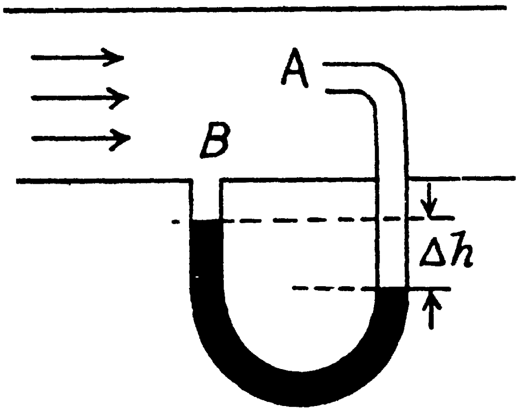
$$v_1 = S_2 \sqrt{\frac{2 g \Delta h}{S_2^2 - S_1^2}}$$



Hence the sought volume of water flowing per see

$$Q = v_1 S_1 = S_1 S_2 \sqrt{\frac{2 g \Delta h}{S_2^2 - S_1^2}}$$

1.317 Applying Bernoulli's theorem for the point A and B,



$$p_A = p_B + \frac{1}{2}\rho v^2 \quad \text{as,} \quad v_A = 0$$

or, $$\frac{1}{2}\rho v^2 = p_A - p_B = \Delta h \rho_0 g$$

So, $$v = \sqrt{\frac{2\Delta h \rho_0 g}{\rho}}$$

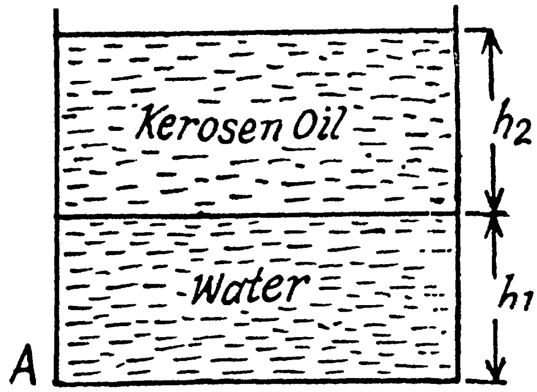
Thus, rate of flow of gas, $Q = S v = S\sqrt{\dfrac{2\Delta h \rho_0 g}{\rho}}$

The gas flows over the tube past it at B. But at A the gas becomes stationary as the gas will move into the tube which already contains gas.

In applying Bernoulli's theorem we should remember that $\frac{p}{\rho} + \frac{1}{2}v^2 + gz$ is constant along a streamline. In the present case, we are really applying Bernoulli's theorem somewhat indirectly. The streamline at A is not the streamline at B. Nevertheless the result is correct. To be convinced of this, we need only apply Bernoull's theorem to the streamline that goes through A by comparing the situation at A with that above B on the same level. In steady conditions, this agrees with the result derived because there cannot be a transverse pressure differential.

1.318 Since, the density of water is greater than that of kerosene oil, it will collect at the bottom. Now, pressure due to water level equals $h_1 \rho_1 g$ and pressure due to kerosene oil level equals $h_2 \rho_2 g$. So, net pressure becomes $h_1 \rho_1 g + h_2 \rho_2 g$.

From Bernoulli's theorem, this pressure energy will be converted into kinetic energy while flowing through the whole A.

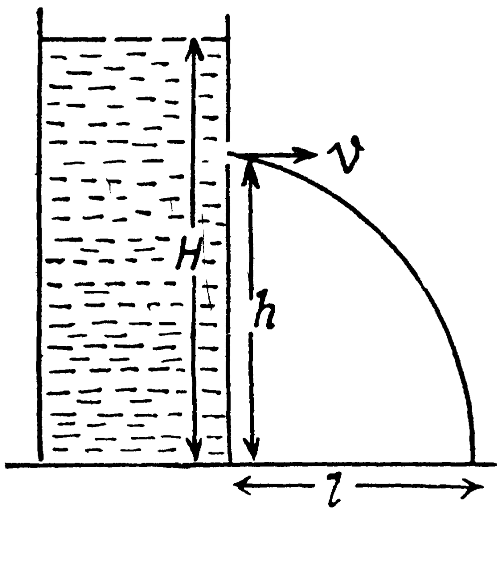


i.e. $$h_1 \rho_1 g + h_2 \rho_2 g = \frac{1}{2}\rho_1 v^2$$

Hence $$v = \sqrt{2\left(h_1 + h_2 \frac{\rho_2}{\rho_1}\right) g} = 3 \text{ m/s}$$

1.319 Let, H be the total height of water column and the hole is made at a height h from the bottom.

Then from Bernoulli's theorem



$$\frac{1}{2}\rho v^2 = (H - h)\rho g$$

or, $v = \sqrt{(H-h)\, 2g}$, which is directed horizontally.

For the horizontal range, $l = v\, t$

$$= \sqrt{2g(H-h)} \cdot \sqrt{\frac{2h}{g}} = 2\sqrt{(Hh - h^2)}$$

Now, for maximum l, $\frac{d\,(Hh - h^2)}{dh} = 0$

which yields $$h = \frac{H}{2} = 25 \text{ cm.}$$

1.320 Let the velocity of the water jet, near the orifice be v', then applying Bernoullis theorem,

$$\frac{1}{2}\rho v^2 = h_0 \rho g + \frac{1}{2}\rho v^2$$

or, $$v' = \sqrt{v^2 - 2g\,h_0} \tag{1}$$

Here the pressure term on both sides is the same and equal to atmospheric pressure. (In the problem book Fig. should be more clear.)

Now, if it rises upto a height h, then at this height, whole of its kinetic energy will be converted into potential energy. So,

$$\frac{1}{2}\rho v'^2 = \rho g h \quad \text{or} \quad h = \frac{v'^2}{2g}$$

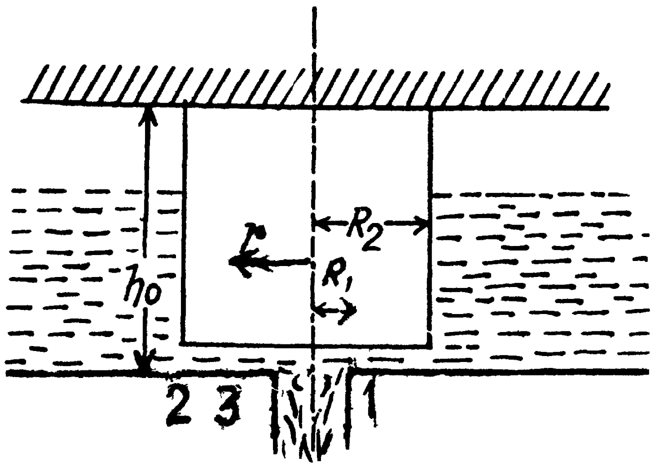
$$= \frac{v^2}{2g} - h_0 = 20 \text{ cm, [using Eq. (1)]}$$

1.321 Water flows through the small clearance into the orifice. Let d be the clearance. Then from the equation of continuity

$$(2\pi R_1 d)\, v_1 = (2\pi r\, d)\, v = (2\pi R_2 d)\, v_2$$

or $$v_1 R_1 = v\, r = v_2 R_2 \tag{1}$$

where v_1, v_2 and v are respectively the inward radial velocities of the fluid at 1, 2 and 3.



Now by Bernoulli's theorem just before 2 and just after it in the clearance

$$p_0 + h\rho g = p_2 + \frac{1}{2}\rho v_2^2 \tag{2}$$

Applying the same theorem at 3 and 1 we find that this also equals

$$p + \frac{1}{2}\rho v^2 = p_0 + \frac{1}{2}\rho v_1^2 \tag{3}$$

(since the pressure in the orifice is p_0)

From Eqs. (2) and (3) we also hence

$$v_1 = \sqrt{2gh} \tag{4}$$

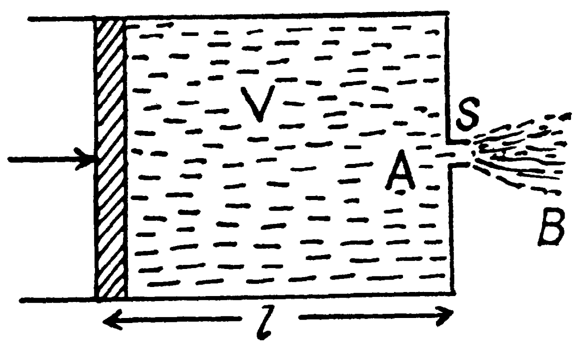
and $$p = p_0 + \frac{1}{2}\rho v_1^2 \left(1 - \left(\frac{v}{v_1}\right)^2\right)$$

$$= p_0 + h\rho g \left(1 - \left(\frac{R_1}{r}\right)^2\right) \quad \text{[Using (1) and (4)]}$$

1.322 Let the force acting on the piston be F and the length of the cylinder be l.

Then, work done $= Fl$ (1)

Applying Bernoulli's theorem for points A and B, $p = \frac{1}{2}\rho v^2$ where ρ is the density and v is the velocity at point B. Now, force on the piston,



$$F = pA = \frac{1}{2}\rho v^2 A \quad (2)$$

where A is the cross section area of piston.

Also, discharge through the orifice during time interval $t = Svt$ and this is equal to the volume of the cylinder, i.e.,

$$V = Svt \quad \text{or} \quad v = \frac{V}{St} \quad (3)$$

From Eq. (1), (2) and (3) work done

$$= \frac{1}{2}\rho v^2 A\, l = \frac{1}{2}\rho A \frac{V^2}{(St)^2} l = \frac{1}{2}\rho V^3/S^3 t^2 \quad (\text{as } Al = V)$$

1.323 Let at any moment of time, water level in the vessel be H then speed of flow of water through the orifice, at that moment will be

$$v = \sqrt{2gH} \quad (1)$$

In the time interval dt, the volume of water ejected through orifice,

$$dV = s\,v\,dt \quad (2)$$

On the other hand, the volume of water in the vessel at time t equals

$$V = SH$$

Differentiating (3) with respect to time,

$$\frac{dV}{dt} = S\frac{dH}{dt} \quad \text{or} \quad dV = S\,dH \quad (4)$$

Eqs. (2) and (4)

$$S\,dH = s\,v\,dt \quad \text{or} \quad dt = \frac{S}{s}\frac{dH}{\sqrt{2gH}}, \text{from (2)}$$

Integrating,
$$\int_0^t dt = \frac{S}{s\sqrt{2h}}\int_h^0 \frac{dh}{\sqrt{H}}$$

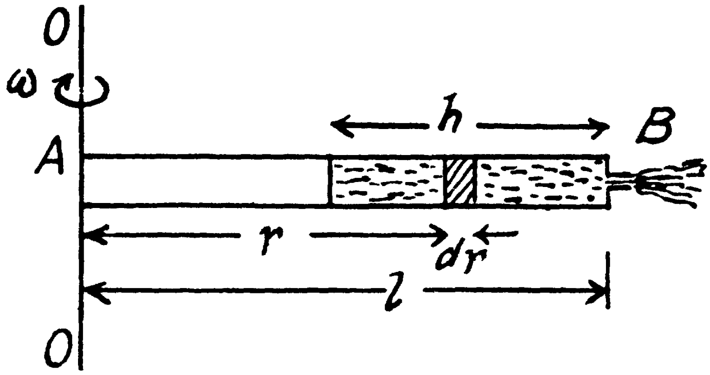
Thus,
$$t = \frac{S}{s}\sqrt{\frac{2h}{g}}$$

1.324 In a rotating frame (with constant angular velocity) the Eulerian equation is

$$-\vec{\nabla}p + \rho\vec{g} + 2\rho(\vec{v}' \times \vec{\omega}) + \rho\omega^2\vec{r} = \rho\frac{d\vec{v}'}{dt}$$

In the frame of rotating tube the liquid in the "column" is practically static because the orifice is sufficiently small. Thus the Eulerian Eq. in projection form along $\vec{r}$ (which is

the position vector of an arbitrary liquid element of lenth dr relative to the rotation axis) reduces to



$$\frac{-dp}{dr} + \rho\,\omega^2 r = 0$$

or, $$dp = \rho\,\omega^2 r\,dr$$

so, $$\int_{p_0}^{p} dp = \rho\omega^2 \int_{(l-h)}^{r} r\,dr$$

Thus $$p(r) = p_0 + \frac{\rho\omega^2}{2}\left[r^2 - (l-h)^2\right] \qquad (1)$$

Hence the pressure at the end B just before the orifice i.e.

$$p(l) = p_0 + \frac{\rho\,\omega^2}{2}(2\,l\,h - h^2) \qquad (2)$$

Then applying Bernoull's theorem at the orifice for the points just inside and outside of the end B

$p_0 + \frac{1}{2}\rho\,\omega^2\,(2\,l\,h - h^2) = p_0 + \frac{1}{2}\rho\,v^2$ (where v is the sought velocity)

So, $$v = \omega h\sqrt{\frac{2l}{h} - 1}$$

1.325 The Euler's equation is $\rho\frac{d\vec{v}}{dt} = \vec{f} - \vec{\nabla}p = -\vec{\nabla}(p + \rho\,gz)$, where z is vertically upwards.

Now $$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\,\vec{v}. \qquad (1)$$

But $$(\vec{v}\cdot\vec{\nabla})\,\vec{v} = \vec{\nabla}\left(\frac{1}{2}v^2\right) - \vec{v}\times \text{Curl}\,\vec{v} \qquad (2)$$

we consider the steady (i.e. $\partial\vec{v}/\partial t = 0$) flow of an incompressible fluid then ρ = constant.

· and as the motion is irrotational Curl $\vec{v} = 0$

So from (1) and (2) $$\rho\,\vec{\nabla}\left(\frac{1}{2}v^2\right) = -\vec{\nabla}(p + \rho\,gz)$$

or, $$\vec{\nabla}\left(p + \frac{1}{2}\rho\,v^2 + \rho gz\right) = 0$$

Hence $$p + \frac{1}{2}\rho\,v^2 + \rho\,gz = \text{constant.}$$

1.326 Let the velocity of water, flowing through A be v_A and that through B be v_B, then discharging rate through $A = Q_A = S\,v_A$ and similarly through $B = S\,v_B$.

Now, force of reaction at A,

$$F_A = \rho\,Q_A\,v_A = \rho\,S\,v_B^2$$

Hence, the net force,

$$F = \rho S (v_A^2 - v_B^2) \quad \text{as} \quad \vec{F}_A \uparrow\downarrow \vec{F}_B \qquad (1)$$

Applying Bernoulli's theorem to the liquid flowing out of A we get

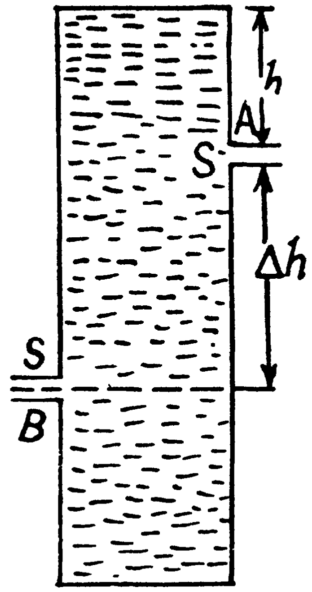
$$p_0 + \rho g h = p_0 + \frac{1}{2}\rho v_A^2$$

and similarly at B

$$p_0 + \rho g (h + \Delta h) = p_0 + \frac{1}{2}\rho v_B^2$$

Hence $\qquad (v_B^2 - v_A^2)\frac{\rho}{2} = \Delta h \rho g$

Thus $\qquad F = 2\rho g S \Delta h = 0{\cdot}50\,\text{N}$



1.327 Consider an element of height dy at a distance y from the top. The velocity of the fluid coming out of the element is

$$v = \sqrt{2gy}$$

The force of reaction dF due to this is $dF = \rho\, dA\, v^2$, as in the previous problem, $= \rho\,(b\,dy)\,2gy$

Integrating $\qquad F = \rho g b \int_{h-1}^{h} 2y\,dy$

$$= \rho g b\,[h^2 - (h-l)^2] = \rho g b l\,(2h - l)$$

(The slit runs from a depth $h - l$ to a depth h from the top.)

1.328 Let the velocity of water flowing through the tube at a certain instant of time be u, then $u = \frac{Q}{\pi r^2}$, where Q is the rate of flow of water and πr^2 is the cross section area of the tube.

From impulse momentum theorem, for the stream of water striking the tube corner, in x–direction in the time interval dt,

$$F_x\, dt = -\rho Q u\, dt \quad \text{or} \quad F_x = -\rho Q u$$

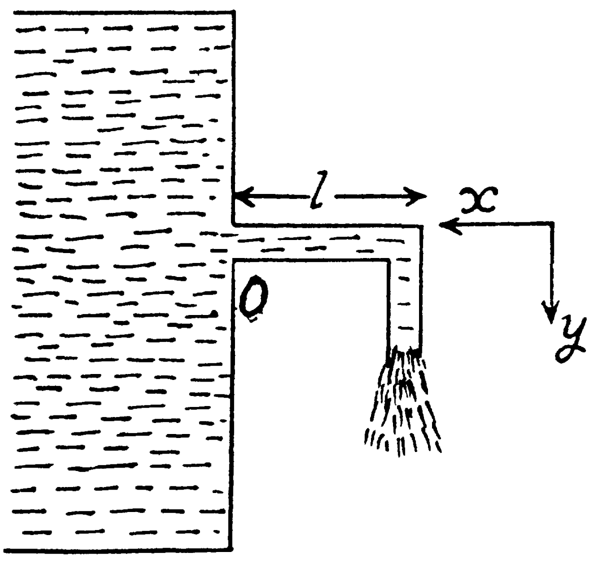
and similarly, $F_y = \rho Q u$

Therefore, the force exerted on the water stream by the tube,

$$\vec{F} = -\rho Q u\,\vec{i} + \rho Q u\,\vec{j}$$

According to third law, the reaction force on the tube's wall by the stream equals $(-\vec{F})$

$$= \rho Q u\,\vec{i} - \rho Q u\,\vec{j}.$$



Hence, the sought moment of force about 0 becomes

$$\vec{N} = l(-\vec{i}) \times (\rho Q u\,\vec{i} - \rho Q u\,\vec{j}) = \rho Q u\, l\,\vec{k} = \frac{\rho Q^2}{\pi r^2}\, l\,\vec{k}$$

and $\qquad |\vec{N}| = \frac{\rho Q^2 l}{\pi r^2} = 0{\cdot}70\,\text{N·m}$

1.329 Suppose the radius at A is R and it decreases uniformaly to r at B where $S = \pi R^2$ and $s = \pi r^2$. Assume also that the semi vectical angle at 0 is α. Then

$$\frac{R}{L_2} = \frac{r}{L_1} = \frac{y}{x}$$

So $$y = r + \frac{R-r}{L_2-L_1}(x - L_1)$$

where y is the radius at the point P distant x from the vertex O. Suppose the velocity with which the liquid flows out is V at A, v at B and u at P. Then by the equation of continuity

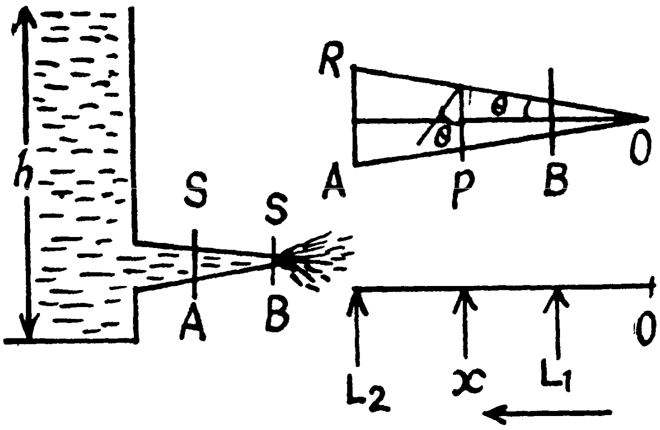
$$\pi R^2 V = \pi r^2 v = \pi y^2 u$$

The velocity v of efflux is given by

$$v = \sqrt{2gh}$$

and Bernoulli's theorem gives

$$p_p + \frac{1}{2}\rho u^2 = p_0 + \frac{1}{2}\rho v^2$$



where p_p is the pressure at P and p_0 is the atmospheric pressure which is the pressure just outside of B. The force on the nozzle tending to pull it out is then

$$F = \int (p_p - p_0)\sin\theta \; 2\pi y ds$$

We have subtracted p_0 which is the force due to atmosphenic pressure the factor $\sin\theta$ gives horizontal component of the force and ds is the length of the element of nozzle surface, $ds = dx \sec\theta$ and

$$\tan\theta = \frac{R-r}{L_2-L_1}$$

Thus

$$F = \int_{L_1}^{L_2} \frac{1}{2}(v^2 - u^2)\rho \; 2\pi y \; \frac{R-r}{L_2-L_1}dx$$

$$= \pi\rho \int_r^R v^2\left(1 - \frac{r^4}{y^4}\right) y\, dy$$

$$= \pi\rho v^2 \frac{1}{2}\left(R^2 - r^2 + \frac{r^4}{R^2} - r^2\right) = \rho gh\left(\frac{\pi(R^2 - r^2)^2}{R^2}\right)$$

$$= \rho gh \;(S - s)^2/S = 6{\cdot}02\,\text{N} \quad \text{on putting the values.}$$

Note : If we try to calculate F from the momentum change of the liquid flowing out we will be wrong even as regards the sign of the force.

There is of course the effect of pressure at S and s but quantitative derivation of F from Newton's law is difficult.

1.330 The Euler's equation is $\rho \frac{d\vec{v}}{dt} = \vec{f} - \vec{\nabla} p$ in the space fixed frame where $\vec{f} = -\rho g \vec{k}$ downward. We assume incompressible fluid so ρ is constant.

Then $\vec{f} = -\vec{\nabla}(\rho g z)$ where z is the height vertically upwards from some fixed origin. We go to rotating frame where the equation becomes

$$\rho \frac{d\vec{v}'}{dt} = -\vec{\nabla}(p + \rho g z) + \rho \omega^2 \vec{r} + 2\rho (\vec{v}' \times \vec{\omega})$$

the additional terms on the right are the well known coriolis and centrifugal forces. In the frame rotating with the liquid $\vec{v}' = 0$ so

$$\vec{\nabla}\left(p + \rho g z - \frac{1}{2}\rho \omega^2 r^2\right) = 0$$

or

$$p + \rho g z - \frac{1}{2}\rho \omega^2 r^2 = \text{constant}$$

On the free surface $p =$ constant, thus

$$z = \frac{\omega^2}{2g} r^2 + \text{constant}$$

If we choose the origin at point $r = 0$ (i.e. the axis) of the free surface then "cosntant" = 0 and

$$z = \frac{\omega^2}{2g} r^2 \quad \text{(The paraboloid of revolution)}$$

At the bottom $z =$ constant

So

$$p = \frac{1}{2}\rho \omega^2 r^2 + \text{constant}$$

If $p = p_0$ on the axis at the bottom, then

$$p = p_0 + \frac{1}{2}\rho \omega^2 r^2.$$

1.331 When the disc rotates the fuild in contact with, corotates but the fluid in contact with the walls of the cavity does not rotate. A velocity gradient is then set up leading to viscous forces. At a distance r from the axis the linear velocity is ωr so there is a velocity gradient $\frac{\omega r}{h}$ both in the upper and lower clearance. The corresponding force on the element whose radial width is dr is

$$\eta\, 2\pi r dr\, \frac{\omega r}{h} \quad \text{(from the formular } F = \eta A \frac{dv}{dx}\text{)}$$

The torque due to this force is

$$\eta\, 2\pi r dr\, \frac{\omega r}{h}\, r$$

and the net torque considering both the upper and lower clearance is

$$2\int_0^R \eta 2\pi r^3\, dr \frac{\omega}{h}$$

$$= \pi R^4 \omega \eta / h$$

So power developed is

$$P = \pi R^4 \omega^2 \eta / h = 9 \cdot 05 \text{ W (on putting the values).}$$

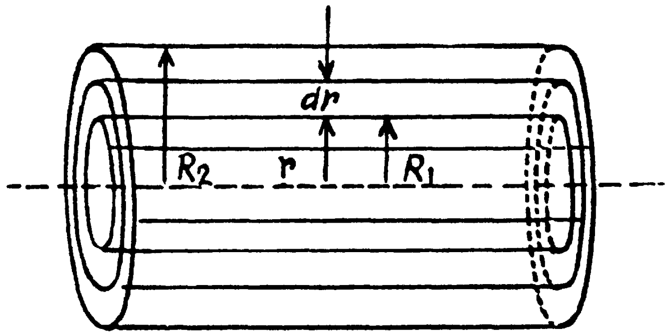
(As instructed end effects i.e. rotation of fluid in the clearance $r > R$ has been neglected.)

1.332 Let us consider a coaxial cylinder of radius r and thickness dr, then force of friction or viscous force on this elemental layer, $F = 2\pi r l \eta \frac{dv}{dr}$.

This force must be constant from layer to layer so that steady motion may be possible.

or, $$\frac{F\,dr}{r} = 2\pi l \eta\, dv. \qquad (1)$$

Integrating,



$$F\int_{R_2}^{r}\frac{dr}{r} = 2\pi l \eta \int_0^v dv$$

or, $$F \ln\left(\frac{r}{R_2}\right) = 2\pi l \eta\, v \qquad (2)$$

Putting $r = R_1$, we get

$$F \ln\frac{R_1}{R_2} = 2\pi l \eta\, v_0$$

From (2) by (3) we get,

$$v = v_0 \frac{\ln r/R_2}{\ln R_1/R_2}$$

Note : The force F is supplied by the agency which tries to carry the inner cylinder with velocity v_0 .

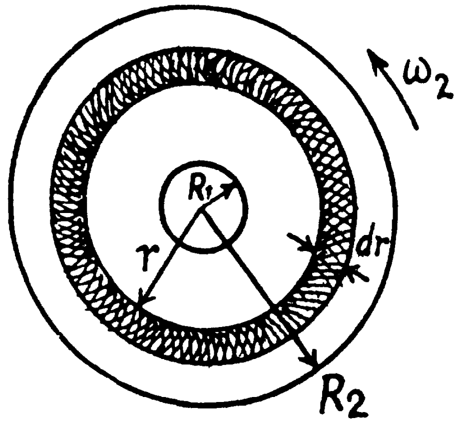
1.333 (a) Let us consider an elemental cylinder of radius r and thickness dr then from Newton's formula

$$F = 2\pi r l \eta\, r \frac{d\omega}{dr} = 2\pi l \eta\, r^2 \frac{d\omega}{dr}$$

and moment of this force acting on the element,

$$N = 2\pi r^2 l \eta \frac{d\omega}{dr} r = 2\pi r^3 l \eta \frac{d\omega}{dr}$$

or, $$2\pi l \eta\, d\omega = N\frac{dr}{r^3} \qquad (2)$$



As in the previous problem N is constant when conditions are steady

Integrating, $$2\pi l \eta \int_0^{\omega} d\omega = N\int_{R_1}^{r}\frac{dr}{r^3}$$

or, $$2\pi l \eta\, \omega = \frac{N}{2}\left[\frac{1}{R_1^2} - \frac{1}{r^2}\right] \qquad (3)$$

Putting $r = R_2,\ \omega = \omega_2$, we get

$$2\pi l \eta\, \omega_2 = \frac{N}{2}\left[\frac{1}{R_1^2} - \frac{1}{R_2^2}\right] \qquad (4)$$

From (3) and (4),

$$\omega = \omega_2 \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\left[\frac{1}{R_1^2} - \frac{1}{r^2}\right]$$

(b) From Eq. (4),

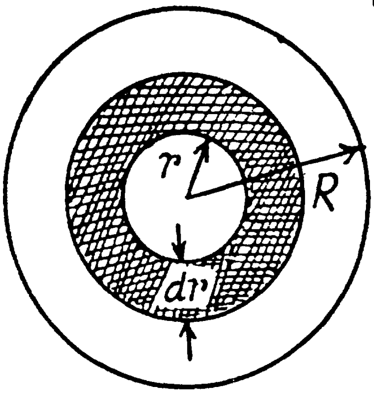
$$N_1 = \frac{N}{l} = 4\pi\eta\,\omega_2 \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}$$

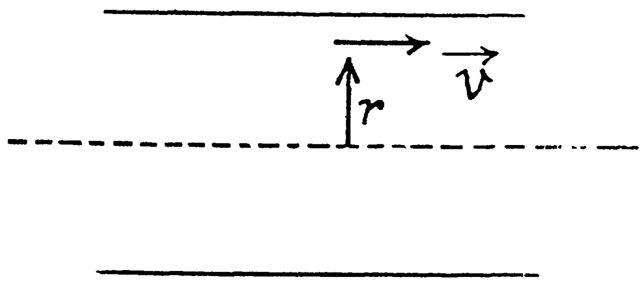
1.334 (a) Let dV be the volume flowing per second through the cylindrical shell of thickness dr then,

$$dV = -(2\pi r\,dr)\,v_0\left(1 - \frac{r^2}{R^2}\right) = 2\pi v_0\left(r - \frac{r^3}{R^2}\right)dr$$

and the total volume,

$$V = 2\pi v_0 \int_0^R \left(r - \frac{r^3}{R^2}\right)dr = 2\pi v_0 \frac{R^2}{4} = \frac{\pi}{2} R^2 v_0$$





(b) Let, dE be the kinetic energy, within the above cylindrical shell. Then

$$dT = \frac{1}{2}(dm)\,v^2 = \frac{1}{2}(2\pi r\,l\,d\,r\rho)\,v^2$$

$$= \frac{1}{2}(2\pi\,l\,\rho)\,r\,dr\,v_0^2\left(1 - \frac{r^2}{R^2}\right) = \pi\,l\,\rho\,v_0\left[r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right]dr$$

Hence, total energy of the fluid,

$$T = \pi\,l\,\rho\,v_0^2 \int_0^R \left(r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right)dr = \frac{\pi R^2 \rho\,l\,v_0^2}{6}$$

(c) Here frictional force is the shearing force on the tube, exerted by the fluid, which equals $-\eta S \frac{dv}{dt}$.

Given, $$v = v_0\left(1 - \frac{r^2}{R^2}\right)$$

So, $$\frac{dv}{dr} = -2 v_0 \frac{r}{R^2}$$

And at $$r = R,\ \frac{dv}{dr} = -\frac{2 v_0}{R}$$

Then, viscous force is given by, $F = -\eta\,(2\pi R l)\left(\frac{dv}{dr}\right)_{r=R}$

$$= -2\pi R\eta\, l\left(-\frac{2v_0}{R}\right) = 4\pi\eta\, v_0\, l$$

(d) Taking a cylindrical shell of thickness dr and radius r viscous force,

$$F = -\eta\,(2\pi r l)\frac{dv}{dr},$$

Let Δp be the pressure difference, then net force on the element $= \Delta p\,\pi r^2 + 2\pi\eta\, l\, r\frac{dv}{dr}$

But, since the flow is steady, $F_{net} = 0$

or, $$\Delta p = \frac{-2\pi l\eta\, r\frac{dv}{dr}}{\pi r^2} = \frac{-2\pi l\,\eta\, r\left(-2v_0\frac{r}{R^2}\right)}{\pi r^2} = 4\eta\, v_0\, l/R^2$$

1.335 The loss of pressure head in travelling a distance l is seen from the middle section to be $h_2 - h_1 = 10$ cm. Since $h_2 - h_1 = h_1$ in our problem and $h_3 - h_2 = 15\text{ cm} = 5 + h_2 - h_1$, we see that a pressure head of 5 cm remains incompensated and must be converted into kinetic energy, the liquid flowing out. Thus

$$\frac{\rho v^2}{2} = \rho g\,\Delta h \quad \text{where } \Delta h = h_3 - h_2$$

Thus $$v = \sqrt{2g\Delta h} = 1\text{ m/s}$$

1.336 We know that, Reynold's number (R_e) is defined as, $R_e = \rho\, v\, l/\eta$, where v is the velocity l is the characteristic length and η the coefficient of viscosity. In the case of circular cross section the chracteristic length is the diameter of cross-section d, and v is taken as average velocity of flow of liquid.

Now, R_{e_1} (Reynold's number at x_1 from the pipe end) $= \frac{\rho d_1 v_1}{m\eta}$ where v_1 is the velocity at distance x_1

and similarly, $$R_{e_2} = \frac{\rho\, d_2 v_2}{\eta} \quad \text{so} \quad \frac{R_{e_1}}{R_{e_2}} = \frac{d_1 v_1}{d_2 v_2}$$

From equation of continuity, $A_1 v_1 = A_2 v_2$

or, $$\pi r_1^2 v_1 = \pi r_2^2 v_2 \quad \text{or} \quad d_1 v_1 r_1 = d_2 v_2 r_2$$

$$\frac{d_1 v_1}{d_2 v_2} = \frac{r_2}{r_1} = \frac{r_0 e^{-\alpha x_2}}{r_0 e^{-\alpha x_1}} = e^{-\alpha\Delta x} \quad (\text{as } x_2 - x_1 = \Delta x)$$

Thus $$\frac{R_{e_2}}{R_{e_1}} = e^{\alpha\Delta x} = 5$$

1.337 We know that Reynold's number for turbulent flow is greater than that on laminar flow.

Now, $$(R_e)_l = \frac{\rho v d}{\eta} = \frac{2\rho_1 v_1 r_1}{\eta_1} \quad \text{and} \quad (R_e)_t = \frac{2\rho_2 v_2 r_2}{\eta}$$

But, $(R_e)_t \geq (R_e)_l$

so $$v_{2_{min}} = \frac{\rho_1 v_1 r_1 \eta_2}{\rho_2 r_2 \eta_1} = 5\ \mu\text{ m/s on putting the values.}$$

1.338 We have $R = \dfrac{v\rho_0 d}{\eta}$ and v is given by

$$6\pi\eta r v = \frac{4\pi}{3} r^2 (\rho - \rho_0) g$$

(ρ = density of lead, ρ_0 = density of glycerine.)

$$v = \frac{2}{9\eta}(\rho - \rho_0) g r^2 = \frac{1}{18\eta}(\rho - \rho_0) g d^2$$

Thus $$\frac{1}{2} = \frac{1}{18\eta^2}(\rho - \rho_o) g \rho_0 d^3$$

and $$d = [9\eta^2/\rho_0(\rho - \rho_0) g]^{1/3} = 5{\cdot}2\text{ mm on putting the values.}$$

1.339 $$m\frac{dv}{dt} = mg - 6\pi\eta r v$$

or $$\frac{dv}{dt} + \frac{6\pi\eta r}{m} v = g$$

or $$\frac{dv}{dt} + kv = g,\ k = \frac{6\pi\eta r}{m}$$

or $$e^{kt}\frac{dv}{dt} + ke^{kt} v = ge^{kt} \text{ or } \frac{d}{dt} e^{kt} v = ge^{kt}$$

or $$v e^{kt} = \frac{g}{k} e^{kt} + C \text{ or } v = \frac{g}{k} + C e^{-kt} \text{ (where } C \text{ is const.)}$$

Since $$v = 0 \text{ for } t = 0,\ 0 = \frac{g}{k} + C$$

So $$C = -\frac{g}{k}$$

Thus $$v = \frac{g}{k}(1 - e^{-kt})$$

The steady state velocity is $\dfrac{g}{k}$.

v differs from $\dfrac{g}{k}$ by n where $e^{-kt} = n$

or $$t = \frac{1}{k}\ln n$$

Thus $$\frac{1}{k} = -\frac{\frac{4\pi}{3} r^3 P}{6\pi\eta r} = -\frac{4 r^2 \rho}{18\eta} = -\frac{d^2\rho}{18\eta}$$

We have neglected buoyancy in olive oil.

1.8 RELATIVISTIC MECHANICS

1.340 From the formula for length contraction

$$\left(l_0 - l_0\sqrt{1-\frac{v^2}{c^2}}\right) = \eta\, l_0$$

So, $$1-\frac{v^2}{c^2} = (1-\eta)^2 \quad \text{or} \quad v = c\sqrt{\eta\,(2-\eta)}$$

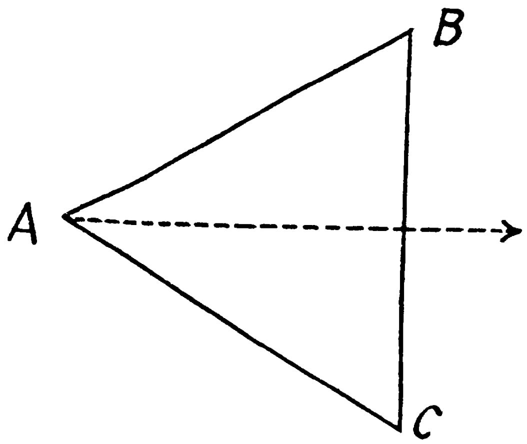
1.341 (a) In the frame in which the triangle is at rest the space coordinates of the vertices are (000), $\left(a\frac{\sqrt{3}}{2}, +\frac{a}{2}, 0\right)\left(a\frac{\sqrt{3}}{2}, -\frac{a}{2}, 0\right)$, all measured at the same time t. In the moving frame the corresponding coordinates at time t' are

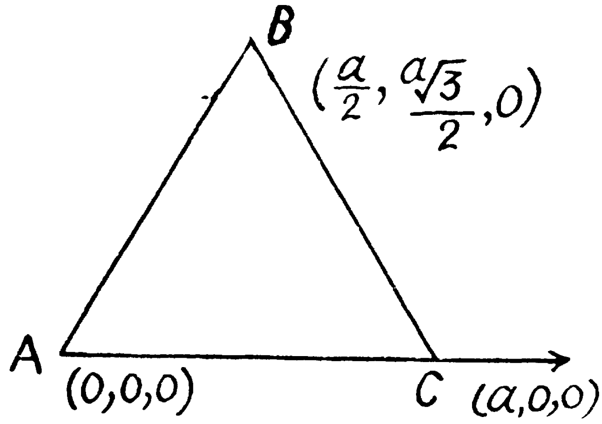
$A: (vt', 0, 0), B: \left(\frac{a}{2}\sqrt{3}\sqrt{1-\beta^2} + vt', \frac{a}{2}, 0\right)$ and $C: \left(\frac{a}{2}\sqrt{3}\sqrt{1-\beta^2} + vt', -\frac{a}{2}, 0\right)$

The perimeter P is then

$$P = a + 2a\left(\frac{3}{4}(1-\beta^2) + \frac{1}{4}\right)^{1/2} = a\left(1+\sqrt{4-3\beta^2}\right)$$

(b) The coordinates in the first frame are shown at time t. The coordinates in the moving frame are,





$A: (vt', 0,0), B: \left(\frac{a}{2}\sqrt{1-\beta^2} + vt', \ a\frac{\sqrt{3}}{2}, 0\right)$, $C: \left(a\sqrt{1-\beta^2} + vt', 0, 0\right)$

The perimeter P is then

$$P = a\sqrt{1-\beta^2} + \frac{a}{2}\left[1-\beta^2+3\right]^{1/2} \times 2 = a\left(\sqrt{1-\beta^2} + \sqrt{4-\beta^2}\right) \text{ here } \beta = \frac{V}{c}$$

1.342 In the rest frame, the coordinates of the ends of the rod in terms of proper length l_0

$$A: (0,0,0)\ B: (l_0\cos\theta_0, l_0\sin\theta_0, 0)$$

at time t. In the laboratory frame the coordinates at time t' are

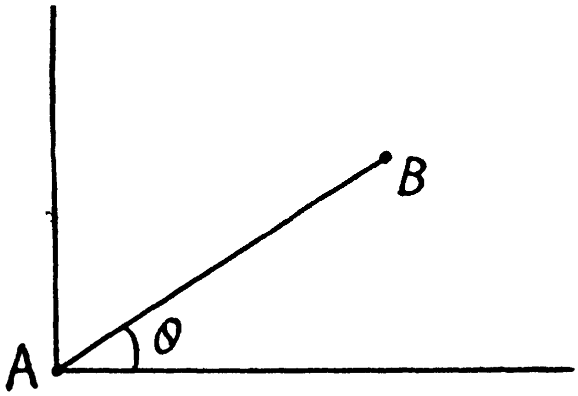
$$A: (vt', 0, 0), B: \left(l_0\cos\theta_0\sqrt{1-\beta^2} + vt', l_0\sin\theta_0, 0\right)$$

Therefore we can write,

$$l\cos\theta_0 = l_0\cos\theta_0\sqrt{1-\beta^2} \text{ and } l\sin\theta = l_0\sin\theta_0$$

Hence $l_0^2 = (l^2)\left(\dfrac{\cos^2\theta + (1-\beta^2)\sin^2\theta}{1-\beta^2}\right)$

or, $= \sqrt{\dfrac{1-\beta^2\sin^2\theta}{1-\beta^2}}$



1.343 In the frame K in which the cone is at rest the coordinates of A are $(0,0,0)$ and of B are $(h, h\tan\theta, 0)$. In the frame K', which is moving with velocity v along the axis of the cone, the coordinates of A and B at time t' are

$A : (-vt', 0,0), B : \left(h\sqrt{1-\beta^2} - vt', h\tan\theta, 0\right)$

Thus the taper angle in the frame K' is

$$\tan\theta' = \frac{\tan\theta}{\sqrt{1-\beta^2}}\left(= \frac{y'_B - y'_A}{x'_B - x'_A}\right)$$

and the lateral surface area is,

$$S = \pi h'^2 \sec\theta' \tan\theta'$$

$$= \pi h^2(1-\beta^2)\frac{\tan\theta}{\sqrt{1-\beta^2}}\sqrt{1+\frac{\tan^2\theta}{1-\beta^2}} = S_0\sqrt{1-\beta^2\cos^2\theta}$$

Here $S_0 = \pi h^2 \sec\theta\tan\theta$ is the lateral surface area in the rest frame and

$$h' = h\sqrt{1-\beta^2}, \; \beta = v/c.$$

1.344 Because of time dilation, a moving clock reads less time. We write,

$$t - \Delta t = t\sqrt{1-\beta^2}, \; \beta = \frac{v}{c}$$

Thus, $$1 - \frac{2\Delta t}{t} + \left(\frac{\Delta t}{t}\right)^2 = 1-\beta^2$$

or, $$v = c\sqrt{\frac{\Delta t}{t}\left(2-\frac{\Delta t}{t}\right)}$$

1.345 In the frame K the length l of the rod is related to the time of flight Δt by

$$l = v\,\Delta t$$

In the reference frame fixed to the rod (frame K')the proper length l_0 of the rod is given by

$$l_0 = v\,\Delta t'$$

But $$l_0 = \frac{l}{\sqrt{1-\beta^2}} = \frac{v\,\Delta t}{\sqrt{1-\beta^2}}, \; \beta = \frac{v}{c}$$

Thus, $$v\,\Delta t' = \frac{v\,\Delta t}{\sqrt{1-\beta^2}}$$

So $$1-\beta^2 = \left(\frac{\Delta t}{\Delta t'}\right)^2 \quad \text{or} \quad v = c\sqrt{1-\left(\frac{\Delta t}{\Delta t'}\right)^2}$$

and $$l_0 = c\sqrt{(\Delta t')^2-(\Delta t)^2} = c\,\Delta t'\sqrt{1-\left(\frac{\Delta t}{\Delta t'}\right)^2}$$

1.346 The distance travelled in the laboratory frame of reference is $v\Delta t$ where v is the velocity of the particle. But by time dilation

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-v^2/c^2}} \quad \text{So} \quad v = c\sqrt{1-(\Delta t_0/\Delta t)^2}$$

Thus the distance traversed is

$$c\Delta t\sqrt{1-(\Delta t_0/\Delta t)^2}$$

1.347 (a) If τ_0 is the proper life time of the muon the life time in the moving frame is

$$\frac{\tau_0}{\sqrt{1-v^2/c^2}} \quad \text{and hence} \quad l = \frac{v\,\tau_0}{\sqrt{1-v^2/c^2}}$$

Thus $$\tau_0 = \frac{l}{v}\sqrt{1-v^2/c^2}$$

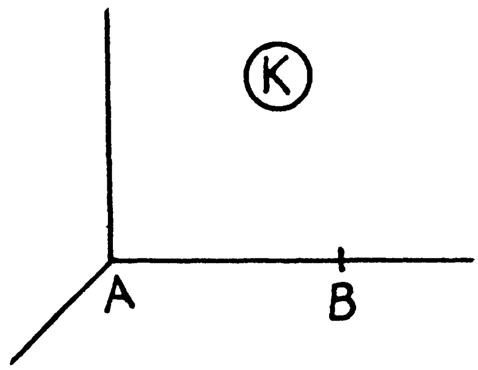
(The words "from the muon's stand point" are not part of any standard terminology)

1.348 In the frame K in which the particles are at rest, their positions are A and B whose coordinates may be taken as,

$$A:(0,0,0),\ B = (l_0\,,0,0)$$

In the frame K' with respect to which K is moving with a velocity v the coordinates of A and B at time t' in the moving frame are

$$A = (vt',0,0)\ B = \left(l_0\sqrt{1-\beta^2}+vt',0,0\right),\ \beta = \frac{v}{c}$$



Suppose B hits a stationary target in K' after time t'_B while A hits it after time $t_B+\Delta t$. Then,

$$l_0\sqrt{1-\beta^2}+vt'_B = v\,(t'_B+\Delta t)$$

So, $$l_0\,\frac{v\,\Delta t}{\sqrt{1-v^2/c^2}}$$

1.349 In the reference frame fixed to the ruler the rod is moving with a velocity v and suffers Lorentz contraction. If l_0 is the proper length of the rod, its measured length will be

$$\Delta x_1 = l_0\sqrt{1-\beta^2}\,,\ \beta = \frac{v}{c}$$

In the reference frame fixed to the rod the ruler suffers Lorentz contraction and we must have

$$\Delta x_2 \sqrt{1-\beta^2} = l_0 \text{ thus } l_0 = \sqrt{\Delta x_1 \Delta x_2}$$

and
$$1-\beta^2 = \frac{\Delta x_1}{\Delta x_2} \quad \text{or} \quad v = c\sqrt{1-\frac{\Delta x_1}{\Delta x_2}}$$

1.350 The coordinates of the ends of the rods in the frame fixed to the left rod are shown. The points B and D coincides when

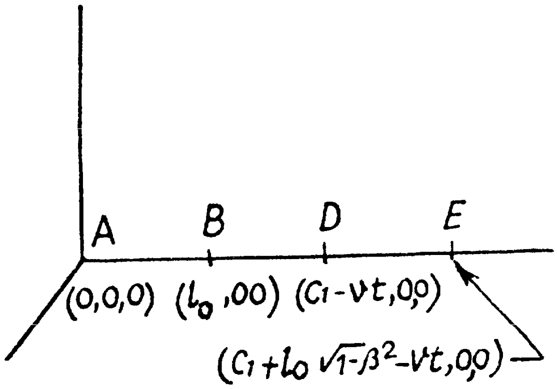
$$l_0 = c_1 - vt_0 \quad \text{or} \quad t_0 = \frac{c_1 - l_0}{v}$$

The points A and E coincide when

$$0 = c_1 + l_0\sqrt{1-\beta^2} - vt_1, \; t_1 = \frac{c_1 + l_0\sqrt{1-\beta^2}}{v}$$

Thus $\Delta t = t_1 - t_0 = \frac{l_0}{v}\left(1 + \sqrt{1-\beta^2}\right)$

or $\left(\frac{v\,\Delta t}{l_0} - 1\right)^2 = 1 - \beta^2 = 1 - \frac{v^2}{c^2}$



From this
$$v = \frac{2c^2\,\Delta t/l_0}{1 + c^2\,\Delta t^2/l_0^2} = \frac{2l_0/\Delta t}{1 + (l_0/c\,\Delta t)^2}$$

1.351 In K_0, the rest frame of the particles, the events corresponding to the decay of the particles are,

$$A : (0,0,0,0) \text{ and } (0, l_0, 0,0) = B$$

In the reference frame K, the corresponding coordintes are by Lorentz transformation

$$A : (0,0,0,0), B : \left(\frac{vl_0}{c^2\sqrt{1-\beta^2}}, \frac{l_0}{\sqrt{1-\beta^2}}, 0,0\right)$$

Now
$$l_0\sqrt{1-\beta^2} = l$$

by Lorentz Fitzgerald contraction formula. Thus the time lag of the decay time of B is

$$\Delta t = \frac{vl_0}{c^2\sqrt{1-\beta^2}} = \frac{v\,l}{c^2(1-\beta^2)} = \frac{vl}{c^2 - v^2}$$

B decays later (B is the forward particle in the direction of motion)

1.352 (a) In the reference frame K with respect to which the rod is moving with velocity v, the coordinates of A and B are

$$A : t, x_A + v(t - t_A), 0, 0$$

$$B : t, x_B + v(t - t_B), 0, 0$$

Thus $l = x_A - x_B - v(t_A - t_B) = l_0\sqrt{1-\beta^2}$

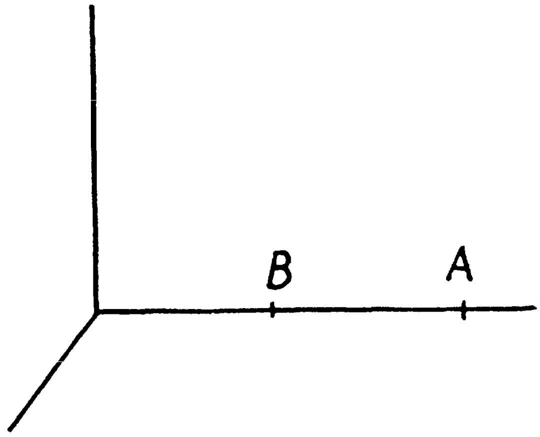
So $$l_0 = \frac{x_A - x_B - v(t_A - t_B)}{\sqrt{1 - v^2/c^2}}$$

(b) $\pm l_0 - v(t_A - t_B) = l = l_0\sqrt{1 - v^2/c^2}$
(since $x_A - x_B$ can be either $+l_0$ or $-l_0$.)

Thus $v(t_A - t_B) = \left(\pm 1 - \sqrt{1 - v^2/c_2}\right) l_o$

i.e. $$t_A - t_B = \frac{l_0}{v}\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right)$$

or $$t_B - t_A = \frac{l_0}{v}\left(1 + \sqrt{1 - v^2/c^2}\right)$$



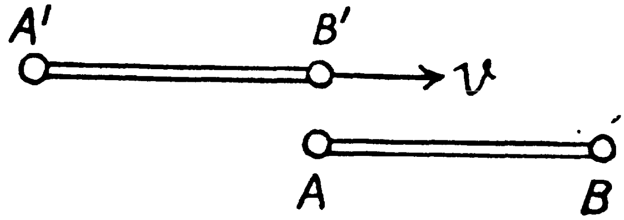
1.353 At the instant the picture is taken the coordintes of A, B, A', B' in the rest frame of AB are

$A : (0, 0, 0, 0)$

$B : (0, l_0, 0, 0)$

$B' : (0, 0, 0, 0)$

$A' : (0_1 - l_0\sqrt{1 - v^2/c^2}, 0, 0)$



In this frame the coordinates of B' at other times are $B' : (t, vt, 0, 0)$. So B' is opposite to B at time $t(B) = \frac{l_0}{v}$. In the frame in which B', A' is at rest the time corresponding this is by Lorentz tranformation.

$$t^0(B') = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(\frac{l_0}{v} - \frac{vl_0}{c^2}\right) = \frac{l_0}{v}\sqrt{1 - v^2/c^2}$$

Similarly in the rest frame of A, B, te coordinates of A at other times are

$$A' : \left(t, -l_0\sqrt{1 - \frac{v^2}{c^2}} + vt, 0, 0\right)$$

A' is opposite tu A at time $t(A) = \frac{l_0}{v}\sqrt{1 - \frac{v^2}{c^2}}$

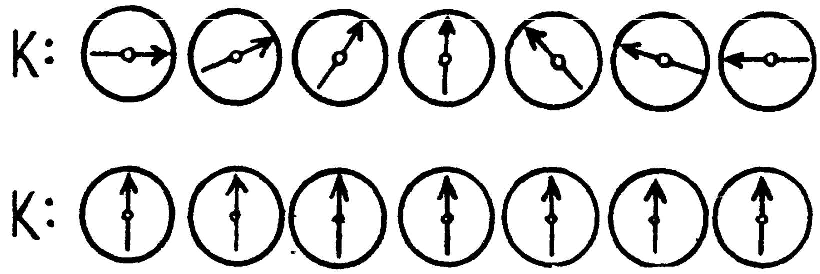
The corresponding time in the frame in which A', B' are at rest is

$$t(A') = \gamma\, t(A) = \frac{l_0}{v}$$

1.354 By Lorentz transformation $t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(t - \frac{vx}{c^2}\right)$

So at time $t = 0,\ t' = \dfrac{vx}{c^2} \dfrac{1}{\sqrt{1 - v^2/c^2}}$

If $x > 0$ $t' < 0$, if $x < 0$, $t' > 0$ and we get the diagram given below "in terms of the K–clock".



The situation in terms of the K' clock is reversed.

1.355 Suppose $x(t)$ is the locus of points in the frame K at which the readings of the clocks of both reference system are permanently identical, then by Lorentz transformation

$$t' = \frac{1}{\sqrt{1 - V^2/c^2}} \left(t - \frac{V x(t)}{c^2} \right) = t$$

So differentiating $x(t) = \dfrac{c^2}{V} \left(1 - \sqrt{1 - \dfrac{V^2}{c^2}} \right) = \dfrac{c}{\beta} \left(1 - \sqrt{1 - \beta^2} \right),\ \beta = \dfrac{V}{c}$

Let $\beta = \tan h\theta,\ 0 \le \theta < \infty$, Then

$$x(t) = \frac{c}{\tan h\theta} \left(1 - \sqrt{1 - \tan h^2 \theta} \right) = c \frac{\cos h\theta}{\sin h\theta} \left(1 - \frac{1}{\cos h\theta} \right)$$

$$= c \frac{\cos h\theta - 1}{\sin h\theta} = c \sqrt{\frac{\cos h\theta - 1}{\cos h\theta + 1}} = c \tan h \frac{\theta}{2} \le v$$

·($\tan h\theta$ is a monotonically increasing function of θ)

1.356 We can take the coordinates of the two events to be

$$A : (0, 0, 0, 0)\quad B : (\Delta t, a, 0, 0)$$

For B to be the effect and A to be cause we must have $\Delta t > \dfrac{|a|}{c}$.

In the moving frame the coordinates of A and B become

$A : (0, 0, 0, 0), B : \left[\gamma \left(\Delta t - \dfrac{aV}{c^2} \right),\ \gamma (a - V \Delta t), 0, 0 \right]$ where $\gamma = \dfrac{1}{\sqrt{1 - \left(\dfrac{V^2}{c^2} \right)}}$

Since

$$(\Delta t')^2 - \frac{a'^2}{c^2} = \gamma^2 \left[\left(\Delta t - \frac{aV}{c^2} \right)^2 - \frac{1}{c^2} (a - V \Delta t)^2 \right] = (\Delta t)^2 - \frac{a^2}{c^2} > 0$$

we must have $\Delta t' > \dfrac{|a'|}{c}$

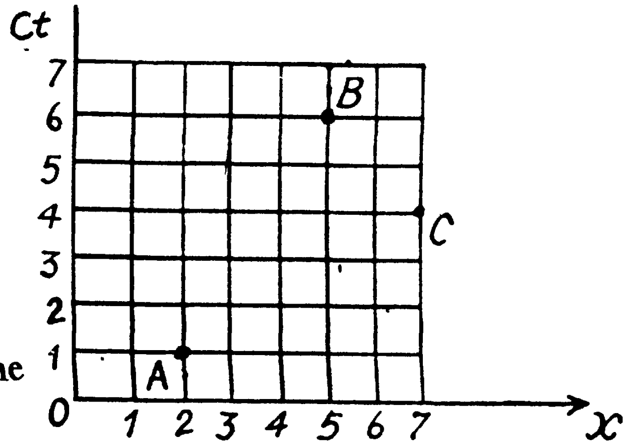
1.357 (a) The four-dimensional interval between A and B (assuming $\Delta y = \Delta z = 0$) is :

$$5^2 - 3^2 = 16 \text{ units}$$

Therefore the time interval between these two events in the reference frame in which the events occurred at the same place is

$$c\,(t'_B - t'_A) = \sqrt{16} = 4\,\text{m}$$

or $\quad t'_B - t'_A = \dfrac{4}{c} = \dfrac{4}{3} \times 10^{-8}\,\text{s}$



(b) The four dimensional interval between A and C is (assuming $\Delta y = \Delta z = 0$)

$$3^2 - 5^2 = -16$$

So the distance between the two events in the frame in which they are simultaneous is 4 units = 4m.

1.358 By the velocity addition formula

$$v_x' = \frac{v_x - V}{1 - \dfrac{V v_x}{c^2}}, \quad v'_y = \frac{v_y \sqrt{1 - V^2/c^2}}{1 - \dfrac{v_x V}{c^2}}$$

and $$v' = \sqrt{v'^2_x + v'^2_y} = \frac{\sqrt{(v_x - V)^2 + v_y^2 (1 - V^2/c^2)}}{1 - \dfrac{v_x V}{c^2}}$$

1.359 (a) By definition the velocity of apporach is

$$v_{\text{approach}} = \frac{dx_1}{dt} - \frac{dx_2}{dt} = v_1 - (-v_2) = v_1 + v_2$$

in the reference frame K.

(b) The relative velocity is obtained by the transformation law

$$v_r = \frac{v_1 - (-v_2)}{1 - \dfrac{v_1(-v_2)}{c^2}} = \frac{v_1 + v_2}{1 + \dfrac{v_1 v_2}{c^2}}$$

1.360 The velocity of one of the rods in the reference frame fixed to the other rod is

$$V = \frac{v + v}{1 + \dfrac{v^2}{c^2}} = \frac{2v}{1 + \beta^2}$$

The length of the moving rod in this frame is

$$l = l_0 \sqrt{1 - \frac{4\,v^2/c^2}{(1 + \beta^2)^2}} = l_0 \frac{1 - \beta^2}{1 + \beta^2}$$

1.361 The approach velocity is defined by

$$\vec{V}_{\text{approach}} = \frac{d\vec{r_1}}{dt} - \frac{d\vec{r_2}}{dt} = \vec{V_1} - \vec{V_2}$$

in the laboratory frame. So $V_{\text{approach}} = \sqrt{v_1^2 + v_2^2}$

On the other hand, the relative velocity can be obtained by using the velocity addition formula and has the components

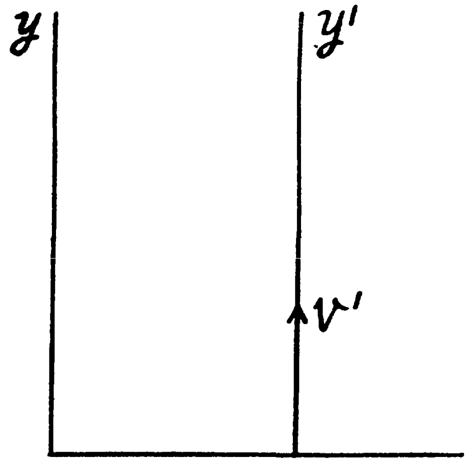
$$\left[-v_1,\ v_2\sqrt{1-\left(\frac{v_1^2}{c^2}\right)}\right] \text{ so } V_r = \sqrt{v_1^2+v_2^2-\frac{v_1 v_2^2}{c^2}}$$

1.362 The components of the velocity of the unstable particle in the frame K are

$$\left(V,\ v'\sqrt{1-\frac{V^2}{c^2}},\ 0\right)$$

so the velocity relative to K is

$$\sqrt{V^2+v'^2-\frac{v'^2 V^2}{c^2}}$$



The life time in this frame dilates to

$$\Delta t_0 \Big/ \sqrt{1-\frac{V^2}{c^2}-\frac{v'^2}{c^2}+\frac{v'^2 V^2}{c^4}}$$

and the distance traversed is

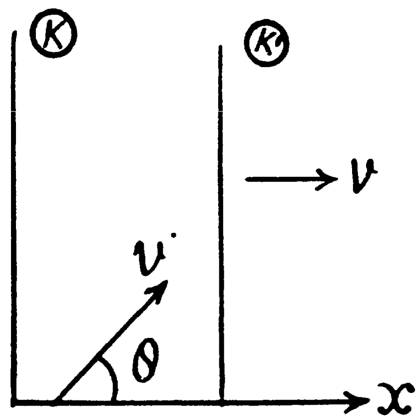
$$\Delta t_0 \frac{\sqrt{V^2+v'^2-(v'^2 V^2)/c^2}}{\sqrt{1-V^2/c^2}\,\sqrt{1-v'^2/c^2}}$$

1.363 In the frame K' the components of the velocity of the particle are

$$v'_x = \frac{v\cos\theta - V}{1-\dfrac{vV\cos\theta}{c^2}}$$

$$v'_y = \frac{v\sin\theta\sqrt{1-V^2/c^2}}{1-\dfrac{vV}{c^2}\cos\theta}$$

Hence, $\tan\theta' = \dfrac{v'_y}{v'_x} = \dfrac{v\sin\theta}{v\cos\theta - V}\sqrt{(1-V^2)/c^2}$



1.364 In K' the coordinates of A and B are

$$A:(t',0,-v't',0);\ B:(t',l,-v't',0)$$

After performing Lorentz transformation to the frame K *we get*

$$A: t=\gamma t' \quad B: t=\gamma\left(t'+\frac{Vl}{c^2}\right)$$

$$x=\gamma V t' \quad x=\gamma(l+Vt')$$

$$y=v't' \quad y=-v't'$$

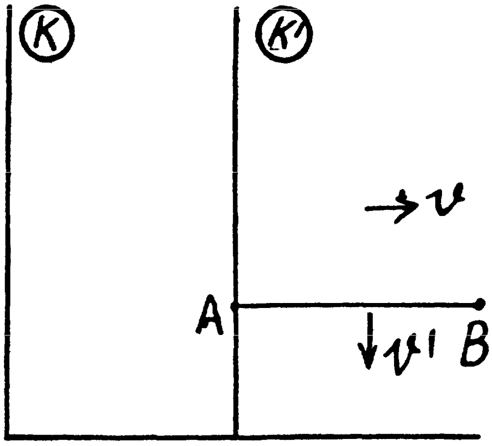
$$z=0 \quad z=0$$

By translating $t' \to t' - \dfrac{Vl}{c^2}$, we can write

the coordinates of B as $B: t=\gamma t'$

$$x = \gamma\, l\left(1 - \frac{V^2}{c^2}\right) + V t' \gamma = l\sqrt{1 - \frac{v^2}{c^2}} + V t' \gamma$$

$$y = -v'\left(t' - \frac{Vl}{c^2}\right),\ z = 0$$



Thus $$\Delta x = l\sqrt{1 - \left(\frac{V^2}{c^2}\right)},\ \Delta y = \frac{v' V l}{c^2}$$

Hence $$\tan\theta' = \frac{v' V}{c^2\sqrt{1 - \frac{v' V}{c^2}}}$$

1.365 $\overset{t}{\underset{\vec{v}}{-}}$ - - - - - - - - - - - - $\overset{l + dt}{\underset{\vec{v} + \vec{w}\,dt}{-}}$ K

In K the velocities at time t and $t + dt$ are respectively v and $v + wdt$ along x – axis which is parallel to the vector $\vec{V}$. In the frame K' moving with velocity $\vec{V}$ with respect to K, the velocities are respectively,

$$\frac{v - V}{1 - \frac{vV}{c^2}} \text{ and } \frac{v + w\,dt - V}{1 - (v + w\,dt)\frac{V}{c^2}}$$

The latter velocity is written as

$$\frac{v - V}{1 - v\frac{V}{c^2}} + \frac{w\,dt}{1 - v\frac{V}{c^2}} + \frac{v - V}{\left(1 - \frac{vV}{c^2}\right)}\frac{wV}{c^2}dt = \frac{v - V}{1 - v\frac{V}{c^2}} + \frac{w\,dt\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \frac{vV}{c^2}\right)^2}$$

Also by Lorentz transformation

$$dt' = \frac{dt - V\,dx/c^2}{\sqrt{1 - V^2/c^2}} = dt\,\frac{1 - vV/c^2}{\sqrt{1 - V^2/c^2}}$$

Thus the acceleration in the K' frame is

$$w' = \frac{dv'}{dt'} = \frac{w}{\left(1 - \frac{vV}{c^2}\right)^3}\left(1 - \frac{V^2}{c^2}\right)^{3/2}$$

(b) In the K frame the velocities of the particle at the time t and $t + di$ are repectively

$$(0, v, 0) \text{ and } (0, v + wdt, 0)$$

where $\vec{V}$ is along x–axis. In the K' frame the velocities are

$$\left(-V, v\sqrt{1 - V^2/c^2}, 0\right)$$

and $$\left(-V, (v + wdt)\sqrt{1 - V^2/c^2}, 0\right) \text{ respectively}$$

Thus the acceleration

$$w' = \frac{w\,dt\sqrt{(1 - V^2/c^2)}}{dt'} = w\left(1 - \frac{V^2}{c^2}\right) \text{ along the } y\text{-axis.}$$

We have used $dt' = \dfrac{dt}{\sqrt{1 - V^2/c^2}}$

1.366 In the instantaneous rest frame $v = V$ and

$$w' = \frac{w}{\left(1 - \frac{V^2}{c^2}\right)^{3/2}} \text{ (from 1.365a)}$$

So,

$$= \frac{dv}{\left(1 - \frac{V^2}{c^2}\right)^{3/2}} = w'\,dt$$

w' is constant by assumption. Thus integration gives

$$v = \frac{w'\,t}{\sqrt{1 + \left(\frac{w'\,t}{c}\right)^2}}$$

Integrating once again $x = \dfrac{c^2}{w'}\left(\sqrt{1 + \left(\frac{w'\,t}{c}\right)^2} - 1\right)$

1.367 The boost time τ_0 in the reference frame fixed to the rocket is related to the time τ elapsed on the earth by

$$\tau_0 = \int_0^\tau \sqrt{1 - \frac{v^2}{c^2}}\,dt = \int_0^\tau \left[1 - \frac{\left(\frac{w'\,t}{c}\right)^2}{1 + \left(\frac{w'\,t}{c}\right)^2}\right]^{1/2} dt$$

$$= \int_0^\tau \frac{dt}{\sqrt{1 + \left(\frac{w'\,t}{c}\right)^2}} = \frac{c}{w'}\int_0^{(w'\tau)/c} \frac{d\xi}{\sqrt{1 + \xi^2}} = \frac{c}{w'}\ln\left[\frac{w'\tau}{c} + \sqrt{1 + \left(\frac{w'\tau}{c}\right)^2}\right]$$

1.368 $m = \dfrac{m_0}{\sqrt{1 - \beta^2}}$

For

$$\beta \to 1, \; \frac{m}{m_0} = \frac{1}{\sqrt{2(1 - \beta)}} = \frac{1}{\sqrt{2}\,\eta}$$

1.369 We define the density ρ in the frame K in such a way that $\rho\,dx\,dy\,dz$ is the rest mass dm_0 of the element. That is $\rho\,dx\,dy\,dz = \rho_0\,dx_0\,dy_0\,dz_0$, where ρ_0 is the proper density dx_0 , dy_0 , dz_0 are the dimensions of the element in the rest frame K_0. Now

$$dy = dy_0,\; dz = dz_0,\; dx = dx_0\sqrt{1 - \frac{v^2}{c^2}}$$

if the frame K is moving with velocity, v relative to the frame K_0. Thus

$$\rho = \frac{\rho_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Defining η by $\rho = \rho_0(1+\eta)$

We get $\quad 1+\eta = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ or, $\dfrac{v^2}{c^2} = 1 - \dfrac{1}{(1+\eta)^2} = \dfrac{\eta(2+\eta)}{(1+\eta)^2}$

or $$v = c\sqrt{\frac{\eta(2+\eta)}{(1+\eta)^2}} = \frac{c\sqrt{\eta(2+\eta)}}{1+\eta}$$

1.370 We have

$$\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} = p \quad \text{or,} \quad \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \sqrt{m_0^2 + \frac{p^2}{c^2}}$$

or $$1-\frac{v^2}{c^2} = \frac{m_0^2 c^2}{m_0^2 c^2 + p^2} = 1 - \frac{p^2}{p^2 + m_0^2 c^2}$$

or $$v = \frac{cp}{\sqrt{p^2 + m_0^2 c^2}} = \frac{c}{\sqrt{1+\left(\frac{m_0 c}{p}\right)^2}}$$

So $$\frac{c-v}{c} = \left[1 - \left(1 + \left(\frac{m_0 c}{p}\right)^2\right)^{-1/2}\right] \times 100\,\% = \frac{1}{2}\left(\frac{m_0 c}{p}\right)^2 \times 100\,\%$$

1.371 By definition of η,

$$\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} = \eta\, m_0 v \quad \text{or} \quad 1-\frac{v^2}{c^2} = \frac{1}{\eta^2}$$

or $$v = c\sqrt{1-\frac{1}{\eta^2}} = \frac{c}{\eta}\sqrt{\eta^2-1}$$

1.372 The work done is equal to change in kinetic energy which is different in the two cases Classically i.e. in nonrelativistic mechanics, the change in kinetic energy is

$$\frac{1}{2} m_0 c^2 \left((0{\cdot}8)^2 - (0{\cdot}6)^2\right) = \frac{1}{2} m_0 c^2\, 0{\cdot}28 = 0{\cdot}14\, m_0 c^2$$

Relativistically it is,

$$\frac{m_0 c^2}{\sqrt{1-(0{\cdot}8)^2}} - \frac{m_0 c^2}{\sqrt{1-(0{\cdot}6)^2}} = \frac{m_0 c^2}{0{\cdot}6} - \frac{m_0 c^2}{0{\cdot}8} = m_0 c^2 (1{\cdot}666 - 1{\cdot}250)$$

$$= 0{\cdot}416\, m_0 c^2 = 0{\cdot}42\, m_0 c^2$$

1.373 $\dfrac{m_0 c^2}{\sqrt{1-\dfrac{v^2}{c^2}}} = 2 m_0 c^2$

or $$\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2} \quad \text{or} \quad 1-\frac{v^2}{c^2} = \frac{1}{4}$$

or $$\frac{v}{c} = \frac{\sqrt{3}}{2} \text{ i.e. } v = c\frac{\sqrt{3}}{2}$$

1.374 Relativistically

$$\frac{T}{m_0 c^2} = \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) = \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4$$

So $$\beta_{rel}^2 \approx \frac{2T}{m_0 c^2} - \frac{3}{4}\left(\beta_{rel}^2\right)^2 \approx \frac{2T}{m_0 c^2} - \frac{3}{4}\left(\frac{2T}{m_0 c^2}\right)^2$$

Thus $$\beta_{rel} = \left[\frac{2T}{m_0 c^2} - 3\frac{T^2}{m_0^2 c^4}\right]^{1/2} = \sqrt{\frac{2T}{m_0 c^2}}\left(1-\frac{3}{4}\frac{T}{m_0 c^2}\right)$$

But Classically, $\beta_{cl} = \sqrt{\dfrac{2T}{m_0 c^2}}$ so $\dfrac{\beta_{rel}-\beta_{cl}}{\beta_{cl}} = \dfrac{3}{4}\dfrac{T}{m_0 c^2} = \varepsilon$

Hence if $$\frac{T}{m_0 c^2} < \frac{4}{3}\varepsilon$$

the velocity β is given by the classical formula with an error less than ε.

1.375 From the formula

$$E = \frac{m_0 c^2}{\sqrt{1-\dfrac{v^2}{c^2}}}, \quad p = \frac{m_0 v}{\sqrt{1-v^2/c^2}}$$

we find $$E^2 = c^2 p^2 + m_0^2 c^4 \quad \text{or} \quad (m_0 c^2 + T)^2 = c^2 p^2 + m_0^2 c^4$$

or $$T(2 m_0 c^2 + T) = c^2 p^2 \quad \text{i.e.} \quad p = \frac{1}{c}\sqrt{T(2m_0 c^2 + T)}$$

1.376 Let the total force exerted by the beam on the target surface be F and the power liberated there be P. Then, using the result of the previous problem we see

$$F = Np = \frac{N}{c}\sqrt{T(T+2m_0 c^2)} = \frac{I}{ec}\sqrt{T(T+2m_0 c^2)}$$

since $I = Ne$, N being the number of particles striking the target per second. Also,

$$P = N\left(\frac{m_0 c^2}{\sqrt{1-v^2/c^2}} - m_0 c^2\right) = \frac{I}{e}T$$

These will be, respectively, equal to the pressure and power developed per unit area of the target if I is current density.

1.377 In the frame fixed to the sphere :- The momentum transferred to the eastically scatterred particle is

$$\frac{2\,mv}{\sqrt{1-\frac{v^2}{c^2}}}$$

The density of the moving element is, from 1.369, $n\,\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

and the momentum transferred per unit time per unit area is

$$p = \text{the pressure} = \frac{2\,mv}{\sqrt{1-\frac{v^2}{c^2}}}\,n\,\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot v = \frac{2\,mn\,v^2}{1-\frac{v^2}{c^2}}$$

In the frame fixed to the gas :- When the sphere hits a stationary particle, the latter recoils with a velocity

$$= \frac{v+v}{1+\frac{v^2}{c^2}} = \frac{2\,v}{1+\frac{v^2}{c^2}}$$

The momentum transferred is $\dfrac{\dfrac{m\cdot 2v}{1+v^2/c^2}}{\sqrt{1-\dfrac{4v^2/c^2}{(1-v^2/c^2)^2}}} = \dfrac{2\,mv}{1-\dfrac{v^2}{c^2}}$

and the pressure is $\dfrac{2\,mv}{1-\dfrac{v^2}{c^2}}\cdot n\cdot v = \dfrac{2\,mnv^2}{1-\dfrac{v^2}{c^2}}$

1.378 The equation of motion is

$$\frac{d}{dt}\left(\frac{m_0\,v}{\sqrt{1-\frac{v^2}{c^2}}}\right) = F$$

Integrating $= \dfrac{v/c}{\sqrt{1-\dfrac{v^2}{c^2}}} = \dfrac{\beta}{\sqrt{1-\beta^2}} = \dfrac{F_t}{m_0\,c}$, using $v = 0$ for $t = 0$

$$\frac{\beta^2}{1-\beta^2} = \left(\frac{Ft}{m_0\,c}\right)^2 \quad \text{or,}\quad \beta^2 = \frac{(Ft)^2}{(Ft)^2+(m_0\,c)^2} \quad \text{or,}\quad v = \frac{Fct}{\sqrt{(m_0\,c)^2+(Ft)^2}}$$

$$\text{or}\quad x = \int \frac{Fct\;dt}{\sqrt{F^2\,t^2+m_0^2\,c^2}} = \frac{c}{F}\int \frac{\xi\,d\xi}{\sqrt{\xi^2+(m_0\,c)^2}} = \frac{c}{F}\sqrt{F^2\,t^2+m_0^2\,c^2} + \text{constant}$$

or using $x = 0$ at $t = 0$, we get, $x = \sqrt{c^2\,t^2+\left(\dfrac{m_0\,c^2}{F}\right)^2} - \dfrac{m_0\,c^2}{F}$

1.379 $x = \sqrt{a^2 + c^2 t^2}$, so $\dot{x} = v = \frac{c^2 t}{a^2 + c^2 t^2}$

or, $$\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{c^2 t}{a} \cdot \text{ Thus } \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = \frac{m_0 c^2}{a} = F$$

1.380 $$\vec{F} = \frac{d}{dt}\left(\frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = m_0 \frac{\dot{\vec{v}}}{\sqrt{1 - \frac{v^2}{c^2}}} + m_0 \frac{\vec{v}}{c^2} \vec{v} \cdot \dot{\vec{v}} \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}$$

Thus $$\vec{F}_\perp = m_0 \frac{\vec{w}}{\sqrt{1 - \beta^2}}, \quad \vec{w} = \dot{\vec{v}}, \ \vec{w} \perp \vec{v}$$

$$\vec{F}_\parallel = m_0 \frac{\vec{w}}{(1 - \beta^2)^{3/2}}, \quad \vec{w} = \dot{\vec{v}}, \ \vec{w} \parallel \vec{v}$$

1.381 By definition,

$$E = m_0 \frac{c^2}{\sqrt{1 - \frac{v_x^2}{c^2}}} = \frac{m_0 c^3 dt}{ds}, \quad p_x = m_0 \frac{v_x}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{c\, m_0\, dx}{ds}$$

where $ds^2 = c^2 dt^2 - dx^2$ is the invariant interval $(dy = dz = 0)$

Thus, $$p'_x = c m_0 \frac{dx'}{ds} = c m_0 \gamma \frac{(dx - V dt)}{ds} = \frac{p_x - VE/c^2}{\sqrt{1 - V^2/c^2}}$$

$$E' = m_0 c^3 \frac{dt'}{ds} - c^3 m_0 \gamma \frac{\left(dt - \frac{V d_x}{c^2}\right)}{ds} = \frac{E - V p_x}{\sqrt{1 - \frac{V^2}{c^2}}},$$

1.382 For a photon moving in the x direction

$$\varepsilon = c p_x, \ p_y = p_z = 0,$$

In the moving frame, $\varepsilon' = \frac{1}{\sqrt{1 - \beta^2}}\left(\varepsilon - V \frac{\varepsilon}{c}\right) = \varepsilon \sqrt{\frac{1 - V/c}{1 + V/c}}$

Note that $\varepsilon' = \frac{\varepsilon}{2}$ if, $\frac{1}{4} = \frac{1 - \beta}{1 + \beta}$ or $\beta = \frac{3}{5}$, $V = \frac{3c}{5}$.

1.383 As before

$$E = m_0 c^3 \frac{dt}{ds}, \ p_x = m_0 c \frac{dx}{ds}.$$

Similarly $$p_y = m_0 c \frac{dy}{ds}, \; p_z = m_0 c \frac{dz}{ds}$$

Then $$E^2 - c^2 p^2 = E^2 - c^2 \left(p_x^2 + p_y^2 + p_x^2\right)$$

$$= m_0^2 c^4 \frac{(c^2 dt^2 - dx^2 - dy^2 - dz^2)}{ds^2} = m_0^2 c^4 \text{ is invariant}$$

1.384 (b) & (a) In the CM frame, the total momentum is zero, Thus

$$\frac{V}{c} = \frac{c p_{1x}}{E_1 + E_2} = \frac{\sqrt{T(T + 2m_0 c^2)}}{T + 2m_0 c^2} = \sqrt{\frac{T}{T + 2m_0 c^2}}$$

where wc have used the result of probiem (1.375)

Then

$$\frac{1}{\sqrt{1 - V^2/c^2}} = \frac{1}{\sqrt{1 - \dfrac{T}{T + 2m_0 c^2}}} = \sqrt{\frac{T + 2m_0 c^2}{2m_0 c^2}}$$

Total energy in the CM frame is

$$\frac{2m_0 c^2}{\sqrt{1 - V^2/c^2}} = 2m_0 c^2 \sqrt{\frac{T + 2m_0 c^2}{2m_0 c^2}} = \sqrt{2\, m_0 c^2 (T + 2m_0 c^2)} = \tilde{T} + 2m_0 c^2$$

So $$\tilde{T} = 2m_0 c^2 \left(\sqrt{1 + \frac{T}{2m_0 c^2}} - 1\right)$$

Also $2\sqrt{c^2 \tilde{p}^2 + m_0^2 c^4} = \sqrt{2m_0 c^2 (T + 2m_0 c^2)}$, $4 c^2 \tilde{p}^2 = 2m_0 c^2 T$, or $\tilde{p} = \sqrt{\frac{1}{2} m_0 T}$

1.385 $M_0 c^2 = \sqrt{E^2 - c^2 p^2}$

$$\sqrt{(2m_0 c^2 + T)^2 - T(2m_0 c^2 + T)} = \sqrt{2m_0 c^2 (2m_0 c^2 + T)} = c\sqrt{2m_0 (2m_0 c^2 + T)}$$

Also $$cp = \sqrt{T(T + 2m_0 c^2)}, \; v = \frac{c^2 p}{E} = c\sqrt{\frac{T}{T + 2m_0 c^2}}$$

1.386 Let T' = kinetic energy of a proton striking another stationary particle of the same rest mass. Then, combined kinetic energy in the CM frame

$$= 2m_0 c^2 \left(\sqrt{1 + \frac{T'}{2m_0 c^2}} - 1\right) = 2T, \; \left(\frac{T}{m_0 c^2} + 1\right)^2 = 1 + \frac{T'}{2m_0 c^2}$$

$$\frac{T'}{2m_0 c^2} = \frac{T(2m_0 c^2 + T)}{m_0^2 c^4}, \; T' = \frac{2T(T + 2m_o c^2)}{m_0 c^2}$$

1.387 We have

$$E_1 + E_2 + E_3 = m_0 c^2, \ \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$$

Hence $$(m_0 c^2 - E_1)^2 - c^2 \vec{p}_1^{\,2} = (E_2 + E_3)^2 - \left(\vec{p}_2 + \vec{p}_3\right)^2 c^2$$

The L.H.S. $= (m_0^2 c^4 - E_1)^2 - c^2 \vec{p}_1^{\,2} = (m_0^2 + m_1^2) c^4 - 2m_0 c^2 E_1$

The R.H.S. is an invariant. We can evaluate it in any frame. Choose the CM frame of the particles 2 and 3.

In this frame R.H.S. $= (E'_2 + E'_3)^2 = (m_2 + m_3)^2 c^4$

Thus $(m_0^2 + m_1^2) c^4 - 2m_0 c^2 E_1 = (m_2 + m_3)^2 c^4$

or $2 m_0 c^2 E_1 \le \left\{ m_0^2 + m_1^2 - (m_2 + m_3)^2 \right\} c^4$, or $E_1 \le \dfrac{m_0^2 + m_1^2 - (m_2 + m_3)^2}{2 m_0} c^2$

1.388 The velocity of ejected gases is u realtive to the rocket. In an earth centred frame it is

$$\frac{v - u}{1 - \dfrac{vu}{c^2}}$$

in the direction of the rocket. The momentum conservation equation then reads

$$(m + dm)(v + dv) + \frac{v - u}{1 - \dfrac{uv}{c^2}} (- dm) = mv$$

or $$mdv - \left(\frac{v - u}{1 - \dfrac{uv}{c^2}} - v \right) dm = 0$$

Here $-dm$ is the mass of the ejected gases. so

$$mdv - \frac{-u + \dfrac{uv^2}{c^2}}{1 - \dfrac{uv}{c^2}} dm = 0, \text{ or } mdv + u\left(1 - \frac{v^2}{c^2}\right) dm = 0$$

(neglecting $1 - \dfrac{uv}{c^2}$ since u is non- relativistic.)

Integrating $\left(\beta = \dfrac{v}{c}\right)$, $\displaystyle\int \frac{d\beta}{1 - \beta^2} + \frac{u}{c} \int \frac{dm}{m} = 0$, $\ln \dfrac{1 + \beta}{1 - \beta} + \dfrac{u}{c} \ln m =$ constant

The constant $= \dfrac{u}{c} \ln m_0$ since $\beta = 0$ initially.

Thus $$\frac{1 - \beta}{1 + \beta} = \left(\frac{m}{m_0}\right)^{u/c} \text{ or } \beta = \frac{1 - \left(\dfrac{m}{m_0}\right)^{u/c}}{1 + \left(\dfrac{m}{m_0}\right)^{u/c}}$$

PART TWO

THERMODYNAMICS AND MOLECULAR PHYSICS

2.1 EQUATION OF THE GAS STATE • PROCESSES

2.1 Let m_1 and m_2 be the masses of the gas in the vessel before and after the gas is released. Hence mass of the gas released,

$$\Delta m = m_1 - m_2$$

Now from ideal gas equation

$$p_1 V = m_1 \frac{R}{M} T_0 \text{ and } p_2 V = m_2 \frac{R}{M} T_0$$

as V and T are same before and after the release of the gas.

so, $$(p_1 - p_2) V = (m_1 - m_2) \frac{R}{M} T_0 = \Delta m \frac{R}{M} T_0$$

or, $$\Delta m = \frac{(p_1 - p_2) V M}{R T_0} = \frac{\Delta p \, V M}{R T_0} \quad (1)$$

We also know $p = \rho \frac{R}{M} T$ so, $\frac{M}{R T_0} = \frac{\rho}{p_0}$ (2)

(where $p_0 =$ standard atmospheric pressure and $T_0 = 273$ K)

From Eqs. (1) and (2) we get

$$\Delta m = \rho V \frac{\Delta p}{p_0} = 1{\cdot}3 \times 30 \times \frac{0{\cdot}78}{1} = 30 \text{ g}$$

2.2 Let m_1 be the mass of the gas enclosed.

Then, $$p_1 V = \nu_1 R T_1$$

When heated, some gas, passes into the evacuated vessel till pressure difference becomes Δp. Let p'_1 and p'_2 be the pressure on the two sides of the valve. Then $p'_1 V = \nu'_1 R T_2$ and

$$p'_2 V = \nu'_2 R T_2 = (\nu_1 - \nu'_1) R T_2$$

$$p'_2 V = \left(\frac{p_1 V}{R T_1} - \frac{p'_1 V}{R T_2}\right) \quad \text{or} \quad p'_2 = \left(\frac{p_1}{T_1} - \frac{p'_1}{T_2}\right) T_2$$

But,
$$p'_1 - p'_2 = \Delta p$$

So,
$$p'_2 = \left(\frac{p_1}{T_1} - \frac{p'_2 + \Delta p}{T_2}\right) T_2$$

$$= \frac{p_1 T_2}{T_1} - p'_2 - \Delta p$$

or,
$$p_2' = \frac{1}{2}\left(\frac{p_1 T_2}{T_1} - \Delta p\right) = 0{\cdot}08 \text{ atm}$$

2.3 Let the mixture contain ν_1 and ν_2 moles of H_2 and H_e respectively. If molecular weights of H_2 and H_e are M_1 and M_2, then respective masses in the mixture are equal to

$$m_1 = \nu_1 M_1 \text{ and } m_2 = \nu_2 M_2$$

Therefore, for the total mass of the mixture we get,

$$m = m_1 + m_2 \quad \text{or} \quad m = \nu_1 M_1 + \nu_2 M_2 \tag{1}$$

Also, if ν is the total number of moles of the mixture in the vessels, then we know,

$$\nu = \nu_1 + \nu_2 \tag{2}$$

Solving (1) and (2) for ν_1 and ν_2, we get,

$$\nu_1 = \frac{(\nu M_2 - m)}{M_2 - M_1}, \qquad \nu_2 = \frac{m - \nu M_1}{M_2 - M_1}$$

Therefore, we get $m_1 = M_1 \cdot \dfrac{(\nu M_2 - m)}{M_2 - M_1}$ and $m_2 = M_2 \dfrac{(m - \nu M_1)}{M_2 - M_1}$

or,
$$\frac{m_1}{m_2} = \frac{M_1}{M_2} \frac{(\nu M_2 - m)}{(m - \nu M_1)}$$

One can also express the above result in terms of the effective molecular weight M of the mixture, defined as,

$$M = \frac{m}{\nu} = m \frac{R T}{pV}$$

Thus,
$$\frac{m_1}{m_2} = \frac{M_1}{M_2} \cdot \frac{M_2 - M}{M - M_1} = \frac{1 - M/M_2}{M/M_1 - 1}$$

Using the data and table, we get :

$$M = 3{\cdot}0 \text{ g and, } \frac{m_1}{m_2} = 0{\cdot}50$$

2.4 We know, for the mixture, N_2 and CO_2 (being regarded as ideal gases, their mixture too behaves like an ideal gas)

$$pV = \nu RT, \quad \text{so} \quad p_0 V = \nu RT$$

where, ν is the total number of moles of the gases (mixture) present and V is the volume of the vessel. If ν_1 and ν_2 are number of moles of N_2 and CO_2 respectively present in the mixture, then

$$\nu = \nu_1 + \nu_2$$

Now number of moles of N_2 and CO_2 is, by definition, given by

$$\nu_1 = \frac{m_1}{M_1} \text{ and, } \nu_2 = \frac{m_2}{M_2}$$

where, m_1 is the mass of N_2 (Moleculer weight = M_1) in the mixture and m_2 is the mass of CO_2 (Molecular weight = M_2) in the mixture.

Therefore density of the mixture is given by

$$\rho = \frac{m_1 + m_2}{V} = \frac{m_1 + m_2}{(\nu RT/P_0)}$$

$$= \frac{p_0}{RT} \cdot \frac{m_1 + m_2}{\nu_1 + \nu_2} = \frac{p_0 (m_1 + m_2) M_1 M_2}{RT (m_1 M_2 + m_2 M_1)}$$

$$= 1{\cdot}5 \text{ kg/m}^3 \text{ on substitution}$$

2.5 (a) The mixture contains ν_1, ν_2 and ν_3 moles of O_2, N_2 and CO_2 respectively. Then the total number of moles of the mixture

$$\nu = \nu_1 + \nu_2 + \nu_3$$

We know, ideal gas equation for the mixture

$$pV = \nu RT \quad \text{or} \quad p = \frac{\nu RT}{V}$$

or, $$p = \frac{(\nu_1 + \nu_2 + \nu_3) RT}{V} = 1{\cdot}968 \text{ atm on substitution}$$

(b) Mass of oxygen (O_2) present in the mixture : $m_1 = \nu_1 M_1$

Mass of nitrogen (N_2) present in the mixture : $m_2 = \nu_2 M_2$

Mass of carbon dioxide (CO_2) present in the mixture : $m_3 = \nu_3 M_3$

So, mass of the mixture

$$m = m_1 + m_2 + m_3 = \nu_1 M_1 + \nu_2 M_2 + \nu_3 M_3$$

Moleculer mass of the mixture : $M = \dfrac{\text{mass of the mixture}}{\text{total number of moles}}$

$$= \frac{\nu_1 M_1 + \nu_2 M_2 + \nu_3 M_3}{\nu_1 + \nu_2 + \nu_3} = 36{\cdot}7 \text{ g/mol. on substitution}$$

2.6 Let p_1 and p_2 be the pressure in the upper and lower part of the cylinder respectively at temperature T_0. At the equilibrium position for the piston :

$p_1 S + mg = p_2 S$ or, $p_1 + \frac{mg}{S} = p_2$ (m is the mass of the piston.)

But $p_1 = \frac{R\,T_0}{\eta\, V_0}$ (where V_0 is the initial volume of the lower part)

So, $$\frac{RT_o}{\eta V_0} + \frac{mg}{S} = \frac{RT_0}{V_0} \quad \text{or,} \quad \frac{mg}{S} = \frac{RT_0}{V_0}\left(1 - \frac{1}{\eta}\right) \tag{1}$$

Let T' be the sought temperature and at this temperature the volume of the lower part becomes V', then according to the problem the volume of the upper part becomes $\eta' V'$

Hence, $$\frac{mg}{S} = \frac{RT'}{V'}\left(1 - \frac{1}{\eta'}\right) \tag{2}$$

From (1) and (2).

$$\frac{RT_0}{V_0}\left(1 - \frac{1}{\eta}\right) = \frac{RT'}{V'}\left(1 - \frac{1}{\eta'}\right) \quad \text{or,} \quad T' = \frac{T_0\left(1 - \frac{1}{\eta}\right)V'}{V_0\left(1 - \frac{1}{\eta'}\right)}$$

As, the total volume must be constant,

$$V_0(1 + \eta) = V'(1 + \eta') \quad \text{or,} \quad V' = \frac{V_0(1 + \eta)}{(1 + \eta')}$$

Putting the value of V' in Eq. (3), we get

$$T' = \frac{T_0\left(1 - \frac{1}{\eta}\right)V_0\frac{(1 + \eta)}{(1 + \eta')}}{V_0\left(1 - \frac{1}{\eta'}\right)}$$

$$= \frac{T_0(\eta^2 - 1)\,\eta'}{(\eta'^2 - 1)\,\eta} = 0{\cdot}42\text{ k K}$$

2.7 Let ρ_1 be the density after the first stroke. The the mass remains constant

$$V\rho = (V + \Delta V)\,\rho_1, \quad \text{or,} \quad \rho_1 = \frac{V\rho}{(V + \Delta V)}$$

Similarly, if ρ_2 is the density after second stroke

$$V\rho_1 = (V + \Delta V)\,\rho_2 \quad \text{or,} \quad \rho_2 = \left(\frac{V}{V + \Delta V}\right)\rho_1 = \left(\frac{V}{V + \Delta V}\right)^2 \rho_0$$

In this way after nth stroke.

$$\rho_n = \left(\frac{V}{V + \Delta V}\right)^n \rho_0$$

Since pressure α density,

$$p_n = \left(\frac{V}{V+\Delta V}\right)^n p_0 \text{ (because temperature is constant.)}$$

It is required by $\frac{p_n}{p_0}$ to be $\frac{1}{\eta}$

so, $$\frac{1}{\eta} = \left(\frac{V}{V+\Delta V}\right)^n \quad \text{or,} \quad \eta = \left(\frac{V+\Delta V}{V}\right)^n$$

Hence $$n = \frac{\ln \eta}{\ln\left(1 + \frac{\Delta V}{V}\right)}$$

2.8 From the ideal gas equation $p = \frac{m}{M}\frac{RT}{V}$

$$\frac{dp}{dt} = \frac{RT}{MV}\frac{dm}{dt} \tag{1}$$

In each stroke, volume v of the gas is ejected, where v is given by

$$v = \frac{V}{m_N}\left[m_{N-1} - m_N\right]$$

In case of continuous ejection, if (m_{N-1}) corresponds to mass of gas in the vessel at time t, then m_N is the mass at time $t + \Delta t$, where Δt, is the time in which volume v of the gas has come out. The rate of evacuation is therefore $\frac{v}{\Delta t}$ i.e.

$$C = \frac{v}{\Delta t} = -\frac{V}{m(t+\Delta t)} \cdot \frac{m(t+\Delta t) - m(t)}{\Delta t}$$

In the limit $\Delta t - 0$, we get

$$C = \frac{V}{m}\frac{dm}{dt} \tag{2}$$

From (1) and (2)

$$\frac{dp}{dt} = -\frac{C}{V}\frac{mRT}{MV} = -\frac{C}{V}p \quad \text{or} \quad \frac{dp}{p} = -\frac{C}{V}dt$$

Integrating $$\int_p^{p_0} \frac{dp}{p} = -\frac{C}{V}\int_t^0 dt \quad \text{or} \quad \ln\frac{p}{p_0} = -\frac{C}{V}t$$

Thus $$p = p_0 e^{-Ct/V}$$

2.9 Let ρ be the instantaneous density, then instantaneous mass $= V_\rho$. In a short interval dt the volume is increased by Cdt.

So, $$V\rho = (V + Cdt)(\rho + d\rho)$$

(because mass remains constant in a short interval dt)

so, $$\frac{d\rho}{\rho} = -\frac{C}{V}dt$$

Since pressure α density $\frac{dp}{p} = -\frac{C}{V}dt$

or $$\int_{p_1}^{p_2} -\frac{dp}{p} = \frac{C}{V}t,$$

or $$t = \frac{V}{C}\ln\frac{p_1}{p_2} = \frac{V}{C}\ln\frac{1}{\eta}\ 1{\cdot}0 \text{ min}$$

2.10 The physical system consists of one mole of gas confined in the smooth vertical tube. Let m_1 and m_2 be the masses of upper and lower pistons and S_1 and S_2 are their respective areas.

For the lower piston

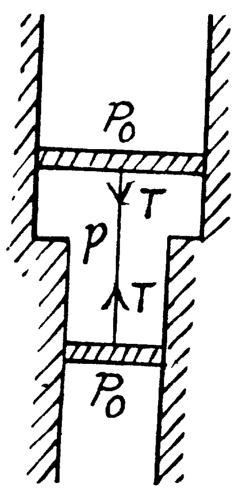
$$p S_2 + m_2 g = p_0 S_2 + T,$$

or, $$T = (p - p_0) S_2 + m_2 g \qquad (1)$$

Similarly for the upper piston

$$p_0 S_1 + T + m_1 g = p S_1,$$

or, $$T = (p - p_0) S_1 - m_1 g \qquad (2)$$



From (1) and (2)

$$(p - p_0)(S_1 - S_2) = (m_1 + m_2) g$$

or, $$(p - p_0)\Delta S = mg$$

so, $$p = \frac{mg}{\Delta S} + p_0 = \text{constant}$$

From the gas law, $p V = \nu R T$

$$p \Delta V = \nu R \Delta T \text{ (because } p \text{ is constant)}$$

So, $$\left(p_0 + \frac{mg}{\Delta S}\right)\Delta S\, l = R\,\Delta T,$$

Hence, $$\Delta T = \frac{1}{R}(p_0 \Delta S + mg)\, l = 0{\cdot}9 \text{ K}$$

2.11 (a) $p = p_0 - \alpha V^2 = p_0 - \alpha\left(\frac{RT}{p}\right)^2$

(as, $V = RT/p$ for one mole of gas)

Thus, $$T = \frac{1}{R\sqrt{\alpha}}\, p\sqrt{p_0 - p} = \frac{1}{R\sqrt{\alpha}}\sqrt{p_0 p^2 - p^3} \qquad (1)$$

For T_{max}, $\frac{d}{dp}\left(p_0 p^2 - p^3\right)$ must be zero

which yields, $$p = \frac{2}{3}p_0 \qquad (2)$$

Hence, $$T_{max} = \frac{1}{R\sqrt{\alpha}} \cdot \frac{2}{3}p_0 \sqrt{p_0 - \frac{2}{3}p_0} = \frac{2}{3}\left(\frac{p_0}{R}\right)\sqrt{\frac{p_0}{3\alpha}}$$

(b) $p = p_0 e^{-\beta V} = p_0 e^{-\beta RT/p}$

so $$\frac{\beta RT}{p} = \ln\frac{p_o}{p}, \text{ and } T = \frac{p}{\beta R}\ln\frac{p_0}{p} \qquad (1)$$

For T_{max} the condition is $\frac{dT}{dp} = 0$, which yields

$$p = \frac{p_0}{e}$$

Hence using this value of p in Eq. (1), we get

$$T_{max} = \frac{p_o}{e\beta R}$$

2.12 $T = T_0 + \alpha V^2 = T_0 + \alpha \frac{R^2 T^2}{p^2}$

(as, $V = RT/p$ for one mole of gas)

So, $$p = \sqrt{\alpha}\, RT\,(T - T_0)^{1/2} \qquad (1)$$

For $$p_{min},\ \frac{dp}{dT} = 0, \text{which gives}$$

$$T = 2T_0 \qquad (2)$$

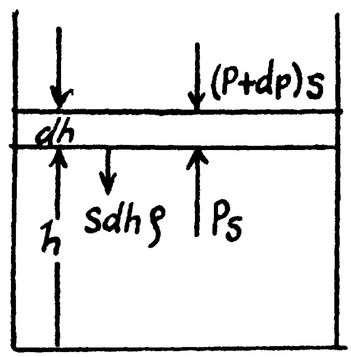
From (1) and (2), we get,

$$p_{min} = \sqrt{\alpha}\, R\, 2T_0\,(2T_0 - T_0)^{-1/2} = 2R\sqrt{\alpha T_0}$$

2.13 Consider a thin layer at a height h and thickness dh. Let p and $dp + p$ be the pressure on the two sides of the layer. The mass of the layer is $Sdh\rho$. Equating vertical downward force to the upward force acting on the layer.

$$Sdh\,\rho\, g + (p + dp)\,S = p\,S$$

So, $$\frac{dp}{dh} = -\rho g \qquad (1)$$



But, $p = \frac{\rho}{M}RT$, we have $dp = \frac{\rho R}{M}dT$,

or, $-\frac{\rho R}{M}dT = \rho g\, dh$

So, $\frac{dT}{dh} = -\frac{gM}{R} = -34$ K/km

That means, temperature of air drops by 34°C at a height of 1 km above bottom.

2.14 We have, $\frac{dp}{dh} = -\rho g$ (See 2.13) (1)

But, from $p = C\rho^n$ (where C is, a const) $\frac{dp}{d\rho} = Cn\,\rho^{n-1}$ (2)

We have from gas low $p = \rho\frac{R}{M}T$, so using (2)

$$C\rho^n = \rho\frac{R}{M}\cdot T, \quad \text{or} \quad T = \frac{M}{R}C\rho^{n-1}$$

Thus,
$$\frac{dT}{d\rho} = \frac{M}{R}\cdot C(n-1)\rho^{n-2} \tag{3}$$

But,
$$\frac{dT}{dh} = \frac{dT}{d\rho}\cdot\frac{d\rho}{dp}\cdot\frac{dp}{dh}$$

So,
$$\frac{dT}{dh} = \frac{M}{R}C(n-1)\rho^{n-2}\frac{1}{Cn\rho^{n-1}}(-\rho g) = \frac{-Mg(n-1)}{nR}$$

2.15 We have, $dp = -\rho g\,dh$ and from gas law $\rho = \frac{M}{RT}p$ (1)

Thus
$$\frac{dp}{p} = -\frac{Mg}{RT}dh$$

Integrating, we get

or,
$$\int_{p_o}^{p}\frac{dp}{p} = -\frac{Mg}{RT}\int_0^h dh \quad \text{or,} \quad \ln\frac{p}{p_0} = -\frac{Mg}{RT}h,$$

(where p_0 is the pressure at the surface of the Earth.)

$$p = p_0\,e^{-Mgh/RT},$$

[Under standard condition, $p_0 = 1$ atm, $T = 273$ K

Pressure at a height of 5 atm $= 1\times e^{-28\times 9{\cdot}81\times 5000/8314\times 273} = 0{\cdot}5$ atm.

Pressure in a mine at a depth of 5 km $= 1\times e^{-28\times 9{\cdot}81\times(-5000)/8314\times 273} = 2$ atm.]

2.16 We have $dp = -\rho g\,dh$ but from gas law $p = \frac{\rho}{M}RT$,

Thus
$$dp = \frac{d\rho}{M}RT \text{ at const. temperature}$$

So,
$$\frac{d\rho}{\rho} = \frac{gM}{RT}dh$$

Integrating within limits $\int_{\rho_0}^{\rho}\frac{d\rho}{\rho} = \int_0^h\frac{gM}{RT}dh$

or,
$$\ln \frac{\rho}{\rho_0} = -\frac{gM}{RT} h$$

So, $\rho = \rho_0 e^{-Mgh/RT}$ and $h = -\frac{RT}{Mg} \ln \frac{\rho}{\rho_0}$

(a) Given $T = 273°K$, $\frac{\rho_0}{\rho} = e$

Thus
$$h = -\frac{RT}{Mg} \ln e^{-1} = 8 \text{ km}.$$

(b) $T = 273° K$ and

$$\frac{\rho_0 - \rho}{\rho_0} = 0.01 \quad \text{or} \quad \frac{\rho}{\rho_0} = 0.99$$

Thus
$$h = -\frac{RT}{Mg} \ln \frac{\rho}{\rho_0} = 0.09 \text{ km on substitution}$$

2.17 From the Barometric formula, we have
$$p = p_0 e^{-Mgh/RT}$$

and from gas law
$$\rho = \frac{pM}{RT}$$

So, at constant temperature from these two Eqs.

$$\rho = \frac{Mp_0}{RT} e^{-Mgh/RT} = \rho_0 e^{-Mgh/RT} \tag{1}$$

Eq. (1) shows that density varies with height in the same manner as pressure. Let us consider the mass element of the gas contained in the column.

$$dm = \rho (Sdh) = \frac{Mp_0}{RT} e^{-Mgh/RT} Sdh$$

Hence the sought mass,

$$m = \frac{Mp_0 S}{RT} \int_0^h e^{-Mgh/RT} dh = \frac{p_0 S}{g} \left(1 - e^{-Mgh/RT}\right)$$

2.18 As the gravitational field is constant the centre of gravity and the centre of mass are same. The location of C.M.

$$h = \frac{\int_0^\infty h\, dm}{\int_0^\infty dm} = \frac{\int_0^\infty h \rho\, dh}{\int_0^\infty \rho\, dh}$$

But from Barometric formula and gas law $\rho = \rho_0 e^{-Mgh/RT}$

So, $$h = \frac{\int_0^{\alpha} h \left(e^{-Mgh/RT}\right) dh}{\int_0^{\alpha} \left(e^{-Mgh/RT}\right) dh} = \frac{RT}{Mg}$$

2.19 (a) We know that the variation of pressure with height of a fluid is given by :

$$dp = -\rho g\, dh$$

But from gas law $\quad p = \frac{\rho}{M} RT \quad$ or, $\quad \rho = \frac{pM}{RT}$

From these two Eqs.

$$dp = -\frac{p\,Mg}{RT}\, dh \qquad (1)$$

or, $$\frac{dp}{p} = \frac{-M\,g\,dh}{R\,T_0\,(1 - ah)}$$

Integrating, $$\int_{p_0}^{p} \frac{dp}{p} = \frac{-Mg}{R\,T_0} \int_0^{h} \frac{dh}{(1 - ah)},$$ we get

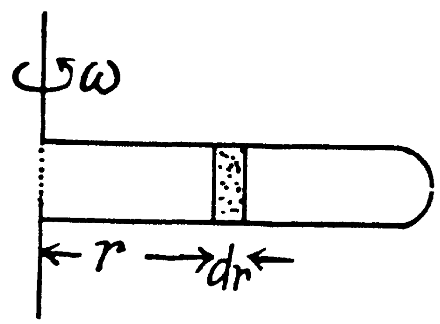
$$\ln \frac{p}{p_0} = \ln\,(1 - ah)^{Mg/aRT_0}$$

Hence, $\quad p = p_0\,(1 - ah)^{Mg/aRT_0}$. Obvionsly $h < \frac{1}{a}$

(b) Proceed up to Eq. (1) of part (a), and then put $T = T_0\,(1 + a\,h)$ and proceed further in the same fashion to get

$$p = \frac{p_0}{(1 + ah)^{Mg/aRT_0}}$$

2.20 Let us consider the mass element of the gas (thin layer) in the cylinder at a distance r from its open end as shown in the figure.



Using Newton's second law for the element

$$F_n = m w_n:$$

$$(p + dp)\,S - pS = (\rho\,S\,dr)\,\omega^2\,r$$

or, $$dp = \rho\,\omega^2\,r\,dr = \frac{pM}{RT}\,\omega^2\,r\,dr$$

So, $$\frac{dp}{p} = \frac{M\omega^2}{RT} r\,dr \quad \text{or,} \quad \int_{p_0}^{p} \frac{dp}{p} = \frac{M\omega^2}{RT} \int_0^r r\,dr,$$

Thus, $$\ln \frac{p}{p_0} = \frac{M\omega^2}{2RT} r^2 \quad \text{or,} \quad p = p_0\, e^{M\omega^2 r^2/2RT}$$

2.21 For an ideal gas law

$$p = \frac{\rho}{M} R\,T$$

So, $$p = 0{\cdot}082 \times 300 \times \frac{500}{44} \text{ atms} = 279{\cdot}5 \text{ atmosphere}$$

For Vander Waal gas Eq.

$$\left(p + \frac{\nu^2 a}{V^2}\right)(V - \nu\,b) = \nu\,R\,T, \quad \text{where } V = \nu V_M$$

or, $$p = \frac{\nu\,R\,T}{V - \nu\,b} - \frac{a\nu^2}{V^2} = \frac{m\,R\,T/M}{V - \dfrac{mb}{M}} - \frac{am^2}{V^2 M^2}$$

$$= \frac{\rho\,R\,T}{M - \rho\,b} - \frac{a\,\rho^2}{M^2} = 79{\cdot}2 \text{ atm}$$

2.22 (a) $$p = \left[\frac{R\,T}{V_M - b} - \frac{a}{V_M^2}\right](1 + \eta) = \frac{R\,T}{V_M}$$

(The pressure is less for a Vander Waal gas than for an ideal gas)

or, $$\frac{a\,(1+\eta)}{V_M^2} = R\,T\left[\frac{-1}{V_M} + \frac{1+\eta}{V_M - b}\right] = R\,T \frac{\eta\,V_M + b}{V_M\,(V_M - b)}$$

or, $$T = \frac{a\,(1+\eta)\,(V_M - b)}{R\,V_M(\eta\,V_M + b)},$$ (here V_M is the molar volume.)

$$= \frac{1{\cdot}35 \times 1{\cdot}1 \times (1 - 0{\cdot}039)}{0{\cdot}082 \times (0{\cdot}139)} \approx 125 \text{ K}$$

(b) The corresponding pressure is

$$p = \frac{RT}{V_M - b} - \frac{a}{V_M^2} = \frac{a\,(1+\eta)}{V_M(\eta\,V_M + b)} - \frac{a}{V_M^2}$$

$$= \frac{a}{V_M^2} \frac{(V_M + \eta\,V_M - \eta\,V_M - b)}{(\eta\,V_M + b)} = \frac{a}{V_M^2} \frac{(V_M - b)}{(V_M + b)}$$

$$= \frac{1{\cdot}35}{1} \times \frac{0{\cdot}961}{0{\cdot}139} \approx 9{\cdot}3 \text{ atm}$$

2.23 $p_1 = RT_1 \dfrac{1}{V-b} - \dfrac{a}{V^2}$, $p_2 = R\,T_2 \dfrac{1}{V-b} - \dfrac{a}{V^2}$

So,
$$p_2 - p_1 = \frac{R\,(T_2 - T_1)}{V-b}$$

or.
$$V - b = \frac{R\,(T_2 - T_1)}{p_2 - p_1} \quad \text{or,} \quad b = V - \frac{R\,(T_2 - T_1)}{p_2 - p_1}$$

Also,
$$p_1 = T_1 \frac{p_2 - p_1}{T_2 - T_1} - \frac{a}{V^2},$$

$$\frac{a}{V^2} = \frac{T_1\,(p_2 - p_1)}{T_2 - T_1} - p_1 = \frac{T_1 p_2 - p_1 T_2}{T_2 - T_1}$$

or,
$$a = V^2 \frac{T_1 p_2 - p_1 T_2}{T_2 - T_1}$$

Using $T_1 = 300\,\text{K}$, $p_1 = 90$ atms, $T_2 = 350\,\text{K}$, $p_2 = 110$ atm, $V = 0{\cdot}250$ litre

$$a = 1{\cdot}87\ \text{atm. litre}^2/\text{mole}^2, \quad b = 0{\cdot}045\ \text{litre/mole}$$

2.24 $p = \dfrac{R\,T}{V-b} - \dfrac{a}{V^2} - V\left(\dfrac{\partial p}{\partial V}\right)_T = \dfrac{R\,T\,V}{(V-b)^2} - \dfrac{2a}{V^2}$

or,
$$\kappa = \frac{-1}{V}\left(\frac{\partial V}{\partial p}\right)_T$$

$$= \left[\frac{R\,T\,V^3 - 2a\,(V-b)^2}{V^2\,(V-b)^2}\right]^{-1} = \frac{V^2\,(V-b)}{\left[RTV^3 - 2a\,(V-b)^2\right]}$$

2.25 For an ideal gas $\kappa_0 = \dfrac{V}{RT}$

Now $\kappa = \dfrac{(V-b)^2}{RTV}\left\{1 - \dfrac{2a\,(V-b)^2}{RTV^3}\right\}^{-1} = \kappa_0\left(1 - \dfrac{b}{V}\right)^2\left\{1 - \dfrac{2a}{RTV}\left(1 - \dfrac{b}{V}\right)^2\right\}^{-1}$

$$= \kappa_0\left\{1 - \frac{2b}{V} + \frac{2a}{RTV}\right\}, \text{ to leading order in } a,\ b$$

Now
$$\kappa > \kappa_0 \quad \text{if} \quad \frac{2a}{RTV} > \frac{2b}{V} \quad \text{or} \quad T < \frac{a}{bR}$$

If a, b do not vary much with temperature, then the effect at high temperature is clearly determined by b and its effect is repulsive so compressibility is less.

2.2 THE FIRST LAW OF THERMODYNAMICS. HEAT CAPACITY

2.26 Internal energy of air, treating as an ideal gas

$$U = \frac{m}{M} C_V T = \frac{m}{M} \frac{R}{\gamma - 1} T = \frac{pV}{\gamma - 1} \qquad (1)$$

Using $\quad C_V = \frac{R}{\gamma - 1}$, since $C_p - C_V = R$ and $\frac{C_p}{C_V} = \gamma$

Thus at constant pressure $U =$ constant, because the volume of the room is a constant.

Puting the value of $p = p_{atm}$ and V in Eq. (1), we get $U = 10$ MJ.

2.27 From energy conservation

$$U_i + \frac{1}{2}(\nu M) v^2 = U_f$$

or,
$$\Delta U = \frac{1}{2} \nu M v^2 \qquad (1)$$

But from $U = \nu \frac{RT}{\gamma - 1}$, $\quad \Delta U = \frac{\nu R}{\gamma - 1} \Delta T$ (from the previous problem) $\qquad$ (2)

Hence from Eqs. (1) and (2).

$$\Delta T = \frac{M v^2 (\gamma - 1)}{2R}$$

2.28 On opening the valve, the air will flow from the vessel at heigher pressure to the vessel at lower pressure till both vessels have the same air pressure. If this air pressure is p, the total volume of the air in the two vessels will be $(V_1 + V_2)$. Also if ν_1 and ν_2 be the number of moles of air initially in the two vessels, we have

$$p_1 V_1 = \nu_1 R T_1 \text{ and } p_2 V_2 = \nu_2 R T_2 \qquad (1)$$

After the air is mixed up, the total number of moles are $(\nu_1 + \nu_2)$ and the mixture is at temperature T.

Hence
$$p (V_1 + V_2) = (\nu_1 + \nu_2) RT \qquad (2)$$

Let us look at the two portions of air as one single system. Since this system is contained in a thermally insulated vessel, no heat exchange is involved in the process. That is, total heat transfer for the combined system $Q = 0$

Moreover, this combined system does not perform mechanical work either. The walls of the containers are rigid and there are no pistons etc to be pushed, looking at the total system, we know $A = 0$.

Hence, internal energy of the combined system does not change in the process. Initially energy of the combined system is equal to the sum of internal energies of the two portions of air :

$$U_i = U_1 + U_2 = \frac{\nu_1 R T_1}{\gamma - 1} + \frac{\nu_2 R T_2}{\gamma - 1} \qquad (3)$$

Final internal energy of $(n_1 + n_2)$ moles of air at temperature T is given by

$$U_f = \frac{(\nu_1 + \nu_2) RT}{\gamma - 1} \tag{4}$$

Therefore, $U_i = U_f$ implies :

$$T = \frac{\nu_1 T_1 + \nu_2 T_2}{\nu_1 + \nu_2} = \frac{p_1 V_1 + p_2 V_2}{(p_1 V_1/T_1) + (p_2 V_2/T_2)} = T_1 T_2 \frac{p_1 V_1 + p_2 V_2}{p_1 V_1 T_2 + p_2 V_2 T_2}$$

From (2), therefore, final pressure is given by :

$$p = \frac{\nu_1 + \nu_2}{V_1 + V_2} RT = \frac{R}{V_1 + V_2}(\nu_1 T_1 + \nu_2 T_2) = \frac{p_1 V_1 + p_2 V_2}{V_1 + V_2}$$

This process in an example of free adiabatic expansion of ideal gas.

2.29 By the first law of thermodynamics,

$$Q = \Delta U + A$$

Here $A = 0$, as the volume remains constant,

So, $$Q = \Delta U = \frac{\nu R}{\gamma - 1} \Delta T$$

From gas law, $$p_0 V = \nu R T_0$$

So, $$\Delta U = \frac{p_0 V \Delta T}{T_0 (\gamma - 1)} = -0{\cdot}25 \text{ kJ}$$

Hence amount of heat lost $= -\Delta U = 0.25$ kJ

2.30 By the first law of thermodynamics $Q = \Delta U + A$

But $$\Delta U = \frac{p \Delta V}{\gamma - 1} = \frac{A}{\gamma - 1} \text{ (as } p \text{ is constant)}$$

$$Q = \frac{A}{\gamma - 1} + A = \frac{\gamma \cdot A}{\gamma - 1} = \frac{1{\cdot}4}{1{\cdot}4 - 1} \times 2 = 7 \text{ J}$$

2.31 Under isobaric process $A = p\Delta V = R\,\Delta T$ (as $\nu = 1$) $= 0{\cdot}6$ kJ

From the first law of thermodynamics

$$\Delta U = Q - A = Q - R\,\Delta T = 1 \text{ kJ}$$

Again increment in internal energy $\Delta U = \dfrac{R\,\Delta T}{\gamma - 1}$, for $\nu = 1$

Thus $$Q - R\,\Delta T = \frac{R\,\Delta T}{\gamma - 1} \quad \text{or} \quad \gamma = \frac{Q}{Q - R\,\Delta T} = 1{\cdot}6$$

2.32 Let $\nu = 2$ moles of the gas. In the first phase, under isochoric process, $A_1 = 0$, therefore from gas law if pressure is reduced n times so that temperature i.e. new temperature becomes T_0/n.

Now from first law of thermodynamics

$$Q_1 = \Delta U_1 = \frac{\nu R\,\Delta T}{\gamma - 1}$$

$$= \frac{\nu R}{\gamma - 1}\left(\frac{T_o}{n} - T_0\right) = \frac{\nu R T_0 (1-n)}{n(\gamma - 1)}$$

During the second phase (under isobaric process),

$$A_2 = p\Delta V = \nu R\, \Delta T$$

Thus from first law of thermodynamics :

$$Q_2 = \Delta U_2 + A_2 = \frac{\nu R\, \Delta T}{\gamma - 1} + \nu R \Delta T$$

$$= \frac{\nu R\left(T_0 - \frac{T_0}{n}\right)\gamma}{\gamma - 1} = \frac{\nu R T_0 (n-1)\gamma}{n(\gamma - 1)}$$

Hence the total amount of heat absorbed

$$Q = Q_1 + Q_2 = \frac{\nu R T_0 (1-n)}{n(\gamma - 1)} + \frac{\nu R T_0 (n-1)\gamma}{n(\gamma - 1)}$$

$$= \frac{\nu R T_0 (n-1)\gamma}{n(\gamma - 1)}(-1+\gamma) = \nu R T_0\left(1 - \frac{1}{n}\right)$$

2.33 Total no. of moles of the mixture $\nu = \nu_1 + \nu_2$

At a certain temperature, $U = U_1 + U_2$ or $\nu C_V = \nu_1 C_{V_1} + \nu_2 C_{V_2}$

Thus $$C_V = \frac{\nu_1 C_{V_1} + \nu_2 C_{V_2}}{\nu} = \frac{\left(\nu_1 \frac{R}{\gamma_1 - 1} + \nu_2 \frac{R}{\gamma_2 - 1}\right)}{\nu}$$

Similarly $$C_p = \frac{\nu_1 C_{p_1} + \nu_2 C_{p_2}}{\nu}$$

$$= \frac{\nu_1 \gamma_1 C_{V_1} + \nu_2 \gamma_2 C_{V_2}}{\nu} = \frac{\left(\nu_1 \frac{\gamma_1 R}{\gamma_1 - 1} + \nu_2 \frac{\gamma_2 R}{\gamma_2 - 1}\right)}{\nu}$$

Thus $$\gamma = \frac{C_p}{C_V} = \frac{\nu_1 \frac{\gamma_1}{\gamma_1 - 1} R + \nu_2 \frac{\gamma_2}{\gamma_2 - 1} R}{\nu_1 \frac{R}{\gamma_1 - 1} + \nu_2 \frac{R}{\gamma_2 - 1}}$$

$$= \frac{\nu_1 \gamma_1 (\gamma_2 - 1) + \nu_2 \gamma_2 (\gamma_1 - 1)}{\nu_1 (\gamma_2 - 1) + \nu_2 (\gamma_1 - 1)}$$

2.34 From the previous problem

$$C_V = \frac{\nu_1 \frac{R}{\gamma_1 - 1} + \nu_2 \frac{R}{\gamma_2 - 1}}{\nu_1 + \nu_2} = 15{\cdot}2\ \text{J/mole.K}$$

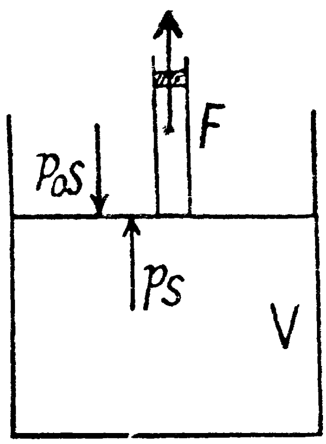
and
$$C_p = \frac{\nu_1 \frac{\gamma_1 R}{\gamma_1 - 1} + \nu_2 \frac{\gamma_2 R}{\gamma_2 - 1}}{\nu_1 + \nu_2} = 23{\cdot}85 \text{ J/mole. K}$$

Now molar mass of the mixture $(M) = \frac{\text{Total mass}}{\text{Total number of moles}} = \frac{20+7}{\frac{1}{2}+\frac{1}{4}} = 36$

Hence
$$c_V = \frac{C_V}{M} = 0{\cdot}42 \text{ J/g}\cdot\text{K and } c_p = \frac{C_p}{M} = 0{\cdot}66 \text{ J/g}\cdot\text{K}$$

2.35 Let S be the area of the piston and F be the force exerted by the external agent. Then, $F + pS = p_0 S$ (Fig.) at an arbitrary instant of time. Here p is the pressure at the instant the volume is V. (Initially the pressure inside is p_0)

$$A \text{ (Work done by the agent)} = \int_{V_0}^{\eta V_0} F\,dx$$



$$= \int_{V_0}^{\eta V_0} (p_0 - p) S \cdot dx = \int_{V_0}^{\eta V_0} (p_0 - p)\, dV$$

$$= p_0(\eta - 1) V_0 - \int_{V_0}^{\eta V_0} p dV = p_0 (\eta - 1) V_0 - \int_{V_0}^{\eta V_0} \nu RT \cdot \frac{dV}{V}$$

$$= (\eta - 1) p_0 V_0 - nRT \ln \eta = (\eta - 1)\nu RT - \nu RT \ln \eta$$

$$= \nu RT (\eta - 1 - \ln \eta) = RT(\eta - 1 - \ln \eta) \text{ (For } \nu = 1 \text{ mole)}$$

2.36 Let the agent move the piston to the right by x. In equilibirium position,

$$p_1 S + F_{agent} = p_2 S, \quad \text{or,} \quad F_{agent} = (p_2 - p_1) S$$

Work done by the agent in an infinitesmal change dx is

$$F_{agent} \cdot dx = (p_2 - p_1) Sdx = (p_2 - p_1)\, dV$$

By applying $pV =$ constant, for the two parts,

$$p_1 (V_0 + Sx) = p_0 V_0 \text{ and } p_2 (V_0 - Sx) = p_0 V_0$$

So,
$$p_2 - p_1 = \frac{p_0 V_0 2Sx}{V_0^2 - S^2 x^2} = \frac{2 p_0 V_0 V}{V_0^2 - V^2} \text{ (where } Sx = V)$$

When the volume of the left end is η times the volume of the right end

$$(V_0 + V) = \eta (V_0 - V), \quad \text{or,} \quad V = \frac{\eta - 1}{\eta + 1} V_0$$

$$A = \int_0^V (p_2 - p_1)\, dV = \int_0^V \frac{2p_0 V_0 V}{V_0^2 - V^2}\, dV = -p_0 V_0 \Big[\ln (V_0^2 - V^2) \Big]_0^V$$

$$= -p_0 V_0 \Big[\ln (V_0^2 - V^2) - \ln V_0^2 \Big]$$

$$= -p_0 V_0 \left[\ln \left\{ V_0^2 - \left(\frac{\eta - 1}{\eta + 1^2} \right) V_0^2 \right\} - \ln V_0^2 \right]$$

$$= -p_0 V_0 \left(\ln \frac{4\eta}{(\eta + 1)^2} \right) = p_0 V_0 \ln \frac{(\eta + 1)^2}{4\eta}$$

2.37 In the isothermal process, heat transfer to the gas is given by

$$Q_1 = \nu R T_0 \ln \frac{V_2}{V_1} = \nu R T_0 \ln \eta \quad \left(\text{For } \eta = \frac{V_2}{V_1} = \frac{p_1}{p_2} \right)$$

In the isochoric process, $A = 0$

Thus heat transfer to the gas is given by

$$Q_2 = \Delta U = \nu C_V \Delta T = \frac{\nu R}{\gamma - 1} \Delta T \quad \left(\text{for } C_V = \frac{R}{\gamma - 1} \right)$$

But $\quad \frac{p_2}{p_1} = \frac{T_0}{T}$, or, $T = T_0 \frac{p_1}{p_2} = \eta T_0 \quad \left(\text{for } \eta = \frac{p_1}{p_2} \right)$

or, $\quad \Delta T = \eta T_0 - T_0 = (\eta - 1) T_0$, so, $Q_2 = \frac{\nu R}{\gamma - 1} \cdot (\eta - 1) T_0$

Thus, net heat transfer to the gas

$$Q = \nu R T_0 \ln \eta + \frac{\nu R}{\gamma - 1} \cdot (\eta - 1) T_0$$

or, $\quad \frac{Q}{\nu R T_0} = \ln \eta + \frac{\eta - 1}{\gamma - 1}$, or, $\frac{Q}{\nu R T_0} - \ln \eta = \frac{\eta - 1}{\gamma - 1}$

or, $\quad \gamma = 1 + \frac{\eta - 1}{\frac{Q}{\nu R T_0} - \ln \eta} = 1 + \frac{6 - 1}{\left(\frac{80 \times 10^3}{3 \times 8{\cdot}314 \times 273} \right) - \ln 6} = 1{\cdot}4$

2.38 (a) From ideal gas law $p = \left(\frac{\nu R}{V} \right) T = kT \left(\text{where } k = \frac{\nu R}{V} \right)$

For isochoric process, obviously $k =$ constant, thus $p = kT$, represents a straight line passing through the origin and its slope becomes k.

For isobaric process $p =$ constant, thus on $p - T$ curve, it is a horizontal straight line parallel to $T -$ axis, if T is along horizontal (or $x -$ axis)

For isothermal process, $T =$ constant, thus on $p - T$ curve, it represents a vertical straight line if T is taken along horizontal (or $x -$ axis)

For adiabatic process $T^\gamma p^{1-\gamma} =$ constant

After diffrentiating, we get $(1 - \gamma) p^{-\gamma} dp \cdot T^\gamma + \gamma p^{1-\gamma} \cdot T^{\gamma - 1} \cdot dT = 0$

$$\frac{dp}{dT} = \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{p^{1-\gamma}}{p^{-\gamma}}\right)\left(\frac{T^{\gamma-1}}{T^{\gamma}}\right) = \left(\frac{\gamma}{\gamma-1}\right)\frac{p}{T}$$

The approximate plots of isochoric, isobaric, isothermal, and adiabatic processess are drawn in the answersheet.

(b) As p is not considered as variable, we have from ideal gas law

$$V = \frac{\nu R}{p} T = k' T \left(\text{where } k' = \frac{\nu R}{p}\right)$$

On $V - T$ co-ordinate system let us, take T along x – axis.

For isochoric process $V =$ constant, thus $k' =$ constant and $V = k'T$ obviously represents a straight line pasing through the origin of the co-ordinate system and k' is its slope.

For isothermal process $T =$ constant. Thus on the stated co- ordinate system it represents a straight line parallel to the V – axis.

For adiabatic process $TV^{\gamma-1} =$ constant

After differentiating, we get $(\gamma - 1)\, V^{\gamma-2}\, dV \cdot T + V^{\gamma-1}\, dT = 0$

$$\frac{dV}{dT} = -\left(\frac{1}{\gamma-1}\right)\cdot\frac{V}{T}$$

The approximate plots of isochoric, isobaric, isothermal and adiabatic processess are drawn in the answer sheet.

2.39 According to $T - p$ relation in adiabatic process, $T^{\gamma} = kp^{\gamma-1}$ (where $k =$ constant)

and $\left(\frac{T_2}{T_1}\right)^{\gamma} = \left(\frac{p_2}{p_1}\right)^{\gamma-1}$ So, $\frac{T^{\gamma}}{T_0^{\gamma}} = \eta^{\gamma-1}\left(\text{for } \eta = \frac{p_2}{p_1}\right)$

Hence $T = T_0 \cdot \eta^{\frac{\gamma-1}{\gamma}} = 290 \times 10^{(1\cdot4-1)/1\cdot4} = 0{\cdot}56 \text{ kK}$

(b) Using the solution of part (a), sought work done

$$A = \frac{\nu R\, \Delta T}{\gamma-1} = \frac{\nu R T_0}{\gamma-1}\left(\eta^{(\gamma-1)/\gamma} - 1\right) = 5{\cdot}61 \text{ kJ} \quad \text{(on substitution)}$$

2.40 Let (p_0, V_0, T_0) be the initial state of the gas.

We know $A_{adia} = \frac{-\nu R \Delta T}{\gamma - 1}$ (work done by the gas)

But from the equation $TV^{\gamma-1} =$ constant, we get $\Delta T = T_0\left(\eta^{\gamma-1} - 1\right)$

Thus $$A_{\text{adia}} = \frac{-\nu R T_0\left(\eta^{\gamma-1} - 1\right)}{\gamma-1}$$

On the other hand, we know $A_{iso} = \nu R T_0 \ln\left(\frac{1}{\eta}\right) = -\nu R\, T_0 \ln \eta$ (work done by the gas)

Thus $$\frac{A_{\text{adia}}}{A_{\text{iso}}} = \frac{\eta^{\gamma-1} - 1}{(\gamma-1)\ln\eta} = \frac{5^{0\cdot4} - 1}{0{\cdot}4 \times \ln 5} = 1{\cdot}4$$

2.41 Since here the piston is conducting and it is moved slowly the temperature on the two sides increases and maintained at the same value.

Elementary work done by the agent = Work done in compression - Work done in expansion i.e. $dA = p_2\,dV - p_1\,dV = (p_2 - p_1)\,dV$

where p_1 and p_2 are pressures at any instant of the gas on expansion and compression side respectively.

From the gas law $p_1(V_0 + Sx) = \nu RT$ and $p_2(V_0 - Sx) = \nu RT$, for each section

(x is the displacement of the piston towards section 2)

So, $$p_2 - p_1 = \nu RT\frac{2\,Sx}{V_0^2 - S^2x^2} = \nu RT\cdot\frac{2V}{V_0^2 - V^2}\ (\text{as } Sx = V)$$

So $$dA = \nu RT\frac{2V}{V_0^2 - V^2}\,dV$$

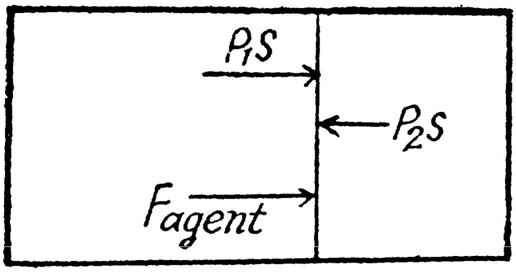
Also, from the first law of thermodynamics

$$dA = -dU = -2\nu\frac{R}{\gamma - 1}\,dT \quad (\text{as } dQ = 0)$$

So, work done on the gas $= -dA = 2\nu\cdot\dfrac{R}{\gamma - 1}\,dT$

Thus $2\nu\dfrac{R}{\gamma - 1}\,dT = \nu RT\dfrac{2V\cdot dV}{V_0^2 - V^2}$,

or, $$\frac{dT}{T} = \gamma - 1\,\frac{V\,dV}{V_0^2 - V^2}$$



When the left end is η times the volume of the right end.

$$(V_0 + V) = \eta\,(V_0 - V) \quad \text{or} \quad V = \frac{\eta - 1}{\eta + 1}V_0$$

On integrating $$\int_{T_0}^{T}\frac{dT}{T} = (\gamma - 1)\int_0^V\frac{V\,dV}{V_0^2 - V^2}$$

or $$\ln\frac{T}{T_0} = (\gamma - 1)\left[-\frac{1}{2}\ln(V_0^2 - V^2)\right]_0^V$$

$$= -\frac{\gamma - 1}{2}\left[\ln(V_0^2 - V^2) - \ln V_0^2 - V^2) - \ln V_0^2\right]$$

$$= \frac{\gamma - 1}{2}\left[\ln V_0^2 - \ln V_0^2\left\{1 - \left(\frac{\eta - 1}{\eta + 1}\right)^2\right\}\right] = \frac{\gamma - 1}{2}\ln\frac{(\eta + 1)^2}{4\,\eta}$$

Hence $$T = T_0\left(\frac{(\eta + 1)^2}{4\eta}\right)^{\frac{\gamma - 1}{2}}$$

2.42 From energy conservation as in the derivation of Bernoullis theorem it reads

$$\frac{p}{\rho}+\frac{1}{2}v^2+gz+u+Q_d = \text{constant} \tag{1}$$

In the Eq. (1) u is the internal energy per unit mass and in this case is the thermal energy per unit mass of the gas. As the gas vessel is thermally insulated $Q_d = 0$, also in our case.

Just inside the vessel $u = \frac{C_V T}{M} = \frac{RT}{M(\gamma-1)}$ also $\frac{p}{\rho} = \frac{RT}{M}$ · Inside the vessel $v = 0$ also. Just outside $p = 0$, and $u = 0$. Ingeneral gz is not very significant for gases.

Thus applying Eq. (1) just inside and outside the hole, we get

$$\frac{1}{2}v^2 = \frac{p}{\rho}+u$$

$$= \frac{RT}{M}+\frac{RT}{M(\gamma-1)} = \frac{\gamma RT}{M(\gamma-1)}$$

Hence $\quad v^2 = \frac{2\gamma RT}{M(\gamma-1)}$ or, $v = \sqrt{\frac{2\gamma RT}{M(\gamma-1)}} = 3.22$ km/s.

Note : The velocity here is the velocity of hydrodynamic flow of the gas into vaccum. This requires that the diameter of the hole is not too small ($D >$ mean free path l). In the opposite case ($D << l$) the flow is called effusion. Then the above result does not apply and kinetic theory methods are needed.

2.43 The differential work done by the gas

$$dA = pdV = \frac{\nu RT^2}{a}\left(-\frac{a}{T^2}\right)dT = -\nu RdT$$

$$\left(\text{as } pV = \nu RT \text{ and } V = \frac{a}{T}\right)$$

So, $$A = -\int_T^{T+\Delta T} \nu RdT = -\nu R\Delta T$$

From the first law of thermodynamics

$$Q = \Delta U + A = \frac{\nu R}{\gamma-1}\Delta T - \nu R\Delta T$$

$$= \nu R\Delta T\cdot\frac{2-\gamma}{\gamma-1} = R\,\Delta T\cdot\frac{2-\gamma}{\gamma-1} \text{ (for } \nu = 1 \text{ mole)}$$

2.44 According to the problem : $A \propto U$ or $dA = aU$ (where a is proportionality constant)

or, $$pdV = \frac{a\,\nu R\,dT}{\gamma-1} \tag{1}$$

From ideal gas law, $pV = \nu RT$, on differentiating

$$pdV + Vdp = \nu RdT \tag{2}$$

Thus from (1) and (2)

$$pdV = \frac{a}{\gamma - 1}(pdV + Vdp)$$

$$\text{or, } pdV\left(\frac{a}{\gamma - 1} - 1\right) + \frac{a}{\gamma - 1} V\,dp = 0$$

or, $\quad pdV(k-1) + kVdp = 0$ (where $k = \frac{a}{\gamma - 1} =$ another constant)

or, $\quad pdV\frac{k-1}{k} + Vdp = 0$

or, $\quad pdVn + Vdp = 0$ (where $\frac{k-1}{k} = n =$ ratio)

Dividing both the sides by pV

$$n\frac{dV}{V} + \frac{dp}{p} = 0$$

On integrating $n \ln V + \ln p = \ln C$ (where C is constant)

or, $\qquad \ln(pV^n) = \ln C \quad$ or, $\quad pV^n = C$ (const.)

2.45 In the polytropic process work done by the gas

$$A = \frac{\nu R\,[T_i - T_f]}{n - 1}$$

(where T_i and T_f are initial and final temperature of the gas like in adiabatic process)

and $$\Delta U = \frac{\nu R}{\gamma - 1}(T_f - T_i)$$

By the first law of thermodynamics $Q = \Delta U + A$

$$= \frac{\nu R}{\gamma - 1}(T_f - T_i) + \frac{\nu R}{n - 1}(T_i - T_f)$$

$$= (T_f - T_i)\,\nu R\left[\frac{1}{\gamma - 1} - \frac{1}{n - 1}\right] = \frac{\nu R\,[n - \gamma]}{(n-1)(\gamma - 1)}\Delta T$$

According to definition of molar heat capacity when number of moles $\nu = 1$ and $\Delta T = 1$ then $Q =$ Molar heat capacity.

Here, $$C_n = \frac{R(n - \gamma)}{(n-1)(\gamma - 1)} < 0 \quad \text{for} \quad 1 < n < \gamma$$

2.46 Let the process be polytropic according to the law $pV^n =$ constant

Thus, $$p_f V_f^n = p_i V_i^n \quad \text{or,} \quad \left(\frac{p_i}{p_f}\right) = \beta$$

So, $$\alpha^n = \beta \quad \text{or} \quad \ln\beta = n\ln\alpha \quad \text{or} \quad n = \frac{\ln\beta}{\ln\alpha}$$

In the polytropic process molar heat capacity is given by

$$C_n = \frac{R(n-\gamma)}{(n-1)(\gamma-1)} = \frac{R}{\gamma-1} - \frac{R}{n-1}$$

$$= \frac{R}{\gamma-1} - \frac{R \ln \alpha}{\ln \beta - \ln \alpha}, \quad \text{where } n = \frac{\ln \beta}{\ln \alpha}$$

So, $$C_n = \frac{8{\cdot}314}{1{\cdot}66-1} - \frac{8{\cdot}314 \ln 4}{\ln 8 - \ln 4} = -42 \text{ J/mol.K}$$

2.47 (a) Increment of internal energy for ΔT, becomes

$$\Delta U = \frac{\nu R \Delta T}{\gamma - 1} = \frac{R\,\Delta T}{\gamma-1} = -324 \text{ J (as } \nu = 1 \text{ mole)}$$

From first law of thermodynamics

$$Q = \Delta U + A = \frac{R\,\Delta T}{\gamma-1} - \frac{R\Delta T}{n-1} = 0{\cdot}11 \text{ kJ}$$

(b) Sought work done, $A_n = \int p dV = \int_{V_i}^{V_f} \frac{k}{V^n} dV$

(where $pV^n = k = p_i V_i^n = p_f V_f^n$)

$$= \frac{k}{1-n}\left(V_f^{1-n} - V_i^{1-n}\right) = \frac{\left(p_f V_f^n V_f^{1-n} - p_i V_i^n V_i^{1-n}\right)}{1-n}$$

$$= \frac{p_f V_f - p_i V_i}{1-n} = \frac{\nu R (T_f - T_i)}{1-n}$$

$$= \frac{\nu R \Delta T}{n-1} = -\frac{R\Delta T}{n-1} = 0{\cdot}43 \text{ kJ (as } \nu = 1 \text{ mole)}$$

2.48 Law of the process is $p = \alpha V$ or $pV^{-1} = \alpha$

so the process is polytropic of index $n = -1$

As $p = \alpha V$ so, $p_i = \alpha V_0$ and $p_f = \alpha \eta V_0$

(a) Increment of the internal energy is given by

$$\Delta U = \frac{\nu R}{\gamma-1}[T_f - T_i] = \frac{p_f V_f - p_i V_i}{\gamma - 1}$$

(b) Work done by the gas is given by

$$A = \frac{p_i V_i - p_f V_f}{n-1} = \frac{\alpha V_0^2 - \alpha\eta V_0 \cdot \eta V_0}{-1-1}$$

$$= \frac{\alpha V_0^2 (1-\eta^2)}{-2} = \frac{1}{2}\alpha V_0^2 (\eta^2 - 1)$$

(c) Molar heat capacity is given by

$$C_n = \frac{R(n-\gamma)}{(n-1)(\gamma-1)} = \frac{R(-1-\gamma)}{(-1-1)(\gamma-1)} = \frac{R}{2}\frac{\gamma+1}{\gamma-1}$$

2.49 (a) $\Delta U = \frac{\nu R}{\gamma - 1} \Delta T$ and $Q = \nu C_n \Delta T$

where C_n is the molar heat capacity in the process. It is given that $Q = -\Delta U$

So, $$C_n \Delta T = \frac{R}{\gamma - 1} \Delta T, \quad \text{or} \quad C_n = -\frac{R}{\gamma - 1}$$

(b) By the first law of thermodynamics, $dQ = dU + dA$,

or, $$2\,dQ = dA \text{ (as } dQ = -dU)$$

$$2\nu C_n\, dT = p dV, \text{ or, } \frac{2R\nu}{\gamma - 1} dT + p dV = 0$$

So, $$\frac{2RV}{\gamma - 1} dT + \frac{\nu RT}{V} dV = 0, \quad \text{or,} \quad \frac{2}{(\gamma - 1)} \frac{dT}{T} + \frac{dV}{V} = 0$$

or, $$\frac{dT}{T} + \frac{\gamma - 1}{2} \frac{dV}{V} = 0, \quad \text{or,} \quad TV^{(\gamma - 1)/2} = \text{constant.}$$

(c) We know $C_n = \frac{(n - \gamma) R}{(n - 1)(\gamma - 1)}$

But from part (a), we have $C_n = -\frac{R}{\gamma - 1}$

Thus $$-\frac{R}{\gamma - 1} = \frac{(n - \gamma) R}{(n - 1)(\gamma - 1)} \text{ which yields}$$

$$n = \frac{1 + \gamma}{2}$$

From part (b); we know $TV^{(\gamma - 1)/2} = \text{constant}$

So, $$\frac{T_o}{T} = \left(\frac{V}{V_0}\right)^{(\gamma - 1)/2} = \eta^{(\gamma - 1)/2} \text{ (where } T \text{ is the final temperature)}$$

Work done by the gas for one mole is given by

$$A = R \frac{(T_0 - T)}{n - 1} = \frac{2RT_0 [1 - \eta^{(1-\gamma)/2}]}{\gamma - 1}$$

2.50 Given $p = a T^{\alpha}$ (for one mole of gas)

So, $$pT^{-\alpha} = a \quad \text{or,} \quad p \left(\frac{pV}{R}\right)^{-\alpha} = a,$$

or, $$p^{1-\alpha} V^{-\alpha} = aR^{-\alpha} \quad \text{or,} \quad pV^{\alpha/(\alpha - 1)} = \text{constant}$$

Here polytropic exponent $n = \frac{\alpha}{\alpha - 1}$

(a) In the polytropic process for one mole of gas :

$$A = \frac{R\Delta T}{1 - n} = \frac{R\Delta T}{\left(1 - \frac{\alpha}{\alpha - 1}\right)} = R\Delta T (1 - \alpha)$$

(b) Molar heat capacity is given by

$$C = \frac{R}{\gamma - 1} - \frac{R}{n - 1} = \frac{R}{\gamma - 1} - \frac{R}{\frac{\alpha}{\alpha - 1} - 1} = \frac{R}{\gamma - 1} + R(1 - \alpha)$$

2.51 Given $U = aV^{\alpha}$

or, $$\nu C_V T = a\,V^{\alpha}, \quad \text{or,} \quad \nu C_V \frac{pV}{\nu R} = a\,V^{\alpha}$$

or, $$aV^{\alpha} \cdot \frac{R}{C_V} \cdot \frac{1}{pV} = 1, \quad \text{or,} \quad V^{\alpha - 1} \cdot p^{-1} = \frac{C_V}{Ra}$$

or $$pV^{1-\alpha} = \frac{Ra}{C_V} = \text{constant} = a\,(\gamma - 1) \left[\text{as } C_V = \frac{R}{\gamma - 1} \right]$$

So polytropric index $n = 1 - \alpha$.

(a) Work done by the gas is given by

$$A = \frac{-\nu R\,\Delta T}{n-1} \text{ and } \Delta U = \frac{\nu\,R\Delta T}{\gamma - 1}$$

Hence $$A = \frac{-\Delta U\,(\gamma - 1)}{n-1} = \frac{\Delta U\,(\gamma - 1)}{\alpha} \quad (\text{as } n = 1 - \alpha)$$

By the first law of thermodynamics, $Q = \Delta U + A$

$$= \Delta U + \frac{\Delta U\,(\gamma - 1)}{\alpha} = \Delta U \left[1 + \frac{\gamma - 1}{\alpha} \right]$$

(b) Molar heat capacity is given by

$$C = \frac{R}{\gamma - 1} - \frac{R}{n-1} = \frac{R}{\gamma - 1} - \frac{R}{1 - \alpha - 1}$$

$$= \frac{R}{\gamma - 1} + \frac{R}{\alpha} \quad (\text{as } n = 1 - \alpha)$$

2.52 (a) By the first law of thermodynamics

$$dQ = dU + dA = \nu C_V dT + pdV$$

Molar specific heat according to definition

$$C = \frac{dQ}{\nu dT} = \frac{C_V dT + pdV}{\nu dT}$$

$$= \frac{\nu C_V dT + \frac{\nu RT}{V} dV}{\nu dT} = C_V + \frac{RT}{V} \frac{dV}{dT},$$

We have $$T = T_0\,e^{\alpha V}$$

After differentiating, we get $dT = \alpha\,T_0\,e^{\alpha\,V} \cdot dV$

So, $$\frac{dV}{DT} = \frac{1}{\alpha\,T_0 e^{\alpha\,V}},$$

Hence $$C = C_V + \frac{RT}{V} \cdot \frac{1}{\alpha\,T_0 e^{\alpha V}} = C_V + \frac{RT_0\,e^{\alpha\,V}}{\alpha\,VT_0\,e^{\alpha\,v}} = C_V + \frac{R}{\alpha\,V}$$

(b) Process is $p = p_0\,e^{\alpha\,V}$

$$p = \frac{RT}{V} = p_0\,e^{\alpha V}$$

$$\text{or,}\quad T = \frac{p_0}{R} e^{\alpha V} \cdot V$$

So, $$C = C_V + \frac{RT}{V}\frac{dV}{dT} = C_V + p_0 e^{\alpha V} \cdot \frac{R}{p_0 e^{\alpha V}(1+\alpha V)} = C_V + \frac{R}{1+\alpha V}$$

2.53 Using 2.52

(a) $$C = C_V + \frac{RT}{V}\frac{dV}{dT} = C_V + \frac{p dV}{dT}\ \text{(for one mole of gas)}$$

We have $p = p_0 + \frac{\alpha}{V}$, or, $\frac{RT}{V} = p_0 + \frac{\alpha}{V}$, or, $RT = p_0 V + \alpha$

Therefore $\quad RdT = p_0\, dV$, So, $\frac{dV}{dT} = \frac{R}{p_0}$

Hence $$C = C_V + \left(p_0 + \frac{\alpha}{V}\right) \cdot \frac{R}{p_0} = \frac{R}{\gamma - 1} + \left(1 + \frac{\alpha}{p_0 V}\right) R$$

$$= \left(R + \frac{R}{\gamma - 1}\right) + \frac{\alpha R}{p_0 V} = \frac{\gamma R}{\gamma - 1} + \frac{\alpha R}{p_0 V}$$

(b) Work done is given by

$$A = \int_{V_1}^{V_2} \left(p_0 + \frac{\alpha}{V}\right) dV = p_0 (V_2 - V_1) + \alpha \ln \frac{V_2}{V_1}$$

$$\Delta U = C_V (T_2 - T_1) = C_V \left(\frac{p_2 V_2}{R} - \frac{p_1 V_1}{R}\right)\ \text{(for one mole)}$$

$$= \frac{R}{(\gamma - 1) R} (p_2 V_2 - p_1 V_1)$$

$$= \frac{1}{\gamma - 1}\left[\left(p_0 + \alpha V_2\right) V_2 - \left(p_0 + \frac{\alpha}{V_1}\right) V_1\right] = \frac{p_0 (V_2 - V_1)}{\gamma - 1}$$

By the first law of thermodynamics $Q = \Delta U + A$

$$= p_0 (V_2 - V_1) + \alpha \ln \frac{V_2}{V_1} + \frac{p_0 (V_2 - V_1)}{(\gamma - 1)}$$

$$= \frac{\gamma p_0 (V_2 - V_1)}{\gamma - 1} + \alpha \ln \frac{V_2}{V_1}$$

2.54 (a) Heat capacity is given by

$$C = C_V + \frac{RT}{V}\frac{dV}{dT}\ \text{(see solution of 2.52)}$$

We have $\quad T = T_0 + \alpha V$ or, $V = \frac{T}{\alpha} - \frac{T_0}{\alpha}$

After differentiating, we get, $\frac{dV}{dT} = \frac{1}{\alpha}$

Hence $$C = C_V + \frac{RT}{V}\cdot\frac{1}{\alpha} = \frac{R}{\gamma-1} + \frac{R\,(T_0+\alpha V)}{V}\cdot\frac{1}{\alpha}$$

$$= \frac{R}{\gamma-1} + R\left(\frac{T_0}{\alpha V}+1\right) = \frac{\gamma R}{\gamma-1} + \frac{RT_0}{\alpha V} = C_V + \frac{RT}{\alpha V} = C_p + \frac{RT_0}{\alpha V}$$

(b) Given $T = T_0 + \alpha V$

As $T = \frac{pV}{R}$ for one mole of gas

$$p = \frac{R}{V}(T_0+\alpha V) = \frac{RT}{V} = \alpha R$$

Now $$A = \int_{V_1}^{V_2} p dV = \int_{V_1}^{V_2}\left(\frac{RT_0}{v} + \alpha R\right) dV \text{ (for one mole)}$$

$$= RT_0 \ln\frac{V_2}{V_1} + \alpha\,(V_2 - V_1)$$

$$\Delta U = C_V(T_2 - T_1)$$

$$= C_V[T_0 + \alpha V_2 - T_0\,\alpha V_1] = \alpha C_V(V_2 - V_1)$$

By the first law of thermodynamics $Q = \Delta U + A$

$$= \frac{\alpha R}{\gamma-1}(V_2 - V_1) + RT_0\ln\frac{V_2}{V_1} + \alpha R\,(V_2 - V_1)$$

$$= \alpha R\,(V_2 - V_1)\left[1 + \frac{1}{\gamma-1}\right] + RT_0\ln\frac{V_2}{V_1}$$

$$= \alpha\,C_p\,(V_2 - V_1 + RT_0\ln\frac{V_2}{V_1}$$

$$= \alpha\,C_p\,(V_2 - V_1) + RT_0\ln\frac{V_2}{V_1}$$

2.55 Heat capacity is given by $C = C_V + \frac{RT}{V}\frac{dV}{dT}$

(a) Given $C = C_V + \alpha T$

So, $C_V + \alpha T = C_V + \frac{RT}{V}\frac{dV}{dT}$ or, $\frac{\alpha}{R}dT = \frac{dV}{V}$

Integrating both sides, we get $\frac{\alpha}{R}T = \ln V + \ln C_0 = \text{In } VC_0, C_0$ is a constant.

Or, $$V\cdot C_0 = e^{\alpha T/R} \quad\text{or}\quad V\cdot e^{\alpha T/R} = \frac{1}{C_0} = \text{constant}$$

(b) $C = C_V + \beta V$

and $$C = C_V + \frac{RT}{V}\frac{dV}{dT} \text{ so, } C_V \frac{RT}{V}\frac{dV}{dT} = C_V + \beta V$$

or, $$\frac{RT}{V}\frac{dV}{dT} = \beta V \text{ or, } \frac{dV}{V^2} = \frac{\beta}{R}\frac{dT}{T} \text{ or, } V^{-2} = \frac{dT}{T}$$

Integrating both sides, we get $\frac{R}{\beta}\frac{V^{-1}}{\beta - 1} = \ln T + \ln C_0 = \ln T{\cdot}C_0$

So, $\ln T \cdot C_0 = -\frac{R}{\beta V}$ $T \cdot C_0 = e^{-R/\beta V}$ or, $T e^{-R/\beta V} = \frac{1}{C_0} =$ constant

(c) $C = C_V + ap$ and $C = C_V + \frac{RT}{V}\frac{dV}{dT}$

So, $$C_V + ap = C_V + \frac{RT}{V}\frac{dV}{dT} \text{ so, } ap = \frac{RT}{V}\frac{dV}{dT}$$

or, $$a\frac{RT}{V} = \frac{RT}{V}\frac{dV}{dT} \text{ (as } p = \frac{RT}{V} \text{ for one mole of gas)}$$

or, $$\frac{dV}{dT} = a \text{ or, } dV = adT \text{ or, } dT = \frac{dV}{a}$$

So,$T = \frac{V}{a} +$ constant or $V - aT =$ constant

2.56 (a) By the first law of thermodynamics $A = Q - \Delta U$

or, $$= CdT - C_V dT = (C - C_V)\, dT \text{ (for one mole)}$$

Given $$C = \frac{\alpha}{T}$$

So, $$A = \int_{T_0}^{\eta T_0} \left(\frac{\alpha}{T} - C_V\right) dT = \alpha \ln \frac{\eta T_0}{T_0} - C_V (\eta T_0 - T_0)$$

$$= \alpha \ln \eta - C_V T_0 (\eta - 1) = \alpha \ln \eta + \frac{RT}{\gamma - 1}(\eta - 1)$$

(b) $C = +\frac{dQ}{dT} = \frac{RT}{V}\frac{dV}{dT} + C_V$

Given $$C = \frac{\alpha}{T}, \text{ so } C_V + \frac{RT}{V}\frac{dV}{dT} = \frac{\alpha}{T}$$

or, $$\frac{R}{\gamma - 1}\frac{1}{RT} + \frac{dV}{V} = \frac{\alpha}{RT^2} dT$$

or, $$\frac{dV}{V} = \frac{\alpha}{RT^2} dT - \frac{1}{\gamma - 1} \cdot \frac{dT}{T}$$

or, $$(\gamma - 1)\frac{dV}{V} = \frac{\alpha(\gamma - 1)}{RT^2} dT - \frac{dT}{T}$$

Integrating both sides, we get

or, $$(\gamma - 1)\ln V = -\frac{\alpha(\gamma-1)}{RT} - \ln T + \ln K$$

or, $$\ln V^{\gamma-1}\frac{T}{K} = \frac{-\alpha(\gamma-1)}{RT}$$

$$\ln V^{\gamma-1}\cdot\frac{pV}{RK} = \frac{-\alpha(\gamma-1)}{pV}$$

or, $$\frac{pV^{\gamma}}{RK} = e^{-\alpha(\gamma-1)/pV}$$

or, $$pV^{\gamma}e^{\alpha(\gamma-1)/pV} = RK = \text{constant}$$

2.57 The work done is

$$A = \int_{V_1}^{V_2} p\,dV = \int_{V_1}^{V_2}\left(\frac{RT}{v-b} - \frac{a}{V^2}\right)dV$$

$$= RT\ln\frac{V_2-b}{V_1-b} + a\left(\frac{1}{V_2} - \frac{1}{V_1}\right)$$

2.58 (a) The increment in the internal energy is

$$\Delta U = \int_{V_1}^{V_1}\left(\frac{\partial U}{\partial V}\right)_T dV$$

But from second law

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - p = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

On the other hand $$p = \frac{RT}{V-b} - \frac{a}{V^2}$$

or, $$T\left(\frac{\partial p}{\partial T}\right)_V = \frac{RT}{V-b} \text{ and } \left(\frac{\partial U}{\partial V}\right)_t = \frac{a}{V^2}$$

So, $$\Delta U = a\left(\frac{1}{V_1} - \frac{1}{V_2}\right)$$

(b) From the first law

$$Q = A + \Delta U = RT\ln\frac{V_2-b}{V_1-b}$$

2.59 (a) From the first law for an adiabatic

$$dQ = dU + p\,dV = 0$$

From the previous problem

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV = C_V dT + \frac{a}{V^2}dV$$

So, $$0 = C_V dT + \frac{RT\,dV}{V-b}$$

This equation can be integrated if we assume that C_V and b are constant then

$$\frac{R}{C_V}\frac{dV}{V-b}+\frac{dT}{T}=0, \quad \text{or,} \quad \ln T+\frac{R}{C_V}\ln(V-b)=\text{constant}$$

or,
$$T(V-b)^{R/C_V}=\text{constant}$$

(b) We use

$$dU=C_V\,dT+\frac{a}{V^2}dV$$

Now,
$$dQ=C_V\,dT+\frac{RT}{V-b}dV$$

So along constant p,
$$C_p=C_V+\frac{RT}{V-b}\left(\frac{\partial V}{\partial T}\right)_p$$

Thus
$$C_p-C_V=\frac{RT}{V-b}\left(\frac{\partial V}{\partial T}\right)_p, \quad \text{But } p=\frac{RT}{V-b}-\frac{a}{V^2}$$

On differentiating,
$$0=\left(-\frac{RT}{(V-b)^2}+\frac{2a}{V^2}\right)\left(\frac{\partial V}{\partial T}\right)_p+\frac{R}{V-b}$$

or,
$$T\left(\frac{\partial V}{\partial T}\right)_p=\frac{RT/V-b}{\dfrac{RT}{(V-b)^2}-\dfrac{2a}{V^3}}=\frac{V-b}{1-\dfrac{2a(V-b)^2}{RTV^3}}$$

and
$$C_p-C_V=\frac{R}{1-\dfrac{2a(V-b)^2}{RTV^3}}$$

2.60 From the first law

$Q=U_f-U_i+A=0$, as the vessels are themally insulated.

As this is free expansion, $A=0$, so, $U_f=U_i$

But
$$U=\nu C_V T-\frac{a\nu^2}{V}$$

So,
$$C_V(T_f-T_i)=\left(\frac{a}{V_1+V_2}-\frac{a}{V_1}\right)\nu=\frac{-aV_2\nu}{V_1(V_1+V_2)}$$

or,
$$\Delta T=\frac{-a(\gamma-1)V_2\nu}{RV_1(V_1+V_2)}$$

Substitution gives $\Delta T=-3\text{ K}$

2.61 $Q=U_f-U_i+A=U_f-U_i$, (as $A=0$ in free expansion).

So at constant temperature.

$$Q=\frac{-a\nu^2}{V_2}-\left(-\frac{a\nu^2}{V_1}\right)=a\nu^2\frac{V_2-V_1}{V_1\cdot V_2}$$

$$=0\cdot33\text{ kJ from the given data.}$$

2.3 KINETIC THEORY OF GASES. BOLTZMANN'S LAW AND MAXWELL'S DISTRIBUTION

2.62 From the formula $p = n\,k\,T$

$$n = \frac{p}{kT} = \frac{4 \times 10^{-15} \times 1{\cdot}01 \times 10^{5}}{1{\cdot}38 \times 10^{-23} \times 300} \text{ per m}^3$$

$$= 1 \times 10^{11} \text{ per m}^3 = 10^5 \text{ per c.c}$$

Mean distance between molecules

$$(10^{-5}\text{ c.c.})^{1/3} = 10^{1/3} \times 10^{-2}\text{ cm} = 0{\cdot}2\text{ mm.}$$

2.63 After dissociation each N_2 molecule becomes two N-atoms and so contributes, 2×3 degrees of freedom. Thus the number of moles becomes

$$\frac{m}{M}(1+\eta) \quad \text{and} \quad p = \frac{mRT}{MV}(1+\eta)$$

Here M is the molecular weight in grams of N_2.

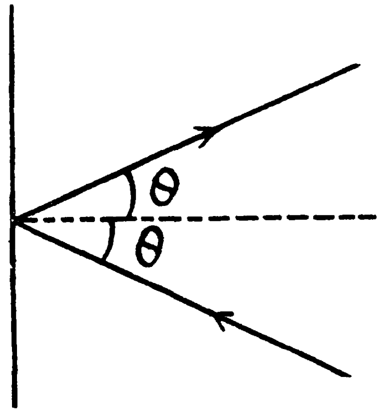
2.64 Let n_1 = number density of He atoms, n_2 = number density of N_2 molecules

Then $$\rho = n_1 m_1 + n_2 m_2$$

where m_1 = mass of He atom, m_2 = mass of N_2 molecule also $p = (n_1 + n_2)\,kT$

From these two equations we get

$$n_1 = \left(\frac{p}{kT} - \frac{\rho}{m_2}\right) \Big/ \left(1 - \frac{m_1}{m_2}\right)$$



2.65 $$p = \frac{nv \times 2\,mv\cos\theta \times dA\cos\theta}{dA}$$

$$= 2\,m\,n\,v^2\cos^2\theta$$

2.66 From the formula

$$v = \sqrt{\frac{\gamma p}{\rho}}, \quad \gamma = \frac{\rho V^2}{p}$$

If i = number of degrees of freedom of the gas then

$$C_p = C_V + RT \text{ and } C_V = \frac{i}{2}RT$$

$$\gamma = \frac{C_p}{C_v} = 1 + \frac{2}{i} \quad \text{or} \quad i = \frac{2}{\gamma - 1} = \frac{2}{\dfrac{\rho v^2}{p} - 1}$$

2.67 $$v_{sound} = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{M}}, \text{ and } v_{rms} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}}$$

so, $$\frac{v_{sound}}{v_{rms}} = \sqrt{\frac{\gamma}{3}} = \sqrt{\frac{i+2}{3i}}$$

(a) For monoatomic gases $i = 3$

$$\frac{v_{\text{sound}}}{v_{\text{rms}}} = \sqrt{\frac{5}{9}} = 0{\cdot}75$$

(b) For rigid diatomic molecules $i = 5$

$$\frac{v_{\text{sound}}}{v_{\text{rms}}} = \sqrt{\frac{7}{15}} = 0{\cdot}68$$

2.68 For a general noncollinear, nonplanar molecule

mean energy $= \frac{3}{2}kT$ (translational) $+ \frac{3}{2}kT$ (rotational) $+ (3N-6)kT$ (vibrational)

$= (3N - 3)\,kT$ per molecule

For linear molecules, mean energy $= \frac{3}{2}kT$ (translational)

$+ kT$ (rotational) $+ (3N-5)\,kT$ (vibrational)

$= \left(3N - \frac{5}{2}\right) kT$ per molecule

Translational energy is a fraction $\frac{1}{2(N-1)}$ and $\frac{1}{2N - \frac{5}{3}}$ in the two cases.

2.69 (a) A diatomic molecule has 3 translational, 2 rotational and one vibrational degrees of freedom. The corresponding energy per mole is

$$\frac{3}{2}RT, \text{ (for translational)} + 2 \times \frac{1}{2}RT, \text{ (for rotational)}$$

$$+ 1 \times RT, \text{ (for vibrational)} = \frac{7}{2}RT$$

Thus, $$C_V = \frac{7}{2}R, \text{ and } \gamma = \frac{C_p}{C_V} = \frac{9}{7}$$

(b) For linear N– atomic molecules energy per mole

$$= \left(3N - \frac{5}{2}\right) RT \text{ as before}$$

So, $$C_V = \left(3N - \frac{5}{2}\right) R \text{ and } \gamma = \frac{6N-3}{6N-5}$$

(c) For noncollinear N– atomic molecules

$C_V = 3\,(N-1)\,R$ as before (2.68) $\quad \gamma = \dfrac{3N-2}{3N-3} = \dfrac{N - 2/3}{N-1}$

2.70 In the isobaric process, work done is

$$A = pdv = RdT \text{ per mole.}$$

On the other hand heat transferred $Q = C_p\,dT$

Now $C_p = (3N-2)\,R$ for non-collinear molecules and $C_p = \left(3N - \frac{3}{2}\right) R$ for linear molecules

Thus $$\frac{A}{Q} = \begin{cases} \dfrac{1}{3N-2} & \text{non collinear} \\ \dfrac{1}{3N-\dfrac{3}{2}} & \text{linear} \end{cases}$$

For monoatomic gases, $c_p = \frac{5}{2}$ and $\frac{A}{Q} = \frac{2}{5}$

2.71 Given specific heats c_p, c_v (per unit mass)

$$M(c_p - c_v) = R \quad \text{or,} \quad M = \frac{R}{c_p - c_v}$$

Also $$\gamma = \frac{c_p}{c_v} = \frac{2}{i} + 1, \quad \text{os,} \quad i = \frac{2}{\dfrac{c_p}{c_v} 1} = \frac{2c_v}{c_p - c_v}$$

2.72 (a) $C_p = 29 \dfrac{J}{°\text{K mole}} = \dfrac{29}{8{\cdot}3} \text{R}$

$$C_v = \frac{20{\cdot}7}{8{\cdot}3} R, \quad \gamma = \frac{29}{20{\cdot}7} = 1{\cdot}4 = \frac{7}{5}$$

$$i = 5$$

(b) In the process $pT =$ const.

$$\frac{T^2}{V} = \text{const}, \quad \text{So} \quad 2\frac{dT}{T} - \frac{dV}{V} = 0$$

Thus $$C dT = C_V dT + p\,dV = C_V dT + \frac{RT}{V} dV = C_V dT + \frac{2RT}{T} dT$$

or $$C = C_V + 2R = \left(\frac{29}{8{\cdot}3}\right) R \quad \text{So} \quad C_V = \frac{12{\cdot}4}{8{\cdot}3} R = \frac{3}{2} R$$

Hence $i = 3$ (monoatomic)

2.73 Obviously

$$\frac{1}{R} C_V = \frac{3}{2}\gamma_1 + \frac{5}{2}\gamma_2$$

(Since a monoatomic gas has $C_V = \frac{3}{2}R$ and a diatomic gas has $C_V = \frac{5}{2}R$. [The diatomic molecule is rigid so no vibration])

$$\frac{1}{R} C_p = \frac{3}{2}\gamma_1 + \frac{5}{2}\gamma_2 + \gamma_1 + \gamma_2$$

Hence $$\gamma = \frac{C_p}{C_V} = \frac{5\gamma_1 + 7\gamma_2}{3\gamma_1 + 5\gamma_2}$$

2.74 The internal energy of the molecules are

$$U = \frac{1}{2} m N < (\vec{u} - \vec{v})^2 > = \frac{1}{2} m N <u^2 - v^2>$$

where $\vec{v}$ = velocity of the vessel, N = number of molecules, each of mass m. When the vessel is stopped, internal energy becomes $\frac{1}{2} mN \langle u^2 \rangle$

So there is an increase in internal energy of $\Delta U = \frac{1}{2} m N v^2$. This will give rise to a rise in temperature of

$$\Delta T = \frac{\frac{1}{2} m N v^2}{\frac{i}{2} R}$$

$$= \frac{m N v^2}{iR}$$

there being no flow of heat. This change of temperature will lead to an excess pressure

$$\Delta p = \frac{R \Delta T}{V} = \frac{m N v^2}{iV}$$

and finally $$\frac{\Delta p}{p} = \frac{M v^2}{i R T} = 2{\cdot}2\ \%$$

where M = molecular weight of N_2, i = number of degrees of freedom of N_2

2.75 (a) From the equipartition theorem

$$\bar{\varepsilon} = \frac{3}{2} k T = 6 \times 10^{-21}\ \text{J}; \text{ and } v_{rms} = \sqrt{\frac{3 k T}{m}} = \sqrt{\frac{3 R T}{M}} = 0{\cdot}47\ \text{km/s}$$

(b) In equilibrium the mean kinetic energy of the droplet will be equal to that of a molecule.

$$\frac{1}{2} \frac{\pi}{6} d^3 \rho v_{rms}^2 = \frac{3}{2} k T \quad \text{or} \quad v_{rms} = 3 \sqrt{\frac{2 k T}{\pi d^3 \rho}} = 0{\cdot}15\ \text{m/s}$$

2.76 Here $i = 5, C_V = \frac{5}{2} R, \gamma = \frac{7}{5}$ given

$$v'_{rms} = \sqrt{\frac{3 R T'}{M}} = \frac{1}{\eta} v_{rms} = \frac{1}{\eta} \sqrt{\frac{3 R T}{M}} \quad \text{or} \quad T' = \frac{1}{\eta^2} T$$

Now in an adiabatic process

$$TV^{\gamma - 1} = T V^{2/i} = \text{constant} \quad \text{or} \quad V T^{i/2} = \text{constant}$$

$$V' \left(\frac{1}{\eta^2} T \right)^{i/2} = V T^{i/2} \quad \text{or} \quad V' \eta^{-i} = V \quad \text{or} \quad V' = \eta^i V$$

The gas must be expanded η^i times, i.e 7·6 times.

2.77 Here $$C_V = \frac{5}{2} \frac{m}{M} R \; (i = 5 \text{ here})$$

m= mass of the gas, M= molecular weight. If v_{rms} increases η times, the temperature will have increased η^2 times. This will require (neglecting expansion of the vessels) a heat flow of amount

$$\frac{5}{2} \frac{m}{M} R (\eta^2 - 1) T = 10\ \text{kJ}.$$

2.78 The root mean square angular velocity is given by

$$\frac{1}{2} I \omega^2 = 2 \times \frac{1}{2} k T \text{ (2 degrees of rotations)}$$

or
$$\omega = \sqrt{\frac{2kT}{I}} = 6{\cdot}3 \times 10^{12} \text{ rad/s}$$

2.79 Under compression, the temperature will rise

$$TV^{\gamma - 1} = \text{constant}, \quad TV^{2/i} = \text{constant}$$

or,
$$T' (\eta^{-1} V_0)^{2/i} = T_0 V_0^{2/i} \quad \text{or,} \quad T' = \eta^{+2/i} T_0$$

So mean kinetic energy of rotation per molecule in the compressed state

$$= kT' = k T_0 \eta^{2/i} = 0{\cdot}72 \times 10^{-20} \text{ J}$$

2.80 No. of collisions $= \frac{1}{4} n <v> = \nu$

Now,
$$\frac{\nu'}{\nu} = \frac{n'}{n} \frac{<v'>}{<v>} = \frac{1}{\eta} \sqrt{\frac{T'}{T}}$$

(When the gas is expanded η times, n decreases by a factor η). Also

$T' (\eta V)^{2/i} = TV^{2/i}$ or $T' = \eta^{2/i} T$ so, $\frac{\nu'}{\nu} = \frac{1}{\eta} \eta^{-1/i} = \eta^{\frac{-i-1}{i}}$

i.e. collisions decrease by a factor $\eta^{\frac{i+1}{i}}$, $i = 5$ here.

2.81 In a polytropic process $pV^n =$ constant., where n is called the polytropic index. For this process

$$pV^n = \text{constant or } TV^{n-1} = \text{constant}$$

$$\frac{dT}{T} + (n-1) \frac{dV}{V} = 0$$

Then
$$dQ = C\,dT = dU + p\,dV = C_V dT + p\,dV$$

$$= \frac{i}{2} R\,dT + \frac{RT}{V} dV = \frac{i}{2} R\,dT - \frac{1}{n-1} R\,dT = \left(\frac{i}{2} - \frac{1}{n-1}\right) R\,dT$$

Now
$$C = R \text{ so } \frac{i}{2} - \frac{1}{n-1} = 1$$

or,
$$\frac{1}{n-1} = \frac{i}{2} - 1 = \frac{i-2}{2} \text{ or } n = \frac{i}{i-2}$$

Now
$$\frac{\nu'}{\nu} = \frac{n'}{n} \frac{<v'>}{<v>} = \frac{1}{\eta} \sqrt{\frac{T'}{T}} = \frac{1}{\eta} \left(\frac{V}{V'}\right)^{\frac{n-1}{2}}$$

$$= \frac{1}{\eta} \left(\frac{1}{\eta}\right)^{\frac{1}{i-2}} = \left(\frac{1}{\eta}\right)^{\frac{i-1}{i-2}} = \eta^{\frac{-i-1}{i-2}} \text{ times} = \frac{1}{2{\cdot}52} \text{ times}$$

2.82 If α is the polytropic index then

$$pV^{\alpha} = \text{constant}, \; TV^{\alpha - 1} = \text{constant.}$$

Now
$$\frac{v'}{v} = \frac{n'}{n}\frac{\langle v' \rangle}{\langle v \rangle} = \frac{V}{V'}\sqrt{\frac{T'}{T}} = \frac{V\,T^{-1/2}}{V'\,T'^{-1/2}} = 1$$

Hence
$$\frac{1}{\alpha - 1} = -\frac{1}{2} \quad \text{or} \quad \alpha = -1$$

Then
$$C = \frac{iR}{2} + \frac{R}{2} = 3R$$

2.83 $v_p = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2RT}{M}} = \sqrt{\frac{2p}{\rho}} = 0{\cdot}45 \text{ km/s},$

$$v_{av} = \sqrt{\frac{8}{\pi}\frac{p}{\rho}} = \cdot 51 \text{ km/s} \quad \text{and} \quad v_{rms} = \sqrt{\frac{3p}{\rho}} = 0{\cdot}55 \text{ km/s}$$

2.84 (a) The formula is

$$df(u) = \frac{4}{\sqrt{\pi}} u^2 e^{-u^2} du, \quad \text{where } u = \frac{v}{v_p}$$

Now Prob
$$\left(\frac{|v - v_p|}{v_p} < \delta\eta \right) = \int_{1-\delta\eta}^{1+\delta\eta} df(u)$$

$$= \frac{4}{\sqrt{\pi}} e^{-1} \times 2\,\delta\eta = \frac{8}{\sqrt{\pi}\,e}\,\delta\eta = 0{\cdot}016\dot{6}$$

(b)
$$\text{Prob}\left(\left| \frac{v - v_{rms}}{v_{rms}} \right| < \delta\eta \right) = \text{Prob}\left(\left| \frac{v}{v_p} - \frac{v_{rms}}{v_p} \right| < \delta\eta\,\frac{v_{rms}}{v_p} \right)$$

$$= \text{Prob}\left(\left| u - \sqrt{\frac{3}{2}} \right| < \sqrt{\frac{3}{2}}\,\delta\eta \right)$$

$$= \int_{\sqrt{\frac{3}{2}} - \sqrt{\frac{3}{2}}\,\delta\eta}^{\sqrt{\frac{3}{2}} + \sqrt{\frac{3}{2}}\,\delta\eta} \frac{4}{\sqrt{\pi}} u^2 e^{-u^2} du$$

$$= \frac{4}{\sqrt{\pi}} \times \frac{3}{2} e^{-3/2} \times 2\sqrt{\frac{3}{2}}\,\delta\eta = \frac{12\sqrt{3}}{\sqrt{2\pi}} e^{-3/2}\,\delta\eta = 0{\cdot}0185$$

2.85 (a) $v_{rms} - v_p = (\sqrt{3} - \sqrt{2})\sqrt{\dfrac{kT}{m}} = \Delta v,$

$$T = \frac{m}{k}\left(\frac{\Delta v}{(\sqrt{3}-\sqrt{2})}\right)^2 \Big/ \mathrm{K} = 384\ \mathrm{K}$$

(b) Clearly v is the most probable speed at this temperature. So

$$\sqrt{\frac{2kT}{m}} = v \quad \text{or} \quad T = \frac{mv^2}{2k} = 342\ \mathrm{K}$$

2.86 (a) We have,

$$\frac{v_1^2}{v_p^2} e^{-v_1^2/v_p^2} = \frac{v_2^2}{v_p^2} e^{-v_2^2/v_p^2} \quad \text{or} \quad \left(\frac{v_1}{v_2}\right)^2 = e^{v_1^2 - v_2^2/v_p^2} \quad \text{or} \quad v_p^2 = \frac{2kT}{m} = \frac{v_1^2 - v_2^2}{(\ln v_1^2/v_2^2)}$$

So
$$T = \frac{m(v_1^2 - v_2^2)}{2k \ln \dfrac{v_1^2}{v_2^2}} = 330\ \mathrm{K}$$

(b) $F(v) = \dfrac{4}{\sqrt{\pi}} \dfrac{v^2}{v_p^2} e^{-v^2/v_p^2} \times \dfrac{1}{v_p} \left(\dfrac{1}{v_p} \text{ comes from } F(v)\,dv = d\,f(u),\ du = \dfrac{dv}{v_p}\right)$

Thus $\dfrac{v^2}{v_{p_1}^3} e^{-v^2/v_{p_1}^2} = v^2/v_{p_2} e^{-v_2^2/v_{p_2}^2}\ \ v_{p_1}^2 = \dfrac{2kT_0}{m},\ v_{p_2}^2 = \dfrac{2kT_0}{m}\eta$ now

$$e^{-\frac{mv^2}{2kT_0}\left(1-\frac{1}{\eta}\right)} = \frac{1}{\eta^{3/2}} \quad \text{or} \quad \frac{mv^2}{2kT_0}\left(1 - \frac{1}{\eta}\right) = \frac{3}{2}\ln\eta$$

Thus
$$v = \sqrt{\frac{3kT_0}{m}}\sqrt{\frac{\ln\eta}{1 - 1/\eta}}$$

2.87 $v_{pN} = \sqrt{\dfrac{2kT}{m_N}} = \sqrt{\dfrac{2RT}{M_N}},\quad v_{p_0} = \sqrt{\dfrac{2RT}{M_0}}$

$$v_{p_N} - v_{p_0} = \Delta v = \sqrt{\frac{2RT}{M_N}}\left(1 - \sqrt{\frac{M_N}{M_0}}\right)$$

$$T = \frac{M_N(\Delta v)^2}{2R\left(1 - \sqrt{\dfrac{M_N}{M_0}}\right)^2} = \frac{m_N(\Delta v)^2}{2k\left(1 - \sqrt{\dfrac{m_N}{M_0}}\right)^2} = 363\ \mathrm{K}$$

2.88 $\dfrac{v^2}{v_{p_H}^3} e^{-v^2/v_{p_H}^2} = \dfrac{v^2}{v_{p_{He}}^3} e^{-v^2/v_{p_{He}}^2}$ or $e^{v^2\left(\frac{m_{He}}{2kT} - \frac{m_H}{2kT}\right)} = \left(\dfrac{m_{He}}{m_H}\right)^{3/2}$

$v^2 = 3kT\dfrac{\ln m_{He}/m_H}{m_{H_e} - m_H}$, Putting the values we get $v = 1{\cdot}60\ \mathrm{km/s}$

2.89 $dN(v) = \frac{N\,4}{\sqrt{\pi}} \frac{v^2\,dv}{v_p^3} e^{-v^2/v_p^2}$

For a given range v to $v + dv$ (i.e. given v and dv) this is maximum when

$$\frac{\delta}{\delta v_p} \frac{dN(v)}{N v^2\,dv} = 0 = \left(-3v_p^{-4} + \frac{2v^2}{v_p^6}\right) e^{-v^2/v_p^2}$$

or,

$$v^2 = \frac{3}{2} v_p^2 = \frac{3kT}{m}. \quad \text{Thus } T = \frac{1}{3}\frac{mv^2}{k}$$

2.90 $d^3 v = 2\pi v_\perp\,dv_\perp\,dv_x$

Thus
$$d\,n(v) = N\left(\frac{m}{2\pi k T}\right)^{3/2} e^{-\frac{m}{2kT}\left(v_x^2 + v_\perp^2\right)} dv_x\,2\pi v_\perp\,d\,v_\perp$$

2.91 $\langle v_x \rangle = 0$ by symmetry

$$\langle |v_x| \rangle = \int_{-\infty}^{\infty} |v_x|\,e^{-\frac{mv_x^2}{2kT}} dv_x \Big/ \int_0^{\infty} e^{\frac{-mv_x^2}{2kT}} dv_x = \int_0^{\infty} v_x\,e^{\frac{-mv_x^2}{2kT}} dv_x \Big/ \int_0^{\infty} e^{\frac{-mv^2}{2kT}} dv_x$$

$$= \sqrt{\frac{2kT}{m}} \int_0^{\infty} u e^{-u^2} du \Big/ \int_0^{\infty} e^{-u^2} du$$

$$= \sqrt{\frac{2kT}{m}} \int_0^{\infty} \frac{1}{2} e^{-x} dx \Big/ \int_0^{\infty} e^{-x} \frac{dx}{2\sqrt{x}}$$

$$= \sqrt{\frac{2kT}{m}}\,\Gamma(1) \Big/ \Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{2kT}{m\pi}}$$

2.92 $\langle v_x^2 \rangle = \int_0^{\infty} v_x^2\,e^{-\frac{mv_x^2}{2kt}} dv_x \Big/ \int_0^{\infty} e^{-\frac{mv_x^2}{2kt}} dv_x$

$$= \frac{2kT}{m} \int_0^{\infty} x e^{-x} \frac{dx}{2\sqrt{x}} \Big/ \int_0^{\infty} e^{-x} \frac{dx}{2\sqrt{x}}$$

$$= \frac{2kT}{m}\,\Gamma\left(\frac{3}{2}\right) \Big/ \Gamma\left(\frac{1}{2}\right) = \frac{kT}{m}$$

2.93 Here νdA = No. of molecules hitting an area dA of the wall per second

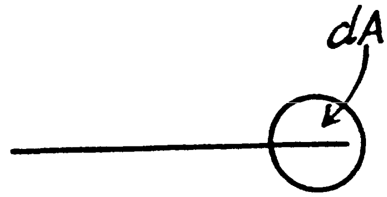
$$= \int_0^{\infty} dN(v_x)\,v_x\,dA$$

or, $$\nu = \int_0^\infty n\left(\frac{m}{2\pi kT}\right)^{1/2} e^{-\frac{mv_x^2}{2kT}} v_x\, dv_x$$

$$= \int_0^\infty \frac{n}{\sqrt{\pi}}\left(\frac{2kT}{m}\right)^{1/2} e^{-u^2}\, u\,du$$

$$= \frac{1}{2} n\sqrt{\frac{2kT}{m\pi}} = n\sqrt{\frac{kT}{2m\pi}} = \frac{1}{4} n <v>,$$

$$\left(\text{where } <v> = \sqrt{\frac{8kT}{m\pi}}\right)$$



2.94 Let, $dn(v_x) = n\left(\frac{m}{2\pi kT}\right)^{1/2} e^{-mv_x^2/2kT}\, dv_x$

be the number of molecules per unit volume with x component of velocity in the range v_x to $v_x + dv_x$

Then $$p = \int_0^\infty 2\, mv_x \cdot v_x\, dn(v_x)$$

$$= \int_0^\infty 2\, mv_x^2\, n\left(\frac{m}{2\pi kT}\right)^{1/2} e^{-mv_x^2/2kT}\, dv_x$$

$$= 2mn\frac{1}{\sqrt{\pi}}\frac{2kT}{m}\int_0^\infty u^2 e^{-u^2}\, du$$

$$= \frac{4}{\sqrt{\pi}} nkT \cdot \int_0^\infty x\, e^{-x}\frac{dx}{2\sqrt{x}} = nkT$$

2.95 $$<\frac{1}{v}> = \int_0^\infty \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}}\, 4\pi v^2\, dv\,\frac{1}{v}$$

$$= \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi\frac{1}{2}\frac{2kT}{m}\int_0^\infty e^{-x}\, dx$$

$$= 2\left(\frac{m}{2\pi kT}\right)^{1/2} = \left(\frac{2m}{\pi kT}\right)^{1/2} = \left(\frac{16}{\pi^2}\frac{m\pi}{8kT}\right)^{1/2} = \frac{4}{\pi <v>}$$

2.96 $dN(v) = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}\, 4\pi v^2\, dv = dN(\varepsilon) = \frac{dN(\varepsilon)}{d\varepsilon} d\varepsilon$

or,
$$\frac{dN(\varepsilon)}{d\varepsilon} = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2RT}\, 4\pi v^2 \frac{dv}{d\varepsilon}$$

Now,
$$\varepsilon = \frac{1}{2} mv^2 \quad \text{so} \quad \frac{dv}{d\varepsilon} = \frac{1}{mv}$$

or,
$$\frac{dN(\varepsilon)}{d\varepsilon} = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\varepsilon/kT}\, 4\pi \sqrt{\frac{2\varepsilon}{m}}\, \frac{1}{m}$$
$$= N\frac{2}{\sqrt{\pi}} (kT)^{-3/2} e^{-\varepsilon/kT} \varepsilon^{1/2}$$

i.e.
$$dN(\varepsilon) = N\frac{2}{\sqrt{\pi}} (kT)^{-3/2} e^{-\varepsilon/kT} \varepsilon^{1/2}\, d\varepsilon$$

The most probable kinetic energy is given from

$\frac{d}{d\varepsilon} \frac{dN(\varepsilon)}{d\varepsilon} = 0$ or, $\frac{1}{2}\varepsilon^{-1/2} e^{-\varepsilon/kT} - \frac{\varepsilon^{1/2}}{kT} e^{\varepsilon/kT} = 0$ or $\varepsilon = \frac{1}{2}kT = \varepsilon_{pr}$

The corresponding velocity is $v = \sqrt{\frac{kT}{m}} \neq v_{pr}$

2.97 The mean kinetic energy is

$$\langle\varepsilon\rangle = \int_0^\infty \varepsilon^{3/2} e^{-\varepsilon/kT}\, d\varepsilon \Big/ \int_0^\infty \varepsilon^{1/2} e^{-\varepsilon/kT}\, d\varepsilon = kT\frac{\Gamma(5/2)}{\Gamma(3/2)} = \frac{3}{2}kT$$

Thus

$$\frac{\delta N}{N} = \int_{\frac{3}{2}kT(1-\delta\eta)}^{\frac{3}{2}(1+\delta\eta)kT} \frac{2}{\sqrt{\pi}} (kT)^{-3/2} e^{-\varepsilon/kT} \varepsilon^{1/2}\, d\varepsilon$$
$$= \frac{2}{\sqrt{\pi}} e^{-3/2} \left(\frac{3}{2}\right)^{3/2} 2\,\delta\eta = 3\sqrt{\frac{6}{\pi}}\, e^{3/2}\,\delta\eta$$

If $\delta\eta = 1\%$ this gives 0·9 %

2.98 $\frac{\Delta N}{N} = \frac{2}{\sqrt{\pi}} (kT)^{-3/2} \int_{\varepsilon_0}^{\infty} \sqrt{\varepsilon}\, e^{-\varepsilon/kT}\, d\varepsilon$

$$\approx \frac{2}{\sqrt{\pi}} (kT)^{-3/2} \sqrt{\varepsilon_0} \int_{\varepsilon_0}^{\infty} e^{-\varepsilon/kT}\, d\varepsilon \; (\varepsilon_0 >> kT)$$

$$= \frac{2}{\sqrt{\pi}} (kT)^{-3/2} \sqrt{\varepsilon_0}\, kT\, e^{-\varepsilon_0/kT} = 2\sqrt{\frac{\varepsilon_0}{\pi kT}}\, e^{-\varepsilon_0/kT}$$

(In evaluating the integral, we have taken out $\sqrt{\varepsilon}$ as $\sqrt{\varepsilon_0}$ since the integral is dominated by the lower limit.)

2.99 (a) $F(v) = Av^3 e^{-mv^2/2kT}$

For the most probable value of the velocity

$$\frac{dF(v)}{dv} = 0 \quad \text{or} \quad 3A v^2 e^{-mv^2/2kT} - A v^3 \frac{2mv}{2kT} e^{-mv^2/2kT} = 0$$

So, $$v_{pr} = \sqrt{\frac{3kT}{m}}$$

This should be compared with the value $v_{pr} = \sqrt{\frac{2kT}{m}}$ for the Maxwellian distribution.

(b) In terms of energy, $\varepsilon = \frac{1}{2} mv^2$

$$F(\varepsilon) = Av^3 e^{-mv^2/2kT} \frac{dv}{d\varepsilon}$$

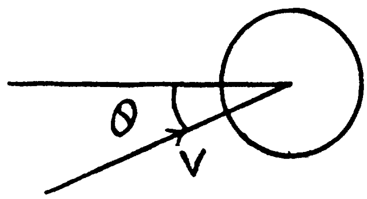
$$= A\left(\frac{2\varepsilon}{m}\right)^{3/2} e^{-\varepsilon/kT} \frac{1}{\sqrt{2m\varepsilon}} = A\frac{2\varepsilon}{m^2} e^{-\varepsilon/kT}$$

From this the probable energy comes out as follows : $F'(\varepsilon) = 0$ implies

$$\frac{2A}{m^2}\left(e^{-\varepsilon/kT} - \frac{\varepsilon}{kT} e^{-\varepsilon/kT}\right) = 0, \quad \text{or,} \quad \varepsilon_{pr} = kT$$

2.100 The number of molecules reaching a unit area of wall at angle between θ and $\theta + d\theta$ to its normal per unit time is

$$dv = \int_{v=0}^{v=\infty} dn(v) \frac{d\Omega}{4\pi} v\cos\theta$$



$$= \int_0^\infty n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} v^3\, dv \sin\theta\cos\theta\, d\theta \times 2\pi$$

$$= n\left(\frac{2kT}{m\pi}\right)^{1/2} \int_0^\infty e^{-x} x\,dx\, \sin\theta\cos\theta\, d\theta = n\left(\frac{2kT}{m\pi}\right)^{1/2} \sin\theta\cos\theta\, d\theta$$

2.101 Similarly the number of molecules reaching the wall (per unit area of the wall with velocities in the interval v to $v + dv$ per unit time is

$$dv = \int_{\theta=0}^{\theta=\pi/2} dn(v) \frac{d\Omega}{4\pi} v\cos 0$$

$$= \int_{\theta=0}^{\theta=\pi/2} n \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} v^3 \, dv \sin\theta \cos\theta \, d\theta \times 2\pi$$

$$= n\pi \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} v^3 \, dv$$

2.102 If the force exerted is F then the law of variation of concentration with height reads

$$n(z) = n_0 e^{-Fz/kT} \quad \text{So,} \quad \eta = e^{F\Delta h/kT} \quad \text{or} \quad F = \frac{kT \ln \eta}{\Delta h} = 9 \times 10^{-20} \text{ N}$$

2.103 Here $F = \frac{\pi}{6} d^3 \Delta\rho \, g = \frac{RT \ln \eta}{N_a h}$ or $N_a = \frac{6RT \ln \eta}{\pi d^3 g \Delta\rho h}$

In the problem, $\frac{\eta}{\eta_0} = 1{\cdot}39$ here

$T = 290\text{K}$, $\eta = 2$, $h = 4 \times 10^{-5}$ m, $d = 4 \times 10^{-7}$ m, $g = 9{\cdot}8$ m/s^2, $\Delta\rho = 0{\cdot}2 \times 10^3$ kg/m^3 and $R = 8{\cdot}31$ J/k

Hence, $N_a = \frac{6 \times 8{\cdot}31 \times 290 \times \ln 2}{\pi \times 64 \times 9{\cdot}8 \; 200 \times 4} \times 10^{26} = 6{\cdot}36 \times 10^{23}$ mole^{-1}

2.104 $\eta = \frac{\text{concetration of } H_2}{\text{concentration of } N_2} = \eta_0 \frac{e^{-M_{H_2} gh/RT}}{e^{-M_{N_2} - gh/RT}} = \eta_0 e^{(M_{N_2} - M_{H_2}) gh/RT}$

So more N_2 at the bottom, $\left(\frac{\eta}{\eta_0} = 1.39 \text{ here}\right)$

2.105 $n_1(h) = n_1 e^{-m_1 gh/kt}$, $n_2(h) = n_2 e^{-m_2 gh/kT}$

They are equal at a height h where $\frac{n_1}{n_2} = e^{gh(m_1 - m_2)/kT}$

$$\text{or} \quad h = \frac{kT}{g} \frac{\ln n_1 - \ln n_2}{m_1 - m_2}$$

2.106 At a temperature T the concentration $n(z)$ varies with height according to

$$n(z) = n_0 e^{-mgz/kT}$$

This means that the cylinder contains $\int_0^\infty n(z) \, dz$

$$= \int_0^\infty n_0 e^{-mgz/kT} \, dz = \frac{n_0 kT}{mg}$$

particles per unit area of the base. Clearly this cannot change. Thus $n_0 kT = p_0 =$ pressure at the bottom of the cylinder must not change with change of temperature.

2.107
$$\langle U\rangle = \frac{\int_0^\infty mgz\, e^{-mgz/kt}\, dz}{\int_0^\infty e^{-mgz/kT}\, dz} = kT \frac{\int_0^\infty x\, e^{-x}\, dx}{\int_0^\infty e^{-x}\, dx} = kT \frac{\Gamma(2)}{\Gamma(1)} = kT$$

When there are many kinds of molecules, this formula holds for each kind and the average energy

$$\langle U\rangle = \frac{\sum f_i\, kT}{\sum f_i} = kT$$

where f_i α fractional concentration of each kind at the ground level.

2.108 The constant acceleration is equivalent to a pseudo force wherein a concentration gradient is set up. Then

$$e^{-M_A w l/RT} = 1 - \eta$$

or
$$w = -\frac{RT \ln (1-\eta)}{M_A\, l} = \frac{\eta RT}{M_A\, l} = 70 \text{ g}$$

2.109 In a centrifuge rotating with angular velocity ω about an axis, there is a centrifugal acceleration $\omega^2 r$ where r is the radial distance from the axis. In a fluid if there are suspended colloidal particles they experience an additional force. If m is the mass of each particle then its volume is $\frac{m}{\rho}$ and the excess force on this particle is

$\frac{m}{\rho}(\rho - \rho_0)\,\omega^2 r$ outward corresponding to a potential energy $-\frac{m}{2\rho}(\rho - \rho_0)\,\omega^2 r^2$

This gives rise to a concentration variation

$$n(r) = n_0 \exp\left(+\frac{m}{2\rho kT}(\rho - \rho_0)\,\omega^2 r^2\right)$$

Thus
$$\frac{n(r_2)}{n(r_1)} = \eta = \exp\left(+\frac{M}{2\rho RT}(\rho - \rho_0)\,\omega^2 \left(r_2^2 - r_1^2\right)\right)$$

where
$$\frac{m}{k} = \frac{M}{R},\ M = N_A m \text{ is the molecular weight}$$

Thus
$$M = \frac{2\rho RT \ln \eta}{(\rho - \rho_0)\,\omega^2 (r_2^2 - r_1^2)}$$

2.110 The potential energy associated with each molecule is : $-\frac{1}{2} m\,\omega^2 r^2$

and there is a concentration variation

$$n(r) = n_0 \exp\left(\frac{m\,\omega^2 r^2}{2kT}\right) = n_0 \exp\left(\frac{M\,\omega^2 r^2}{2RT}\right)$$

Thus
$$\eta = \exp\left(\frac{M\,\omega^2 l^2}{2RT}\right) \quad \text{or} \quad \omega = \sqrt{\frac{2RT}{M l^2} \ln \eta}$$

Using $M = 12 + 32 = 44$ gm, $l = 100$ cm, $R = 8{\cdot}31 \times 10^7 \frac{\text{erg}}{°\text{K}}$, $T = 300$, we get $\omega = 280$ radians per second.

2.111 Here $n(r) = n_0 \exp\left(-\frac{ar^2}{kT}\right)$

(a) The number of molecules located at the distance between r and $r + dr$ is

$$4\pi r^2\, dr\, n(r) = 4\pi n_0 \exp\left(-\frac{ar^2}{kT}\right) r^2\, dr$$

(b) r_{pr} is given by $\frac{d}{dr} r^2 n(r) = 0$ or, $2r - \frac{2ar^3}{kT} = 0$ or $r_{pr} = \sqrt{\frac{kT}{a}}$

(c) The fraction of molecules lying between r and $r + dr$ is

$$\frac{dN}{N} = \frac{4\pi r^2\, dr\, n_0 \exp(-ar^2/kT)}{\int_0^\infty 4\pi r^2\, dr\, n_0 \exp(-ar^2/kT)}$$

$$\int_0^\infty 4\pi r^2\, dr \exp\left(-\frac{ar^2}{kT}\right) = \left(\frac{kT}{a}\right)^{3/2} 4\pi \int_0^\infty x \frac{dx}{2\sqrt{x}} \exp(-x)$$

$$= \left(\frac{kT}{a}\right)^{3/2} 2\pi\, \Gamma\left(\frac{3}{2}\right) = \left(\frac{\pi kT}{a}\right)^{3/2}$$

Thus $$\frac{dN}{N} = \left(\frac{a}{\pi kT}\right)^{3/2} 4\pi r^2\, dr \exp\left(\frac{-ar^2}{kT}\right)$$

(d) $$dN = N\left(\frac{a}{\pi kT}\right)^{3/2} 4\pi r^2\, dr\, \exp\left(\frac{-ar^2}{kT}\right),$$

So $$n(r) = N\left(\frac{a}{\pi kT}\right)^{1/2} \exp\left(\frac{-ar^2}{kT}\right)$$

When T decreases η times $n(0) = n_0$ will increase $\eta^{3/2}$ times.

2.112 Write $U = ar^2$ or $r = \sqrt{\frac{U}{a}}$, so $dr = \sqrt{\frac{1}{a}} \frac{dU}{2\sqrt{U}} = \frac{dU}{2\sqrt{a\,U}}$

so $dN = n_0 4\pi \frac{U}{a} \frac{dU}{2\sqrt{aU}} \exp\left(\frac{U}{kT}\right)$

$= 2\pi n_0 a^{-3/2}\, U^{1/2} \exp\left(\frac{-U}{kT}\right) dU$

The most probable value of U is given by

$$\frac{d}{dU}\left(\frac{dN}{dU}\right) = 0 = \left(\frac{1}{2\sqrt{u}} - \frac{U^{1/2}}{kT}\right) \exp\left(\frac{-U}{kT}\right) \quad \text{or,} \quad U_{pr} = \frac{1}{2} kT$$

From 2.111 (b), the potential energy at the most probable distance is kT.

2.4 THE SECOND LAW OF THERMODYNAMICS. ENTROPY

2.113 The efficiency is given by

$$\eta = \frac{T_1 - T_2}{T_1}, \; T_1 > T_2$$

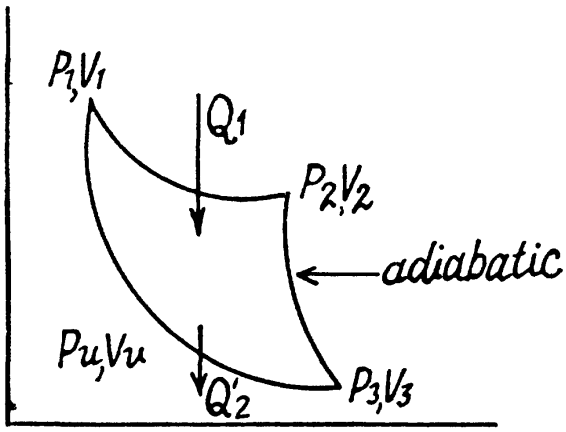
Now in the two cases the efficiencies are

$$\eta_h = \frac{T_1 + \Delta T - T_2}{T_1 + \Delta T}, \quad T_1 \text{ increased}$$

$$\eta_l = \frac{T_1 - T_2 + \Delta T}{T_1}, \; T_2 \text{ decreased}$$

Thus $$\eta_h < n_l$$

2.114 For H_2, $\gamma = \frac{7}{5}$



$$p_1 V_1 = p_2 V_2, \; p_3 V_3 = p_4 V_4$$

$$p_2 V_2^\gamma = p_3 V_3^\gamma, \; p_1 V_1^\gamma = p_4 V_4^\gamma$$

Define n by $V_3 = n V_2$

Then $p_3 = p_2 n^{-\gamma}$ so

$$p_4 V_4 = p_3 V_3 = p_2 V_2 n^{1-\gamma} = p_1 V_1 n^{1-\gamma}$$

$$p_4 V_4^\gamma = p_1 V_1^\gamma \text{ so } V_4^{1-\gamma} = V_1^{1-\gamma} n^{1-\gamma} \text{ or } V_4 = nV_1$$

Also $$Q_1 = p_2 V_2 \ln \frac{V_2}{V_1}, \; Q'_2 = p_3 V_3 \ln \frac{V_3}{V_4} n^{1-\gamma} = p_2 v_2 \ln \frac{V_3}{v_4}$$

Finally $$\eta = 1 - \frac{Q_2}{Q_1} = 1 - n^{1-\gamma} = 0{\cdot}242$$

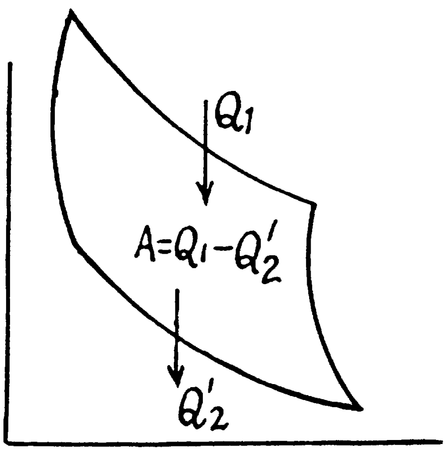
(b) Define n by $p_3 = \frac{p_2}{n}$

$$p_2 V_2^\gamma = \frac{p_2}{n} V_3^\gamma \quad \text{or} \quad V_3 = n^{1/\gamma} V_2$$

So we get the formulae here by $n \to n^{1/\gamma}$ in the previous case.

$$\eta = 1 - n^{(1/\gamma) - 1} = 1 - n^{-\frac{2}{7}} \approx 0{\cdot}18$$

2.115 Used as a refrigerator, the refrigerating efficiency of a heat engine is given by



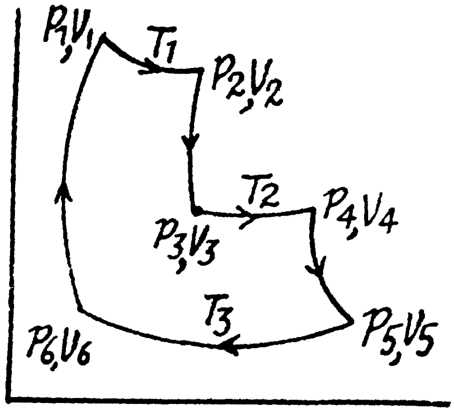
$$\varepsilon = \frac{Q_2'}{A} = \frac{Q_2'}{Q_1 - Q_2} = \frac{Q_2'/Q_1}{1 - \frac{Q_2'}{Q_1}} = \frac{1 - \eta}{\eta} = 9 \text{ here},$$

where η is the efficiency of the heat engine.

2.116 Given $V_2 = n\,V_1,\ V_4 = n\,V_3$

$Q_1 =$ Heat taken at the upper temperature

$$= RT_1 \ln n + R\,T_2 \ln n = R\,(T_1 + T_2) \ln n$$



Now $\quad T_1 V_2^{\gamma-1} = T_2 V_3^{\gamma-1}$ or $V_3 = \left(\dfrac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}} V_2$

Similarly $\quad V_5 = \left(\dfrac{T_2}{T_3}\right)^{\frac{1}{\gamma-1}} V_4,\ V_6 = \left(\dfrac{T_1}{T_3}\right)^{\frac{1}{\gamma-1}} V_1$

Thus $Q_2 =$ heat ejected at the lower temperature $= -RT_3 \ln \dfrac{V_6}{V_5}$

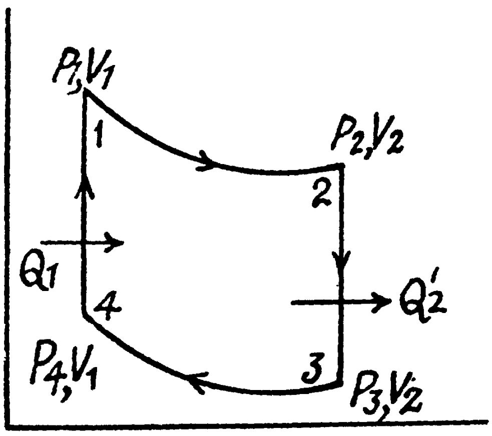
$$= -R\,T_3 \ln \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}} \frac{V_1}{V_4} = -RT_3 \ln \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}} \frac{V_2}{n^2\,V_3}$$

$$= -RT_3 \ln \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}} \frac{1}{n^2} \left(\frac{T_1}{T_2}\right)^{-\frac{1}{\gamma-1}} = 2\,R\,T_3 \ln n$$

Thus $$\eta = 1 - \frac{2T_3}{T_1 + T_2}$$

2.117 $Q'_2 = C_V (T_2 - T_3) = \dfrac{C_V}{R} V_2 (p_2 - p_3)$

$$Q_1 = \frac{C_V}{R} V_1 (p_1 - p_4)$$



Thus $\quad \eta = 1 - \dfrac{V_2 (p_2 - p_3)}{V_1 (p_1 - p_4)}$

On the other hand,

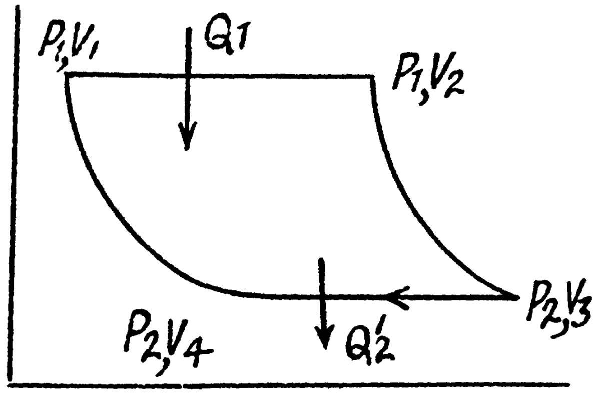
$6p_1 V_1^{\gamma} = p_2 V_2^{\gamma},\ p_3 V_2^{\gamma} = p_4 V_1^{\gamma}$ also $V_2 = n\,V_1$

Thus $\quad p_1 = p_2 n^{\gamma},\ p_4 = p_3 n^{\gamma}$

and $\ \eta = 1 - n^{1-\gamma}$, with $\gamma = \dfrac{7}{5}$ for N_2 this is $\eta = 0{\cdot}602$

2.118 $Q_1 = \dfrac{C_p}{R} p_1 (V_2 - V_1),\ Q'_2 = \dfrac{C_p}{R} p_2 (V_3 - V_4)$

So $\quad \eta = 1 - \dfrac{p_2 (V_3 - V_4)}{p_1 (V_2 - V_1)}$



Now $p_1 = n\,p_2,\ p_1 V_2^{\gamma}$ or $V_3 = n^{\frac{1}{\gamma}} V_2$

$p_2 V_4^{\gamma} = p_1 V_1^{\gamma}$ or $V_4 = n^{\frac{1}{\gamma}} V_1$

so $\ \eta = 1 - \dfrac{1}{n} \cdot n^{\frac{1}{\gamma}} = 1 - n^{\frac{1}{\gamma} - 1}$

2.119 Since the absolute temperature of the gas rises n times both in the isochoric heating and in the isobaric expansion

$$p_1 = np_2 \text{ and } V_2 = n\,V_1 . \text{ Heat taken is}$$

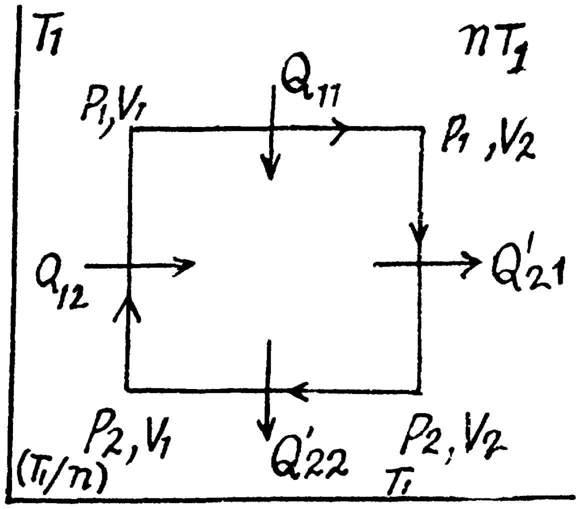
$$Q_1 = Q_{11} + Q_{12}$$

where $Q_{11} = C_p\,(n-1)\,T_1$ and $Q_{12} = C_V\,T_1\left(1 - \frac{1}{n}\right)$

Heat rejected is

$$Q'_2 = Q'_{21} + Q'_{22} \text{ where}$$

$$Q'_{21} = C_V\,T_1\,(n-1)\,,\ Q'_{22} = C_p\,T_1\left(1 - \frac{1}{n}\right)$$



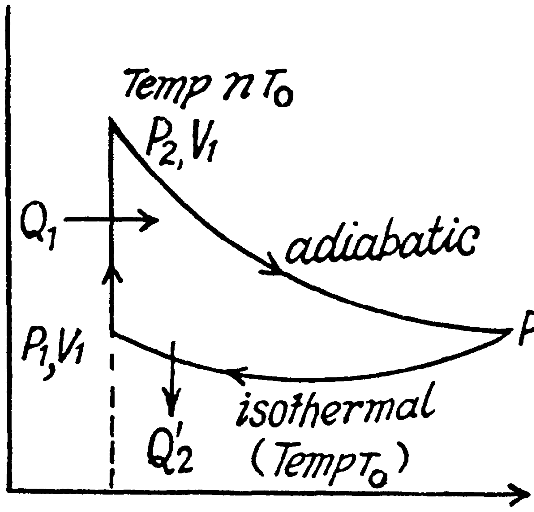
Thus $\eta = 1 - \dfrac{Q'_2}{Q_1} = 1 - \dfrac{C_V\,(n-1) + C_p\left(1 - \frac{1}{n}\right)}{C_p\,(n-1) + C_V\left(1 - \frac{1}{n}\right)}$

$$= 1 - \frac{n - 1 + \gamma\left(1 - \frac{1}{n}\right)}{\gamma\,(n-1) + \left(1 - \frac{1}{n}\right)} = 1 - \frac{1 + \frac{\gamma}{n}}{\gamma + \frac{1}{n}} = 1 - \frac{n + \gamma}{1 + n\gamma}$$

2.120 (a) Here $p_2 = n\,p_1\,,\ p_1\,V_1 = p_0\,V_0\,,$

$$n\,p_1\,V_1^\gamma = p_0\,V_0^\gamma$$

$$Q'_2 = RT_0 \ln \frac{V_0}{V_1},\ Q_1 = C_V\,T_0\,(n-1)$$



But $n\,V_1^{\gamma-1} = V_0^{\gamma-1}$ or, $V_1 = V_0\,n^{\frac{-1}{\gamma-1}}$

$$Q'_2 = RT_0 \ln n^{\frac{1}{\gamma-1}} = \frac{RT_0}{\gamma-1} \ln n$$

Thus $\eta = 1 - \dfrac{\ln n}{n-1}$, on using $C_V = \dfrac{R}{\gamma-1}$

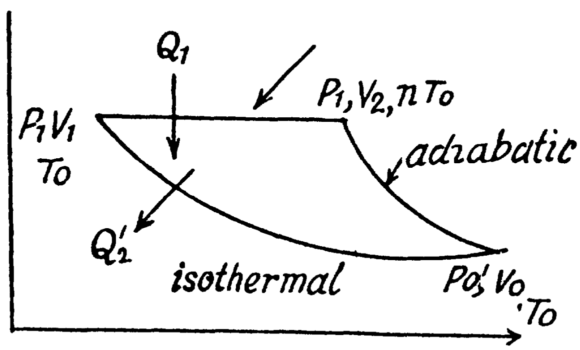
(b) Here $V_2 = nV_1\,,\ p_1\,V_1 = p_0\,V_0$

and $\quad p_1\,(n\,V_1)^\gamma = p_0\,V_0^\gamma$

i.e. $n^\gamma\,V_1^{\gamma-1} = V_0^{\gamma-1}$ or $V_1 = n^{-\frac{\gamma}{\gamma-1}}\,V_0$

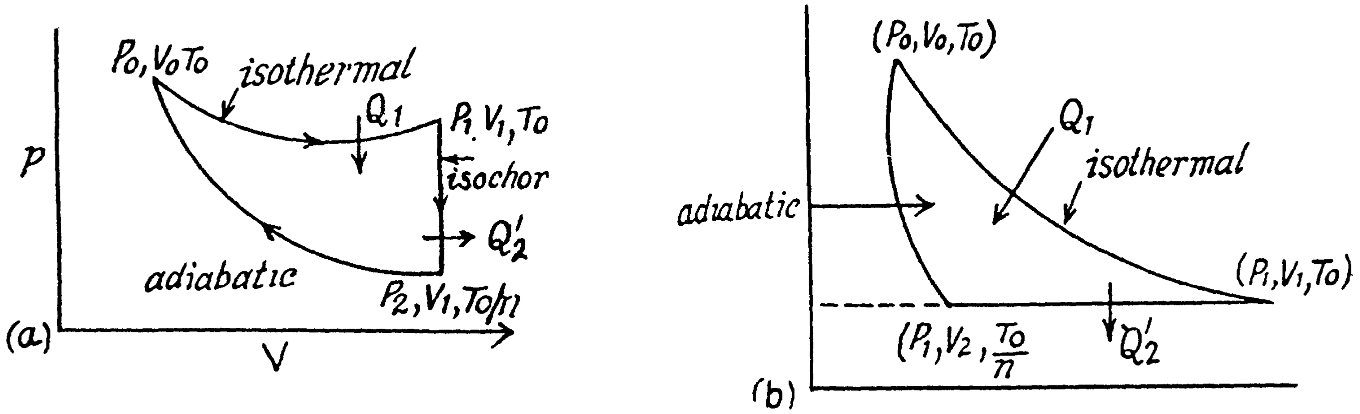
Also $Q_1 = C_p\,T_0\,(n-1)\,,\ Q'_2 = RT_0 \ln \dfrac{V_0}{V_1}$

or $Q'_2 = RT_0 \ln n^{\frac{\gamma}{\gamma-1}} = \dfrac{R\gamma}{\gamma-1}\,T_0 \ln n = C_p\,T_0 \ln n$



Thus $$\eta = 1 - \frac{\ln n}{n-1}$$

2.121 Here the isothermal process proceeds at the maximum temperature instead of at the minimum temperature of the cycle as in 2.120.



(a) Here $p_1 V_1 = p_0 V_0$, $p_2 = \dfrac{p_1}{n}$

$$p_2 V_1^\gamma = p_0 V_0^\gamma \quad \text{or} \quad p_1 V_1^\gamma = n p_0 V_0^\gamma$$

i.e.

$$V_1^{\gamma-1} = n V_0^{\gamma-1} \quad \text{or} \quad V_1 = V_0 n^{\frac{1}{\gamma-1}}$$

$$Q'_2 = C_V T_0 \left(1 - \frac{1}{n}\right), \; Q_1 = R T_0 \ln \frac{V_1}{V_0} = \frac{R T_0}{\gamma - 1} \ln n = C_V T_0 \ln n.$$

Thus

$$\eta = 1 - \frac{Q'_2}{Q_1} = 1 - \frac{n-1}{n \ln n}$$

(b) Here $V_2 = \dfrac{V_1}{n}$, $p_0 V_0 = p_1 V_1$

$$p_0 V_0^\gamma = p_1 V_2^\gamma = p_1 n^{-\gamma} V_1^\gamma = V_0^{\gamma-1} n^{-\gamma} V_1^{\gamma-1} \quad \text{or} \quad V_1 = n^{(\gamma/\gamma-1)} V_0$$

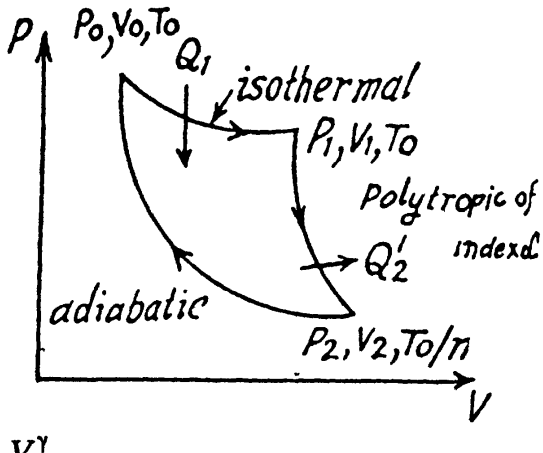
$$Q'_2 = C_p T_0 \left(1 - \frac{1}{n}\right), \; Q_1 = R T_0 \ln \frac{V_1}{V_0} = \frac{R\gamma}{\gamma - 1} T_0 \ln n = C_p T_0 \ln n$$

Thus

$$\eta = 1 - \frac{n-1}{n \ln n}$$

2.122 The section from (p_1, V_1, T_0) to $(p_2, V_2, T_0/n)$ is a polytropic process of index α. We shall assume that the corresponding specific heat C is + ve.

Here, $dQ = C dT = C_V dT + p dV$



Now pV^α = constant or $TV^{\alpha-1}$ = constant.

so $p dV = \dfrac{RT}{V} dV = -\dfrac{R}{\alpha - 1} dT$

Then $C = C_V - \dfrac{R}{\alpha - 1} = R\left(\dfrac{1}{\gamma - 1} - \dfrac{1}{\alpha - 1}\right)$

We have $p_1 V_1 = R T_0 = p_2 V_2 = \dfrac{R T_0}{n} = \dfrac{p_1 V_1}{n}$

$$p_0 V_0 = p_1 V_1 = n p_2 V_2, \; p_0 V_0^\gamma = p_2 V_2^\gamma,$$

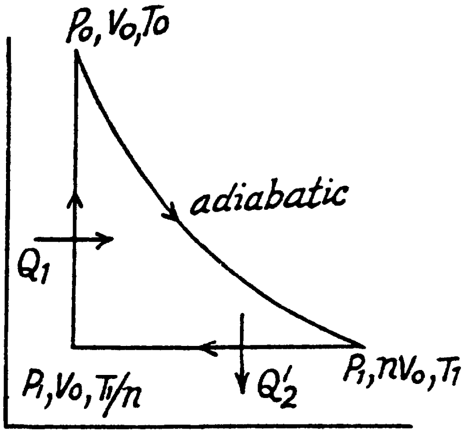
$$p_1 V_1^{\alpha} = p_2 V_2^{\alpha} \quad \text{or} \quad V_0^{\gamma-1} = \frac{1}{n} V_2^{\gamma-1} \quad \text{or} \quad V_2 = V_0 \, n^{\frac{1}{\gamma-1}}$$

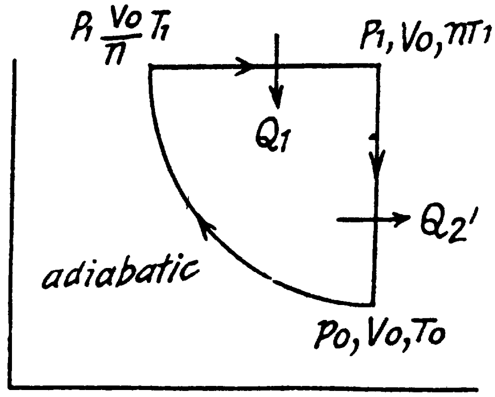
$$V_1^{\alpha-1} = \frac{1}{n} V_2^{\alpha-1} \quad \text{or} \quad V_1 = n^{-\frac{1}{\alpha-1}} V_2 = n^{\frac{1}{\gamma-1} - \frac{1}{\alpha-1}} V_0$$

Now $Q'_2 = CT_0\left(1 - \frac{1}{n}\right)$, $Q_1 = RT_0 \ln \frac{V_1}{V_0} = RT_0\left(\frac{1}{\gamma-1} - \frac{1}{\alpha-1}\right) \ln n = CT_0 \ln n$

Thus $$\eta = 1 - \frac{n-1}{n \ln n}$$

2.123





(a) Here $Q'_2 = C_p\left(T_1 - \frac{T_1}{n}\right) = C_p T_1\left(1 - \frac{1}{n}\right)$, $Q_1 = C_V\left(T_0 - \frac{T_1}{n}\right)$

Along the adiabatic line

$$T_0 V_0^{\gamma-1} = T_1 (n V_0)^{\gamma-1} \quad \text{or,} \quad T_0 = T_1 n^{\gamma-1}.$$

so $$Q_1 = C_V \frac{T_1}{n}(n^{\gamma} - 1). \text{ Thus } \eta = 1 - \frac{\gamma(n-1)}{n^{\gamma-1}}$$

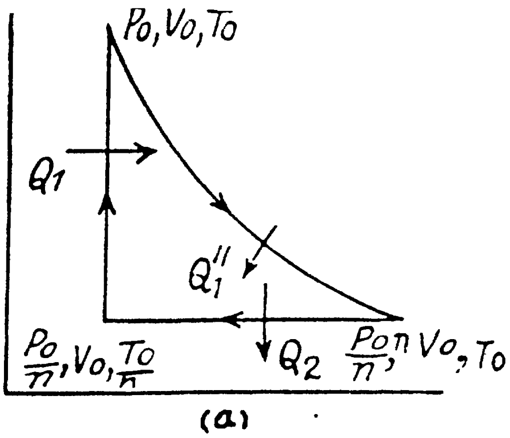
(b) Here $Q'_2 = C_V (n T_1 - T_0)$, $Q_1 = C_p \cdot T_1 (n-1)$

Along the adiabatic line $TV^{\gamma-1} =$ constant

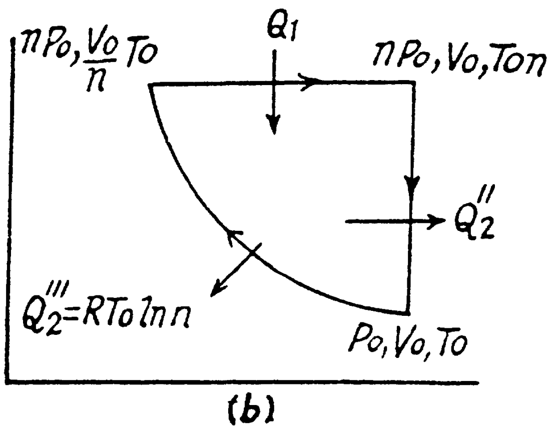
$$T_0 V_0^{\gamma-1} = T_1 \left(\frac{V_0}{n}\right)^{\gamma-1} \quad \text{or} \quad T_1 = n^{\gamma-1} T_0$$

Thus $$\eta = 1 - \frac{n^{\gamma} - 1}{\gamma n^{\gamma-1}(n-1)}$$

2.124



(a)



(b)

(a) $Q'_2 = C_p T_0\left(1 - \frac{1}{n}\right)$, $Q''_1 = RT_0 \ln n$, $Q'_1 = C_V T_0\left(1 - \frac{1}{n}\right)$, $Q_1 = Q'_1 + Q''_1$

So $$\eta = 1 - \frac{Q'_2}{Q_1} = 1 - \frac{C_p\left(1 - \frac{1}{n}\right)}{C_V\left(1 - \frac{1}{n}\right) + R \ln n}$$

$$= 1 - \frac{\gamma}{1 + \frac{R}{C_V}\frac{n \ln n}{n-1}} = 1 - \frac{\gamma (n-1)}{n - 1 + (\gamma - 1)\, n \ln n}$$

(b) $Q_1 = C_p T_0 (n-1), Q''_2 = C_V T_0 (n-1) \quad Q'''_2 = RT_0 \ln n\,,\ Q'_2 = Q''_2 + Q'''_2$

So $$\eta = 1 - \frac{Q'_2}{Q_1} = 1 - \frac{n - 1 + (\gamma - 1)\ln n}{\gamma (n-1)}$$

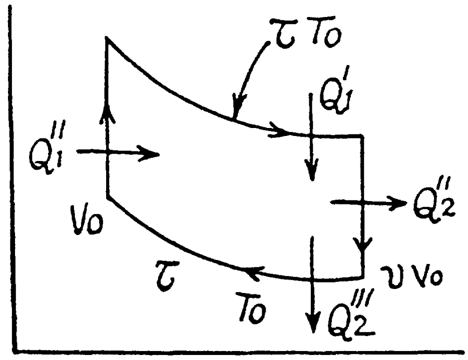
2.125 We have

$Q'_1 = \tau RT_0 \ln \nu,\ Q''_2 = C_V T_0 (\tau - 1)\, Q_1 = Q'_1 + Q''_1$ and

$Q'''_2 = RT_0 \ln \nu,\ Q''_1 = C_V T_0 (\tau - 1)$

as well as $Q_1 = Q_1' + Q_1''$ and

$Q_2' = Q_2'' + Q_2'''$



So $$\eta = 1 - \frac{Q'_2}{Q_1} + 1 = \frac{C_V(\tau - 1) + R \ln \nu}{C_V(\tau - 1) + \tau R \ln \nu}$$

$$= 1 - \frac{\frac{\tau - 1}{\gamma - 1} + \ln \nu}{\frac{\tau - 1}{\gamma - 1} + \tau \ln \nu} = \frac{(\tau - 1)\ln \nu}{\tau \ln \nu + \frac{\tau - 1}{\gamma - 1}}$$

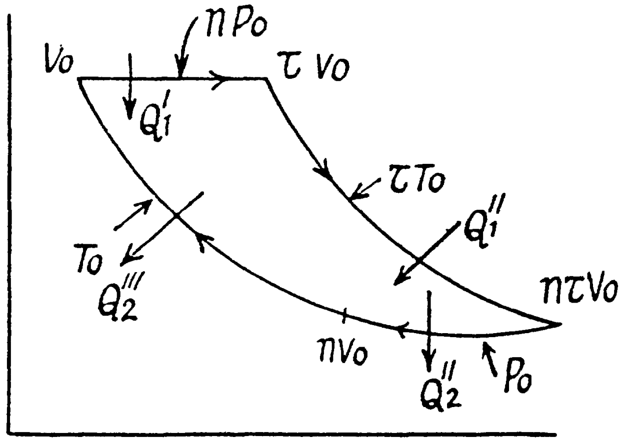
2.126 Here $Q_1'' = C_p T_0 (\tau - 1)$, $Q'1'' = \tau RT_0 \ln n$ and

$Q_2'' = C^P T_0 (\tau - 1)$, $Q_2''' = RT_0 \ln n$

in addition to we have

$Q_1 = Q_1' + Q_1''$ and

$Q_2' = Q_2'' + Q_2'''$



So $$\eta = 1 - \frac{Q'_2}{Q_1} = 1 - \frac{C_p(\tau - 1) + R \ln n}{C_p(\tau - 1) + \tau R \ln n}$$

$$= 1 - \frac{\tau - 1 + \left(1 - \frac{1}{\gamma}\right)\ln n}{\tau - 1 + \left(1 - \frac{1}{\gamma}\right)\tau \ln n}$$

$$= 1 - \frac{\tau - 1 + \left(1 - \frac{1}{\gamma}\right)\ln n}{\tau - 1 + \left(1 - \frac{1}{\gamma}\right)\tau \ln n} = \frac{(\tau - 1)\ln n}{\tau \ln n + \frac{\gamma(\tau - 1)}{\gamma - 1}}$$

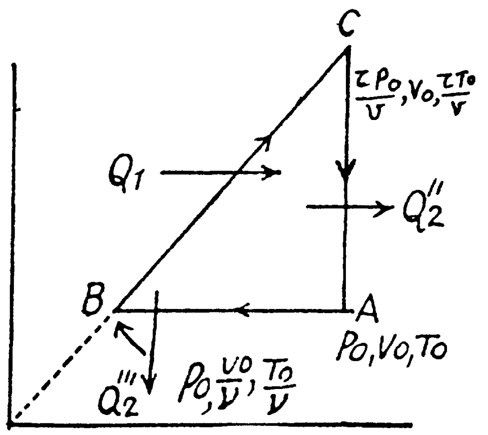
2.127 Because of the linearity of the section BC whose equation is

$$\frac{p}{p_0} = \frac{\nu V}{V_0} (\Rightarrow p = \alpha V)$$

We have $\frac{\tau}{\nu} = \nu$ or $\nu = \sqrt{\tau}$

Here $\quad Q''_2 = C_V T_0 (\sqrt{\tau} - 1),$

$$Q'''_2 = C_p T_0 \left(1 - \frac{1}{\sqrt{\tau}}\right) = C_p \frac{T_0}{\sqrt{\tau}} (\sqrt{\tau} - 1)$$



Thus $\quad Q'_2 = Q''_2 + Q'''_2 = \frac{RT_0}{\gamma - 1} (\sqrt{\tau} - 1) \left(1 + \frac{\gamma}{\sqrt{\tau}}\right)$

Along BC, the specific heat C is given by

$$C dT = C_V dT + p dV = C_V dT + d\left(\frac{1}{2} \alpha V^2\right) = \left(C_V + \frac{1}{2} R\right) dT$$

Thus $$Q_1 = \frac{1}{2} R T_0 \frac{\gamma + 1}{\gamma - 1} \frac{\tau - 1}{\sqrt{\tau}}$$

Finally $$\eta = 1 - \frac{Q'_2}{Q_1} = 1 - 2 \frac{\sqrt{\tau} + \gamma}{\sqrt{\tau} + 1} \frac{1}{\gamma + 1} = \frac{(\gamma - 1)(\sqrt{\tau} - 1)}{(\gamma + 1)(\sqrt{\tau} + 1)}$$

2.128 We write Claussius inequality in the form

$$\int \frac{đ_1 Q}{T} - \int \frac{đ_2 Q}{T} \le 0$$

where $đQ$ is the heat transeferred to the system but $đ_2 Q$ is heat rejected by the system, both are $+ve$ and this explains the minus sign before $đ_2 Q$,

In this inequality $T_{max} > T > T_{min}$ and we can write

$$\int \frac{đ_1 Q}{T_{max}} - \int \frac{đ_2 Q}{T_{min}} < 0$$

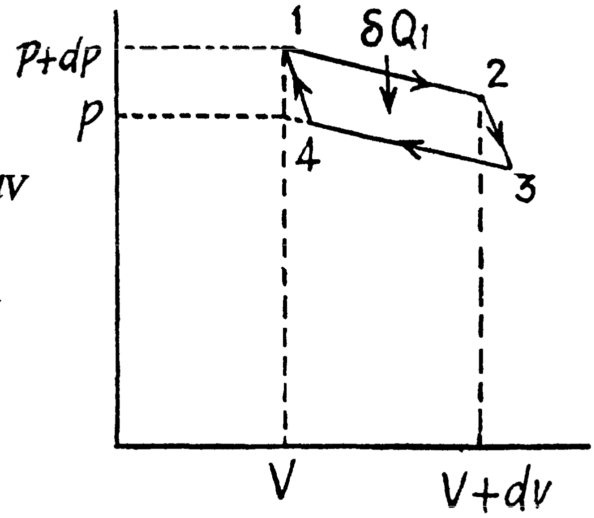
Thus $$\frac{Q_1}{T_{max}} < \frac{Q'_2}{T_{min}} \quad \text{or} \quad \frac{T_{min}}{T_{max}} < \frac{Q'_2}{Q_1}$$

or $$\eta = 1 - \frac{Q'_2}{Q_1} < 1 - \frac{T_{min}}{T_{max}} = \eta_{carnot}$$

2.129 We consider an infinitesimal carnot cycle with isothermal process at temperatures $T + dT$ and T.

Let δA be the work done in the cycle and δQ, be the heat received at the higher temperature. Then by Carnot's theorem

$$\frac{\delta A}{\delta Q_1} = \frac{dT}{T}$$



On the other hand $\delta A = dp\, dV = \left(\frac{\partial p}{\partial T}\right)_V dT\, dV$

while $\delta Q_1 = dU_{12} + p dV = \left[\left(\frac{\partial U}{\partial V}\right)_T + p\right] dV$

Hence $\left(\frac{\partial U}{\partial V}\right)_T + p = T\left(\frac{\partial p}{\partial T}\right)_V$

2.130 (a) In an isochoric process the entropy change will be

$$\Delta S = \int_{T_i}^{T_f} \frac{C_V dT}{T} = C_V \ln \frac{T_f}{T_i} = C_V \ln n = \frac{R \ln n}{\gamma - 1}$$

For carbon dioxide $\gamma = 1{\cdot}30$

so, $\Delta S = 19{\cdot}2$ Joule/°K – mole

(b) For an isobaric process,

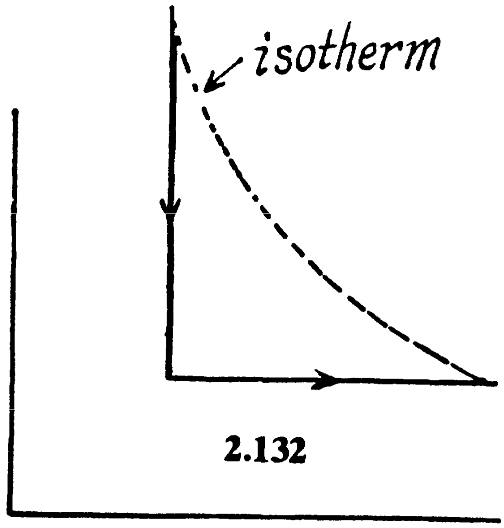
$$\Delta S = C_p \ln \frac{T_f}{T_i} = C_p \ln n = \frac{\gamma R \ln n}{\gamma - 1}$$

$= 25$ Joule/°K – mole

2.131 In an isothermal expansion

$$\Delta S = \nu R \ln \frac{V_f}{V_i}$$

so, $\frac{V_f}{V_i} = e^{\Delta S/\nu R} = 2{\cdot}0$ times



2.132

2.132 The entropy change depends on the final & initial states only, so we can calculate it directly along the isotherm, it is $\Delta S = 2R \ln n = 20$ J/°K (assuming that the final volume is n times the inital volume)

2.133 If the initial temperature is T_0 and volume is V_0 then in adiabatic expansion.

$$T V^{\gamma - 1} = T_0 V_0^{\gamma - 1}$$

so, $$T = T_0 n^{1-\gamma} = T_1 \text{ where } n = \frac{V_1}{V_0}$$

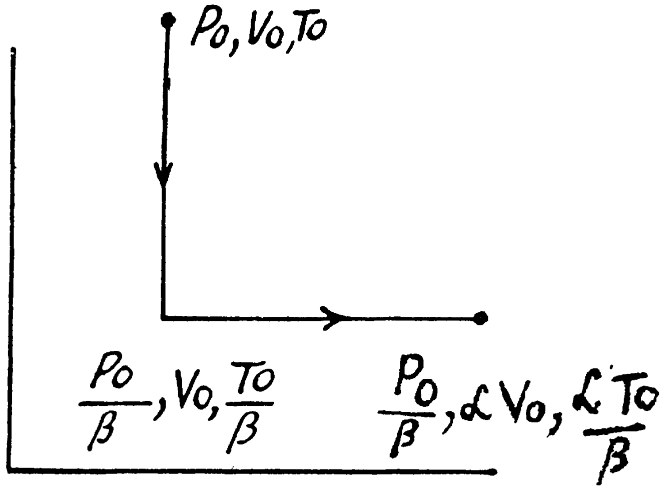
V_1 being the volume at the end of the adiabatic process. There is no entropy change in this process. Next the gas is compressed isobarically and the net entropy change is

$$\Delta S = \left(\frac{m}{M} C_p\right) \ln \frac{T_f}{T_1}$$

But $$\frac{V_1}{T_1} = \frac{V_0}{T_f}, \quad \text{or} \quad T_f = T_1 \frac{V_0}{V_1} = T_0 n^{-\gamma}$$

So $$\Delta S = \left(\frac{m}{M} C_p\right) \ln \frac{1}{n} = -\frac{m}{M} C_p \ln n = -\frac{m}{M} \frac{R\gamma}{\gamma - 1} \ln n = -9{\cdot}7 \text{ J/K}$$

2.134 The entropy change depends on the initial and final state only so can be calculated for any process whatsoever.



We choose to evaluate the entropy change along the pair of lines shown above. Then

$$\Delta S = \int_{T_0}^{T_0/\beta} \frac{\nu C_V \, dT}{T} + \int_{T_0/\beta}^{\alpha T_0/\beta} \nu C_p \frac{dT}{T}$$

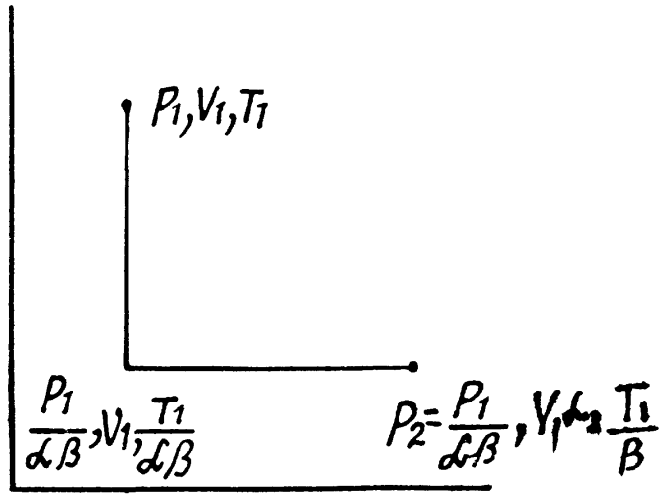
$$= (-C_V \ln \beta + C_p \ln \alpha)\,\nu = \frac{\nu R}{\gamma - 1}(\gamma \ln \alpha - \ln \beta) \approx -11 \frac{\text{Joule}}{{}^\circ\text{K}}$$

2.135 To calculate the required entropy difference we only have to calculate the entropy difference for a process in which the state of the gas in vessel 1 is changed to that in vessel 2.

$$\Delta S = \nu \left(\int_{T_1}^{T_1/\alpha\beta} C_V \frac{dT}{T} + \int_{T_1/\alpha\beta}^{T_1/\beta} C_p \frac{dT}{T} \right)$$

$$= \nu\,(C_p \ln \alpha - C_V \ln \alpha\beta)$$

$$= \nu \left(R \ln \alpha - \frac{R}{\gamma - 1} \ln \beta\right) = \nu R \left(\ln \alpha - \frac{\ln \beta}{\gamma - 1}\right)$$



With $\gamma = \frac{5}{3}$, $\alpha = 2$ and $\beta = 1{\cdot}5$, $\nu = 1{\cdot}2$,

this gives $\Delta S = 0{\cdot}85$ Joule/°K

2.136 For the polytropic process with index n

$$p V^n = \text{constant}$$

Along this process (See 2.122)

$$C = R\left(\frac{1}{\gamma - 1} - \frac{1}{n - 1}\right) = \frac{n - \gamma}{(\gamma - 1)(n - 1)} \cdot R$$

So $$\Delta S = \int_{T_0}^{\tau T_0} C \frac{dT}{T} = \frac{n - \gamma}{(\gamma - 1)(n - 1)} R \ln \tau$$

2.137 The process in question may be written as

$$\frac{p}{p_0} = \alpha \frac{V}{V_0}$$

where α is a constant and p_0, V_0 are some reference values. For this process (see 2.127) the specific heat is

$$C = C_V + \frac{1}{2} R = R\left(\frac{1}{\gamma - 1} + \frac{1}{2}\right) = \frac{1}{2} R \frac{\gamma + 1}{\gamma - 1}$$

Along the line volume increases α times then so does the pressure. The temperature must then increase α^2times. Thus

$$\Delta S = \int_{T_0}^{\alpha^2 T_0} \nu C \frac{dT}{T} = \frac{\nu R}{2} \frac{\gamma + 1}{\gamma - 1} \ln \alpha^2 = \nu R \frac{\gamma + 1}{\gamma - 1} \ln \alpha$$

if $\qquad \nu = 2, \ \gamma = \frac{5}{3}, \ \alpha = 2, \Delta S = 46{\cdot}1$ Joule/°K

2.138 Let (p_1, V_1) be a reference point on the line

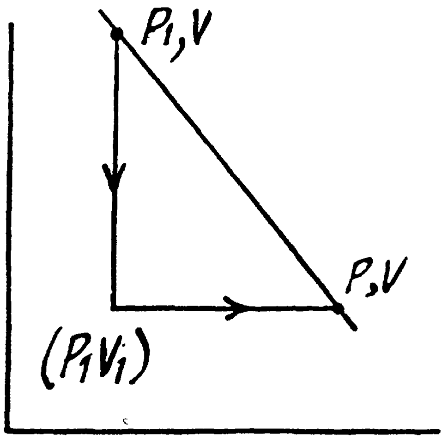
$$p = p_0 - \alpha V$$

and let (p, V) be any other point.

The entropy difference

$$\Delta S = S(p, V) - S(p_1, V_1)$$

$$= C_V \ln \frac{p}{p_1} + C_p \ln \frac{V}{V_1} = C_V \ln \frac{p_0 - \alpha V}{p_1} + C_p \ln \frac{V}{V_1}$$



For an exetremum of ΔS

$$\frac{\partial \Delta S}{\partial V} = \frac{-\alpha C_V}{p_0 - \alpha V} + \frac{C_p}{V} = 0$$

or $\ C_p (p_0 - \alpha V) - \alpha V C_V = 0$

or $\ \gamma (p_0 - \alpha V) - \alpha V = 0\ $ or $\ V = V_m = \dfrac{\gamma p_0}{\alpha(\gamma + 1)}$

This gives a maximum of ΔS because $\dfrac{\partial^2 \Delta S}{\partial V^2} < 0$

(**Note** :- a maximum of ΔS is a maximum of $S(p, V)$)

2.139 Along the process line : $S = aT + C_V \ln T$

or the specific heat is : $C = T \dfrac{dS}{dT} = aT + C_V$

On the other hand : $dQ = CdT = C_V dT + pdV$ for an ideal gas.

Thus, $\qquad pdV = \dfrac{RT}{V} dV = aT\, dT$

or $$\frac{R}{a}\frac{dV}{V} = dT \quad \text{or,} \quad \frac{R}{a}\ln V + \text{constant} = T$$

Using $T = T_0$ when $V = V_0$, we get, $T = T_0 + \frac{R}{a}\ln\frac{V}{V_0}$

2.140 For a Vander Waal gas

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT$$

The entropy change along an isotherm can be calculated from

$$\Delta S = \int_{V_1}^{V_2} \left(\frac{\partial S}{\partial V}\right)_T dV$$

It follows from (2.129) that

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V = \frac{R}{V - b}$$

assuming a, b to be known constants.

Thus $$\Delta S = R\ln\frac{V_2 - b}{V_1 - b}$$

2.141 We use, $$\Delta S = \int_{V_1, T_1}^{V_2, T_2} dS\,(V, T) = \int_{T_1}^{T_2} \left(\frac{\partial S}{\partial T}\right)_{V_1} dT + \int_{V_1}^{V_2} \left(\frac{\partial S}{\partial V}\right)_{T = T_2} dV$$

$$= \int_{T_1}^{T_2} \frac{C_V\,dT}{T} + \int_{V_1}^{V_2} \frac{R}{V - b}\,dV = C_V\ln\frac{T_2}{T_1} + R\ln\frac{V_2 - b}{V_1 - b}$$

assuming C_V, a, b to be known constants.

2.142 We can take $S \to 0$ as $T \to 0$ Then

$$S = \int_0^T C\frac{dT}{T} = \int_0^T aT^2\,dT = \frac{1}{3}aT^3$$

2.143 $$\Delta S = \int_{T_1}^{T_2} \frac{C\,dT}{T} = \int_{T_1}^{T_2} \frac{m\,(a + bT)}{T}\,dT = mb\,(T_2 - T_1) + ma\ln\frac{T_2}{T_1}$$

2.144 Here $T = a\,S^n$ or $S = \left(\frac{T}{a}\right)^{\frac{1}{n}}$

Then $$C = T\frac{1}{n}\frac{T^{\frac{1}{n}-1}}{a^{1/n}} = \frac{S}{n}$$

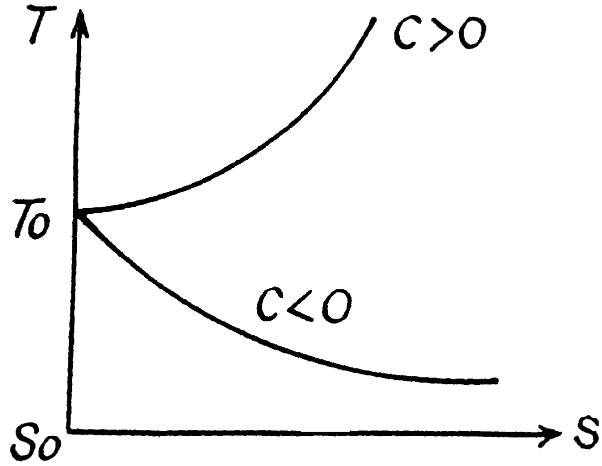
Clearly $$C < 0 \text{ if } n < 0.$$

2.145 We know,

$$S - S_0 = \int_{T_0}^{T} \frac{CdT}{T} = C\ln\frac{T}{T_0}$$

assuming C to be a known constant.

Then $\quad T = T_0 \exp\left(\frac{S - S_0}{C}\right)$



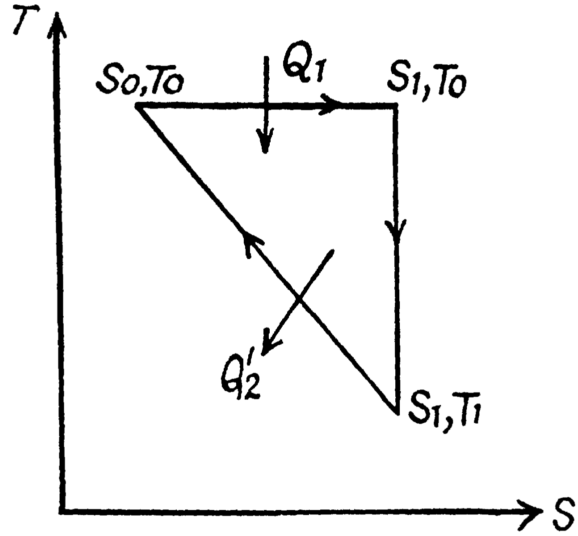
2.146 (a) $C = T\frac{dS}{dT} = -\frac{\alpha}{T}$

(b) $Q = \int_{T_1}^{T_2} CdT = \alpha\ln\frac{T_1}{T_2}$

(c) $W = \Delta Q - \Delta U \qquad = \alpha\ln\frac{T_1}{T_2} + C_V(T_1 - T_2)$

Since for an ideal gas C_V is constant and $\Delta U = C_V(T_2 - T_1)$

(U does not depend on V)



2.147 (a) We have from the definition

$Q = \int TdS =$ area under the curve

$Q_1 = T_0(S_1 - S_0)$

$Q'_2 = \frac{1}{2}(T_0 + T_1)(S_1 - S_0)$

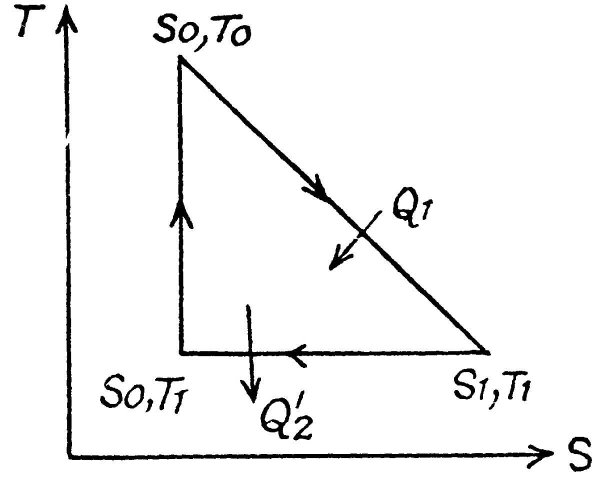
Thus, using $T_1 = \frac{T_0}{n}$,

$$\eta = 1 - \frac{T_0 + T_1}{2T_0} = 1 - \frac{1 + \frac{1}{n}}{2} = \frac{n-1}{2n}$$

(b) Here $Q_1 = \frac{1}{2}(S_1 - S_0)(T_1 + T_0)$

$Q'_2 = T_1(S_1 - S_0)$

$$\eta = 1 - \frac{2T_1}{T_1 + T_0} = \frac{T_0 - T_1}{T_0 - T_1} = \frac{n-1}{n+1}$$



2.148 In this case, called free expansion no work is done and no heat is exchanged. So internal energy must remain unchanged $U_f = U_i$. For an ideal gas this implies constant temperature $T_f = T_i$. The process is irreversible but the entropy change can be calculated by considering a reversible isothermal process. Then, as before

$$\Delta S = \int \frac{dQ}{T} = \int_{V_1}^{V_2} \frac{pdV}{T} = \nu R \ln n = 20{\cdot}1 \text{ J/K}$$

2.149 The process consists of two parts. The first part is free expansion in which $U_f = U_i$. The second part is adiabatic compression in which work done results in change of internal energy. Obviously,

$$0 = U_F - U_f + \int_{V_f}^{V_0} pdV, \; V_f = 2\,V_0$$

Now in the first part $p_f = \frac{1}{2} p_0$, $V_f = 2V_0$, because there is no change of temperature.

In the second part, $p\,V^\gamma = \frac{1}{2} p_0\,(2V_0)^\gamma = 2^{\gamma-1} p_0\,V_0^\gamma$

$$\int_{2V_0}^{V_0} pdV = \int_{2V_0}^{V_0} \frac{2^{\gamma-1} p_0\,V_0^\gamma}{V^\gamma}\,dV = \left[\frac{2^{\gamma-1} p_0\,V_0^\gamma}{-\gamma+1}\,V^{1-\gamma}\right]_{2V_0}^{V_0}$$

$$= 2^{\gamma-1} p_0\,V_0^\gamma\,V_0^{-\gamma+1}\,\frac{2^{-\gamma+1}-1}{\gamma-1} = -\frac{(2^{\gamma-1}-1}{\gamma-1}RT$$

Thus $$\Delta U = U_F - U_i = \frac{R\,T_0}{\gamma-1}(2^{\gamma-1}-1)$$

The entropy change $\Delta S = \Delta S_I + \Delta S_{II}$

$\Delta S_I = R \ln 2$ and $\Delta S_{II} = 0$ as the process is reversible adiabatic. Thus $\Delta S = R \ln 2$.

2.150 In all adiabatic processes

$$Q = U_f - U_i + A = 0$$

by virtue of the first law of thermodynamics. Thus,

$$U_f = U_i - A$$

For a slow process, $A' = \int_{V_0}^{V} pdV$ where for a quasistatic adiabatic process $pV^\gamma =$ constant.

On the other hand for a fast process the external work done is $A'' < A'$. In fact $A'' = 0$ for free expansion. Thus U'_f (slow) $< U''_f$ (fast)

Since U depends on temperature only, $T'_f < T''_f$

Consequently, $p''_f > p'_f$

(From the ideal gas equation $pV = RT$)

2.151 Let $V_1 = V_0$, $V_2 = n\,V_0$

Since the temperature is the same, the required entropy change can be calculated by considering isothermal expansion of the gas in either parts into the whole vessel.

Thus $$\Delta S = \Delta S_I + \Delta S_{II} = \nu_1 R \ln \frac{V_1 + V_2}{V_1} + \nu_2 R \ln \frac{V_1 + V_2}{V_2}$$

$$= \nu_1 R \ln (1 + n) + \nu_2 R \ln \frac{1 + n}{n} = 5{\cdot}1 \text{ J/K}$$

2.152 Let c_1 = specific heat of copper specific heat of water = c_2

Then $$\Delta S = \int_{7+273}^{T_0} \frac{c_2 m_2\, dT}{T} - \int_{T_0}^{97+273} \frac{m_1 c_1\, dT}{T} = m_2 c_2 \ln \frac{T_0}{280} - m_1 c_1 \ln \frac{370}{T_0}$$

T_0 is found from

$$c_2 m_2 (T_0 - 280) = m_1 c_1 (370 - T_0) \quad \text{or} \quad T_0 = \frac{280\, m_2 c_2 + 370\, m_1 c_1}{c_2 m_2 + m_1 c_1}$$

using $$c_1 = 0{\cdot}39 \text{ J/g °K},\ c_2 = 4{\cdot}18 \text{ J/g °K},$$

$$T_0 = 300\text{°K and } \Delta S = 28{\cdot}4 - 24{\cdot}5 = 3{\cdot}9 \text{ J / °K}$$

2.153 For an ideal gas the internal energy depends on temperature only. We can consider the process in question to be one of simultaneous free expansion. Then the total energy $U = U_1 + U_2$. Since

$U_1 = C_V T_1$, $U_2 = C_V T_2$, $U = 2C_V \dfrac{T_1 + T_2}{2}$ and $(T_1 + T_2)/2$ is the final temperature. The entropy change is obtained by considering isochoric processes because in effect, the gas remains confined to its vessel.

$$\Delta S = \int_{T_1}^{(T_1+T_2)/2} \frac{C_V\, dT}{T} - \int_{(T_1+T_2)/2}^{T_2} C_V \frac{dT}{T} = C_V \ln \frac{(T_1 + T_2)^2}{4\, T_1 T_2}$$

Since $$(T_1 + T_2)^2 = (T_1 - T_2)^2 + 4\, T_1 T_2\,,\ \Delta S > 0$$

2.154 (a) Each atom has a probability $\frac{1}{2}$ to be in either campartment. Thus

$$p = 2^{-N}$$

(b) Typical atomic velocity at room temperature is $\sim 10^5$ cm/s so it takes an atom 10^{-5} sec to cross the vessel. This is the relevant time scale for our problem. Let $T = 10^{-5}$ sec, then in time t there will be t/T crossing or arrangements of the atoms. This will be large enough to produce the given arrangement if

$$\frac{t}{\tau} 2^{-N} \sim 1 \quad \text{or} \quad N \sim \frac{\ln t/\tau}{\ln 2} \sim 75$$

2.155 The statistical weight is

$$N_{C_{N/2}} = \frac{N!}{N/2\,!\,\frac{N}{2}!} = \frac{10\times9\times8\times7\times6}{8\times4\times3\times2} = 252$$

The probability distribution is

$$N_{C_{N/2}}\, 2^{-N} = 252\times2^{-10} = 24{\cdot}6\ \%$$

2.156 The probabilites that the half A contains n molecules is

$$N_{C_n}\times 2^{-N} = \frac{N\,!}{n\,!\,(N-n)\,!}\,2^{-N}$$

2.157 The probability of one molecule being confined to the marked volume is

$$p = \frac{V}{V_0}$$

We can choose this molecule in many (N_{C_1}) ways. The probability that n molecules get confined to the marked volume is cearly

$$N_{C_n}\, p^n\,(1-p)^{N-n} = \frac{N\,!}{n\,!\,(N-n)\,!}\,p^n\,(1-p)^{N-n}$$

2.158 In a sphere of diameter d there are

$$N = \frac{\pi d^3}{6}\, n_0 \quad \text{molecules}$$

where $n_0 =$ Loschmidt's number = No. of molecules per unit volume (1 cc) under NTP. The relative fluctuation in this number is

$$\frac{\partial N}{N} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}} = \eta$$

$$\text{or} \quad \frac{1}{\eta^2} = \frac{\pi}{6}d^3\, n_0 \quad \text{or} \quad d^3 = \frac{6}{\pi\, n_0\, \eta^2} \quad \text{or} \quad d = \left(\frac{6}{\pi\,\eta^2\, n_0}\right)^{1/3} = 0{\cdot}41\ \mu\text{m}$$

The average number of molecules in this sphere is $\frac{1}{\eta^2} = 10^6$

2.159 For a monoatomic gas $C_V = \frac{3}{2}R$ per mole

The entropy change in the process is

$$\Delta S = S - S_0 = \int_{T_0}^{T_0+\Delta T} C_V\frac{dT}{T} = \frac{3}{2}R\ln\left(1+\frac{\Delta T}{T_0}\right)$$

Now from the Boltzmann equation

$$S = k\ln\Omega$$

$$\frac{\Omega}{\Omega_0} = e^{(S-S_0)/k} = \left(1+\frac{\Delta T}{T_0}\right)^{\frac{3N_A}{2}} = \left(1+\frac{1}{300}\right)^{\frac{3\times6}{2}\times10^{23}} = 10^{13}\times10^{21}$$

Thus the statistical weight increases by this factor.

2.5 LIQUIDS. CAPILLARY EFFECTS

2.160 (a) $\Delta p = \alpha\left(\frac{1}{d/2} + \frac{1}{d/2}\right) = \frac{4\alpha}{d}$

$$= \frac{4 \times 490 \times 10^{-3}}{1{\cdot}5 \times 10^{-6}} \frac{\text{N}}{\text{m}^2} = 1{\cdot}307 \times 10^6 \frac{\text{N}}{\text{m}^2} = 13 \text{ atmosphere}$$

(b) The soap bubble has two surfaces

so
$$\Delta p = 2\,\alpha\left(\frac{1}{d/2} + \frac{1}{d/2}\right) = \frac{8\alpha}{d}$$

$$= \frac{8 \times 45}{3 \times 10^{-3}} \times 10^{-3} = 1{\cdot}2 \times 10^{-3} \text{ atomsphere.}$$

2.161 The pressure just inside the hole will be less than the outside pressure by $4\,\alpha/d$. This can support a height h of Hg where

$$\rho\, g\, h = \frac{4\alpha}{d} \quad \text{or} \quad h = \frac{4\alpha}{\rho\, g\, d}$$

$$= \frac{4 \times 490 \times 10^{-3}}{13{\cdot}6 \times 10^3 \times 9{\cdot}8 \times 70 \times 10^{-6}} = \frac{200}{13{\cdot}6 \times 70} \approx {\cdot}21 \text{ m of Hg}$$

2.162 By Boyle's law

$$\left(p_0 + \frac{8\alpha}{d}\right)\frac{4\pi}{3}\left(\frac{d}{2}\right)^3 = \left(\frac{p_0}{n} + \frac{8\alpha}{\eta d}\right)\frac{4\pi}{3}\left(\frac{\eta\, d}{2}\right)^3$$

or
$$p_0\left(1 - \frac{\eta^3}{n}\right) = \frac{8\alpha}{d}(\eta^2 - 1)$$

Thus
$$\alpha = \frac{1}{8} p_0\, d\left(1 - \frac{\eta^3}{n}\right)(\eta^2 - 1)$$

2.163 The pressure has terms due to hydrostatic pressure and capillarity and they add

$$p = p_0 + \rho g h + \frac{4\alpha}{d}$$

$$= \left(1 + \frac{5 \times 9{\cdot}8 \times 10^3}{10^5} + \frac{4 \times {\cdot}73 \times 10^{-3}}{4 \times 10^{-6}} \times 10^{-5}\right) \text{atoms} = 2{\cdot}22 \text{ atom.}$$

2.164 By Boyle's law

$$\left(p_0 + h\, g\, \rho + \frac{4\alpha}{d}\right)\frac{\pi}{6} d^3 = \left(p_0 + \frac{4\alpha}{nd}\right)\frac{\pi}{6} n^3 d^3$$

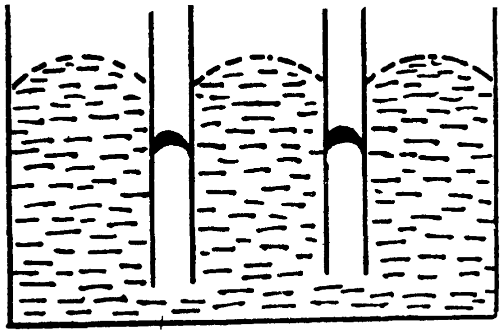
or
$$\left[hg\rho - p_0 (n^3 - 1)\right] = \frac{4\alpha}{d}(n^2 - 1)$$

or
$$h = \left[p_0 (n^3 - 1) + \frac{4\alpha}{d}(n^2 - 1)\right] \Big/ g\,\rho = 4{\cdot}98 \text{ meter of water}$$

2.165 Clearly

$$\Delta h\, \rho\, g = 4\,\alpha\,|\cos\theta|\left(\frac{1}{d_1} - \frac{1}{d_2}\right)$$

$$\Delta h = \frac{4\,\alpha\,|\cos\theta|\,(d_2 - d_1)}{d_1\, d_2\, \rho g} = 11 \text{ mm}$$



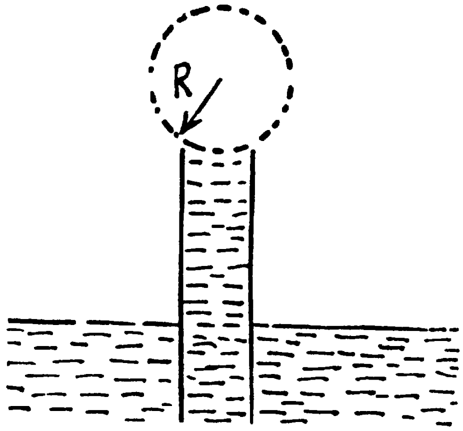
2.166 In a capillary with diameter $d = 0{\cdot}5$ mm water will rise to a height

$$\frac{2\alpha}{\rho g r} = \frac{4\alpha}{\rho g d}$$

$$= \frac{4 \times 73 \times 10^{-3}}{10^3 \times 9{\cdot}8 \times 0{\cdot}5 \times 10^{-3}} = 59{\cdot}6 \text{ mm}$$

Since this is greater than the height (= 25 mm) of the tube, a meniscus of radius R will be formed at the top of the tube, where

$$R = \frac{2\alpha}{\rho gh} = \frac{2 \times 73 \times 10^{-3}}{10^3 \times 9{\cdot}8 \times 25 \times 10^{-3}} \approx 0{\cdot}6 \text{ mm}$$



2.167 Initially the pressure of air in the cppillary is p_0 and it's length is l. When submerged under water, the pressure of air in the portion above water must be $p_0 + 4\frac{\alpha}{d}$, since the level of water inside the capillary is the same as the level outside. Thus by Boyle's law

$$\left(p_0 + \frac{4\alpha}{d}\right)(l - x) = p_0\, l$$

or $$\frac{4\alpha}{d}(l - x) = p_0 x \quad \text{or} \quad x = \frac{l}{1 + \frac{p_0 d}{4\alpha}}$$

2.168 We have by Boyle's law

$$\left(p_0 - \rho g h + \frac{4\,\alpha\cos\theta}{d}\right)(l - h) = p_0\, l$$

or, $$\frac{4\,\alpha\cos\theta}{d} = \rho\, gh + \frac{p_0\, h}{l - h}$$

Hence, $$\alpha = \left(\rho\, gh + \frac{p_0\, h}{l - h}\right)\frac{d}{4\cos\theta}$$

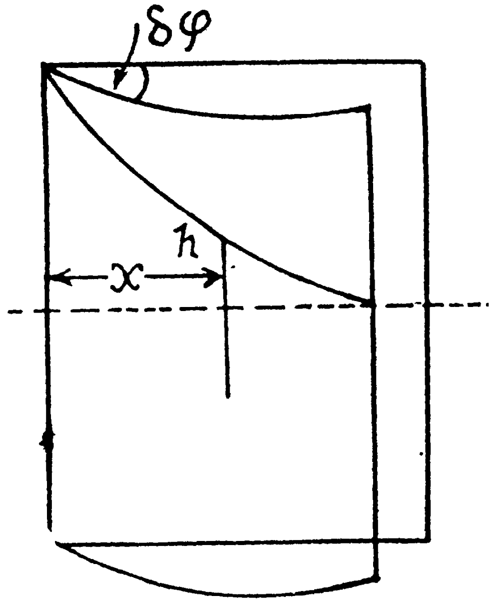
2.169 Suppose the liquid rises to a height h. Then the total energy of the liquid in the capillary is

$$E\,(h) = \frac{\pi}{4}\,(d_2^2 - d_1^2)\, h \times \rho g \times \frac{h}{2} - \pi\,(d_2 - d_1)\,\alpha h$$

Minimising E we get

$$h = \frac{4\alpha}{\rho g\,(d_2 - d_1)} = 6 \text{ cm.}$$

2.170 Let h be the height of the water level at a distance x from the edge. Then the total energy of water in the wedge above the level outside is.



$$E = \int x\,\delta\varphi \cdot dx \cdot h \cdot \rho\, g \frac{h}{2} - 2\int dx \cdot h \cdot \alpha \cos\theta$$

$$= \int dx \frac{1}{2} x \rho g \delta\varphi \left(h^2 - 2\frac{2\alpha\cos\theta}{x\rho g\delta\varphi} h\right)$$

$$= \int dx \frac{1}{2} x \rho g \delta\varphi \left[\left(h - \frac{2\alpha\cos\theta}{x\rho g\delta\varphi}\right)^2 - \frac{4\alpha^2\cos^2\theta}{x^2\rho^2 g^2\delta\varphi^2}\right]$$

This is minimum when $h = \dfrac{2\alpha\cos\theta}{x\rho g\delta\varphi}$

2.171 From the equation of continuity

$$\frac{\pi}{4} d^2 \cdot v = \frac{\pi}{4}\left(\frac{d}{n}\right)^2 \cdot V \quad \text{or} \quad V = n^2 v.$$

We then apply Bernoulli's theorem

$$\frac{p}{\rho} + \frac{1}{2} v^2 + \Phi = \text{constant}$$

The pressure p differs from the atmospheric pressure by capillary effects. At the upper section

$$p = p_0 + \frac{2\alpha}{d}$$

neglecting the curvature in the vertical plane. Thus,

$$\frac{p_0 + \dfrac{2\alpha}{d}}{\rho} + \frac{1}{2} v^2 + gl = \frac{p_0 + \dfrac{2n\alpha}{d}}{\rho} + \frac{1}{2} n^4 v^2$$

or

$$v = \sqrt{\frac{2gl - \dfrac{4\alpha}{\rho d}(n-1)}{n^4 - 1}}$$

Finally, the liquid coming out per second is,

$$V = \frac{1}{4}\pi d^2 \sqrt{\frac{2gl - \dfrac{4\alpha}{\rho d}(n-1)}{n^4 - 1}}$$

2.172 The radius of curvature of the drop is R_1 at the upper end of the drop and R_2 at the lower end. Then the pressure inside the drop is $p_0 + \dfrac{2\alpha}{R_1}$ at the top end and $p_0 + \dfrac{2\alpha}{R_2}$ at the bottom end. Hence

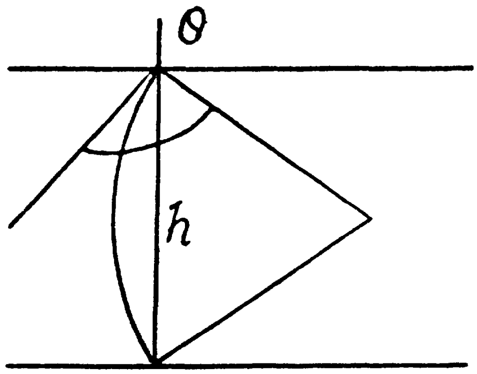
$$p_0 + \frac{2\alpha}{R_1} = p_0 + \frac{2\alpha}{R_2} + \rho g h \quad \text{or} \quad \frac{2\alpha(R_2 - R_1)}{R_1 R_2} = \rho g h$$

To a first approximation $R_1 \approx R_2 \approx \dfrac{h}{2}$ so $R_2 - R_1 \approx \dfrac{1}{8}\rho g h^3/\alpha. \approx 0{\cdot}20$ mm

if $\quad h = 2{\cdot}3$ mm, $\alpha = 73$ mN/m

2.173 We must first calculate the pressure difference inside the film from that outside. This is

$$p = \alpha\left(\frac{1}{r_1}+\frac{1}{r_2}\right).$$



Here $2\,r_1\,|\cos\theta| = h$ and $r_2 \sim -R$ the radius of the tablet and can be neglected. Thus the total force exerted by mercury drop on the upper glass plate is

$$\frac{2\pi R^2\alpha\,|\cos\theta|}{h} \text{ typically}$$

We should put h/n for h because the tablet is compresed n times. Then since Hg is nearly, incompressible, $\pi R^2 h$ = constants so $R \to R\sqrt{n}$. Thus,

$$\text{total force} = \frac{2\pi R^2\alpha\,|\cos\theta|}{h}n^2$$

Part of the force is needed to keep the Hg in the shape of a table rather than in the shape of infinitely thin sheet. This part can be calculated being putting $n = 1$ above. Thus

$$mg + \frac{2\pi R^2\alpha\,|\cos\theta|}{h} = \frac{2\pi R^2\alpha\,|\cos\theta|}{h}n^2$$

or $$m = \frac{2\pi R^2\alpha\,|\cos\theta|}{hg}(n^2-1) = 0{\cdot}7\text{ kg}$$

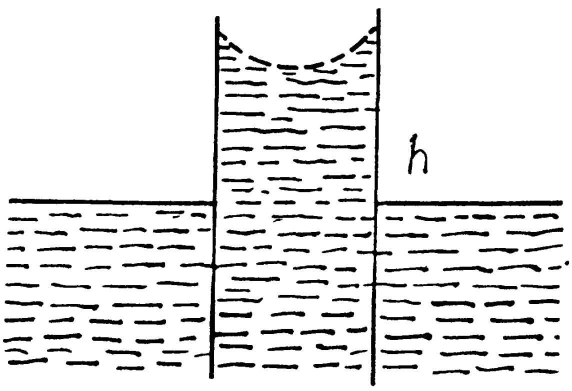
2.174 The pressure inside the film is less than that outside by an amount $\alpha\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$ where r_1 and r_2 are the principal radii of curvature of the meniscus. One of these is small being given by $h = 2\,r_1\cos\theta$ while the other is large and will be ignored. Then $F \approx \frac{2A\cos\theta}{h}\alpha$ where A = area of the water film between the plates.

Now $A = \frac{m}{\rho h}$ so $F = \frac{2m\alpha}{\rho h^2}$ when θ (the angle of contact) = 0

2.175 This is analogous to the previous problem except that : $A = \pi R^2$

So $$F = \frac{2\pi R^2\alpha}{h} = 0{\cdot}6\text{ kN}$$

2.176 The energy of the liquid between the plates is



$$E = l\,d\,h\,\rho\,g\frac{h}{2} - 2\,\alpha\,l\,h = \frac{1}{2}\rho\,g\,l\,d\,h^2 - 2\,\alpha\,l\,h$$

$$= \frac{1}{2}\rho\,g\,l\,d\left(h-\frac{2\alpha}{\rho g d}\right)^2 - \frac{2\alpha^2 l}{\rho g d}$$

This energy is minimum when, $h = \frac{2\alpha}{\rho g d}$ and the minimum potential energy is then $E_{\min} = -\frac{2\alpha^2 l}{\rho g d}$

The force of attraction between the plates can be obtained from this as

$$F = \frac{-\partial E_{\min}}{\partial d} = -\frac{2\alpha^2 l}{\rho g d^2} \text{ (minus sign means the force is attractive.)}$$

Thus $$F = -\frac{\alpha l h}{d} = 13\text{ N}$$

2.177 Suppose the radius of the bubble is x at some instant. Then the pressure inside is $p_0 + \frac{4\alpha}{x}$. The flow through the capillary is by Poiseuille's equation,

$$Q = \frac{\pi r^4}{8\eta l}\frac{4\alpha}{x} = -4\pi^2\frac{dx}{dt}$$

Integrating $\frac{\pi r^4 \alpha}{2\eta l} t = \pi (R^4 - x^4)$ where we have used the fact that $t = 0$ where $x = R$.

This gives $t = \frac{2\eta l R^4}{\alpha r^4}$ as the life time of the bubble corresponding to $x = 0$

2.178 If the liquid rises to a height h, the energy of the liquid column becomes

$$E = \rho g \pi r^2 h \cdot \frac{h}{2} - 2\pi r h \alpha = \frac{1}{2}\rho g \pi \left(r h - 2\frac{\alpha}{\rho g}\right)^2 - \frac{2\pi\alpha^2}{\rho g}$$

This is minimum when $rh = \frac{2\alpha}{\rho g}$ and that is relevant height to which water must rise.

At this point, $$E_{min} = -\frac{2\pi\alpha^2}{\rho g}$$

Since $E = 0$ in the absence of surface tension a heat $Q = \frac{2\pi\alpha^2}{\rho g}$ must have been liberated.

2.179 (a) The free energy per unit area being α,

$$F = \pi \alpha d^2 = 3\ \mu J$$

(b) $F = 2\pi\alpha d^2$ because the soap bubble has two surfaces. Substitution gives $F = 10\ \mu J$

2.180 When two mercury drops each of diameter d merge, the resulting drop has diameter d_1

where $$\frac{\pi}{6}d_1^3 = \frac{\pi}{6}d^3 \times 2 \quad \text{or,} \quad d_1 = 2^{1/3} d$$

The increase in free energy is

$$\Delta F = \pi 2^{2/3} d^2 \alpha - 2\pi d^2 \alpha = 2\pi d^2 \alpha (2^{-1/3} - 1) = -1{\cdot}43\ \mu J$$

2.181 Work must be done to stretch the soap film and compress the air inside. The former is simply $2\alpha \times 4\pi R^2 = 8\pi R^2 \alpha$, there being two sides of the film. To get the latter we note that the compression is isothermal and work done is

$$-\int_{V_i = V_0}^{V_f = V} p dV \quad \text{where} \quad V_0 p_0 = \left(p_0 + \frac{4\alpha}{R}\right)\cdot V, \ V = \frac{4\pi}{3}R^3$$

or $$V_0 = \frac{pV}{p_0},\ p = p_0 + \frac{4\alpha}{R}$$

and minus sign is needed becaue we are calculating work done on the system. Thus since pV remains constants, the work done is

$$pV \ln\frac{V_0}{V} = pV\ln\frac{p}{p_0}$$

So $$A' = 8\pi R^2 \alpha + pV\ln\frac{p}{p_0}$$

2.182 When heat is given to a soap bubble the temperature of the air inside rises and the bubble expands but unless the bubble bursts, the amount of air inside does not change. Further we shall neglect the variation of the surface tension with temperature. Then from the gas equations

$$\left(p_0 + \frac{4\alpha}{r}\right)\frac{4\pi}{3}r^3 = \nu R T, \ \nu = \text{Constant}$$

Differentiating

$$\left(p_0 + \frac{8\alpha}{3r}\right) 4\pi r^2 dr = \nu R\, dT$$

or
$$dV = 4\pi r^2 dr = \frac{\nu R dT}{p_0 + \dfrac{8\alpha}{3r}}$$

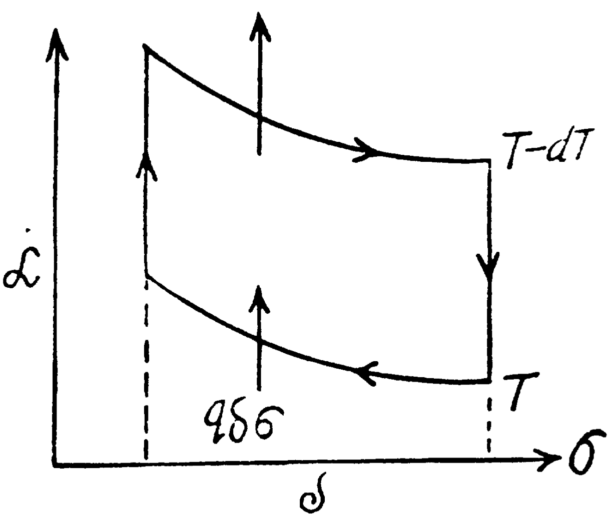
Now from the first law

$$đQ = \nu C dT = \nu C_V dT + \frac{\nu R\, dT}{p_0 + \dfrac{8\alpha}{3r}} \cdot \left(p_0 + \frac{4\alpha}{r}\right)$$

or
$$C = C_V + R\frac{p_0 + \dfrac{4\alpha}{r}}{p_0 \dfrac{8\alpha}{3r}}$$

using
$$C_p = C_V + R\,, \ C = C_p + \frac{\dfrac{1}{2}R}{1 + \dfrac{3p_0 r}{8\alpha}}$$

2.183 Consider an infinitesimal Carnot cycle with isotherms at $T - dT$ and T. Let A be the work done during the cycle. Then



$$A = [\alpha(T - dT) - \alpha(T)]\,\delta\sigma = -\frac{d\alpha}{dT}dT\,\delta\sigma$$

Where $\delta\sigma$ is the change in the area of film (we are considering only one surface).

Then $\eta = \dfrac{A}{Q_1} = \dfrac{dT}{T}$ by Carnot therom.

or $\dfrac{-\dfrac{d\alpha}{dT}dT\,\delta\sigma}{q\,\delta\sigma} = \dfrac{dT}{T}$ or $q = -T\dfrac{d\alpha}{dT}$

2.184 As before we can calculate the heat required. It, is taking into account two sides of the soap film

$$\delta q = -T\frac{d\alpha}{dT}\delta\sigma \times 2$$

Thus
$$\Delta S = \frac{\delta q}{T} = -2\frac{d\alpha}{dT}\delta\sigma$$

Now $\quad \Delta F = 2\alpha\,\delta\sigma$ so, $\Delta U = \Delta F + T\Delta S = 2\left(\alpha - T\dfrac{d\alpha}{dT}\right)\delta\sigma$

2.6 PHASE TRANSFORMATIONS

2.185 The condensation takes place at constant pressure and temperature and the work done is

$$p\,\Delta V$$

where ΔV is the volume of the condensed vapour in the vapour phase. It is

$$p\,\Delta V = \frac{\Delta m}{M}RT = 120{\cdot}6\text{ J}$$

where $M = 18$ gm is the molecular weight of water.

2.186 The specific volume of water (the liquid) will be written as V'_l. Since $V'_v >> V'_l$, most of the weight is due to water. Thus if m_l is mass of the liquid and m_v that of the vapour then

$$m = m_l + m_v$$

$$V = m_l V'_l + m_v V'_v \quad \text{or} \quad V - mV'_l = m_v (V'_v - V'_l)$$

So $m_v = \dfrac{V - mV'_l}{V'_v - V'_1} = 20$ gm in the present case. Its volume is $m_v V'_v = 1{\cdot}01$

2.187 The volume of the condensed vapour was originally $V_0 - V$ at temperature $T = 373$ K. Its mass will be given by

$p(V_0 - V) = \dfrac{m}{M}RT$ or $m = \dfrac{Mp(V_0 - V)}{RT} = 2$ gm where $p =$ atmospheric pressure

2.188 We let $V'_l =$ specific volume of liquid. $V'_v = N V'_l =$ specific volume of vapour.
Let $V =$ Original volume of the vapour. Then

$$M\frac{pV}{RT} = m_l + m_v = \frac{V}{N V'_l} \quad \text{or} \quad \frac{V}{n} = (m_l + Nm_v)\,V'_l$$

So $\quad (N-1)\,m_l V'_l = V\left(1 - \dfrac{1}{n}\right) = \dfrac{V}{n}(n-1) \quad \text{or} \quad \eta = \dfrac{m_l V'_l}{V/n} = \dfrac{n-1}{N-1}$

In the case when the final volume of the substance corresponds to the midpoint of a horizontal portion of the isothermal line in the p, v diagram, the final volume must be $(1+N)\dfrac{V'_l}{2}$ per unit mass of the substance. Of this the volume of the liquid is $V'_1/2$ per unit total mass of the substance.

Thus
$$\eta = \frac{1}{1+N}$$

2.189 From the first law of thermodynamics

$$\Delta U + A = Q = m\,q$$

where q is the specific latent heat of vaporization

Now
$$A = p(V'_v - V'_l)\,m = m\frac{RT}{M}$$

Thus
$$\Delta U = m\left(q - \frac{RT}{M}\right)$$

For water this gives $\approx 2{\cdot}08 \times 10^6$ Joules.

2.190 Some of the heat used in heating water to the boiling temperature

$T = 100°C = 373$ K. The remaining heat

$$= Q - m\,c\,\Delta T$$

($c =$ specific heat of water, $\Delta T = 100\,K$) is used to create vapour. If the piston rises to a height h then the volume of vapour will be $\approx sh$(neglecting water). Its mass will be $\frac{p_0\,sh}{RT} \times M$ and heat of vapourization will be $\frac{p_0\,sh\,Mq}{RT}$. To this must be added the work done in creating the saturated vapour $\approx p_0\,sh$. Thus

$$Q - m\,c\,\Delta T \approx p_0\,S\,h\left(1 + \frac{qM}{RT}\right) \quad \text{or} \quad h = \frac{Q - m\,c\,\Delta T}{p_0\,s\left(1 + \frac{qM}{RT}\right)} = 20 \text{ cm}$$

2.191 A quantity $\frac{mc\,(T - T_0)}{q}$ of saturated vapour must condense to heat the water to boiling point $T = 373°$K

(Here $c =$ specific heat of water, $T_0 = 295$ K $=$ initial water temperature).

The work done in lowering the piston will then be

$$\frac{mc\,(T - T_0)}{q} \times \frac{RT}{M} = 25 \text{ J},$$

since work done per unit mass of the condensed vapour is $p\,V = \frac{RT}{M}$

2.192 Given $\Delta P = \frac{\rho_v}{\rho_l}\frac{2\alpha}{r} = \frac{\rho_v}{\rho_l} \times \frac{4\alpha}{d} = \eta\,p_{vap} = \eta\,\frac{\frac{m}{M}RT}{V_{vap}} = \frac{\eta\,RT}{M}\rho_v$

or
$$d = \frac{4\,\alpha\,M}{\rho_1\,RT\,\eta}$$

For water $\alpha = 73$ dynes/cm, $M = 18$gm, $\rho_l =$ gm/cc, $T = 300$ K, and with $\eta \approx 0{\cdot}01$, we get

$$d \approx 0{\cdot}2\,\mu\,\text{m}$$

2.193 In equilibrium the number of "liquid" molecules evaporoting must equals the number of "vapour" molecules condensing. By kinetic theory, this number is

$$\eta \times \frac{1}{4} n <v> = \eta \times \frac{1}{4} n \times \sqrt{\frac{8}{\pi}\frac{kT}{m}}$$

Its mass is

$$\mu = m \times \eta \times n \times \sqrt{\frac{kT}{2\pi m}} = \eta\,n\,k\,T\sqrt{\frac{m}{2\pi kt}}$$

$$= \eta\,p_0\sqrt{\frac{M}{2\pi RT}} = 0{\cdot}35 \text{ g/cm}^2\cdot\text{s}.$$

where p_0 is atmospheric pressure and $T = 373$ K and $M =$ molecular weight of water.

2.194 Here we must assume that μ is also the rate at which the tungsten filament loses mass when in an atmosphere of its own vapour at this temperature and that η (of the previous problem) $\approx$ 1. Then

$$p = \mu \sqrt{\frac{2\pi RT}{M}} = 0{\cdot}9 \text{ n Pa}$$

from the previous problem where $p =$ pressure of the saturated vapour.

2.195 From the Vander Waals equation

$$p = \frac{RT}{V-b} - \frac{a}{V^2}$$

where $V =$ Volume of one gm mole of the substances.

For water $V = 18$ c.c. per mole $= 1{\cdot}8 \times 10^{-2}$ litre per mole

$$a = 5{\cdot}47 \text{ atmos} \cdot \frac{\text{litre}^2}{\text{mole}^2}$$

If molecular attraction vanished the equation will be

$$p' = \frac{RT}{V-b}$$

for the same specific volume. Thus

$$\Delta p = \frac{a}{V^2} = \frac{5{\cdot}47}{1{\cdot}8 \times 1{\cdot}8} \times 10^4 \text{ atmos} \approx 1{\cdot}7 \times 10^4 \text{ atmos}$$

2.196 The internal pressure being $\frac{a}{V^2}$, the work done in condensation is

$$\int_{V_l}^{V_g} \frac{a}{V^2}\, dV = \frac{a}{V_l} - \frac{a}{V_g} \approx \frac{a}{V_l}$$

This by assumption is Mq, M being the molecular weight and V_l, V_g being the molar volumes of the liquid and gas.

Thus
$$p_i = \frac{a}{V_l^2} = \frac{Mq}{V_l} = \rho q$$

where ρ is the density of the liquid. For water $p_i \approx 3{\cdot}3 \times 10^{13}$ atm

2.197 The Vandar Waal's equation can be written as (for one mole)

$$p(V) = \frac{RT}{V-b} - \frac{a}{V^2}$$

At the critical point $\left(\frac{\partial p}{\partial V}\right)_T$ and $\left(\frac{\partial^2 p}{\partial V^2}\right)_T$ vanish. Thus

$$0 = -\frac{RT}{(V-b)^2} + \frac{2a}{V^3} \quad \text{or} \quad \frac{RT}{(V-b)^2} = \frac{2a}{V^3}$$

$$0 = \frac{2RT}{(V-b)^3} - \frac{6a}{V^4} \quad \text{or} \quad \frac{RT}{(V-b)^3} = \frac{3a}{V^4}$$

Solving these simultaneously we get on division

$$V - b = \frac{2}{3} V, \quad V = 3b = V_{MCr}$$

This is the critical molar volume. Putting this back

$$\frac{R\,T_{Cr}}{4b^2} = \frac{2a}{27b^3} \quad \text{or} \quad T_{Cr} = \frac{8a}{27\,bR}$$

Finally $$p_{Cr} = \frac{RT_{Cr}}{V_{MCr} - b} - \frac{a}{V_{MCr}^2} = \frac{4a}{27b^2} - \frac{a}{9b^2} = \frac{a}{27b^2}$$

From these we see that $$\frac{P_{Cr}\,V_{MCr}}{RT_{Cr}} = \frac{a/9b}{8a/27\,b} = \frac{3}{8}$$

2.198 $$\frac{P_{Cr}}{RT_{Cr}} = \frac{a/27\,b^2}{8\,a/27\,b} = \frac{1}{8\,b}$$

Thus $$b = R\frac{T_{Cr}}{8\,p_{Cr}} = \frac{0{\cdot}082 \times 304}{73 \times 8} = 0{\cdot}043 \text{ litre/mol}$$

and $$\frac{(RT_{Cr})^2}{p_{Cr}} = \frac{64a}{27} \quad \text{or} \quad a = \frac{27}{64}(RT_{Cr})^2/p_{Cr} = 3{\cdot}59\,\frac{\text{atm}\cdot\text{litre}^2}{(\text{mol})^2}$$

2.199 Specific volume is molar volume divided by molecular weight. Thus

$$V'_{Cr} = \frac{V_{MCr}}{M} = \frac{3\,RT_{Cr}}{8\,M\,p_{Cr}} = \frac{3 \times {\cdot}082 \times 562}{8 \times 78 \times 47}\,\frac{\text{litre}}{g} = 4{\cdot}71\,\frac{\text{cc}}{g}$$

2.200 $$\left(p + \frac{a}{V_M^2}\right)(V_m - b) = RT$$

or $$\frac{p + \frac{a}{V_m^2}}{P_{Cr}} \times \frac{V_m - b}{V_{MCr}} = \frac{8}{3}\frac{T}{T_{Cr}}$$

or $$\left(\pi + \frac{a}{p_{Cr}\,V_m^2}\right) \times \left(\nu - \frac{b}{V_{MCr}}\right) = \frac{8}{3}\tau,$$

where $$\pi = \frac{p}{p_{Cr}},\ \nu = \frac{V_m}{V_{MCr}},\ \tau = \frac{T}{T_{Cr}}$$

or $$\left(\pi + \frac{27\,b^2}{V_M^2}\right)\left(\nu - \frac{1}{3}\right) = \frac{8}{3}\tau, \quad \text{or} \quad \left(\pi + \frac{3}{\nu^2}\right)\left(\nu - \frac{1}{3}\right) = \frac{8}{3}\tau$$

When $$\pi = 12 \text{ and } \nu = \frac{1}{2},\ \tau = \frac{3}{8} \times 24 \times \frac{1}{6} = \frac{3}{2}$$

2.201 (a) The ciritical Volume V_{MCr} is the maximum volume in the liquid phase and the mimimum volume in the gaseous. Thus

$$V_{\max} = \frac{1000}{18} \times 3 \times {\cdot}030 \text{ litre} = 5 \text{ litre}$$

(b) The critical pressure is the maximum possible pressure in the vapour phase in equilibrium with liquid phase. Thus

$$p_{\max} = \frac{a}{27\,b^2} = \frac{5{\cdot}47}{27 \times {\cdot}03 \times {\cdot}03} = 225\text{atmosphere}$$

2.202 $T_{Cr} = \frac{8}{27}\frac{a}{bR} = \frac{8}{27} \times \frac{3{\cdot}62}{{\cdot}043 \times {\cdot}082} = 304\text{ K}$

$$\rho_{Cr} = \frac{M}{3b} = \frac{44}{3 \times 43}\text{gm/c.c.} = 0{\cdot}34\text{ gm/c.c.}$$

2.203 The vessel is such that either vapour or liquid of mass m occupies it at critical point. Then its volume will be

$$v_{Cr} = \frac{m}{M} V_{MCr} = \frac{3}{8}\frac{RT_{Cr}}{p_{Cr}}\frac{m}{M}$$

The corresponding volume in liquid phase at room temperature is

$$V = \frac{m}{\rho}$$

where ρ = density of liquid ether at room tmeperature. Thus

$$\eta = \frac{V}{V_{Cr}} = \frac{8\,Mp_{Cr}}{3\,RT_{Cr}\,\rho} \approx 0{\cdot}254$$

using the given data (and ρ = 720gm per litre)

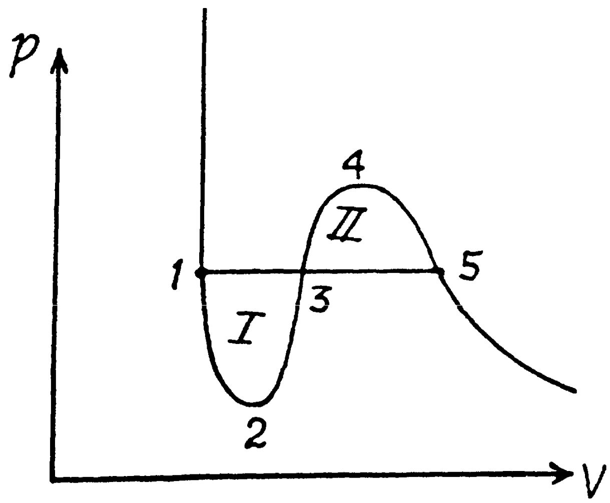
2.204 We apply the relation (T = constant)

$$T\oint dS = \oint dU + \oint p\,dV$$

to the cycle 1234531.

Here $\oint dS = \oint dU = 0$

So $\oint p\,dV = 0$



This implies that the areas I and II are equal. This reasoning is inapplicable to the cycle 1231, for example. This cycle is irreversible because it involves the irreversible transition from a single phase to a two-phase state at the point 3.

2.205 When a portion of supercool water turns into ice some heat is liberated, which should heat it upto ice point. Neglecting the variation of specific heat of water, the fraction of water turning inot ice is clearly

$$f = \frac{c\,|t|}{q} = 0{\cdot}25$$

where c = specific heat of water and q = latent heat of fusion of ice, Clearly $f = 1$ at $t = -80°C$

2.206 From the Claussius-Clapeyron (C–C)equations

$$\frac{dT}{dp} = \frac{T\left(V'_2 - V'_1\right)}{q_{12}}$$

$q_{12}$is the specific latent heat absorbed in $1 \rightarrow 2$ (1 = solid, 2 = liquid)

$$\Delta T = \frac{T\left(V'_w - V'_{ice}\right)}{q_{12}} \Delta p = -\frac{273 \times \cdot 091}{333} \times 1 \frac{\text{atm} \times \text{cm}^3 \times \text{K}}{\text{joule}}$$

$$1 \frac{\text{atm} \times \text{cm}^3}{\text{Joule}} \approx \frac{10^5 \frac{\text{N}}{\text{m}^2} \times 10^{-6}\,\text{m}^3}{\text{Joule}} = 10^{-1}, \; \Delta T = -\cdot 0075\text{ K}$$

2.207 Here 1 = liquid, 2 = Steam

$$\Delta T = \frac{T\left(V'_s - V'_{liq}\right)}{q_{12}} \Delta P$$

or $$V'_s \approx \frac{q_{12}}{T}\frac{\Delta T}{\Delta p} = \frac{2250}{373} \times \frac{0 \cdot 9}{3 \cdot 2} \times 10^{-3}\,\text{m}^3/\text{g} = 1 \cdot 7\,\text{m}^3/\text{kg}$$

2.208 From C-C equations

$$\frac{dp}{dT} = \frac{q_{12}}{T\left(V'_2 - V'_1\right)} \approx \frac{q_{12}}{T V'_2}$$

Assuming the saturated vapour to be ideal gas

$$\frac{1}{V'_2} = \frac{mp}{RT}, \text{ Thus } \Delta p = \frac{Mq}{RT^2} p\,\Delta T$$

and $$p \approx p_0\left(1 + \frac{Mq}{RT^2}\,\Delta T\right) \approx 1 \cdot 04 \text{ atmosphere}$$

2.209 From C-C equation, neglecting the voolume of the liquid

$$\frac{dp}{dT} \approx \frac{q_{12}}{TV'_2} \approx \frac{Mq}{RT^2}p, \; (q = q_{12})$$

or $$\frac{dp}{p} = \frac{Mq}{RT}\frac{dT}{T}$$

Now $$pV = \frac{m}{M}RT \text{ or } m = \frac{MpV}{RT} \text{ for a perfect gas}$$

So $$\frac{dm}{m} = \frac{dp}{p} - \frac{dT}{T} \; (V \text{ is Const = specific volume})$$

$$= \left(\frac{Mq}{RT} - 1\right)\frac{dT}{T} = \left(\frac{18 \times 2250}{8 \cdot 3 \times 373} - 1\right) \times \frac{1 \cdot 5}{373} \approx 4 \cdot 85\ \%$$

2.210 From C-C equation

$$\frac{dp}{dT} \approx \frac{q}{T V'_2} \approx \frac{Mq}{R T^2} p$$

Integrating $\ln p \approx$ constant $- \dfrac{Mq}{RT}$

So
$$p \approx p_0 \exp\left[\frac{Mq}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]$$

This is reasonable for $|T - T_0| << T_0$, and far below critical temperature.

2.211 As before (2.206) the lowering of melting point is given by

$$\Delta T = -\frac{T \Delta V'}{q} p$$

The superheated ice will then melt in part. The fraction that will melt is

$$\eta = \frac{C T \Delta V'}{q^2} p \approx \cdot 03$$

2.212 (a) The equations of the transition lines are

$$\log p = 9{\cdot}05 - \frac{1800}{T} : \text{ Solid gas}$$

$$= 6{\cdot}78 - \frac{1310}{T} : \text{ Liquid gas}$$

At the triple point they intersect. Thus

$$2{\cdot}27 = \frac{490}{T_{tr}} \quad \text{or} \quad T_{tr} = \frac{490}{2{\cdot}27} = 216\text{ K}$$

corresponding p_{tr} is 5.14 atmosphere.

In the formula $\log p = a - \frac{b}{T}$, we compare b with the corresponding term in the equation in 2.210. Then

$$\ln p = a \times 2{\cdot}303 - \frac{2{\cdot}303\, b}{T} \quad \text{So,} \quad 2{\cdot}303 = \frac{Mq}{R}$$

or,
$$q_{sublimation} = \frac{2{\cdot}303 \times 1800 \times 8{\cdot}31}{44} = 783\text{ J/gm}$$

$$q_{liquid-gas} = \frac{2{\cdot}303 \times 1310 \times 8{\cdot}31}{44} = 570\text{ J/gm}$$

Finally $q_{solid-liquid} = 213$ J/gm on subtraction

2.213
$$\Delta S = \int_{T_1}^{T_2} mc \frac{dT}{T} + \frac{mq}{T_2} \approx m\left(c \ln\frac{T_2}{T_1} + \frac{q}{T_2}\right)$$

$$= 10^3\left(4{\cdot}18 \ln\frac{373}{283} + \frac{2250}{373}\right) \approx 7{\cdot}2\text{ kJ/K}$$

2.214 $\Delta S = \frac{q_m}{T_1} + c \ln \frac{T_2}{T} + \frac{q_v}{T_2}$

$$= \frac{333}{273} + 4{\cdot}18 \ln \frac{373}{283} \approx 8{\cdot}56 \text{ J/°K}$$

2.215 c = specific heat of copper = $0{\cdot}39 \frac{\text{J}}{\text{g} \cdot \text{K}}$. Suppose all ice does not melt, then

heat rejected = $90 \times 0{\cdot}39\,(90 - 0) = 3159$ J

heat gained by ice = $50 \times 2{\cdot}09 \times 3 + x \times 333$

Thus $x = 8{\cdot}5$ gm

The hypothesis is correct and final temperature will be $T = 273$K.

Hence change in entropy of copper piece

$$= mc \ln \frac{273}{363} = -10 \text{ J/K.}$$

2.216 (a) Here $t_2 = 60$°C. Suppose the final temperature is t °C. Then

$$\text{heat lost by water} = m_2\, c\, (t_2 - t)$$

$$\text{heat gained by ice} = m_1 q_m + m_1 c\,(t - t_1), \text{ if all ice melts}$$

In this case $m_1 q_m = m_2 \times 4{\cdot}18\,(60 - t)$, for $m_1 = m_2$

So the final temperature will be 0°C and only some ice will melt.

Then $100 \times 4{\cdot}18\,(60) = m'_1 \times 333$

$m'_1 = 75{\cdot}3$ gm = amount of ice that will melt

Finally $\Delta S = 75{\cdot}3 \times \frac{333}{273} + 100 \times 4{\cdot}18 \ln \frac{273}{333}$

$$\Delta S = \frac{m'_1 q_m}{T_1} + m_2 c \ln \frac{T_1}{T_2}$$

$$= m_2 c \frac{(T_2 - T_1)}{T_1} - m_2 \ln \frac{T_2}{T_1}$$

$$= m_2 C \left[\frac{T_2}{T_1} - 1 - \ln \frac{T_2}{T_1}\right] = 8{\cdot}8 \text{ J/K}$$

(b) If $m_2 c t_2 > m_1 q_m$ then all ice will melt as one can check and the final temperature can be obtained like this

$$m_2 c\,(T_2 - T) = m_1 q_m + m_1 c\,(T - T_1)$$

$$(m_2 T_2 + m_1 T_1)\, c - m_1 q_m = (m_1 + m_2)\, cT$$

or $$T = \frac{m_2 T_2 + m_1 T_1 - \frac{m_1 q_m}{c}}{m_1 + m_2} = 280 \text{ K}$$

and $$\Delta S = \frac{m_1 q}{T_1} + c\left(m_1 \ln \frac{T}{T_1} - m_2 \ln \frac{T_2}{T}\right) = 19 \text{ J/K}$$

2.217 $\Delta S = -\dfrac{m\, q_1}{T_2} - mc\ln\dfrac{T_2}{T_1} + \dfrac{M\, q_{ice}}{T_1}$

where $$M\, q_{ice} = m\left(q_2 + c\,(T_2 - T_1)\right)$$

$$= mq_2\left(\frac{1}{T_1} - \frac{1}{T_2}\right) + mc\left(\frac{T_2}{T_1} - 1 - \frac{T_2}{T_1}\right)$$

$$= 0{\cdot}2245 + 0{\cdot}2564 \approx 0{\cdot}48 \text{ J/K}$$

2.218 When heat dQ is given to the vapour its temperature will change by dT, pressure by dp and volume by dV, it being assumed that the vapour remains saturated.

Then by C-C equation

$$\frac{dp}{dT} = \frac{q}{TV'}\,(V'_{\text{vapour}} >> V'_{\text{Liq}}), \quad \text{or} \quad dp = \frac{q}{TV'}\,dT$$

on the other hand, $pV' = \dfrac{RT}{M}$

So $$pdV' + V'\,dp = \frac{RdT}{M},$$

Hence $$pdV' = \left(\frac{R}{M} - \frac{q}{T}\right)dT$$

finally $$dQ = CdT = dU + pdV'$$

$$= C_V\,dT + \left(\frac{T}{M} - \frac{q}{T}\right)dT = C_p\,dT - \frac{q}{T}\,dT$$

(C_p, C_V refer to unit mass here). Thus

$$C = C_p - \frac{q}{T}$$

For water $C_p = \dfrac{R\gamma}{\gamma - 1}\cdot\dfrac{1}{M}$ with $\gamma = 1{\cdot}32$ and $M = 18$

So $$C_p = 1{\cdot}90 \text{ J/gm K}$$

and $$C = -4{\cdot}13 \text{ J/gm°K} = -74 \text{ J/mole K}$$

2.219 The required entropy change can be calculated along a process in which the water is heated from T_1 to T_2 and then allowed to evaporate. The entropy change for this is

$$\Delta S = C_p \ln\frac{T_2}{T_1} + \frac{qM}{T_2}$$

where q = specific latent heat of vaporization.

2.7 TRANSPORT PHENOMENA

2.220 (a) The fraction of gas molecules which traverses distances exceeding the mean free path without collision is just the probability to traverse the distance $s = \lambda$ without collision.

Thus
$$P = e^{-1} = \frac{1}{e} = 0{\cdot}37$$

(b) This probability is
$$P = e^{-1} - e^{-2} = 0{\cdot}23$$

2.221 From the formula
$$\frac{1}{\eta} = e^{-\Delta l/\lambda} \quad \text{or} \quad \lambda = \frac{\Delta l}{\ln \eta}$$

2.222 (a) Let $P(t)$ = probability of no collision in the interval $(0, t)$. Then
$$P(t + dt) = P(t)(1 - \alpha\, dt)$$

or
$$\frac{dP}{dt} = -\alpha P(t) \quad \text{or} \quad P(t) = e^{-\alpha t}$$

where we have used $P(0) = 1$

(b) The mean interval between collision is also the mean interval of no collision. Then
$$\langle t \rangle = \frac{\int_0^\infty t e^{-\alpha t}\, dt}{\int_0^\infty e^{-\alpha t}\, dt} = \frac{1}{\alpha}\frac{\Gamma(2)}{\Gamma(1)} = \frac{1}{\alpha}$$

2.223 (a) $\lambda = \dfrac{1}{\sqrt{2}\,\pi d^2 n} = \dfrac{kT}{\sqrt{2}\,\pi d^2 p}$
$$= \frac{1{\cdot}38 \times 10^{-23} \times 273}{\sqrt{2}\,\pi (0{\cdot}37 \times 10^{-9})^2 \times 10^5} = 6{\cdot}2 \times 10^{-8}\text{ m}$$
$$\tau = \frac{\lambda}{\langle v \rangle} = \frac{6{\cdot}2 \times 10^{-8}}{454} s = {\cdot}136 \text{ n s}$$

$\lambda = 6.2 \times 10^6$ m

(b) $\eta = 1{\cdot}36 \times 10^4$ s $= 3{\cdot}8$ hours

2.224 The mean distance between molecules is of the order
$$\left(\frac{22{\cdot}4 \times 10^{-3}}{6{\cdot}0 \times 10^{23}}\right)^{1/3} = \left(\frac{224}{6}\right)^{1/3} \times 10^{-9}\text{ meters} \approx 3{\cdot}34 \times 10^{-9}\text{ meters}$$

This is about 18.5 times smaller than the mean free path calculated in 2.223 (a) above.

2.225 We know that the Vander Waal's constant b is four times the molecular volume. Thus
$$b = 4 N_A \frac{\pi}{6} d^3 \quad \text{or} \quad d = \left(\frac{3b}{2\pi N_A}\right)^{1/3}$$

Hence
$$\lambda = \left(\frac{kT_0}{\sqrt{2}\,\pi p_0}\right)\left(\frac{2\pi N_A}{3b}\right)^{2/3}$$

2.226 The volocity of sound in N_2 is

$$\sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$$

so,
$$\frac{1}{v} = \sqrt{\frac{\gamma RT_0}{M}} = \frac{RT_0}{\sqrt{2}\,\pi d^2 p_0 N_A}$$

or,
$$v = \pi d^2 p_0 N_A \sqrt{\frac{2\gamma}{MRT_0}}$$

2.227 (a) $\lambda > l$ if $p < \dfrac{kT}{\sqrt{2}\,\pi d^2 l}$

Now
$$\frac{kT}{\sqrt{2}\,\pi d^2 l} \text{ for } O_2 \text{ of } O \text{ is } 0{\cdot}7 \text{ Pa.}$$

(b) The corresponding n is obtained by dividing by kT and is $1{\cdot}84 \times 10^{20}$ per $m^3 = 1{\cdot}84^{14}$per c.c. and the corresponding mean distance is $\dfrac{l}{n^{1/3}}$.

$$= \frac{10^{-2}}{(0{\cdot}184)^{1/3} \times 10^5} = 1{\cdot}8 \times 10^{-}\ m \approx 0{\cdot}18\ \mu\text{m}.$$

2.228 (a) $v = \dfrac{1}{\tau} = \dfrac{1}{\lambda/\langle v \rangle} = \dfrac{\langle v \rangle}{\lambda}$

$$= \sqrt{2}\,\pi d^2 n \langle v \rangle = {\cdot}74 \times 10^{10}\,\text{s}^{-1} \text{ (see 2.223)}$$

(b) Total number of collisions is

$$\frac{1}{2} nv \approx 1{\cdot}0 \times 10^{29}\ \text{s cm}^{-3}$$

Note, the factor $\frac{1}{2}$ · When two molecules collide we must not count it twice.

2.229 (a) $\lambda = \dfrac{1}{\sqrt{2}\,\pi d^2 n}$

d is a constant and n is a constant for an isochoric process so λ is constant for an isochoric process.

$$v = \frac{\langle v \rangle}{\lambda} = \frac{\sqrt{\frac{8RT}{M\pi}}}{\lambda}\ \alpha \sqrt{T}$$

(b) $\lambda = \dfrac{1}{\sqrt{2}\,\pi d^2} \dfrac{kT}{p}\ \alpha\ T$ for an isobaric process.

$$v = \frac{\langle v \rangle}{\lambda}\ \alpha\ \frac{\sqrt{T}}{T} = \frac{1}{\sqrt{T}} \text{ for an isobaric process.}$$

2.230 (a) In an isochoric process λ is constant and

$$v\ \alpha \sqrt{T}\ \alpha \sqrt{pV}\ \alpha \sqrt{p}\ \alpha \sqrt{n}$$

(b) $\lambda = \dfrac{kT}{\sqrt{2}\,\pi d^2 p}$ must decrease n times in an isothermal process and v must increase n times because $\langle v \rangle$ is constant in an isothermal process.

2.231 (a) $\lambda \propto \frac{1}{n} \Rightarrow \frac{1}{N/V} = \frac{V}{N}$

Thus $$\lambda \propto V \text{ and } v \propto \frac{T^{1/2}}{V}$$

But in an adiabatic process $\left(\gamma = \frac{7}{5} \text{ here}\right)$

$$TV^{\gamma - 1} = \text{constant so } TV^{2/5} = \text{constant}$$

or $$T^{1/2} \propto V^{-1/5} \text{ Thus } v \propto V^{-6/5}$$

(b) $\lambda \propto \frac{T}{p}$

But $p\left(\frac{T}{p}\right)^{\gamma} = \text{constant}$ or $\frac{T}{p} \propto p^{-1/\gamma}$ or $T \propto p^{1-1/\gamma}$

Thus $$\lambda \propto p^{-1/\gamma} = p^{-5/7}$$

$$v = \frac{<v>}{\lambda} \propto \frac{p}{\sqrt{T}} \propto p^{1/2 + \frac{1}{2\gamma}} = p^{\frac{\gamma+1}{2\gamma}} = p^{6/7}$$

(c) $\lambda \propto V$

But $$TV^{2/5} = \text{constant or } V \propto T^{-5/2}$$

Thus $$\lambda \propto T^{-5/2}$$

$$v \propto \frac{T^{1/2}}{V} \propto T^3$$

2.232 In the polytropic process of index n

$$pV^n = \text{constant}, TV^{n-1} = \text{constant and } p^{1-n}T^n = \text{constant}$$

(a) $\lambda \propto V$

$$v \propto \frac{T^{1/2}}{V} = V^{\frac{1-n}{2}} V^{-1} = V^{\frac{-n+1}{2}}$$

(b) $\lambda \propto \frac{T}{p}$, $T^n \propto p^{n-1}$ or $T \propto p^{1-\frac{1}{n}}$

so $$\lambda \propto p^{-1/n}$$

$$v = \frac{<v>}{\lambda} \propto \frac{p}{\sqrt{T}} \propto p^{1-\frac{1}{2}+\frac{1}{2n}} = p^{\frac{n+1}{2n}}$$

(c) $\lambda \propto \frac{T}{p}$, $p \propto T^{\frac{n}{n-1}}$

$$\lambda \propto T^{1-\frac{n}{n1}} = T^{-\frac{1}{n-1}} = T^{\frac{1}{1-n}}$$

$$v \propto \frac{p}{\sqrt{T}} \propto T^{\frac{n}{n-1}-\frac{1}{2}} = T^{\frac{n+1}{2(n-1)}}$$

2.233 (a) The number of collisions between the molecules in a unit volume is

$$\frac{1}{2} n\, \nu = \frac{1}{\sqrt{2}} \pi d^2 n^2 <v> \alpha \frac{\sqrt{T}}{V^2}$$

This remains constant in the poly process pV^{-3} = constant

Using (2.122) the molar specific heat for the polytropic process

$$pV^{\alpha} = \text{ constant,}$$

is
$$C = R\left(\frac{1}{\gamma - 1} - \frac{1}{\alpha - 1}\right)$$

Thus
$$C = R\left(\frac{1}{\gamma - 1} + \frac{1}{4}\right) = R\left(\frac{5}{2} + \frac{1}{4}\right) = \frac{11}{4} R$$

It can also be written as $\frac{1}{4} R\,(1 + 2\,i)$ where $i = 5$

(b) In this case $\frac{\sqrt{T}}{V}$ = constant and so pV^{-1} = constant

so
$$C = R\left(\frac{1}{\gamma - 1} + \frac{1}{2}\right) = R\left(\frac{5}{2} + \frac{1}{2}\right) = 3R$$

It can also be written as $\frac{R}{2}(i + 1)$

2.234 We can assume that all molecules, incident on the hole, leak out. Then,

$$-dN = -d\,(nV) = \frac{1}{4} n <v> S\, dt$$

or
$$dn = -n \frac{dt}{4v/S <v>} = -n \frac{dt}{\tau}$$

Integrating $\quad n = n_0\, e^{-t/\tau}$. Hence $<v> = \sqrt{\frac{8RT}{\pi M}}$

2.235 If the temperature of the compartment 2 is η times more than that of compartment 1, it must contain $\frac{1}{\eta}$ times less number of molecules since pressure must be the same when the big hole is open. If M = mass of the gas in 1 than the mass of the gas in 2 must be $\frac{M}{\eta}$. So immediately after the big hole is closed.

$$n_1^0 = \frac{M}{mV},\; n_2^0 = \frac{M}{mV\eta}$$

where m = mass of each molecule and n_1^0, n_2^0 are concentrations in 1 and 2. After the big hole is closed the pressures will differ and concentration will become n_1 and n_2 where

$$n_1 + n_2 = \frac{M}{m\, V\eta}(1 + \eta)$$

On the other hand

$$n_1 <v_1> = n_2 <v_2> \quad \text{i.e. } n_1 = \sqrt{\eta}\, n_2$$

Thus $$n_2'(1+\sqrt{\eta}) = \frac{m}{mV\eta}(1+\eta) = n_2^0(1+\eta)$$

So $$n_2 = n_2^0 \frac{1+\eta}{1+\sqrt{\eta}}$$

2.236 We know

$$\eta = \frac{1}{3}<v>\lambda\rho = \frac{1}{3}<v>\frac{1}{\sqrt{2}\,\pi d^2}m\,\alpha\sqrt{T}$$

Thus η changing α times implies T changing α^2times.
On the other hand

$$D = \frac{1}{3}<v>\lambda = \frac{1}{3}\sqrt{\frac{8kT}{\pi m}}\frac{kT}{\sqrt{2}\,\pi d^2 p}$$

Thus D changing β times means $\frac{T^{3/2}}{p}$ changing β times

So p must change $\frac{\alpha^3}{\beta}$ times

2.237 $D\,\alpha\,\frac{\sqrt{T}}{n} \propto V\sqrt{T}\,,\ \eta \propto \sqrt{T}$

(a) D will increase n times

η will remain constant if T is constant

(b) $D\,\alpha\,\frac{T^{3/2}}{p}\,\alpha\,\frac{(pV)^{3/2}}{p} = p^{1/2}\,V^{3/2}$

$$\eta\,\alpha\,\sqrt{pV}$$

Thus D will increase $n^{3/2}$ times, η will increase $n^{1/2}$ times, if p is constant

2.238 $D\,\alpha\,V\sqrt{T}\,,\ \eta\,\alpha\,\sqrt{T}$

In an adiabatic process

$$TV^{\gamma-1} = \text{constant, or } T \propto V^{1-\gamma}$$

Now V is decreased $\frac{1}{n}$ times. Thus

$$D\,\alpha\,V^{\frac{3-\gamma}{2}} = \left(\frac{1}{n}\right)^{\frac{3-\gamma}{2}} = \left(\frac{1}{n}\right)^{4/5}$$

$$\eta\,\alpha\text{ of }V^{\frac{1-\gamma}{2}} = \left(\frac{1}{n}\right)^{-1/5} = n^{1/5}$$

So D decreases $n^{4/5}$times and η increase $n^{1/5}$times.

2.239 (a) $D\,\alpha\,V\sqrt{T}\,\alpha\,\sqrt{pV^3}$

Thus D remains constant in the process pV^3 = constant

So polytropic index $n = 3$

(b) $\eta\,\alpha\,\sqrt{T}\,\alpha\,\sqrt{pV}$

So η remains constant in the isothermal process

$$pV = \text{constant}, \ n = 1, \text{ here}$$

(c) Heat conductivity $\kappa = \eta\, C_V$

and C_V is a constant for the ideal gas

Thus $n = 1$ here also,

2.240 $$\eta = \frac{1}{3}\sqrt{\frac{8\,kT}{\pi\, m}}\,\frac{m}{\sqrt{2}\,\pi\, d^2} = \frac{2}{3}\sqrt{\frac{m\,kT}{\pi^3}}\,\frac{1}{d^2}$$

or $$d = \left(\frac{2}{3\,\eta}\right)^{1/2}\left(\frac{m\,kT}{\pi^3}\right)^{1/4} = \left(\frac{2}{3\times 18{\cdot}9}\times 10^6\right)^{1/2}\left(\frac{4\times 8{\cdot}31\times 273\times 10^{-3}}{\pi^3\times 36\times 10^{46}}\right)^{1/4}$$

$$= 10^{-10}\left(\frac{2}{3\times 18{\cdot}9}\right)^{1/2}\left(\frac{4\times 83{\cdot}1\times 273}{\pi^3\times {\cdot}36}\right)^{1/4} \approx 0{\cdot}178 \text{ nm}$$

2.241 $$\kappa = \frac{1}{3} < v > \lambda\, \rho\, c_V$$

$$= \frac{1}{3}\sqrt{\frac{8\,kT}{m\,\pi}}\,\frac{1}{\sqrt{2}\,\pi\, d^2\, n}\, mn\, \frac{C_V}{M}$$

$\left(C_V \text{ is the specific heat capacity which is } \frac{C_V}{M}\right)$. Now C_V is the same for all monoatomic gases such as He and A. Thus

$$\kappa \ \alpha\ \frac{1}{\sqrt{M}\, d^2}$$

or $$\frac{\kappa_{He}}{\kappa_A} = 8{\cdot}7 = \frac{\sqrt{M_A}\, d_A^2}{\sqrt{M_{H_e}}\, d_{H_e}^2} = \sqrt{10}\,\frac{d_A^2}{d_{H_e}^2}$$

$$\frac{d_A}{d_{H_e}} = \sqrt{\frac{8{\cdot}7}{\sqrt{10}}} = 1{\cdot}658 \approx 1{\cdot}7$$

2.242 In this case

$$N_1\,\frac{r_2^2 - r_1^2}{r_1^2\, r_2^2} = 4\,\pi\,\eta\,\omega$$

or $$N_1\,\frac{2\,R\,\Delta R}{R^4} = 4\,\pi\,\eta\,\omega \quad \text{or} \quad N_1 = \frac{2\,\pi\,\eta\,\omega\, R^3}{\Delta R}$$

To decrease N_1, n times η must be decreased n times. Now η does not depend on pressure until the pressure is so low that the mean free path equals, say, $\frac{1}{2}\Delta R$. Then the mean free path is fixed and η decreases with pressure. The mean free path equals $\frac{1}{2}\Delta R$ when

$$\frac{1}{\sqrt{2}\,\pi\, d^2\, n_0} = \Delta R \quad (n_0 = \text{concentration})$$

Corresponding pressure is $p_0 = \dfrac{\sqrt{2}\,k\,T}{\pi\,d^2\,\Delta R}$

The sought pressure is n times less

$$p = \frac{\sqrt{2}\,k\,T}{\pi\,d^2\,n\,\Delta R} = 70{\cdot}7 \times \frac{10^{-23}}{10^{-18} \times 10^{-3}} \approx 0{\cdot}71 \text{ Pa}$$

The answer is qualitative and depends on the choice $\frac{1}{2}\Delta R$ for the mean free path.

2.243 We neglect the moment of inertia of the gas in a shell. Then the moment of friction forces on a unit length of the cylinder must be a constant as a function of r.

So, $$2\,\pi\,r^3\,\eta\,\frac{d\omega}{dr} = N_1 \quad \text{or} \quad \omega\,(r) = \frac{N_1}{4\,\pi\,\eta}\left(\frac{1}{r_1^2} - \frac{1}{r^2}\right)$$

and $$\omega = \frac{N_1}{4\,\pi\,\eta}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) \quad \text{or} \quad \eta = \frac{N_1}{4\,\pi\,\omega}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

2.244 We consider two adjoining layers. The angular velocity gradient is $\frac{\omega}{h}$. So the moment of the frictional force is

$$N = \int_0^a r \cdot 2\,\pi\,r\,dr \cdot \eta\,r\,\frac{\omega}{h} = \frac{\pi\,\eta\,a^4\,\omega}{2h}$$

2.245 In the ultrararefied gas we must determine η by taking $\lambda = \frac{1}{2}h$. Then

$$\eta = \frac{1}{3}\sqrt{\frac{8kT}{m\pi}} \times \frac{1}{2}h \times \frac{mp}{kT} = \frac{1}{3}\sqrt{\frac{2M}{\pi\,RT}}\,hp$$

so, $$N = \frac{1}{3}\,\omega\,a^4\,p\,\sqrt{\frac{\pi\,M}{2RT}}$$

2.246 Take an infinitesimal section of length dx and apply Poiseuilles equation to this. Then

$$\frac{dV}{dt} = \frac{-\pi\,a^4}{8\eta}\,\frac{\partial p}{\partial x}$$

From the formula $$pV = RT \cdot \frac{m}{M}$$

$$pdV = \frac{RT}{M}\,dm$$

or $$\frac{dm}{dt} = \mu = -\frac{\pi\,a^4\,M}{8\,\eta\,RT}\,\frac{pdp}{dx}$$

This equation implies that if the flow is isothermal then $p\dfrac{dp}{dx}$ must be a constant and so

equals $\dfrac{\left|p_2^2 - p_1^2\right|}{2\,l}$ in magnitude.

Thus, $$\mu = \frac{\pi\,a^4\,M}{16\,\eta\,R\,T}\,\frac{\left|p_2^2 - p_1^2\right|}{l}$$

2.247 Let $T =$ temperature of the interface.

Then heat flowing from left = heat flowing into right in equilibrium.

Thus, $$\kappa_1 \frac{T_1 - T}{l_1} = \kappa_2 \frac{T - T_2}{l_2} \quad \text{or} \quad T = \frac{\left(\frac{\kappa_1 T_1}{l_1} + \frac{\kappa_2 T_2}{l_2}\right)}{\left(\frac{\kappa_1}{l_1} + \frac{\kappa_2}{l_2}\right)}$$

2.248 We have

$$\kappa_1 \frac{T_1 - T}{l_1} = \kappa_2 \frac{T - T_2}{l_2} = \kappa \frac{T_1 - T_2}{l_1 + l_2}$$

or using the previous result

$$\frac{\kappa_1}{l_1}\left(T_1 - \frac{\frac{\kappa_1 T_1}{l_1} + \frac{\kappa_2 T_2}{l_2}}{\frac{\kappa_1}{l_1} + \frac{\kappa_1}{l_2}}\right) = \kappa \frac{T_1 - T_2}{l_1 + l_2}$$

or $$\frac{\kappa_1}{l_1} \frac{\frac{\kappa_2}{l_2}(T_1 - T_2)}{\frac{\kappa_1}{l_1} + \frac{\kappa_2}{l_2}} = \kappa \frac{T_1 - T_2}{l_1 + l_2} \quad \text{or} \quad \kappa = \frac{l_1 + l_2}{\frac{l_1}{\kappa_1} + \frac{l_2}{\kappa_2}}$$

2.249 By definition the heat flux (per unit area) is

$$\dot{Q} = -K \frac{dT}{dx} = -\alpha \frac{d}{dx} \ln T = \text{constant} = +\alpha \frac{\ln T_1/T_2}{l}$$

Integrating $$\ln T = \frac{x}{l} \ln \frac{T_2}{T_1} + \ln T_1$$

where $T_1 =$ temperature at the end $x = 0$

So $$T = T_1 \left(\frac{T_2}{T_1}\right)^{x/l} \quad \text{and} \quad \dot{Q} = \frac{\alpha \ln T_1/T_2}{l}$$

2.250 Suppose the chunks have temperatures T_1, T_2 at time t and $T_1 - dT_1$, $T_2 + dT_2$ at time $dt + t$.

Then $$C_1 \, dT_1 = C_2 \, dT_2 = \frac{\kappa S}{l}(T_1 - T_2) \, dt$$

Thus $$d\,\Delta T = -\frac{\kappa S}{l}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) \Delta T \, dt \text{ where } \Delta T = T_1 - T_2$$

Hence $$\Delta T = (\Delta T)_0 \, e^{-t/\tau} \quad \text{where } \frac{1}{\tau} = \frac{\kappa s}{l}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

2.251 $\dot{Q} = \kappa \frac{\partial T}{\partial x} = -A\sqrt{T}\frac{\partial T}{\partial x}$

$$= -\frac{2}{3}A\frac{\partial T^{3/2}}{\partial x}, \ (A = \text{constant})$$

$$= \frac{2}{3}A\frac{(T_1^{3/2} - T_2^{3/2})}{l}$$

Thus $$T^{3/2} = \text{constant} - \frac{x}{l}\left(T_1^{3/2} - T_2^{3/2}\right)$$

or using $$T = T_1 \text{ at } x = 0$$

$$T^{3/2} = T_1^{3/2} + \frac{x}{l}\left(T_2^{3/2} - T_1^{3/2}\right) \text{ or } \left(\frac{T}{T_1}\right)^{3/2} = 1 + \frac{x}{l}\left(\left(\frac{T_2}{T}\right)^{3/2} - 1\right)$$

$$T = T_1\left[1 + \frac{x}{l}\left\{\left(\frac{T_2}{T_1}\right)^{3/2} - 1\right\}\right]^{2/3}$$

2.252 $\kappa = \frac{1}{3}\sqrt{\frac{8RT}{\pi M}}\frac{1}{\sqrt{2}\,\pi d^2 n}mn\frac{R\frac{i}{2}}{M} = \frac{R^{3/2}\,i\,T^{3/2}}{3\pi^{3/2}d^2\sqrt{M}\,N_A}$

Then from the previous problem

$$q = \frac{2i\,R^{3/2}(T_2^{3/2} - T_1^{3/2})}{9\pi^{3/2}d^2\sqrt{M}\,N_A\,l}, \ i = 3 \text{ here.}$$

2.253 At this pressure and average temparature $= 27°C = 300\text{K} = T = \frac{(T_1 + T_2)}{2}$

$\lambda = \frac{1}{\sqrt{2}\,\pi d^2}\frac{\kappa T}{p} = 2330 \times 10^{-5}\text{m} = 23{\cdot}3\text{mm} >> 5{\cdot}0\text{mm} = l$

The gas is ultrathin and we write $\lambda = \frac{1}{2}l$ here

Then $$q = \kappa\frac{dT}{dx} = \kappa\,\frac{T_2 - T_1}{l}$$

where $$\kappa = \frac{1}{3} < v > \times \frac{1}{2}l \times \frac{MP}{RT} \times \frac{R}{\gamma - 1} \times \frac{1}{M} = \frac{p < v >}{6T(\gamma - 1)}l$$

and $$q = \frac{p < v >}{6T(\gamma - 1)}(T_2 - T_1)$$

where $< v > = \sqrt{\frac{8RT}{M\pi}}$. We have used $T_2 - T_1 << \frac{T_2 + T_1}{2}$ here.

2.254 In equilibrium $2\pi r\,\kappa\,\dfrac{dT}{dr} = -A =$ constant. So $T = B - \dfrac{A}{2\pi\kappa}\ln r$

But $T = T_1$ when $r = R_1$ and $T = T_2$. when $r = R_2$.

From this we find $T = T_1 + \dfrac{T_2 - T_1}{\ln \dfrac{R_2}{R_1}} \ln \dfrac{r}{r_1}$

2.255 In equilibrium $4\pi\, r^2\, \kappa\,\dfrac{dT}{dr} = -A =$ constant

$$T = B + \frac{A}{4\pi\kappa}\frac{1}{r}$$

Using $T = T_1$ when $r = R_1$ and $T = T_2$ when $r = R_2$,

$$T = T_1 + \frac{T_2 - T_1}{\dfrac{1}{R_2} - \dfrac{1}{R_1}}\left(\frac{1}{r} - \frac{1}{R_1}\right)$$

2.256 The heat flux vector is $-\kappa$ grad T and its divergence equals w. Thus

$$\nabla^2 T = -\frac{w}{\kappa}$$

or $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\frac{w}{\kappa} \text{ in cylindrical coordinates.}$$

or $$T = B + A\ln r - \frac{\omega}{2x}r^2$$

Since T is finite at $r = 0, A = 0$. Also $T = T_0$ at $r = R$

so $$B = T_0 + \frac{w}{4\kappa}R^2$$

Thus $$T = T_0 + \frac{w}{4\kappa}(R^2 - r^2)$$

r here is the distance from the axis of wire (axial radius).

2.257 Here again

$$\nabla^2 T = -\frac{w}{\kappa}$$

So in spherical polar coordinates,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = -\frac{w}{\kappa} \text{ or } r^2\frac{\partial T}{\partial r} = -\frac{w}{3\kappa}r^3 + A$$

or $$T = B - \frac{A}{r} - \frac{w}{6\kappa}r^2$$

Again $$A = 0 \text{ and } B = T_0 + \frac{w}{6\kappa}R^2$$

so finally $$T = T_0 + \frac{w}{6\kappa}(R^2 - r^2)$$

PART THREE

ELECTRODYNAMICS

3.1 CONSTANT ELECTRIC FIELD IN VACUUM

3.1 F_{el} (for electorns) $= \dfrac{q^2}{4\pi\varepsilon_0 r^2}$ and $F_{gr} = \dfrac{\gamma m^2}{r^2}$

Thus $\dfrac{F_{el}}{F_{gr}}$ (for electrons) $= \dfrac{q^2}{4\pi\varepsilon_0\gamma m^2}$

$$= \frac{(1\cdot602\times10^{-19}\,\mathrm{C})^2}{\left(\frac{1}{9\times10^9}\right)\times6\cdot67\times10^{-11}\,\mathrm{m}^3/(\mathrm{kg}\cdot\mathrm{s}^2)\times(9\cdot11\times10^{-31}\,\mathrm{kg})^2} = 4\times10^{42}$$

Similarly $\dfrac{F_{el}}{F_{gr}}$ (for proton) $= \dfrac{q^2}{4\pi\varepsilon_0\gamma m^2}$

$$= \frac{(1\cdot602\times10^{-19}\,\mathrm{C})^2}{\left(\frac{1}{9\times10^9}\right)\times6\cdot67\times10^{-11}\,\mathrm{m}^3/(\mathrm{kg}\cdot\mathrm{s}^2)\times(1\cdot672\times10^{-27}\,\mathrm{kg})^2} = 1\times10^{36}$$

For $F_{el} = F_{gr}$

$$\frac{q^2}{4\pi\varepsilon_0 r^2} = \frac{\gamma m^2}{r^2} \quad\text{or}\quad \frac{q}{m} = \sqrt{4\pi\varepsilon_0\gamma}$$

$$= \sqrt{\frac{6\cdot67\times10^{-11}\,\mathrm{m}^3\,(\mathrm{kg}-\mathrm{s}^2)}{9\times10^9}} = 0\cdot86\times10^{-10}\,\mathrm{C/kg}$$

3.2 Total number of atoms in the sphere of mass 1 gm $= \dfrac{1}{63\cdot54}\times6\cdot023\times10^{23}$

So the total nuclear charge $\lambda = \dfrac{6\cdot023\times10^{23}}{63\cdot54}\times1\cdot6\times10^{-19}\times29$

Now the charge on the sphere = Total nuclear charge – Total electronic charge

$$= \frac{6{\cdot}023 \times 10^{23}}{63{\cdot}54} \times 1{\cdot}6 \times 10^{-19} \times \frac{29 \times 1}{100} = 4{\cdot}298 \times 10^{2}\ \text{C}$$

Hence force of interaction between these two spheres,

$$F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{[4{\cdot}398 \times 10^{2}]^2}{1^2}\ \text{N} = 9 \times 10^9 \times 10^4 \times 19{\cdot}348\ \text{N} = 1{\cdot}74 \times 10^{15}\ \text{N}$$

3.3 Let the balls be deviated by an angle θ, from the vertical when separtion between them equals x.

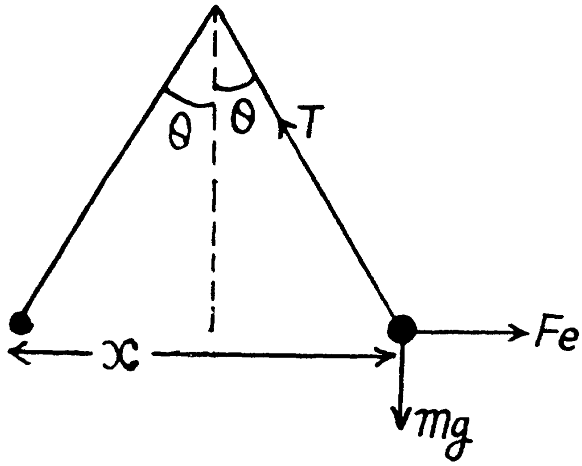
Applying Newton's second law of motion for any one of the sphere, we get,

$$T\cos\theta = mg \qquad (1)$$

and
$$T\sin\theta = F_e \qquad (2)$$

From the Eqs. (1) and (2)

$$\tan\theta = \frac{F_e}{mg} \qquad (3)$$



But from the figure

$$\tan\theta = \frac{x}{2\sqrt{l^2 - \left(\frac{x}{2}\right)^2}} = \frac{x}{2l} \text{ as } x << l \qquad (4)$$

From Eqs. (3) and (4)

$$F_e = \frac{mg\,x}{2l} \quad \text{or} \quad \frac{q^2}{4\pi\varepsilon_0 x^2} = \frac{mg\,x}{2l}$$

Thus
$$q^2 = \frac{2\pi\varepsilon_0\, mg\, x^3}{l} \qquad (5)$$

Differentiating Eqn. (5) with respect to time

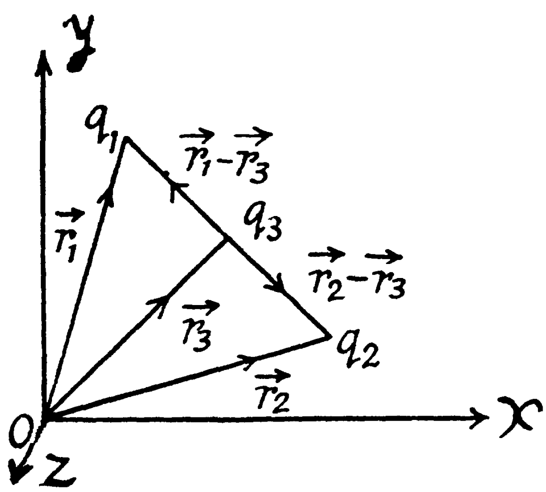
$$2q\frac{dq}{dt} = \frac{2\pi\varepsilon_0\, mg}{l} 3x^2 \frac{dx}{dt}$$

According to the problem $\frac{dx}{dt} = v = a/\sqrt{x}$ (approach velocity is $\frac{dx}{dt}$)

so, $$\left(\frac{2\pi\varepsilon_0\, mg}{l} x^3\right)^{1/2} \frac{dq}{dt} = \frac{3\pi\varepsilon_0\, mg}{l} x^2 \frac{a}{\sqrt{x}}$$

Hence, $$\frac{dq}{dt} = \frac{3}{2} a \sqrt{\frac{2\pi\varepsilon_0\, mg}{l}}$$

3.4 Let us choose coordinate axes as shown in the figure and fix three charges, q_1, q_2 and q_3 having position vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ respectively.



Now, for the equilibrium of q_3

$$\frac{+q_2 q_3 (\vec{r}_2 - \vec{r}_3)}{|\vec{r}_2 - \vec{r}_3|^3} + \frac{q_1 q_3 (\vec{r}_1 - \vec{r}_3)}{|\vec{r}_1 - \vec{r}_3|^3} = 0$$

or, $$\frac{q_2}{|\vec{r_2}-\vec{r_3}|^2} = \frac{q_1}{|\vec{r_1}-\vec{r_3}|^2}$$

because $$\frac{\vec{r_2}-\vec{r_3}}{|\vec{r_2}-\vec{r_3}|} = -\frac{\vec{r_1}-\vec{r_3}}{|\vec{r_1}-\vec{r_3}|}$$

or, $$\sqrt{q_2}\,(\vec{r_1}-\vec{r_3}) = \sqrt{q_1}\,(\vec{r_3}-\vec{r_2})$$

or, $$\vec{r_3} = \frac{\sqrt{q_2}\,\vec{r_1}+\sqrt{q_1}\,\vec{r_2}}{\sqrt{q_1}+\sqrt{q_2}}$$

Also for the equilibrium of q_1,

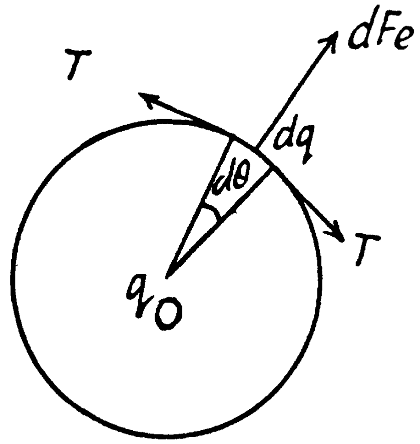
$$\frac{q_3\,(\vec{r_3}-\vec{r_1})}{|\vec{r_3}-\vec{r_1}|^3} + \frac{q_2\,(\vec{r_2}-\vec{r_1})}{|\vec{r_2}-\vec{r_1}|^3} = 0$$

or, $$q_3 = \frac{-q_2}{|\vec{r_2}-\vec{r_1}|^2}\,|\vec{r_1}-\vec{r_3}|^2$$

Substituting the value of $\vec{r_3}$, we get,

$$q_3 = \frac{-q_1 q_2}{(\sqrt{q_1}+\sqrt{q_2})^2}$$

3.5 When the charge q_0 is placed at the centre of the ring, the wire get stretched and the extra tension, produced in the wire, will balance the electric force due to the charge q_0. Let the tension produced in the wire, after placing the charge q_0, be T. From Newton's second law in projection form $F_n = mw_n$.



$$Td\,\theta - \frac{1}{4\pi\varepsilon_0}\frac{q_0}{r^2}\left(\frac{q}{2\pi r}\,r\,d\,\theta\right) = (dm)\,0,$$

or, $$T = \frac{q\,q_0}{8\pi^2\varepsilon_0 r^2}$$

3.6 Sought field strength

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r_0}|^2}$$

$= 4{\cdot}5$ kV/m on putting the values.

3.7 Let us fix the coordinate system by taking the point of intersection of the diagonals as the origin and let $\vec{k}$ be directed normally, emerging from the plane of figure.

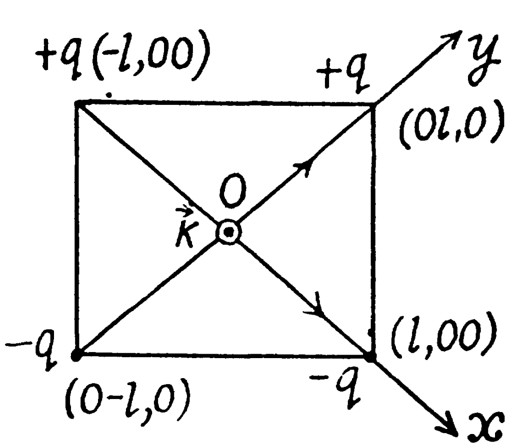
Hence the sought field strength :

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{l\vec{i}+x\vec{k}}{(l^2+x^2)^{3/2}} + \frac{-q}{4\pi\varepsilon_0}\frac{l(-\vec{i})+x\vec{k}}{(l^2+x^2)^{3/2}}$$

$$+\frac{-q}{4\pi\varepsilon_0}\frac{l\vec{j}+x\vec{k}}{(l^2+x^2)^{3/2}} + \frac{q}{4\pi\varepsilon_0}\frac{l(-\vec{j})+x\vec{k}}{(l^2+x^2)^{3/2}}$$

$$= \frac{q}{4\pi\varepsilon_0(l^2+x^2)^{3/2}}[2\,l\vec{i} - 2l\vec{j}\,]$$

Thus $E = \dfrac{ql}{\sqrt{2}\,\pi\varepsilon_0(l^2+x^2)^{3/2}}$



3.8 From the symmetry of the problem the sought field.

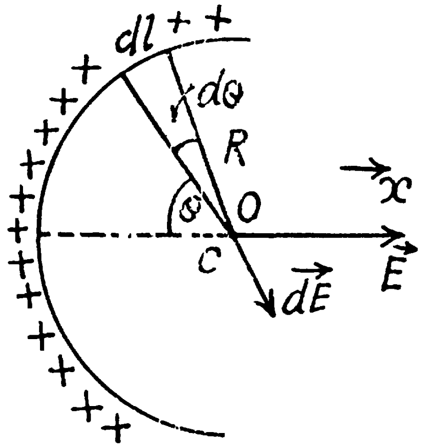
$$E = \int dE_x$$

where the projection of field strength along x – axis due to an elemental charge is

$$dE_x = \frac{dq\cos\theta}{4\pi\varepsilon_0 R^2} = \frac{qR\cos\theta\,d\theta}{4\pi^2\varepsilon_0 R^3}$$

Hence

$$E = \frac{q}{4\pi^2\varepsilon_0 R^2}\int_{\pi/2}^{\pi/2}\cos\theta\,d\theta\;\frac{q}{2\pi^2\varepsilon_0 R^2}$$



3.9 From the symmetry of the condition, it is clear that, the field along the normal will be zero

i.e. $E_n = 0$ and $E = E_l$

Now $dE_l = \dfrac{dq}{4\pi\varepsilon_0(R^2+l^2)}\cos\theta$

But $dq = \dfrac{q}{2\pi R}dx$ and $\cos\theta = \dfrac{l}{(R^2+l^2)^{1/2}}$

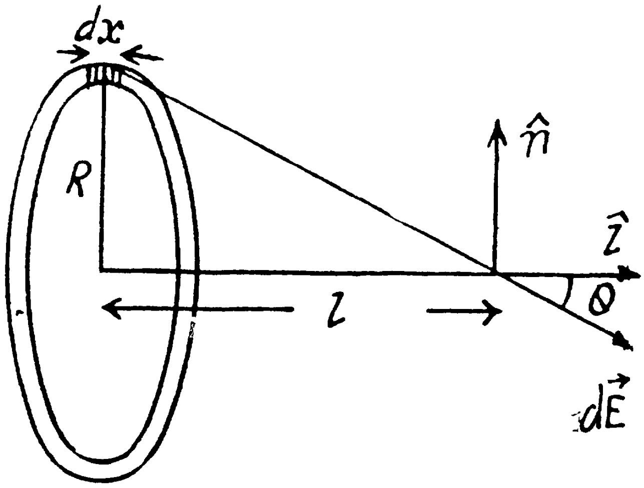
Hence

$$E = \int dE_l = \int_0^{2\pi R}\frac{ql}{2\pi R}\cdot\frac{dx}{4\pi\varepsilon_0(R^2+l^2)^{3/2}}$$

or $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{ql}{(l^2+R^2)^{3/2}}$

and for $l >> R$, the ring behaves like a point charge, reducing the field to the value,

$$E \approx \frac{1}{4\pi\varepsilon_0}\frac{q}{l^2}$$



For E_{max}, we should have $\frac{dE}{dl} = 0$

So, $(l^2+R^2)^{3/2} - \frac{3}{2} l (l^2+R^2)^{1/2} 2l = 0$ or $l^2 + R^2 - 3l^2 = 0$

Thus $l = \frac{R}{\sqrt{2}}$ and $E_{max} = \frac{q}{6\sqrt{3}\,\pi\,\varepsilon_0 R^2}$

3.10 The electric potential at a distance x from the given ring is given by,

$$\varphi(x) = \frac{q}{4\pi\varepsilon_0 x} - \frac{q}{4\pi\varepsilon_0 (R^2+x^2)^{1/2}}$$

Hence, the field strength along x-axis (which is the net field strength in our case),

$$E_x = -\frac{d\varphi}{dx} = \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2} - \frac{qx}{4\pi\varepsilon_0 (R^2+x^2)^{3/2}}$$

$$= \frac{\frac{q}{4\pi\varepsilon_0} x^3 \left[\left(1+\frac{R^2}{x^2}\right)^{3/2} - 1\right]}{x^2 (R^2+x^2)^{3/2}}$$

$$= \frac{\frac{q}{4\pi\varepsilon_0} x^3 \left[1 + \frac{3}{2}\frac{R^2}{x^2} + \frac{3}{8}\frac{R^4}{x^4} + \ldots\right]}{x^2 (R^2+x^2)^{3/2}}$$

Neglecting the higher power of R/x, as $x >> R$.

$$E = \frac{3qR^2}{8\pi\varepsilon_0 x^4}.$$

Note : Instead of $\varphi(x)$, we may write $E(x)$ directly using 3.9

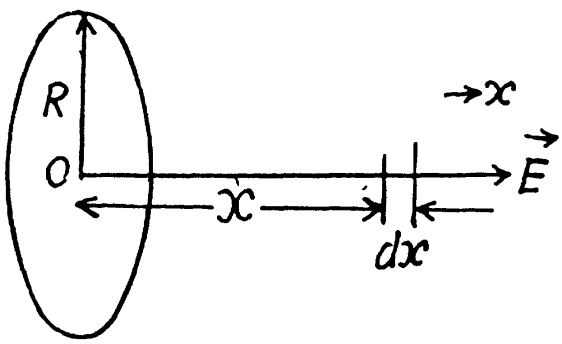
3.11 From the solution of 3.9, the electric field strength due to ring at a point on its axis (say x-axis) at distance x from the centre of the ring is given by :

$$E(x) = \frac{qx}{4\pi\varepsilon_0 (R^2+x^2)^{3/2}}$$

And from symmetry $\vec{E}$ at every point on the axis is directed along the x–axis (Fig.).

Let us consider an element (dx) on thread which carries the charge $(\lambda\, dx)$. The electric force experienced by the element in the field of ring.

$$dF = (\lambda\, dx) E(x) = \frac{\lambda qx\, dx}{4\pi\varepsilon_0 (R^2+x^2)^{3/2}}$$



Thus the sought interaction

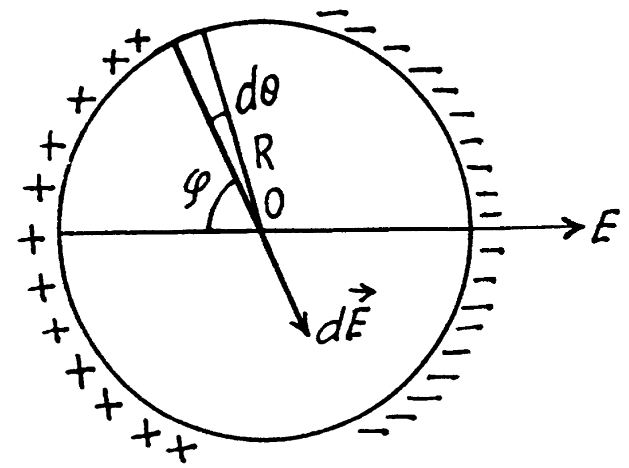
$$F = \int_0^\infty \frac{\lambda q x\, dx}{4\pi\varepsilon_0 (R^2+x^2)^{3/2}}$$

On integrating we get, $F = \frac{\lambda q}{4\pi\varepsilon_0 R}$

3.12 (a) The given charge distribution is shown in Fig. The symmetry of this distribution implies that vector $\vec{E}$ at the point O is directed to the right, and its magnitude is equal to the sum of the projection onto the direction of $\vec{E}$ of vectors $d\vec{E}$ from elementary charges dq. The projection of vector $d\vec{E}$ onto vector $\vec{E}$ is

$$dE \cos\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R^2}\cos\varphi,$$

where $dq = \lambda R\, d\varphi = \lambda_0\, R\cos\varphi\, d\varphi$.



Integrating (1) over φ between 0 and 2π we find the magnitude of the vector E:

$$E = \frac{\lambda_0}{4\pi\varepsilon_0 R}\int_0^{2\pi}\cos^2\varphi\, d\varphi = \frac{\lambda_0}{4\varepsilon_0 R}.$$

It should be noted that this integral is evaluated in the most simple way if we take into account that $\langle\cos^2\varphi\rangle = 1/2$. Then

$$\int_0^{2\pi}\cos^2\varphi\, d\varphi = \langle\cos^2\varphi\rangle\, 2\pi = \pi.$$

(b) Take an element S at an azimuthal angle φ from the x-axis, the element subtending an angle $d\varphi$ at the centre.
The elementary field at P due to the element is

$$\frac{\lambda_0\cos\varphi\, d\varphi\, R}{4\pi\varepsilon_0(x^2+R^2)} \text{ along } SP \text{ with components}$$

$$\frac{\lambda_0\cos\varphi\, d\varphi\, R}{4\pi\varepsilon_0(x^2+R^2)} \times \{\cos\theta \text{ along } OP, \sin\theta \text{ along } OS\}$$

where $$\cos\theta = \frac{x}{(x^2+R^2)^{1/2}}$$

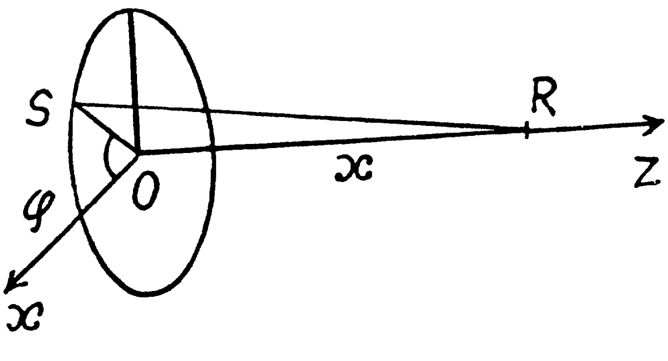
The component along OP vanishes on integration as $\int_0^{2\pi}\cos\varphi\, d\varphi = 0$

The component alon OS can be broken into the parts along OX and OY with

$$\frac{\lambda_0 R^2\cos\varphi\, d\varphi}{4\pi\varepsilon_0(x^2+R^2)^{3/2}} \times \{\cos\varphi \text{ along } OX, \sin\varphi \text{ along } OY\}$$

On integration, the part along OY vanishes.
Finally

$$E = E_x = \frac{\lambda_0 R^2}{4\varepsilon_0(x^2+R^2)^{3/2}}$$



For $x \gg R$

$$E_x = \frac{p}{4\pi\varepsilon_0 x^3} \text{ where } p = \lambda_0\pi R^2$$

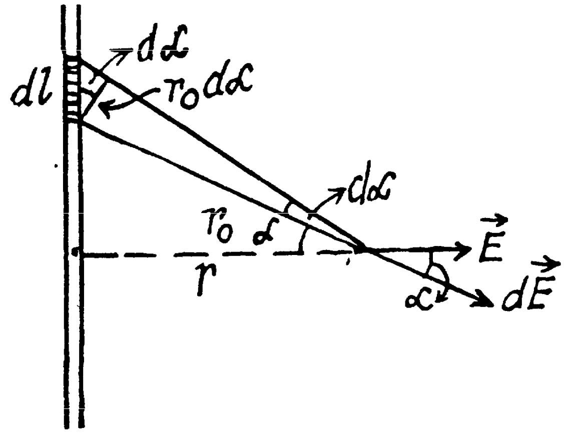
3.13 (a) It is clear from symmetry considerations that vector $\vec{E}$ must be directed as shown in the figure. This shows the way of solving this problem : we must find the component dE_r of the field created by the element dl of the rod, having the charge dq and then integrate the result over all the elements of the rod. In this case

$$dE_r = dE\cos\alpha = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dl}{r_0^2}\cos\alpha,$$

where $\lambda = \dfrac{q}{2a}$ is the linear charge density. Let us reduce this equation of the form convenient for integration. Figure shows that $dl\cos\alpha = r_0\, d\alpha$ and $r_0 = \dfrac{r}{\cos\alpha}$;

Consequently,

$$dE_r = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, r_0\, d\alpha}{r_0^2} = \frac{\lambda}{4\pi\varepsilon_0 r}\cos\alpha\, d\alpha$$



This expression can be easily integrated :

$$E = \frac{\lambda}{4\pi\varepsilon_0 r}\, 2\int_0^{\alpha_0}\cos\alpha\, d\alpha = \frac{\lambda}{4\pi\varepsilon_0 r}\, 2\sin\alpha_0$$

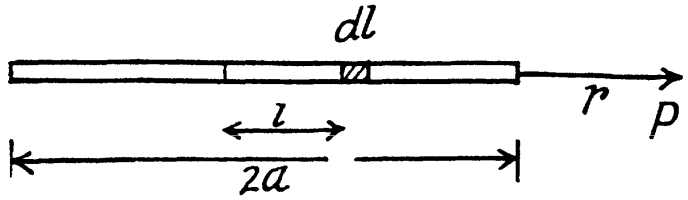
where α_0 is the maximum value of the angle α,

$$\sin\alpha_0 = a\Big/\sqrt{a^2+r^2}$$

Thus, $E = \dfrac{q/2a}{4\pi\varepsilon_0 r}\, 2\dfrac{a}{\sqrt{a^2+r^2}} = \dfrac{q}{4\pi\varepsilon_0 r\sqrt{a^2+r^2}}$

Note that in this case also $E \approx \dfrac{q}{4\pi\varepsilon_0 r^2}$ for $r >> a$ *as of the field of a point charge.*

(b) Let, us consider the element of length dl at a distance l from the centre of the rod, as shown in the figure.



Then field at P, due to this element.

$$dE = \frac{\lambda\, dl}{4\pi\varepsilon_0 (r-l)^2},$$

if the element lies on the side, shown in the diagram, and $dE = \dfrac{\lambda\, dl}{4\pi\varepsilon_0 (r+l)^2}$, if it lies on other side.

Hence $$E = \int dE = \int_0^a \frac{\lambda\, dl}{4\pi\varepsilon_0 (r-l)^2} + \int_0^a \frac{\lambda\, dl}{4\pi\varepsilon_0 (r+l)^2}$$

On integrating and putting $\lambda = \dfrac{q}{2a}$, we get, $E = \dfrac{q}{4\pi\varepsilon_0}\dfrac{1}{(r^2-a^2)}$

For $r >> a$, $E \approx \dfrac{q}{4\pi\varepsilon_0 r^2}$

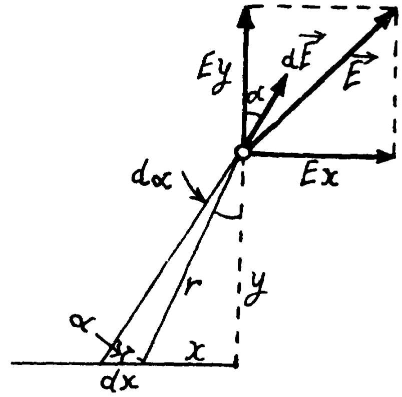
3.14 The problem is reduced to finding E_x and E_y viz. the projections of $\vec{E}$ in Fig, where it is assumed that $\lambda > 0$.

Let us start with E_x. The contribution to E_x from the charge element of the segment dx is

$$dE_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx}{r^2}\sin\alpha \qquad (1)$$

Let us reduce this expression to the form convenient for integration. In our case, $dx = r\, d\alpha/\cos\alpha$, $r = y/\cos\alpha$. Then

$$dE_x = \frac{\lambda}{4\pi\varepsilon_0 y}\sin\alpha\, d\alpha.$$



Integrating this expression over α between 0 and $\pi/2$, we find

$$E_x = \lambda/4\pi\varepsilon_0 y.$$

In order to find the projection E_y it is sufficient to recall that dE_y differs from dE_x in that $\sin\alpha$ in (1) is simply replaced by $\cos\alpha$.

This gives

$dE_y = (\lambda\cos\alpha\, d\alpha)/4\pi\varepsilon_0 y$ and $E_y = \lambda/4\pi\varepsilon_0 y$.

We have obtained an interesting result :

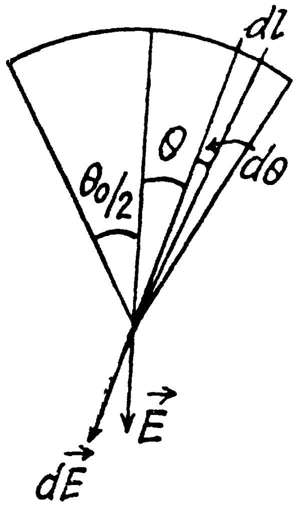
$E_x = E_y$ independently of y,

i.e. $\vec{E}$ is oriented at the angle of 45° to the rod. The modulus of $\vec{E}$ is

$$E = \sqrt{E_x^2 + E_y^2} = \lambda\sqrt{2}/4\pi\varepsilon_0 y.$$

3.15 (a) Using the solution of 3.14, the net electric field strength at the point O due to straight parts of the thread equals zero. For the curved part (arc) let us derive a general expression i.e. let us calculate the field strength at the centre of arc of radius R and linear charge density λ and which subtends angle θ_0 at the centre.

From the symmetry the sought field strength will be directed along the bisector of the angle θ_0 and is given by



$$E = \int_{-\theta_0/2}^{+\theta_0/2} \frac{\lambda\,(R\, d\theta)}{4\pi\varepsilon_0 R^2}\cos\theta = \frac{\lambda}{2\pi\varepsilon_0 R}\sin\frac{\theta_0}{2}$$

In our problem $\theta_0 = \pi/2$, thus the field strength due to the turned part at the point

$E_0 = \dfrac{\sqrt{2}\,\lambda}{4\pi\varepsilon_0 R}$ which is also the sought result.

(b) Using the solution of 3.14 (a), net field strength at O due to stright parts equals $\sqrt{2}\left(\dfrac{\sqrt{2}\,\lambda}{4\pi\varepsilon_0 R}\right) = \dfrac{\lambda}{2\pi\varepsilon_0 R}$ and is directed vertically down. Now using the solution of 3.15

(a), field strength due to the given curved part (semi-circle) at the point O becomes $\frac{\lambda}{2\pi\varepsilon_0 R}$ and is directed vertically upward. Hence the sought net field strengh becomes zero.

3.16 Given charge distribution on the surface $\sigma = \vec{a}\cdot\vec{r}$ is shown in the figure. Symmetry of this distribution implies that the sought $\vec{E}$ at the centre O of the sphere is opposite to $\vec{a}$

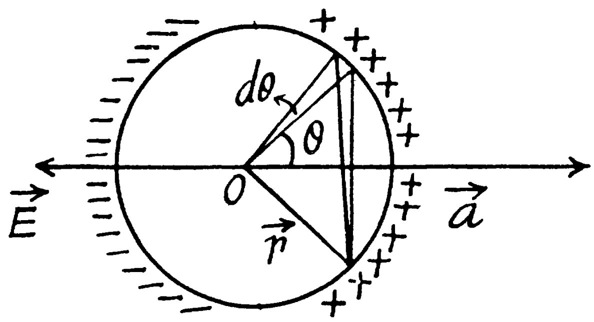
$dq = \sigma\,(2\pi r \sin\theta)\, r\, d\theta = (\vec{a}\cdot\vec{r})\, 2\pi r^2 \sin\theta\, d\theta = 2\pi a r^3 \sin\theta \cos\theta\, d\theta$

Again from symmetry, field strength due to any ring element $d\vec{E}$ is also opposite to $\vec{a}$ i.e. $d\vec{E} \uparrow\downarrow \vec{a}$. Hence

$$d\vec{E} = \frac{dq\, r\cos\theta}{4\pi\varepsilon_0 (r^2\sin^2\theta + r^2\cos^2\theta)^{3/2}} \frac{-\vec{a}}{a} \text{(Using the result of 3.9)}$$

$$= \frac{(2\pi a r^3 \sin\theta\cos\theta\, d\theta)\, r\cos\theta}{4\pi\varepsilon_0 r^3} \frac{(-\vec{a})}{a}$$

$$= \frac{-\vec{a}\, r}{2\varepsilon_0} \sin\theta \cos^2 d\theta$$



Thus

$$\vec{E} = \int d\vec{E} = \frac{(-\vec{a})\, r}{2\varepsilon_0} \int_0^{\pi} \sin\theta \cos^2\theta\, d\theta$$

Integrating, we get $\vec{E} = -\frac{\vec{a}\, r}{2\varepsilon_0}\frac{2}{3} = -\frac{\vec{a}\, r}{3\varepsilon_0}$

3.17 We start from two charged spherical balls each of radius R with equal and opposite charge densities $+\rho$ and $-\rho$. The centre of the balls are at $+\frac{\vec{a}}{2}$ and $-\frac{\vec{a}}{2}$ respectively so the equation of their surfaces are $\left|\vec{r} - \frac{\vec{a}}{2}\right| = R$ or $r - \frac{a}{2}\cos\theta \approx R$ and $r + \frac{a}{2}\cos\theta \approx R$, considering a to be small. The distance between the two surfaces in the radial direction at angle θ is $|\, a\cos\theta\, |$ and does not depend on the azimuthal angle. It is seen from the diagram that the surface of the sphere has in effect a surface density $\sigma = \sigma_0 \cos\theta$ when

$$\sigma_0 = \rho a.$$

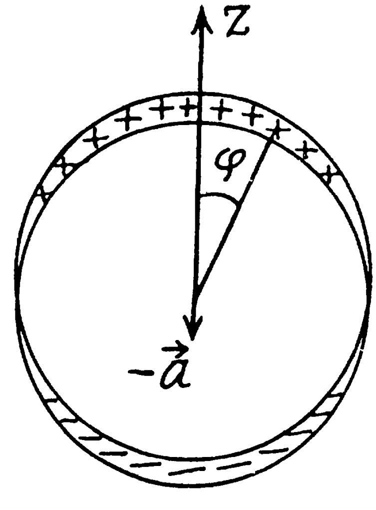
Inside any uniformly charged spherical ball, the field is radial and has the magnitude given by Gauss's theorm

$$4\pi r^2 E = \frac{4\pi}{3} r^3 \rho/\varepsilon_0$$

or
$$E = \frac{\rho r}{3\varepsilon_0}$$

In vector notation, using the fact the V must be measured from the centre of the ball, we get, for the present case

$$\vec{E} = \frac{\rho}{3\varepsilon_0}\left(\vec{r} - \frac{\vec{a}}{2}\right) - \frac{\rho}{3\varepsilon_0}\left(\vec{r} + \frac{\vec{a}}{2}\right)$$



$$= -\rho \vec{a}/3\varepsilon_0 = \frac{\sigma_0}{3\varepsilon_0}\vec{k}$$

When $\vec{k}$ is the unit vector along the polar axis from which θ is measured.

3.18 Let us consider an elemental spherical shell of thickness dr. Thus surface charge density of the shell $\sigma = \rho\, dr = (\vec{a} \cdot \vec{r})\, dr$.

Thus using the solution of 3.16, field strength due to this sperical shell

$$d\vec{E} = -\frac{\vec{a}\, r}{3\,\varepsilon_0}\, dr$$

Hence the sought field strength

$$\vec{E} = -\frac{\vec{a}}{3\,\varepsilon_0}\int_0^R r\, dr = -\frac{\vec{a} R^2}{6\,\varepsilon_0}.$$

3.19 From the solution of 3.14 field strength at a perpendicular distance $r < R$ from its left end

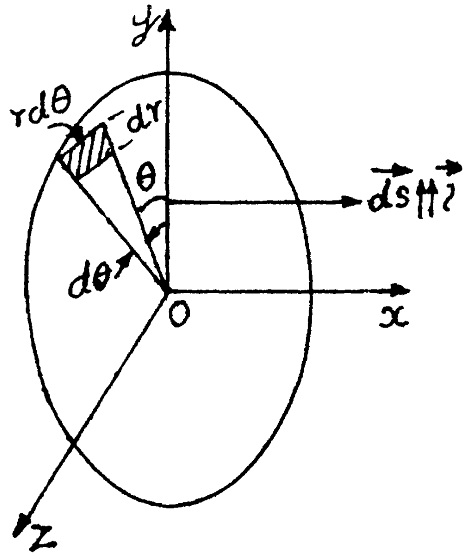
$$\vec{E}(r) = \frac{\lambda}{4\pi\varepsilon_0 r}(-\vec{i}) + \frac{\lambda}{4\pi\varepsilon_0 r}(\hat{e}_r)$$

Here $\hat{e}_r$ is a unit vector along radial direction.

Let us consider an elemental surface, $dS = dy\, dz = dz\,(r\, d\theta)$ a figure. Thus flux of $\vec{E}(r)$ over the element $d\vec{S}$ is given by

$$d\Phi = \vec{E} \cdot d\vec{S} = \left[\frac{\lambda}{4\pi\varepsilon_0 r}(-\vec{i}) + \frac{\lambda}{4\pi\varepsilon_0 r}(\hat{e}_r)\right] \cdot dr\,(r\, d\theta)\,\vec{i}$$

$$= -\frac{\lambda}{4\pi\varepsilon_0}\, dr\, d\theta \left(\text{as } \vec{e}_r \perp \vec{i}\right)$$



The sought flux, $\Phi = -\frac{\lambda}{4\pi\varepsilon_0}\int_0^R dr \int_0^{2\pi} d\theta = -\frac{\lambda R}{2\,\varepsilon_0}.$

If we have taken $d\vec{S} \uparrow\uparrow (-\vec{i})$, then Φ were $\frac{\lambda R}{2\,\varepsilon_0}$

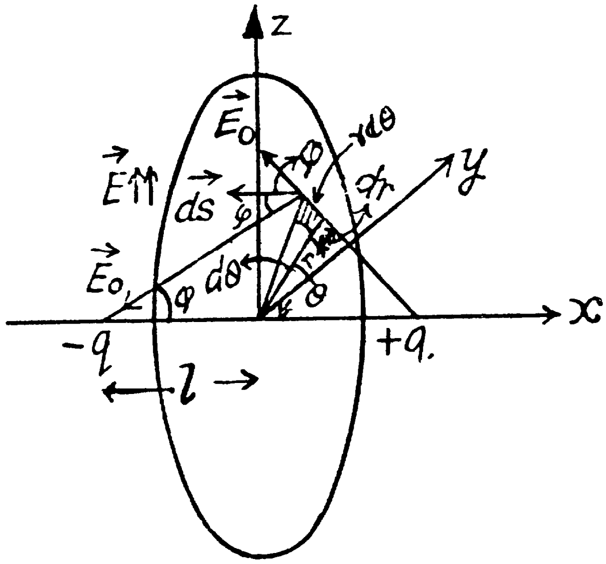
Hence $|\Phi| = \frac{\lambda R}{2\,\varepsilon_0}$

3.20 Let us consider an elemental surface area as shown in the figure. Then flux of the vector $\vec{E}$ through the elemental area,

$$d\Phi = \vec{E} \cdot d\vec{S} = E\, dS = 2E_0 \cos\varphi\, dS \;(\text{as } \vec{E} \uparrow\uparrow d\vec{S})$$

$$= \frac{2q}{4\pi\varepsilon_0 (l^2 + r^2)} \frac{l}{(l^2 + r^2)^{1/2}} (r\, d\theta)\, dr = \frac{2ql\, r\, dr\, d\theta}{4\pi\varepsilon_0 (r^2 + l^2)^{3/2}}$$

where $E_0 = \dfrac{q}{4\pi\varepsilon_0(l^2+r^2)}$ is magnitude of field strength due to any point charge at the point of location of considered elemental area.



Thus $\Phi = \dfrac{2ql}{4\pi\varepsilon_0}\displaystyle\int_0^R \frac{r\,dr}{(r^2+l^2)^{3/2}}\int_0^{2\pi} d\theta$

$$= \frac{2ql\times 2\pi}{4\pi\varepsilon_0}\int_0^R \frac{r\,dr}{(r^2+l^2)^{3/2}} = \frac{q}{\varepsilon_0}\left[1-\frac{l}{\sqrt{l^2+R^2}}\right]$$

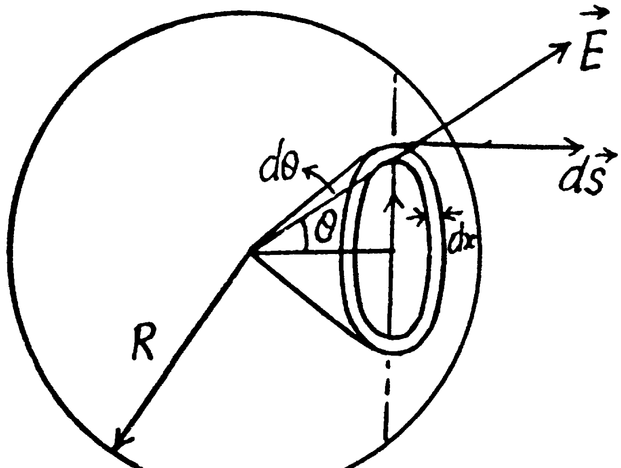
It can also be solved by considering a ring element or by using solid angle.

3.21 Let us consider a ring element of radius x and thickness dx, as shown in the figure. Now, flux over the considered element,

$$d\Phi = \vec{E}\cdot\vec{dS} = E_r\,dS\cos\theta$$

But $E_r = \dfrac{\rho r}{3\varepsilon_0}$ from Gauss's theorem,

and $\quad dS = 2\pi x\,dx\;,\;\cos\theta = \dfrac{r_0}{r}$



Thus $\quad d\Phi = \dfrac{\rho r}{3\varepsilon_0}2\pi x\,dx\dfrac{r_0}{r} = \dfrac{\rho r_0}{3\varepsilon_0}2\pi x\,dx$

Hence sought flux

$$\Phi = \frac{2\pi\rho r_0}{3\varepsilon_0}\int_0^{\sqrt{R^2-r_0^2}} x\,dx \quad = \frac{2\pi\rho r_0}{3\varepsilon_0}\frac{(R^2-r_0^2)}{2} = \frac{\pi\rho r_0}{3\varepsilon_0}(R^2-r_0^2)$$

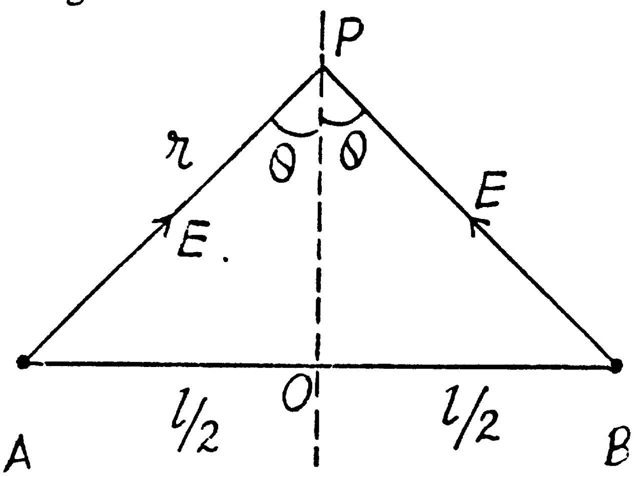
3.22 The field at P due to the threads at A and B are both of magnitude $\dfrac{\lambda}{2\pi\varepsilon_0(x^2+l^2/4)^{1/2}}$

and directed along AP and BP. The resultant is along OP with

$$E = \frac{2\lambda\cos\theta}{2\pi\varepsilon_0(\pi^2+\pi^{1/2})^{1/2}} = \frac{\lambda x}{\pi\varepsilon_0(x^2+l^2/4)}$$

$$= \frac{\lambda}{\pi\varepsilon_0\left[x+\dfrac{l^2}{4x}-2\cdot\dfrac{l}{2\sqrt{x}}\cdot\sqrt{x}+l\right]}$$

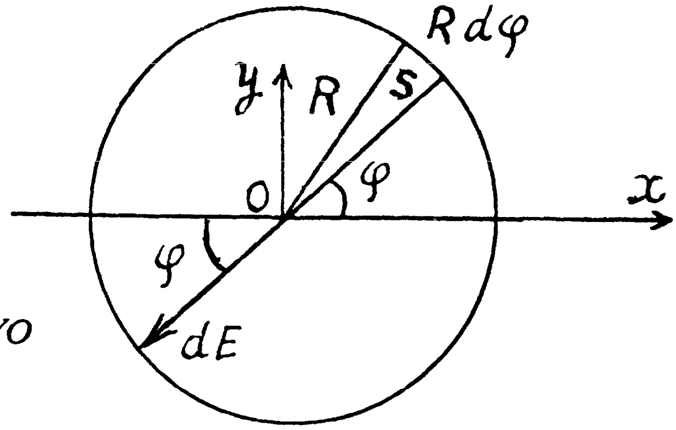
$$= \frac{\lambda}{\pi\varepsilon_0\left[\left(\sqrt{x}-\dfrac{l}{2\sqrt{x}}\right)^2+l\right]}$$



This is maximum when $x = l/2$ and then $E = E_{max} = \dfrac{\lambda}{\pi\varepsilon_0 l}$

3.23 Take a section of the cylinder perpendicular to its axis through the point where the electric field is to be calculated. (All points on the axis are equivalent.) Consider an element S with azimuthal angle φ. The length of the element is $R\,d\varphi$, R being the radius of cross section of the cylinder. The element itself is a section of an infinite strip. The electric field at O due to this strip is

$$\frac{\sigma_0 \cos\varphi\,(R\,d\varphi)}{2\pi\varepsilon_0 R} \text{ along } SO$$



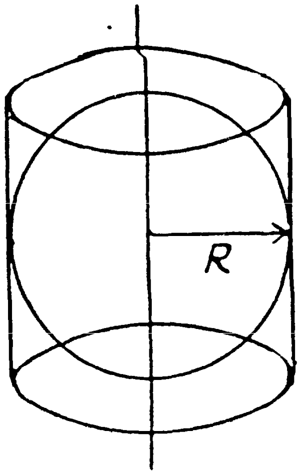
This can be resolved into

$$\frac{\sigma_0 \cos\varphi\, d\varphi}{2\pi\varepsilon_0}\begin{cases}\cos\varphi \text{ along } OX \text{ towards } O\\ \sin\varphi \text{ along } YO\end{cases}$$

On integration the component along YO vanishes. What remains is

$$\int_0^{2\pi} \frac{\sigma_0 \cos^2\varphi\, d\varphi}{2\pi\varepsilon_0} = \frac{\sigma_0}{2\varepsilon_0} \text{ along } XO \text{ i.e. along the direction } \varphi = \pi.$$

3.24 Since the field is axisymmetric (as the field of a uniformly charged filament), we conclude that the flux through the sphere of radius R is equal to the flux through the lateral surface of a cylinder having the same radius and the height $2R$, as arranged in the figure.



Now, $\quad \Phi = \oint \vec{E}\cdot d\vec{S} = E_r S$

But $\quad E_r = \frac{a}{R}$

Thus $\quad \Phi = \frac{a}{R}S = \frac{a}{R}\,2\pi R\cdot 2R = 4\pi a R$

3.25 (a) Let us consider a sphere of radius $r < R$ then charge, inclosed by the considered sphere,

$$q_{inclosed} = \int_0^r 4\pi r^2\, dr\, \rho = \int_0^r 4\pi r^2 \rho_0 \left(1 - \frac{r}{R}\right) dr \qquad (1)$$

Now, applying Gauss' theorem,

$E_r\, 4\pi r^2 = \frac{q_{inclosed}}{\varepsilon_0}$, (where E_r is the projection of electric field along the radial line.)

$$= \frac{\rho_0}{\varepsilon_0}\int_0^r 4\pi r^2 \left(1 - \frac{r}{R}\right) dr$$

or, $$E_r = \frac{\rho_0}{3\varepsilon_0}\left[r^2 - \frac{3r^2}{4R}\right]$$

And for a point. outside the sphere $r > R$.

$$q_{inclosed} = \int_0^R 4\pi r^2 \, dr \, \rho_0 \left(1 - \frac{r}{R}\right) \text{(as there is no charge outside the ball)}$$

Again from Gauss' theorem,

$$E_r \, 4\pi r^2 = \int_0^R \frac{4\pi r^2 \, dr \, \rho_0 \left(1 - \frac{r}{R}\right)}{\varepsilon_0}$$

or,
$$E_r = \frac{\rho_0}{r^2 \varepsilon_0}\left[\frac{R^3}{3} - \frac{R^4}{4R}\right] = \frac{\rho_0 R^3}{12 r^2 \varepsilon_0}$$

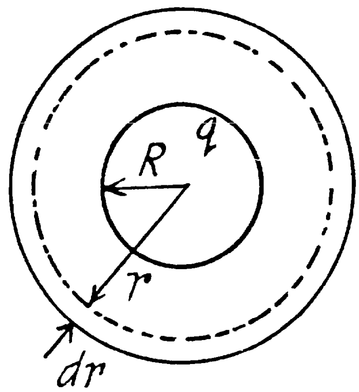
(b) As magnitude of electric field decreases with increasing r for $r > R$, field will be maximum for $r < R$. Now, for E_r to be maximum,

$$\frac{d}{dr}\left(r - \frac{3r^2}{4R}\right) = 0 \quad \text{or} \quad 1 - \frac{3r}{2R} = 0 \quad \text{or} \quad r = r_m = \frac{2R}{3}$$

Hence
$$E_{\max} = \frac{\rho_0 R}{9 \varepsilon_0}$$

3.26 Let the charge carried by the sphere be q, then using Gauss' theorem for a spherical surface having radius $r > R$, we can write.

$$E \, 4\pi r^2 = \frac{q_{inclosed}}{\varepsilon_0} = \frac{q}{\varepsilon_0} + \frac{1}{\varepsilon_0}\int_R^r \frac{\alpha}{r} 4\pi r^2 \, dr$$



On integrating we get,

$$E \, 4\pi r^2 = \frac{(q - 2\pi\alpha R^2)}{\varepsilon_0} + \frac{4\pi\alpha r^2}{2\varepsilon_0}$$

The intensity E does not depend on r when the experession in the parentheses is equal to zero. Hence

$$q = 2\pi\alpha R^2 \text{ and } E = \frac{\alpha}{2\varepsilon_0}$$

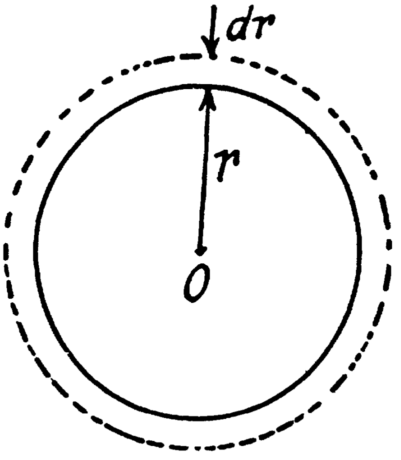
3.27 Let us consider a spherical layer of radius r and thickness dr, having its centre coinciding with the centre of the system. Then using Gauss' theorem for this surface,

$$E_r \, 4\pi r^2 = \frac{q_{inclosed}}{\varepsilon_0} = \int_0^r \frac{\rho \, dV}{\varepsilon_0}$$

$$= \frac{1}{\varepsilon_0}\int_0^r \rho_0 e^{-\alpha r^3} 4\pi r^2 \, dr$$

After integration

$$E_r 4\pi r^2 = \frac{\rho\, 4\pi}{3\,\varepsilon_0\,\alpha}\,[1 - e^{-\alpha r^3}]$$



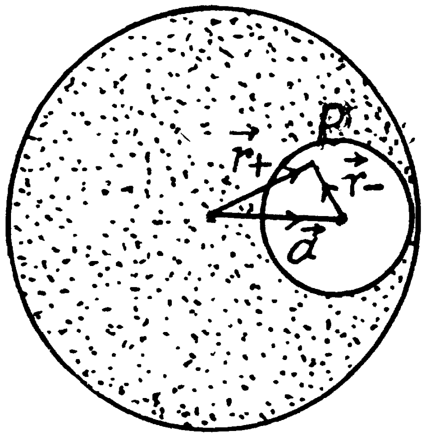
or, $$E_r = \frac{\rho_0}{3\,\varepsilon\,\alpha\, r^2}\,[1 - e^{-\alpha r^3}]$$

Now when $\alpha r^3 << 1$, $E_r \approx \dfrac{\rho_0 r}{3\,\varepsilon_0}$

And when $\alpha r^3 >> 1$, $E_r \approx \dfrac{\rho_0}{3\,\varepsilon_0\,\alpha\, r^2}$

3.28 Using Gauss theorem we can easily show that the electric field strength within a uniformly charged sphere is $\vec{E} = \left(\dfrac{\rho}{3\,\varepsilon_0}\right)\vec{r}$

The cavity, in our problem, may be considered as the superposition of two balls, one with the charge density ρ and the other with $-\rho$.



Let P be a point inside the cavity such that its position vector with respect to the centre of cavity be $\vec{r}_-$ and with respect to the centre of the ball $\vec{r}_+$. Then from the principle of superposition, field inside the cavity, at an arbitrary point P,

$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

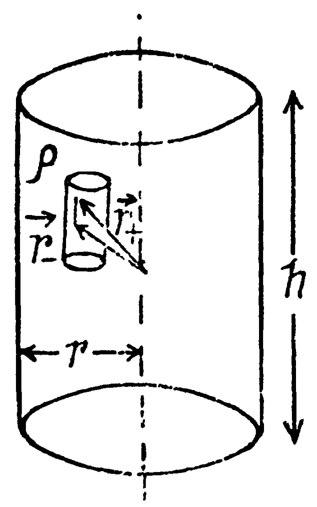
$$= \frac{\rho}{3\,\varepsilon_0}(\vec{r}_+ - \vec{r}_-) = \frac{\rho}{3\,\varepsilon_0}\vec{a}$$

Note : Obtained expression for $\vec{E}$ shows that it is valid regardless of the ratio between the radii of the sphere and the distance between their centres.

3.29 Let us consider a cylinderical Gaussian surface of radius r and height h inside an infinitely long charged cylinder with charge density ρ. Now from Gauss theorem :

$$E_r\, 2\pi r h = \frac{q_{inclosed}}{\varepsilon_0}$$

(where E_r is the field inside the cylinder at a distance r from its axis.)



or, $E_r\, 2\pi r h = \dfrac{\rho\,\pi r^2 h}{\varepsilon_0}$ or $E_r = \dfrac{\rho r}{2\,\varepsilon_0}$

Now, using the method of 3.28 field at a point P, inside the cavity, is

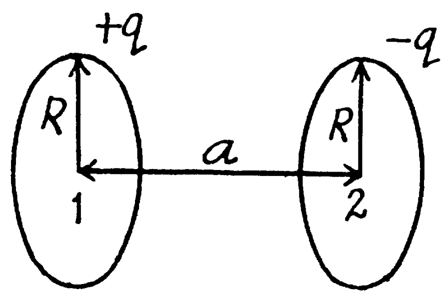
$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho}{2\varepsilon_0}(\vec{r}_+ - \vec{r}_-) = \frac{\rho}{2\varepsilon_0}\vec{a}$$

3.30 The arrangement of the rings are as shown in the figure. Now, potential at the point 1, φ_1 = potential at 1 due to the ring 1 + potential at 1 due to the ring 2.

$$= \frac{q}{4\pi\varepsilon_0 R} + \frac{-q}{4\pi\varepsilon_0 (R^2+a^2)^{1/2}}$$

Similarly, the potential at point 2,

$$\varphi_2 = \frac{-q}{4\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0 (R^2+a^2)^{1/2}}$$



Hence, the sought potential difference,

$$\varphi_1 - \varphi_2 = \Delta\varphi = 2\left(\frac{q}{4\pi\varepsilon_0 R} + \frac{-q}{4\pi\varepsilon_0 (R^2+a^2)^{1/2}}\right)$$

$$= \frac{q}{2\pi\varepsilon_0 R}\left(1 - \frac{1}{\sqrt{1+(a/R)^2}}\right)$$

3.31 We know from Gauss theorem that the electric field due to an infiniety long straight wire, at a perpendicular distance r from it equals, $E_r = \frac{\lambda}{2\pi\varepsilon_0 r}$. So, the work done is

$$\int_1^2 E_r\, dr = \int_x^{\eta x} \frac{\lambda}{2\pi\varepsilon_0 r}\, dr$$

(where x is perpendicular distance from the thread by which point 1 is removed from it.)

Hence $$\Delta\varphi_{12} = \frac{\lambda}{2\pi\varepsilon_0}\ln\eta$$

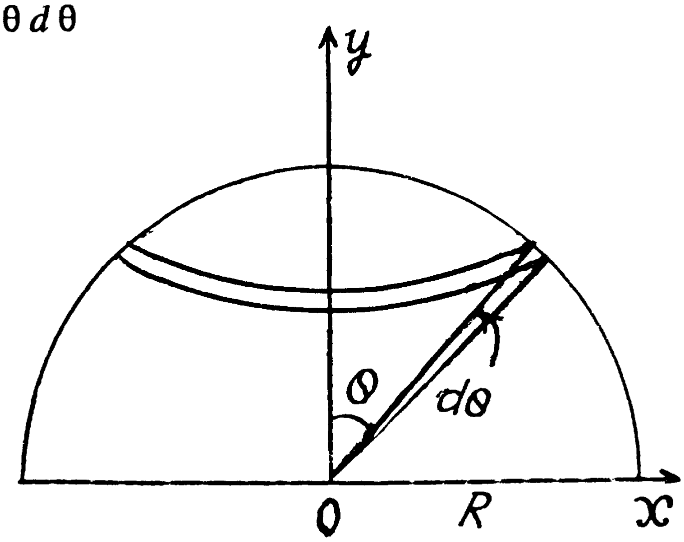
3.32 Let us consider a ring element as shown in the figure. Then the charge, carried by the element, $dq = (2\pi R\sin\theta) R\, d\theta\, \sigma$,

Hence, the potential due to the considered element at the centre of the hemisphere,

$$d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R} = \frac{2\pi\sigma R\sin\theta\, d\theta}{4\pi\varepsilon_0} = \frac{\sigma R}{2\varepsilon_0}\sin\theta\, d\theta$$

So potential due to the whole hemisphere

$$\varphi = \frac{R\sigma}{2\varepsilon_0}\int_0^{\pi/2}\sin\theta\, d\theta = \frac{\sigma R}{2\varepsilon_0}$$

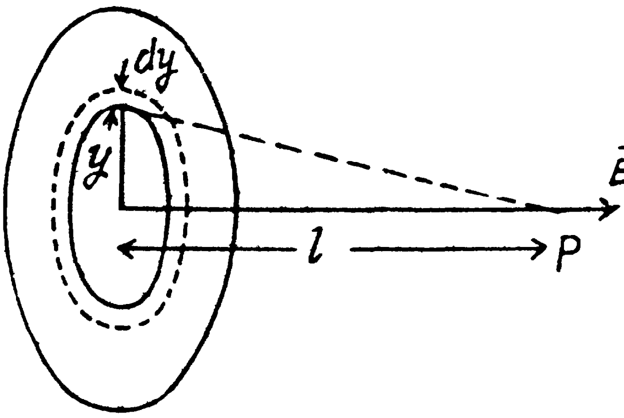


Now from the symmetry of the problem, net electric field of the hemisphere is directed towards the negative y-axis. We have

$$dE_y = \frac{1}{4\pi\varepsilon_0}\frac{dq\cos\theta}{R^2} = \frac{\sigma}{2\varepsilon_0}\sin\theta\cos\theta\, d\theta$$

Thus $$E = E_y' = \frac{\sigma}{2\varepsilon_0}\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta = \frac{\sigma}{4\varepsilon_0}\int_0^{\pi/2}\sin 2\theta\, d\theta = \frac{\sigma}{4\varepsilon_0},$$ along YO

3.33 Let us consider an elementary ring of thickness dy and radius y as shown in the figure. Then potential at a point P, at distance l from the centre of the disc, is



$$d\varphi = \frac{\sigma\, 2\pi y\, dy}{4\pi\varepsilon_0 (y^2 + l^2)^{1/2}}$$

Hence potential due to the whole disc,

$$\varphi = \int_0^R \frac{\sigma\, 2\pi y\, dy}{4\pi\varepsilon_0 (y^2 + l^2)^{1/2}} = \frac{\sigma l}{2\varepsilon_0}\left(\sqrt{1 + (R/l)^2} - 1\right)$$

From symmetry

$$E = E_l = -\frac{d\varphi}{dl}$$

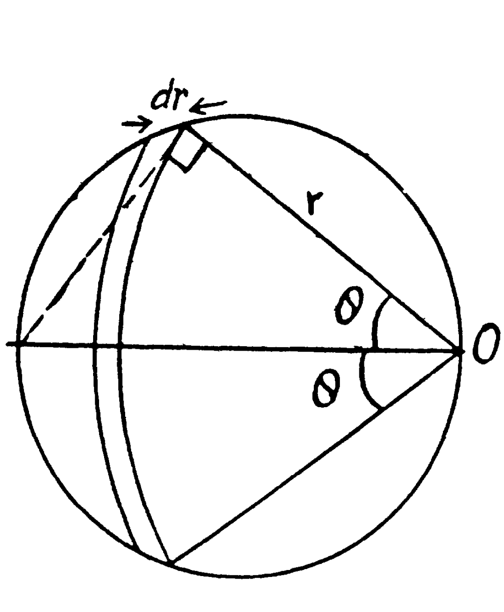
$$= -\frac{\sigma}{2\varepsilon_0}\left[\frac{2l}{2\sqrt{R^2 + l^2}} - 1\right] = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{1}{\sqrt{1 + (R/l)^2}}\right]$$

when $l \to 0$, $\varphi \approx \frac{\sigma R}{2\varepsilon_0}$, $E = \frac{\sigma}{2\varepsilon_0}$ and when $l >> R$,

$$\varphi \approx \frac{\sigma R^2}{4\varepsilon_0 l},\quad E = \frac{\sigma R^2}{4\varepsilon_0 l^2}$$

3.34 By definition, the potential in the case of a surface charge distribution is defined by integral $\varphi = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, dS}{r}$. In order to simplify integration, we shall choose the area element dS in the form of a part of the ring of radius r and width dr in (Fig.). Then $dS = 2\theta\, r\, dr$, $r = 2R\cos\theta$ and $dr = -2R\sin\theta\, d\theta$. After substituting these expressions into integral $\varphi = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, dS}{r}$, we obtain the expression for φ at the point O:

$$\varphi = -\frac{\sigma R}{\pi\varepsilon_0}\int_{\pi/2}^{0} \theta \sin\theta\, d\theta.$$



We integrate by parts, denoting $\theta = u$ and $\sin\theta\, d\theta = dv$:

$\int \theta \sin\theta\, d\theta = -\theta\cos\theta$

$+\int \cos\theta\, d\theta = -\theta\cos\theta + \sin\theta$

which gives -1 after substituting the limits of integration. As a result, we obtain

$$\varphi = \sigma R/\pi\varepsilon_0.$$

3.35 In accordance with the problem $\varphi = \vec{a} \cdot \vec{r}$

Thus from the equation : $\vec{E} = -\vec{\nabla}\varphi$

$$\vec{E} = -\left[\frac{\partial}{\partial x}(a_x x)\vec{i} + \frac{\partial}{\partial_y}(a_y y)\vec{j} + \frac{\partial}{\partial_z}(a_z z)\vec{k}\right] = -[a_x \vec{i} + a_y \vec{j} + {}_z \vec{k}\,] = -\vec{a}$$

3.36 (a) Given, $\varphi = a(x^2 - y^2)$

So, $$\vec{E} = -\vec{\nabla}\varphi = -2a(x\vec{i} - y\vec{j}\,)$$

The sought shape of field lines is as shown in the figure (a) of answersheet assuming $a > 0$:

(b) Since $\varphi = axy$

So, $$\vec{E} = -\vec{\nabla}\varphi = -ay\vec{i} - ax\vec{j}$$

Plot as shown in the figure (b) of answersheet.

3.37 Given, $\varphi = a(x^2 + y^2) + bz^2$

So, $$\vec{E} = -\vec{\nabla}\varphi = -[2ax\vec{i} + 2ay\vec{j} + 2bz\vec{k}]$$

Hence $$|\vec{E}| = 2\sqrt{a^2(x^2 + y^2) + b^2 z^2}$$

Shape of the equipotential surface :

Put $$\vec{\rho} = x\vec{i} + y\vec{j} \text{ or } \rho^2 = x^2 + y^2$$

Then the equipotential surface has the equation

$$a\rho^2 + bz^2 = \text{constant} = \varphi$$

If $a > 0$, $b > 0$ then $\varphi > 0$ and the equation of the equipotential surface is

$$\frac{\rho}{\varphi/a} + \frac{z^2}{\varphi/b} = 1$$

which is an ellipse in ρ, z coordinates. In three dimensions the surface is an ellipsoid of revolution with semi- axis $\sqrt{\varphi/a}$, $\sqrt{\varphi/a}$, $\sqrt{\varphi/b}$.

If $a > 0$, $b < 0$ then φ can be ≥ 0. If $\varphi > 0$ then the equation is

$$\frac{\rho^2}{\varphi/a} - \frac{z^2}{\varphi/|b|} = 1$$

This is a single cavity hyperboloid of revolution about z axis. If $\varphi = 0$ then

$$a\rho^2 - |b|z^2 = 0$$

or $$z = \pm\sqrt{\frac{a}{|b|}}\,\rho$$

is the equation of a right circular cone.

If $\varphi < 0$ then the equation can be written as

$$|b|z^2 - a\rho^2 = |\varphi|$$

or $$\frac{z^2}{|\varphi|/|b|} - \frac{\rho^2}{|\varphi|/a} = 1$$

This is a two cavity hyperboloid of revolution about z-axis.

3.38 From Gauss' theorem intensity at a point, inside the sphere at a distance r from the centre is given by, $E_r = \dfrac{\rho\, r}{3\,\varepsilon_0}$ and outside it, is given by $E_r = \dfrac{1}{4\,\pi\,\varepsilon_0}\dfrac{q}{r^2}$.

(a) Potential at the centre of the sphere,

$$\varphi_0 = \int_0^\infty \vec{E}\cdot d\vec{r} = \int_0^R \frac{\rho r}{3\,\varepsilon_0}\,dr + \int_R^\infty \frac{q}{4\,\pi\,\varepsilon_0\, r^2}\,dr = \frac{\rho}{3\,\varepsilon}\frac{R^2}{2} + \frac{q}{4\,\pi\,\varepsilon_0 R}$$

as
$$= \frac{q}{8\,\pi\,\varepsilon_0 R} + \frac{q}{4\,\pi\,\varepsilon_0 R} = \frac{3q}{8\,\pi\,\varepsilon_0 R}\quad \left(\text{as } \rho = \frac{3\,q}{4\,\pi\, R^3}\right)$$

(b) Now, potential at any point, inside the sphere, at a distance r from it s centre.

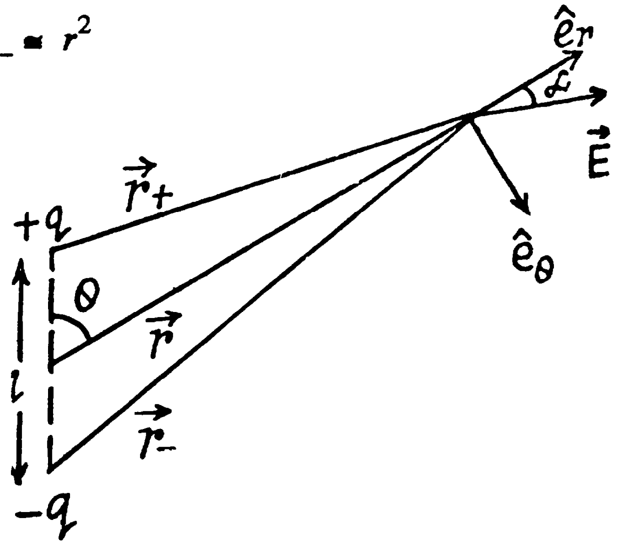
$$\varphi(r) = \int_r^R \frac{\rho}{3\,\varepsilon_0}\, r\, dr \int_r^\infty \frac{q}{4\,\pi\,\varepsilon_0}\frac{dr}{r^2}$$

On integration : $\varphi(r) = \dfrac{3\,q}{8\,\pi\,\varepsilon_0 R}\left[1 - \dfrac{r^2}{3\,R^2}\right] = \varphi_0\left[1 - \dfrac{r^2}{3\,R^2}\right]$

3.39 Let two charges $+q$ and $-q$ be separated by a distance l. Then electric potential at a point at distance $r >> l$ from this dipole,

$$\varphi(r) = \frac{+q}{4\,\pi\,\varepsilon_0\, r_+} + \frac{-q}{4\,\pi\,\varepsilon_0\, r_-} = \frac{q}{4\,\pi\,\varepsilon_0}\left(\frac{r_- - r_+}{r_+ r_-}\right) \tag{1}$$

But $\quad r_- - r_+ = l\cos\theta$ and $r_+ r_- \approx r^2$



From Eqs. (1) and (2),

$$\varphi(r) = \frac{q\, l\cos\theta}{4\,\pi\,\varepsilon_0\, r^2} = \frac{p\cos\theta}{4\,\pi\,\varepsilon_0\, r^2}\ \varphi = \frac{\vec{p}\cdot\vec{r}}{4\,\pi\,\varepsilon_0\, r^3},$$

where p is magnitude of electric moment vector.

Now, $\quad E_r = -\dfrac{\partial\varphi}{\partial r} = \dfrac{2\,p\cos\theta}{4\,\pi\,\varepsilon_0\, r^3}$

and $\quad E_\theta = -\dfrac{\partial\varphi}{r\,\partial\theta} = \dfrac{p\sin\theta}{4\,\pi\,\varepsilon_0\, r^3}$

So $\quad E = \sqrt{E_r^2 + E_\theta^2} = \dfrac{p}{4\,\pi\,\varepsilon_0\, r^3}\sqrt{4\cos^2\theta + \sin^2\theta}$

3.40 From the results, obtained in the previous problem,

$$E_r = \frac{2\,p\cos\theta}{4\,\pi\,\varepsilon_0\, r^3} \text{ and } E_\theta = \frac{p\sin\theta}{4\,\pi\,\varepsilon_0\, r^3}$$

From the given figure, it is clear that,

$$E_z = E_r\cos\theta - E_\theta\sin\theta = \frac{p}{4\,\pi\,\varepsilon_0\, r^3}(3\cos^2\theta - 1)$$

and $\qquad E_{\perp} = E_r \sin\theta + E_0 \cos\theta = \dfrac{3\,p \sin\theta \cos\theta}{4\pi\varepsilon_0 r^3}$

When $\qquad \vec{E} \perp \vec{p},\ |\vec{E}| = E_{\perp}$ and $E_z = 0$

So $\qquad 3\cos^2\theta = 1$ and $\cos\theta = \dfrac{1}{\sqrt{3}}$

Thus $\vec{E}_{\perp}\vec{p}$ at the points located on the lateral surface of the cone, having its axis, coinciding with the direction of z–axis and semi vertex angle $\theta = \cos^{-1} 1/\sqrt{3}$.

3.41 Let us assume that the dipole is at the centre of the one equipotential surface which is spherical (Fig.). On an equipotential surface the net electric field strength along the tangent of it becomes zero. Thus

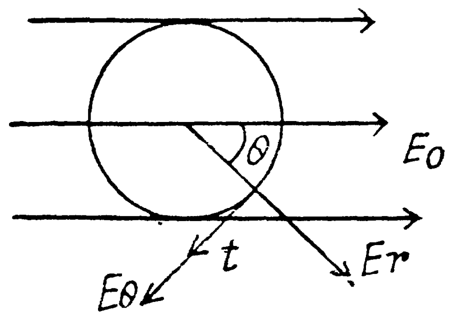
$$-E_0 \sin\theta + E_\theta = 0 \quad \text{or} \quad -E_0 \sin\theta + \frac{p\sin\theta}{4\pi\varepsilon_0 r^3} = 0$$

Hence $\qquad r = \left(\dfrac{p}{4\pi\varepsilon_0 E_0}\right)^{1/3}$

Alternate : Potential at the point, near the dipole is given by,

$$\varphi = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} - \vec{E}_0\cdot\vec{r} + \text{constant},$$

$$= \left(\frac{p}{4\pi\varepsilon_0 r^3} - E_0\right)\cos\theta + \text{Const}$$



For φ to be constant,

$$\frac{p}{4\pi\varepsilon_0 r^3} - E_0 = 0 \quad \text{or,} \quad \frac{p}{4\pi\varepsilon_0 r^3} = E_0$$

Thus $\qquad r = \left(\dfrac{p}{4\pi\varepsilon_0 E_0}\right)^{1/3}$

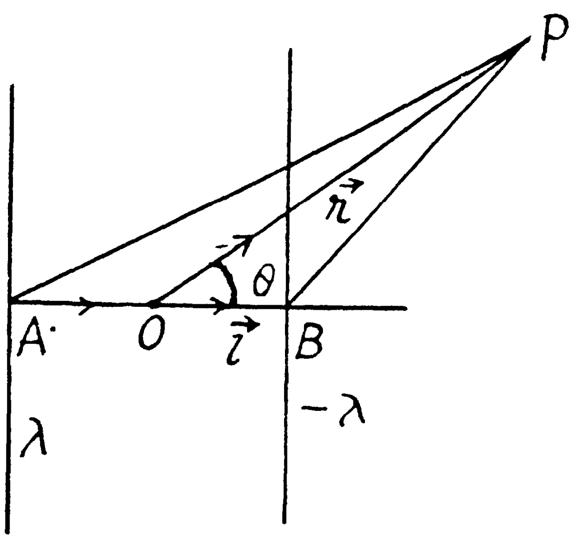
3.42 Let P be a point, at distace $r >> l$ and at an angle to θ the vector $\vec{l}$ (Fig.).

$$\text{Thus } \vec{E} \text{ at } P = \frac{\lambda}{2\pi\varepsilon_0}\,\frac{\vec{r}+\frac{\vec{l}}{2}}{\left|\vec{r}+\frac{\vec{l}}{2}\right|^2} - \frac{\lambda}{2\pi\varepsilon_0}\,\frac{\vec{r}-\frac{\vec{l}}{2}}{\left|\vec{r}-\frac{\vec{l}}{2}\right|^2}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{\vec{r}+\vec{l}/2}{r^2 + \frac{l^2}{4} + r\,l\cos\theta} - \frac{\vec{r}-\vec{l}/2}{r^2 + \frac{l^2}{4} - r\,l\cos\theta}\right]$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{\vec{l}}{r^2} - \frac{2\,l\,\vec{r}}{r^3}\cos\theta\right)$$

Hence $E = |\vec{E}| = \dfrac{\lambda\, l}{2\pi\varepsilon_0 r^2},\ r >> l$



Also, $$\varphi = \frac{\lambda}{2\pi\varepsilon_0}\ln|\vec{r}+\vec{l}/2| - \frac{\lambda}{2\pi\varepsilon_0}\ln|\vec{r}-\vec{l}/2]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{r^2 + r\,l\cos\theta + l^2/4}{r^2 - r\,l\cos\theta + l^2/4} = \frac{\lambda\,l\cos\theta}{2\pi\varepsilon_0 r}, \quad r >> l$$

3.43 The potential can be calculated by superposition. Choose the plane of the upper ring as $x = l/2$ and that of the lower ring as $x = -l/2$.

Then $$\varphi = \frac{q}{4\pi\varepsilon_0[R^2+(x-l/2)^2]^{1/2}} - \frac{q}{4\pi\varepsilon_0[R^2+(x+l/2)^2]^{1/2}}$$

$$= \frac{q}{4\pi\varepsilon_0[R^2+x^2-lx]^{1/2}} - \frac{q}{4\pi\varepsilon_0[R^2+x^2+lx]^{1/2}}$$

$$= \frac{q}{4\pi\varepsilon_0(R^2+x^2)^{1/2}}\left(1+\frac{lx}{2(R^2+x^2)}\right) - \frac{q}{4\pi\varepsilon_0(R^2+x^2)^{1/2}}\left(1-\frac{lx}{2(R^2+x^2)}\right)$$

$$= \frac{q\,lx}{4\pi\varepsilon_0(R^2+x^2)^{3/2}}$$

For $$|x| >> R, \; \varphi = \frac{q\,l}{4\pi x^2}$$

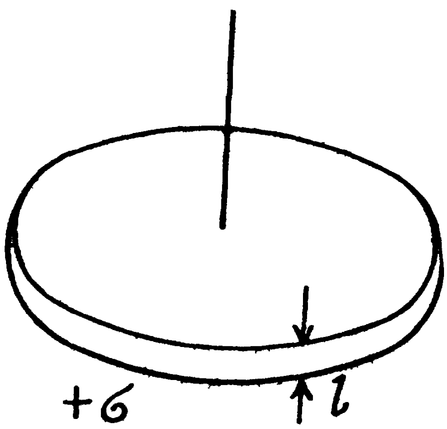
The electric field is $E = -\dfrac{\partial\varphi}{\partial x}$

$$= -\frac{ql}{4\pi\varepsilon_0(R^2+x^2)^{3/2}} + \frac{3}{2}\frac{ql}{(R^2+x^2)^{5/2}\,4\pi\varepsilon_0}\times 2x = \frac{ql\,(2x^2-R^2)}{4\pi\varepsilon_0(R^2+x^2)^{5/2}}$$

For $$|x| >> R, \; E = \frac{ql}{2\pi\varepsilon_0 x^3}.$$ The plot is as given in the book.

3.44 The field of a pair of oppositely charged sheets with holes can by superposition be reduced to that of a pair of uniform opposite charged sheets and discs with opposite charges. Now the charged sheets do not contribute any field outside them. Thus using the result of the previous problem

$$\varphi = \int_0^R \frac{(-\sigma)\,l\,2\pi r\,dr\,x}{4\pi\varepsilon_0(r^2+x^2)^{3/2}}$$



$$= -\frac{\sigma x l}{4\varepsilon_0}\int_{x^2}^{R^2+x^2}\frac{dy}{y^{3/2}} = \frac{\sigma x l}{2\varepsilon_0\sqrt{R^2+x^2}}$$

$$E_x = -\frac{\partial\varphi}{\partial x} = -\frac{\sigma l}{2\varepsilon_0}\left[\frac{1}{\sqrt{R^2+x^2}} - \frac{x^2}{(R^2+x^2)^{3/2}}\right] = -\frac{\sigma l R^2}{2\varepsilon_0(R^2+x^2)^{3/2}}.$$

The plot is as shown in the answersheet.

3.45 For $x > 0$ we can use the result as given above and write

$$\varphi = \pm \frac{\sigma l}{2\varepsilon_0}\left(1 - \frac{|x|}{(R^2 + x^2)^{1/2}}\right)$$

for the solution that vanishes at α. There is a discontinuity in potential for $|x| = 0$. The solution for negative x is obtained by $\sigma \to -\sigma$. Thus

$$\varphi = -\frac{\sigma l x}{2\varepsilon_0 (R + x^2)^{1/2}} + \text{constant}$$

Hence ignoring the jump

$$E = -\frac{\partial \varphi}{\partial x} = \frac{\sigma l R^2}{2\varepsilon_0 (R^2 + x^2)^{3/2}}$$

for large $|x|$ $\quad \varphi = \pm \dfrac{p}{4\pi\varepsilon_0 x^2}$ and $E = \dfrac{p}{2\pi\varepsilon_0 |x|^3}$ (where $p = \pi R^2 \sigma l$)

3.46 Here $E_r = \dfrac{\lambda}{2\pi\varepsilon_0 r}$, $E_\theta = E_\varphi = 0$ and $\vec{F} = p\dfrac{\partial \vec{E}}{\partial l}$

(a) $\vec{p}$ along the thread.

$\vec{E}$ does not change as the point of observation is moved along the thread.

$$\vec{F} = 0$$

(b) $\vec{p}$ along $\vec{r}$,

$$\vec{F} = F_r \vec{e}_r = \frac{\lambda p}{2\pi\varepsilon_0 r^2}\vec{e}_r = -\frac{\lambda \vec{p}}{2\pi\varepsilon_0 r^2} \quad \left(\text{On using } \frac{\partial}{\partial r}\vec{e}_r = 0\right)$$

(c) $\vec{p}$ along $\vec{e}_\theta$

$$\vec{F} = p\frac{\partial}{r\,\partial\theta}\frac{\lambda}{2\pi\varepsilon_0 r}e_r$$

$$= \frac{p\lambda}{2\pi\varepsilon_0 r^2}\frac{\partial \vec{e}_r}{\partial\theta} = \frac{p\lambda}{2\pi\varepsilon_0 r^2}\vec{e}_\theta = \frac{\vec{p}\lambda}{2\pi\varepsilon_0 r^2}.$$

3.47 Force on a dipole of moment p is given by,

$$F = \left|\varphi \frac{\partial \vec{E}}{\partial l}\right|$$

In our problem, field, due to a dipole at a distance l, where a dipole is placed,

$$|\vec{E}| = \frac{p}{2\pi\varepsilon_0 l^3}$$

Hence, the force of interaction,

$$F = \frac{3p^2}{2\pi\varepsilon_0 l^4} = 2{\cdot}1 \times 10^{-16}\,\text{N}$$

3.48 $-d\varphi = \vec{E}\cdot d\vec{r} = a(y\,dx + x\,dy) = a\,d(xy)$

On integrating, $\quad \varphi = -a\,xy + C$

3.49 $-d\varphi = \vec{E}\cdot d\vec{r} = [2a\,xy\,\vec{i} + 2(x^2 - y^2)\vec{j}\,]\cdot[dx\,\vec{i} + dy\,\vec{j}\,]$

or, $\quad d\varphi = 2a\,x\,y\,dx + a(x^2 - y^2)\,dy = ad(x^2 y) - ay^2\,dy$

On integrating, we get,

$$\varphi = ay\left(\frac{y^2}{3} - x^2\right) + C$$

3.50 Given, again

$$-d\varphi = \vec{E}\cdot d\vec{r} = (ay\vec{i} + (ax+bz)\vec{j} + by\vec{k})\cdot(dx\,\vec{i} + dy\,\vec{j} + dx\,\vec{k})$$

$$= a\,(y\,dx + ax\,dy) + b\,(z\,dy + y\,dz) = ad\,(x\,y) + bd\,(yz)$$

On integrating,

$$\varphi = -(a\,xy + b\,yz) + C$$

3.51 Field intensity along x-axis.

$$E_x = -\frac{\partial\varphi}{\partial x} = 3\,a\,x^2 \tag{1}$$

Then using Gauss's theorem in differential from

$$\frac{\partial E_x}{\partial x} = \frac{\rho\,(x)}{\varepsilon_0} \quad \text{so,} \quad \rho\,(x) = 6a\,\varepsilon_0\,x.$$

3.52 In the space between the plates we have the Poisson equation

$$\frac{\partial^2\varphi}{\partial x^2} = -\frac{\rho_0}{\varepsilon_0}$$

or,
$$\varphi = -\frac{\rho_0}{2\,\varepsilon_0}x^2 + A\,x + B$$

where ρ_0 is the constant space charge density between the plates.

We can choose $\quad \varphi\,(0) = 0$ so $B = 0$

Then
$$\varphi\,(d) = \Delta\,\varphi = Ad - \frac{\rho_0\,d^2}{2\,\varepsilon_0} \quad \text{or,} \quad A = \frac{\Delta\,\varphi}{d} + \frac{\rho_0\,d}{2\,\varepsilon_0}$$

Now
$$E = -\frac{\partial\,\varphi}{\partial\,x} = \frac{\rho_0}{\varepsilon_0}x - A = 0 \quad \text{for } x = 0$$

if
$$A = \frac{\Delta\,\varphi}{d} + \frac{\rho_0\,d}{2\,\varepsilon_0} = 0$$

then
$$\rho_0 = -\frac{2\,\varepsilon_0\,\Delta\,\varphi}{d^2}$$

Also
$$E\,(d) = \frac{\rho_0\,d}{\varepsilon_0}.$$

3.53 Field intensity is along radial line and is

$$E_r = -\frac{\partial\,\varphi}{\partial r} = -2\,a\,r \tag{1}$$

From the Gauss' theorem,

$$4\,\pi\,r^2\,E_r = \int\frac{dq}{\varepsilon_{0,}}$$

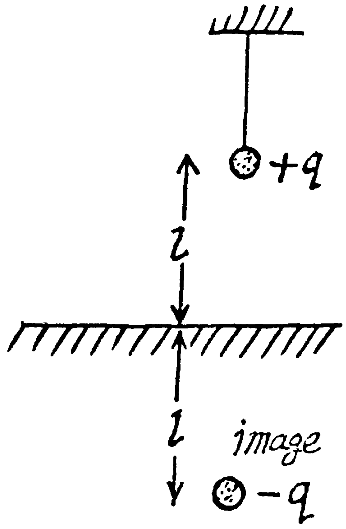
where dq is the charge contained between the sphere of radii r and $r + dr$.

Hence
$$4\,\pi\,r^2\,E_r = 4\,\pi\,r^2 \times (-2ar) = \frac{4\pi}{\varepsilon_0}\int_0^r r'^2\,\rho\,(r')\,dr' \tag{2}$$

Differentiating (2) $\rho = -6\,\varepsilon_0\,a$

3.2 CONDUCTORS AND DIELECTRICS IN AN ELECTRIC FIELD

3.54 When the ball is charged, for the equilibrium of ball, electric force on it must counter balance the excess spring force, exerted, on the ball due to the extension in the spring.



Thus $F_{el} = F_{spr}$

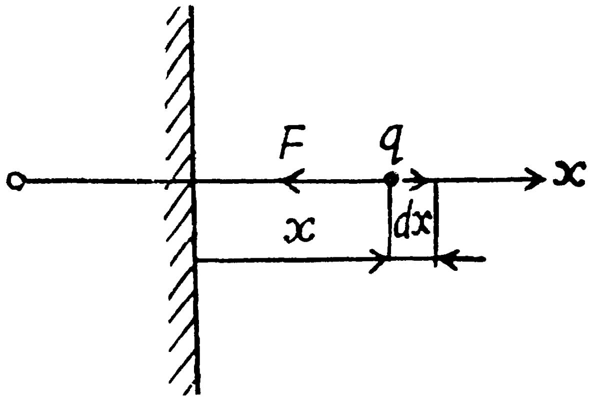
or, $\dfrac{q^2}{4\pi\varepsilon_0 (2l)^2} = \kappa x$, (The force on the charge q might be considered as arised from attraction by the electrical image)

or, $q = 4l\sqrt{\pi\varepsilon_0 \kappa x}$,

sought charge on the sphere.

3.55 By definition, the work of this force done upon an elementry displacement dx (Fig.) is given by

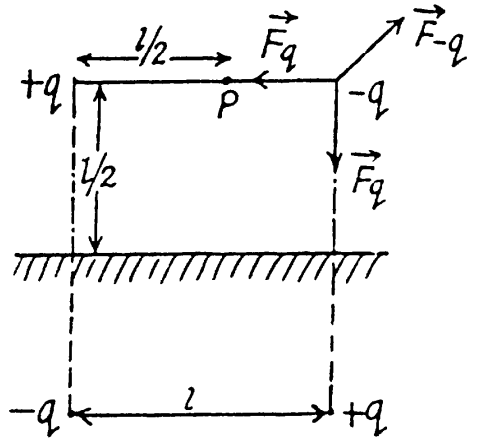
$$dA = F_x\, dx = -\frac{q^2}{4\pi\varepsilon_0 (2x)^2}\, dx,$$



where the expression for the force is obtained with the help of the image method. Integrating this equation over x between l and ∞, we find

$$A = -\frac{q^2}{16\pi\varepsilon_0}\int_l^\infty \frac{dx}{x^2} = -\frac{q^2}{16\pi\varepsilon_0 l}.$$

3.56 (a) Using the concept of electrical image, it is clear that the magnitude of the force acting on each charge,



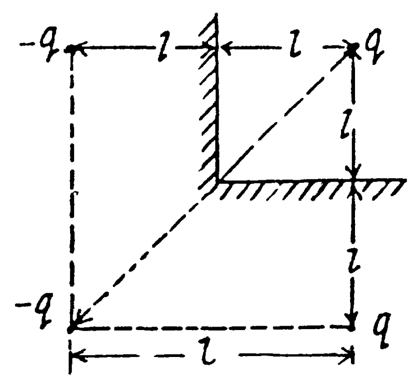
$$|\vec{F}| = \sqrt{2}\,\frac{q^2}{4\pi\varepsilon_0 l^2} - \frac{q^2}{4\pi\varepsilon_0 (\sqrt{2}\,l)^2}$$

$$= \frac{q^2}{8\pi\varepsilon_0 l^2}(2\sqrt{2} - 1)$$

(b) Also, from the figure, magnitude of electrical field strength at P

$$E = 2\left(1 - \frac{1}{5\sqrt{5}}\right)\frac{q}{\pi\varepsilon_0 l^2}$$

3.57 Using the concept of electrical image, it is easily seen that the force on the charge q is,



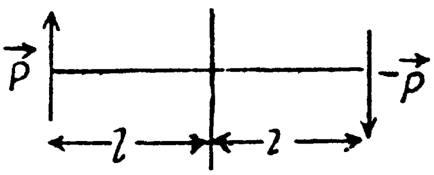
$$F = \frac{\sqrt{2}\,q^2}{4\pi\varepsilon_0 (2l)^2} + \frac{(-q)^2}{4\pi\varepsilon_0 (2\sqrt{2}\,l)^2}$$

$$= \frac{(2\sqrt{2} - 1)\,q^2}{32\pi\varepsilon_0 l^2} \quad \text{(It is attractive)}$$

3.58 Using the concept of electrical image, force on the dipole $\vec{p}$,

$\vec{F} = p\frac{\partial \vec{E}}{\partial l}$, where $\vec{E}$ is field at the location of

$\vec{p}$ due to $(-\vec{p})$

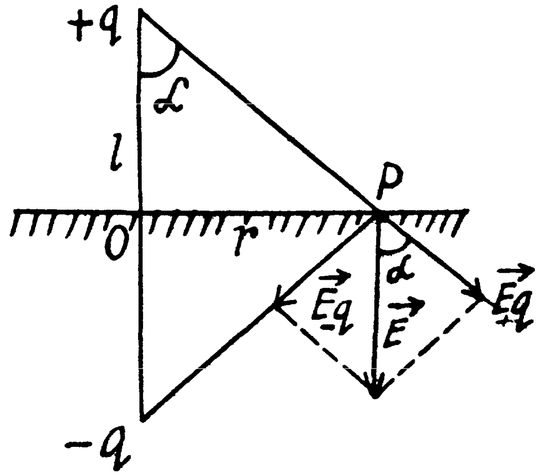


or, $|\vec{F}| = \left|\frac{\partial \vec{E}}{\partial l}\right| p = \frac{3p^2}{32\pi\varepsilon_0 l^4}$

as, $|\vec{E}| = \frac{p}{4\pi\varepsilon_0 (2l)^3}$

3.59 To find the surface charge density, we must know the electric field at the point P (Fig.) which is at a distance r from the point O.

Using the image mirror method, the field at P,



$$E = 2E\cos\alpha = 2\frac{q}{4\pi\varepsilon_0 x^2}\frac{l}{x} = \frac{ql}{2\pi\varepsilon_0 (l^2+r^2)^{3/2}}$$

Now from Gauss' theorem the surface charge density on conductor is connected with the electric field near its surface (in vaccum) through the relation $\sigma = \varepsilon_0 E_n$, where E_n is the projection of $\vec{E}$ onto the outward normal $\vec{n}$ (with respect to the conductor).

As our field strength $\vec{E} \uparrow\downarrow \vec{n}$, so

$$\sigma = -\varepsilon_0 E = -\frac{ql}{2\pi(l^2+r^2)^{3/2}}$$

3.60 (a) The force F_1 on unit length of the thread is given by

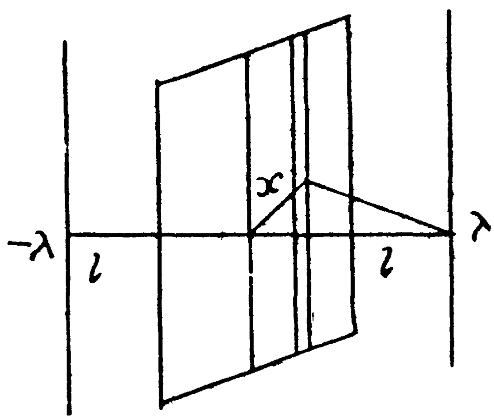
$$F_1 = \lambda E_1$$

where E_1 is the field at the thread due to image charge :

$$E_1 = \frac{-\lambda}{2\pi\varepsilon_0 (2l)}$$

Thus $$F_1 = \frac{-\lambda^2}{4\pi\varepsilon_0 l}$$

minus singn means that the force is one of attraction.



(b) There is an image thread with charge density $-\lambda$ behind the conducting plane. We calculate the electric field on the conductor. It is

$$E(x) = E_n(x) = \frac{\lambda l}{\pi\varepsilon_0 (x^2+l^2)}$$

on considering the thread and its image.

Thus

$$\sigma(x) = \varepsilon_0 E_n = \frac{\lambda l}{\pi(x^2+l^2)}$$

3.61 (a) At O,

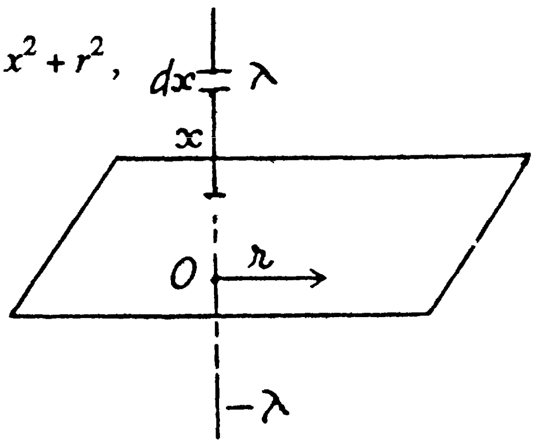
$$E_n(O) = 2\int_l^{\infty} \frac{\lambda\, dx}{4\pi\varepsilon_0 x^2} = \frac{\lambda}{2\pi\varepsilon_0 l}$$

So $\quad \sigma(O) = \varepsilon_0 E_n = \dfrac{\lambda}{2\pi l}$

(b) $$E_n(r) = 2\int_l^{\infty} \frac{\lambda\, dx}{4\pi\varepsilon_0 (x^2+r^2)} \frac{x}{(x^2+r^2)^{1/2}} = \frac{\lambda}{2\pi\varepsilon_0}\int_l^{\infty} \frac{x\, dx}{(x^2+r^2)^{3/2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\int_{l^2+r^2}^{\infty} \frac{dy}{y^{3/2}}, \text{ on putting } y = x^2+r^2,$$

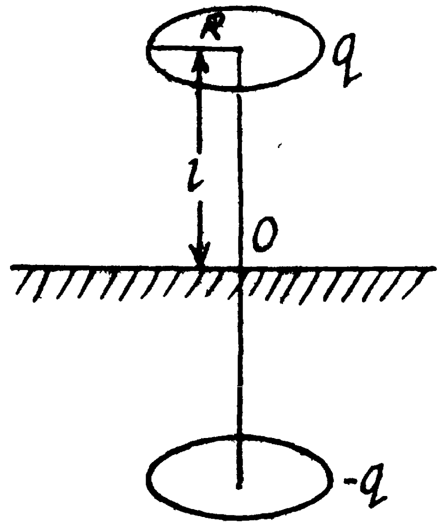
$$= \frac{\lambda}{2\pi\varepsilon_0\sqrt{l^2+r^2}}$$



Hence $\quad \sigma(r) = \varepsilon_0 E_n = \dfrac{\lambda}{2\pi\sqrt{l^2+r^2}}$

3.62 It can be easily seen that in accordance with the image method, a charge $-q$ must be located on a similar ring but on the other side of the conducting plane. (Fig.) at the same perpendicular distance. From the solution of 3.9 net electric field at O,

$\vec{E} = 2\dfrac{ql}{4\pi\varepsilon_0(R^2+l^2)^{3/2}}(-\vec{n})$ where $\vec{n}$ is outward normal with respect to the conducting plane.



Now $\quad E_n = \dfrac{\sigma}{\varepsilon_0}$

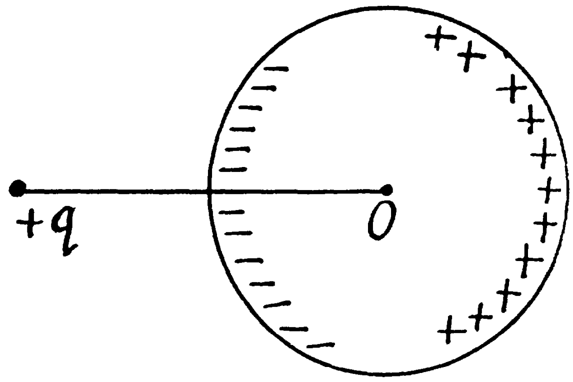
Hence $\quad \sigma = \dfrac{-ql}{2\pi(R^2+l^2)^{3/2}}$

where minus sign indicates that the induced carge is opposite in sign to that of charge $q > 0$.

3.63 Potential φ is the same for all the points of the sphere. Thus we calculate its value at the centre O of the sphere. Thus we can calculate its value at the centre O of the sphere, because only for this point, it can be calculated in the most simple way.

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{l} + \varphi' \tag{1}$$

where the first term is the potential of the charge q, while the second is the potential due to the charges induced on the surface of the sphere. But since all induced charges are at the same distance equal to the radius of the circle from the point C and the total induced charge is equal to zero, $\varphi' = 0$, as well. Thus equation (1) is reduced to the form,



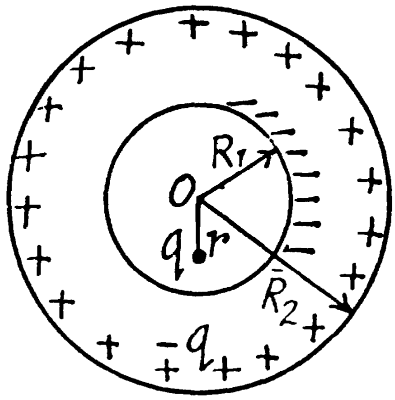
$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}$$

3.64 As the sphere has conducting layers, charge $-q$ is induced on the inner surface of the sphere q and consequently charge $+q$ is induced on the outer layer as the sphere as a whole is uncharged.

Hence, the potential at O is given by,

$$\varphi_0 = \frac{q}{4\pi\varepsilon_0 r} + \frac{(-q)}{4\pi\varepsilon_0 R_1} + \frac{q}{4\pi\varepsilon_0 R_2}$$

It should be noticed that the potential can be found in such a simple way only at O, since all the induced charges are at the same distance from this point, and their distribution, (which is unknown to us), does not play any role.



3.65 Potential at the inside sphere,

$$\varphi_a = \frac{q_1}{4\pi\varepsilon_0 a} + \frac{q_2}{4\pi\varepsilon_0 b}$$

Obviously $\qquad \varphi_a = 0$ for $q_2 = -\frac{b}{a}q_1 \qquad (1)$

When $\qquad r \geq b,$

$$\varphi_r = \frac{q_1}{4\pi\varepsilon_0 r} + \frac{q_2}{4\pi\varepsilon_0 r} = \frac{q_1}{4\pi\varepsilon_0}\left(1 - \frac{b}{a}\right) \Big/ r, \text{ using Eq. (1).}$$

And when $r \leq b$

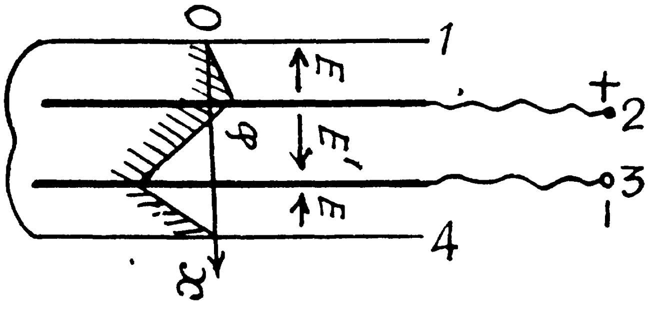
$$\varphi_r = \frac{q_1}{4\pi\varepsilon_0 r} + \frac{q_2}{4\pi\varepsilon_0 b} = \frac{q_1}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{a}\right)$$

3.66 (a) As the metallic plates 1 and 4 are isolated and conncted by means of a conductor, $\varphi_1 = \varphi_4$. Plates 2 and 3 have the same amount of positive and negative charges and due to induction, plates 1 and 4 are respectively negatively and positively charged and in addition to it all the four plates are located a small but at equal distance d relative to each

other, the magnitude of electric field strength between 1 - 2 and 3 - 4 are both equal in magnitude and direction (say $\vec{E}$). Let $\vec{E'}$ be the field strength between the plates 2 and 3, which is directed form 2 to 3. Hence $\vec{E'} \uparrow\downarrow \vec{E}$ (Fig.).

According to the problem

$$E'\,d = \Delta\varphi = \varphi_2 - \varphi_3 \qquad (1)$$



In addition to

$$\varphi_1 - \varphi_4 = 0 = (\varphi_1 - \varphi_2) + (\varphi_2 - \varphi_3) + (\varphi_3 - \varphi_4)$$

or, $\quad 0 = -Ed + \Delta\varphi - Ed$

or, $\quad \Delta\varphi = 2Ed$ or $E = \dfrac{\Delta\varphi}{2d}$

Hence $\quad E = \dfrac{E'}{2} = \dfrac{\Delta\varphi}{2d} \qquad (2)$

(b) Since $E \alpha \sigma$, we can state that according to equation (2) for part (a) the charge on the plate 2 is divided into two parts; such that $1/3$ rd of it lies on the upper side and $2/3$ rd on its lower face.

Thus charge density of upper face of plate 2 or of plate 1 or plate 4 and lower face of $3\sigma = \varepsilon_0 E = \dfrac{\varepsilon_0\,\Delta\varphi}{2d}$ and charge density of lower face of 2 or upper face of 3

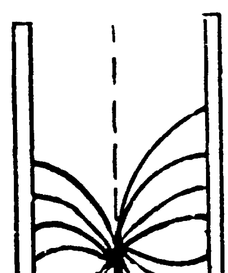
$$\sigma' = \varepsilon_0 E' = \varepsilon_0 \frac{\Delta\varphi}{d}$$

Hence the net charge density of plate 2 or 3 becomes $\sigma + \sigma' = \dfrac{3\,\varepsilon_0\,\Delta\,\varphi}{2\,d}$, which is obvious from the argument.

3.67 The problem of point charge between two conducting planes is more easily tackled (if we want only the total charge induced on the planes) if we replace the point charge by a uniformly charged plane sheet.

Let σ be the charge density on this sheet and E_1, E_2 outward electric field on the two sides of this sheet.

Then $\quad E_1 + E_2 = \dfrac{\sigma}{\varepsilon_0}$



The conducting planes will be assumed to be grounded. Then $E_1 x = E_2 (l - x)$.

Hence $\quad E_1 = \dfrac{\sigma}{l\,\varepsilon_0}(l - x),\; E_2 = \dfrac{\sigma}{l\,\varepsilon_0}x$

This means that the induced charge density on the plane conductors are

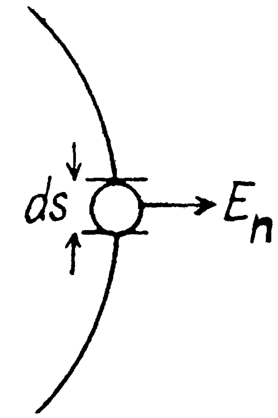
$$\sigma_1 = -\frac{\sigma}{l}(l - x),\; \sigma_2 = -\frac{\sigma}{l}x$$

Hence $q_1 = -\dfrac{q}{l}(l - x),\; q_2 = -\dfrac{q}{l}\,x$

3.68 Near the conductor $E = E_n = \dfrac{\sigma}{\varepsilon_0}$

This field can be written as the sum of two parts E_1 and E_2. E_1 is the electric field due to an infinitesimal area dS.

Very near it $E_1 = \pm \dfrac{\sigma}{2\,\varepsilon_0}$

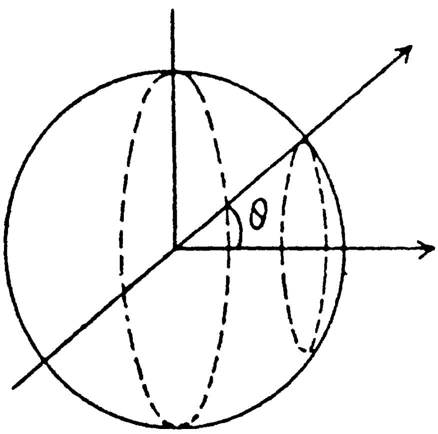


The remaining part contributes $E_2 = \dfrac{\sigma}{2\,\varepsilon_0}$ on both sides. In calculating the force on the element dS we drop E_1 (because it is a self-force.) Thus

$$\frac{dF}{dS} = \sigma \cdot \frac{\sigma}{2\,\varepsilon_0} = \frac{\sigma^2}{2\,\varepsilon_0}$$

3.69 The total force on the hemisphere is

$$F = \int_0^{\pi/2} \frac{\sigma^2}{2\,\varepsilon_0} \cdot \cos\theta \cdot 2\,\pi R \sin\theta\, R\, d\theta$$



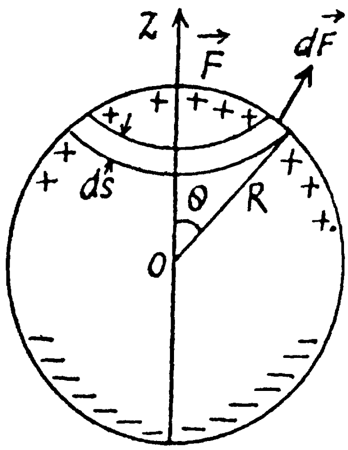
$$= \frac{2\,\pi R^2 \sigma^2}{2\,\varepsilon_0} \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta$$

$$= \frac{2\,\pi R^2}{2\,\varepsilon_0} \times \frac{1}{2} \times \left(\frac{q}{4\,\pi R^2}\right)^2 = \frac{q^2}{32\,\pi\,\varepsilon_0 R}.$$

3.70 We know that the force acting on the area element dS of a conductor is,

$$d\vec{F} = \frac{1}{2} \sigma \vec{E}\, dS \qquad (1)$$

It follows from symmetry considerations that the resultant force $\vec{F}$ is directed along the z-axis, and hence it can be represented as the sum (integral) of the projection of elementary forces (1) onto the z-axis :



$$dF_z = dF \cos\theta \qquad (2)$$

For simplicity let us consider an element area $dS = 2\,\pi R \sin\theta\, R\, d\theta$(Fig.). Now considering that $E = \sigma/\varepsilon_0$. Equation (2) takes the from

$$dF_z = \frac{\pi\,\sigma^2 R^2}{\varepsilon_0} \sin\theta \cos\theta\, d\theta$$

$$= -\left(\frac{\pi\,\sigma_0^2 R^2}{\varepsilon_0}\right) \cos^3\theta\, d\cos\theta$$

Integrating this expression over the half sphere (i.e. with respect to cos θ between 1 and 0),

we obtain $$F = F_z = \frac{\pi \sigma_0^2 R^2}{4 \varepsilon_0}$$

3.71 The total polarization is $P = (\varepsilon - 1) \varepsilon_0 E$. This must equals $\frac{n_0 p}{N}$ where n_0 is the concerntation of water molecules. Thus

$$N = \frac{n_0 p}{(\varepsilon - 1) \varepsilon_0 E} = 2{\cdot}93 \times 10^3 \text{ on putting the values}$$

3.72 From the general formula

$$\vec{E} = \frac{1}{4 \pi \varepsilon_0} \frac{3 \vec{p} \cdot \vec{r} \vec{r} - \vec{p} r^2}{r^5}$$

$$\vec{E} = \frac{1}{4 \pi \varepsilon_0} \frac{2 \vec{p}}{l^3}, \text{ where } r = l \text{ and } \vec{r} \uparrow\uparrow \vec{p}$$

This will cause the induction of a dipole moment.

$$\vec{p}_{ind} = \beta \frac{1}{4 \pi \varepsilon_0} \frac{2 \vec{p}}{l^3} \times \varepsilon_0$$

Thus the force,

$$\vec{F} = \frac{\beta}{4 \pi} \frac{2 p}{l^3} \frac{\partial}{\partial l} \frac{1}{4 \pi \varepsilon_0} \frac{2p}{l^3} = \frac{3 \beta p^2}{4 \pi^2 \varepsilon_0 l^7}$$

3.73 The electric field E at distance x from the centre of the ring is,

$$E(x) = \frac{qx}{4 \pi \varepsilon_0 (R^2 + x^2)^{3/2}}$$

The induced dipole moment is $p = \beta \varepsilon_0 E = \frac{q \beta x}{4 \pi (R^2 + x^2)^{3/2}}$

The force on this molecule is

$$F = p \frac{\partial}{\partial x} E = \frac{q \beta x}{4 \pi (R^2 + x^2)^{3/2}} \frac{q}{4 \pi \varepsilon_0} \frac{\partial}{\partial x} \frac{x}{(R^2 + x^2)^{3/2}} = \frac{q^2 \beta}{16 \pi^2 \varepsilon_0} \frac{x (R^2 - 2x^2)}{(R^2 + x^2)^4}$$

This vanishes for $x = \frac{\pm R}{\sqrt{2}}$ (apart from $x = 0$, $x = \infty$)

It is maximum when

$$\frac{\partial}{\partial x} \frac{x (R^2 - x^2 \times 2)}{(R^2 + x^2)^4} = 0$$

or, $$(R^2 - 2 x^2)(R^2 + x^2) - 4 x^2 (R^2 + x^2) - 8 x^2 (R^2 - 2 x^2) = 0$$

or, $$R^4 - 13 x^2 R^2 + 10 x^4 = 0 \text{ or, } x^2 = \frac{R^2}{20} \left(13 \pm \sqrt{129}\right)$$

or, $x = \frac{R}{\sqrt{20}} \sqrt{13 \pm \sqrt{129}}$ (on either side), Plot of $F_x(x)$ is as shown in the answersheet.

3.74 Inside the ball

$$\vec{D}(\vec{r}\,) = \frac{q}{4\pi}\frac{\vec{r}}{r^3} = \varepsilon\,\varepsilon_0\vec{E}.$$

Also $$\varepsilon_0\vec{E}+\vec{P} = \vec{D} \text{ or } \vec{P} = \frac{\varepsilon-1}{\varepsilon}\vec{D} = \frac{\varepsilon-1}{\varepsilon}\frac{q}{4\pi}\frac{\vec{r}}{r^3}$$

Also, $$q' = -\oint \vec{P}\cdot d\vec{S} = -\frac{\varepsilon-1}{\varepsilon}\frac{q}{4\pi}\int d\Omega = -\frac{\varepsilon-1}{\varepsilon}q$$

3.75 $D_{diel} = \varepsilon\,\varepsilon_0 E_{diel} = D_{conductor} = \sigma$ or, $E_{diel} = \dfrac{\sigma}{\varepsilon\,\varepsilon_0}$

$$P_n = (\varepsilon-1)\,\varepsilon_0 E_{diel} = \frac{\varepsilon-1}{\varepsilon}\sigma$$

$$\sigma' = -P_n = -\frac{\varepsilon-1}{\varepsilon}\sigma$$

This is the surface density of bound charges.

3.76 From the solution of the previous problem q'_{in} = charge on the interior surface of the conductor

$$= -(\varepsilon-1)/\varepsilon\int\sigma\,dS = -\frac{\varepsilon-1}{\varepsilon}q$$

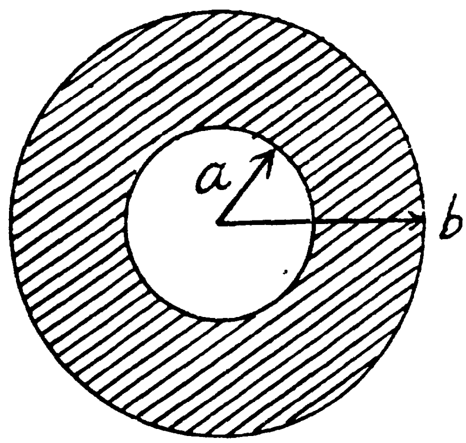
Since the dielectric as a whole is neutral there must be a total charge equal to $q'_{outer} = +\dfrac{\varepsilon-1}{\varepsilon}q$ on the outer surface of the dielectric.

3.77 (a) Positive extraneous charge is distributed uniformly over the internal surface layer. Let σ_0 be the surface density of the charge.

Clearly, $E = 0$, for $r < a$

For $a < r$

$\varepsilon_0 E\times 4\pi r^2 = 4\pi a^2\sigma_0$ by Gauss theorem.



or, $$E = \frac{\sigma_0}{\varepsilon_0\,\varepsilon}\left(\frac{a}{r}\right)^2,\ a < r < b$$

For $r > b$, similarly

$$E = \frac{\sigma_0}{\varepsilon_0}\left(\frac{a}{r}\right)^2,\ r > b$$

Now, $$E = -\frac{\partial\varphi}{\partial r}.$$

So by integration from infinity where $\varphi(\infty) = 0$,

$$\varphi = \frac{\sigma_0\,a^2}{\varepsilon_0\,r}\ r > b$$

$a < r < b \quad \varphi = \dfrac{\sigma_0 a^2}{\varepsilon \varepsilon r} + B$, B is a constant

or by continuity, $\varphi = \dfrac{\sigma_0 a^2}{\varepsilon_0 \varepsilon}\left(\dfrac{1}{r} - \dfrac{1}{b}\right) + \dfrac{\sigma_0 a^2}{\varepsilon_0 b}, \; a < r < b$

For $\quad r < a. \;\; \varphi = A =$ Constant

By continuity, $\varphi = \dfrac{\sigma_0 a^2}{\varepsilon_0 \varepsilon}\left(\dfrac{1}{a} - \dfrac{1}{b}\right) + \dfrac{\sigma_0 a^2}{\varepsilon_0 b}$

(b) Positive extraneous charge is distributed uniformly over the internal volume of the dielectric

Let $\rho_0 =$ Volume density of the charge in the dielectric, for $a < r < b$.

$$E = 0, \;\; r < a$$

$$\varepsilon_0 \, \varepsilon \, 4 \pi r^2 E = \frac{4 \pi}{3} (r^3 - a^3) \, \rho_0 \,, \; (a < r < b)$$

or,
$$E = \frac{\rho_0}{3 \, \varepsilon_0 \, \varepsilon}\left(r - \frac{a^3}{r^2}\right)$$

$$E = \frac{4 \pi}{3} (b^3 - a^3) \, \rho_0 \big/ \varepsilon_0 \, 4 \pi r^2, \;\; r > b$$

or,
$$E = \frac{(b^3 - a^3) \, \rho_0}{3 \, \varepsilon_0 r^2} \quad \text{for } r > b$$

By integration,

$$\varphi = \frac{(b^3 - a^3) \, \rho_0}{3 \, \varepsilon_0 r} \quad \text{for } r > b$$

or,
$$\varphi = B - \frac{\rho_0}{3 \, \varepsilon_0 \, \varepsilon}\left(\frac{r^2}{2} + \frac{a^3}{r}\right), \;\; a < r < b$$

By continuity

$$\frac{b^3 - a^3}{3 \, \varepsilon_0 b} \rho_0 = B - \frac{\rho_0}{3 \, \varepsilon_0 \, \varepsilon}\left(\frac{b^2}{2} + \frac{a^3}{b}\right)$$

or,
$$B = \frac{\rho_0}{3 \, \varepsilon_0 \, \varepsilon}\left\{\frac{\varepsilon \, (b^3 - a^3)}{b} + \left(\frac{b^2}{2} + \frac{a^3}{b}\right)\right\}$$

Finally
$$\varphi = B - \frac{\rho_0}{3 \, \varepsilon_0 \, \varepsilon}\left(\frac{a^2}{2} + a^2\right) = B - \frac{\rho_0 \, a^2}{2 \, \varepsilon_0 \, \varepsilon}, \; r < a$$

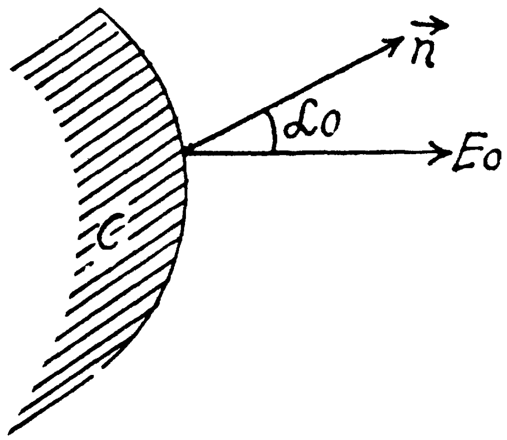
On the basis of obtained expressions $E\,(r)$ and $(\varphi)\,(r)$ can be plotted as shown in the answer-sheet.

3.78 Let the field in the dielectric be $\vec{E}$ making an angle α with $\vec{n}$. Then we have the boundary conditions,

$E_0 \cos \alpha_0 = \varepsilon E \cos \alpha$ and $E_0 \sin c_0 = E \sin \alpha$

So $E = E_0 \sqrt{\sin^2 \alpha_0 + \frac{1}{\varepsilon^2} \cos^2 \alpha_0}$ and $\tan \alpha = \varepsilon \tan \alpha_0$

In the dielectric the normal component of the induction vector is

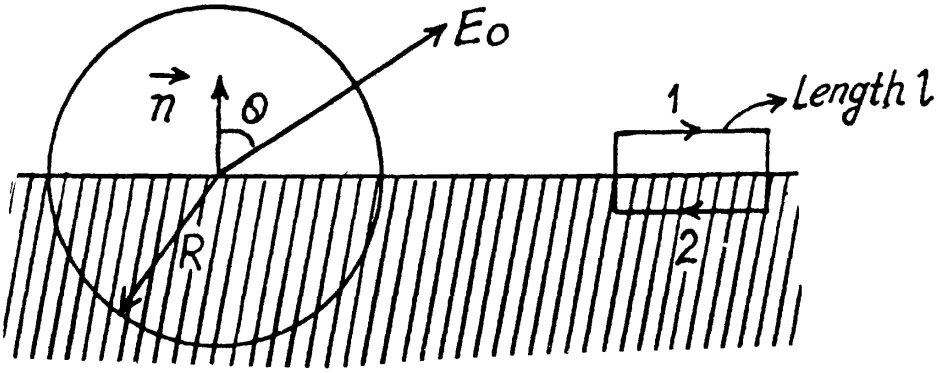


$$D_n = \varepsilon_0 \varepsilon E_n = \varepsilon_0 \varepsilon E \cos \alpha = \varepsilon_0 E_0 \cos \alpha_0$$

$$\sigma' = P_n = D_n - \varepsilon_0 E_n = \left(1 - \frac{1}{\varepsilon}\right) \varepsilon_0 E_0 \cos \alpha_0$$

or, $$\sigma' = \frac{\varepsilon - 1}{\varepsilon} \varepsilon_0 E_0 \cos \alpha_0$$

3.79 From the previous problem, $\sigma' = \varepsilon_0 \frac{\varepsilon - 1}{\varepsilon} E_0 \cos \theta$



(a) Then $\oint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} Q = \pi R^2 E_0 \cos \theta \frac{\varepsilon - 1}{\varepsilon}$

(b) $\oint \vec{D} \cdot d\vec{l} = (D_{1t} - D_{2t})\, l = (\varepsilon_0 E_0 \sin \theta - \varepsilon \varepsilon_0 E_0 \sin \theta) = -(\varepsilon - 1)\, \varepsilon_0 E_0\, l \sin \theta$

3.80 (a) $\operatorname{div} \vec{D} = \frac{\partial D_x}{\partial x} = \rho$ and $D = \rho\, l$

$$E_x = \frac{\rho\, l}{\varepsilon \varepsilon_0},\ l < d \text{ and } E_x = \frac{\rho\, d}{\varepsilon_0} \text{ constant for } l > d$$

$\varphi(x) = -\frac{\rho\, l^2}{2\varepsilon \varepsilon_0}$, $l < d$ and $\varphi(x) = A - \frac{\rho\, ld}{\varepsilon_0}$, $l > d$ then $\varphi(x) = \frac{\rho\, d}{\varepsilon_0}\left(d - \frac{d}{2\varepsilon} - l\right)$, by continuity.

On the basis of obtained expressions $E_x(x)$ and $\varphi(x)$ can be plotted as shown in the figure of answersheet.

(b) $\rho' = -\operatorname{div}\vec{P} = -\operatorname{div}\,(\varepsilon - 1)\,\varepsilon_0 \vec{E} = -\rho\dfrac{(\varepsilon - 1)}{\varepsilon}$

$\sigma' = P_{1n} - P_{2n}$, where n is the normal from 1 to 2.

$= P_{1n}$, ($\vec{P}_2 = 0$ as 2 is vacuum.)

$= (\rho d - \rho\, d/\varepsilon) = \rho\, d\dfrac{\varepsilon - 1}{\varepsilon}$

3.81 $\operatorname{div}\vec{D} = \dfrac{l}{r^2}\dfrac{\partial}{\partial r} r^2 D_r = \rho$

$r^2 D_r = \rho\dfrac{r^3}{3} + A \quad D_r = \dfrac{1}{3}\rho r + \dfrac{A}{r^2},\ r < R$

$A = 0$ as $D_r \neq \infty$ at $r = 0$,Thus, $E_r = \dfrac{\rho r}{3\varepsilon\varepsilon_0}$

For $\quad r > R,\ D_r = \dfrac{B}{r^2}$

By continuity of D_r at $r = R$; $B = \dfrac{\rho R^3}{3}$

so, $\quad E_r = \dfrac{\rho R^3}{3\varepsilon_0 r^2},\ r > R$

$$\varphi = \frac{\rho R^3}{3\varepsilon_0 r},\ r > R \text{ and } \varphi = -\frac{\rho r^2}{6\varepsilon\varepsilon_0} + C,\ r < R$$

$$C = +\frac{\rho R^2}{3\varepsilon_0} + \frac{\rho R^2}{6\varepsilon\varepsilon_0},\ \text{by continuity of } \varphi.$$

See answer sheet for graphs of $E(r)$ and $\varphi(r)$

(b) $\rho' = \operatorname{div}\vec{P} = -\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left\{\dfrac{r^3}{3}\rho\left(1 - \dfrac{1}{\varepsilon}\right)\right\} = -\dfrac{\rho(\varepsilon - 1)}{\varepsilon}$

$\sigma' = P_{1r} - P_{2r} = P_{1r} = \dfrac{1}{3}\rho R\left(1 - \dfrac{1}{\varepsilon}\right)$

3.82 Because there is a discontinuity in polarization at the boundary of the dielectric disc, a bound surface charge appears, which is the source of the electric field inside and outside the disc.

We have for the electric field at the origin.

$$\vec{E} = -\int \frac{\sigma'\, dS}{4\pi\varepsilon_0 r^3}\vec{r},$$

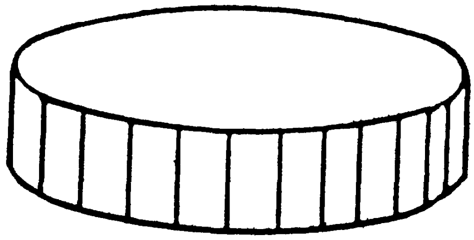
where $\vec{r} =$ radius vector to the origin from the element dS.

$\sigma' = P_n = P\cos\theta$ on the curved surface

($P_n = 0$ on the flat surface.)

Here θ = angle between $\vec{r}$ and $\vec{P}$

By symmetry, $\vec{E}$ will be parallel to $\vec{P}$. Thus



$$E = -\int_0^{2\pi} \frac{P\cos\theta\, R\, d\theta \cdot \cos\theta}{4\pi\varepsilon_0 R^2} \cdot d$$

where, $r = R$ if $d \ll R$.

So, $$E = -\frac{Pd}{4\varepsilon_0 R} \text{ and } \vec{E} = -\frac{\vec{P} d}{4\varepsilon_0 R}$$

3.83. Since there are no free extraneous charges anywhere

$$\operatorname{div}\vec{D} = \frac{\partial D_x}{\partial x} = 0 \text{ or, } D_x = \text{Constant}$$

But $D_x = 0$ at ∞, so, $D_x = 0$, every where.

Thus, $$\vec{E} = -\frac{\vec{P}_0}{\varepsilon_0}\left(1 - \frac{x^2}{d^2}\right) \text{ or, } E_x = -\frac{P_0}{\varepsilon_0}\left(1 - \frac{x^2}{d^2}\right)$$

So, $$\varphi = \frac{P_0 x}{\varepsilon_0} - \frac{P_0 x^3}{3\varepsilon_0 d^2} + \text{constant}$$

Hence,

$$\varphi(+d) - \varphi(-d) = \frac{2P_0 d}{\varepsilon_0} - \frac{2P_0 d^3}{3d^2\varepsilon_0} = \frac{4P_0 d}{3\varepsilon_0}$$

3.84 (a) We have $D_1 = D_2$, or, $\varepsilon E_2 = E_1$

Also, $$E_1\frac{d}{2} + E_2\frac{d}{2} = E_0 d \text{ or, } E_1 + E_2 = 2E_0$$

Hence, $$E_2 = \frac{2E_0}{\varepsilon + 1} \text{ and } E_1 = \frac{2\varepsilon E_0}{\varepsilon + 1} \text{ and } D_1 = D_2 = \frac{2\varepsilon\varepsilon_0 E_0}{\varepsilon + 1}$$

(b) $D_1 = D_2$, or, $\varepsilon E_2 = E_1 = \dfrac{\sigma}{\varepsilon_0} = E_0$

Thus, $$E_1 = E_0,\ E_2 = \frac{E_0}{\varepsilon} \text{ and } D_1 = D_2 = \varepsilon_0 E_0$$

3.85 (a) Constant voltage acros the plates;

$$E_1 = E_2 = E_0\,,\ D_1 = \varepsilon_0 E_0,\ D_2 = \varepsilon_0\,\varepsilon\,E_0.$$

(b) Constant charge across the plates;

$$E_1 = E_2\,,\ D_1 = \varepsilon_0 E_1,\ D_2 = \varepsilon\,\varepsilon_0 E_2 = \varepsilon D_1$$

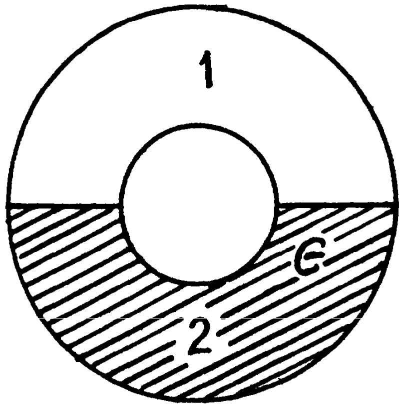
$$E_1(1+\varepsilon) = 2E_0 \quad \text{or} \quad E_1 = E_2 = \frac{2E_0}{\varepsilon+1}$$

3.86 At the interface of the dielectric and vacuum,

$$E_{1t} = E_{2t}$$

The electric field must be radial and

$$E_1 = E_2 = \frac{A}{\varepsilon_0\,\varepsilon\,r^2},\ a<r<b$$



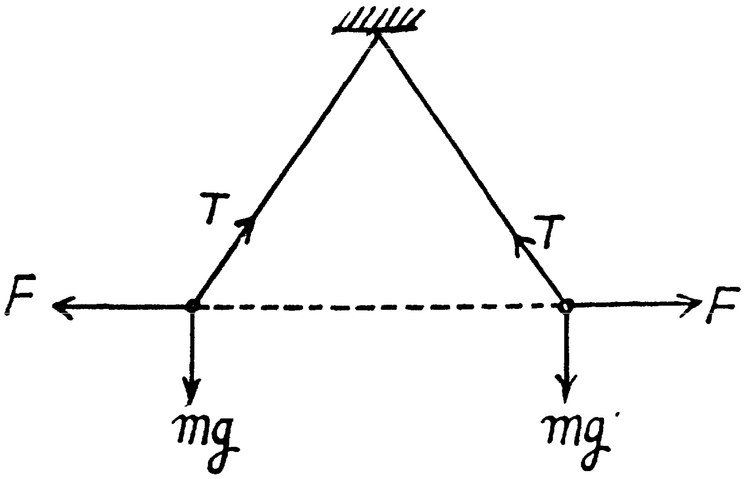
Now, $q = \dfrac{A}{R^2}(2\pi R^2) + \dfrac{A}{\varepsilon R^2}(2\pi R^2)$

$$= A\left(1+\frac{1}{\varepsilon}\right)2\pi$$

or, $\quad E_1 = E_2 = \dfrac{q}{2\pi\varepsilon_0 r^2(1+\varepsilon)}$

3.87 In air the forces are as shown. In K–oil,

$F \to F' = F/\varepsilon$ and $mg \to mg\left(1-\dfrac{\rho_0}{\rho}\right)$.



Since the inclinations do not change

$$\frac{1}{\varepsilon} = 1 - \frac{\rho_0}{\rho}$$

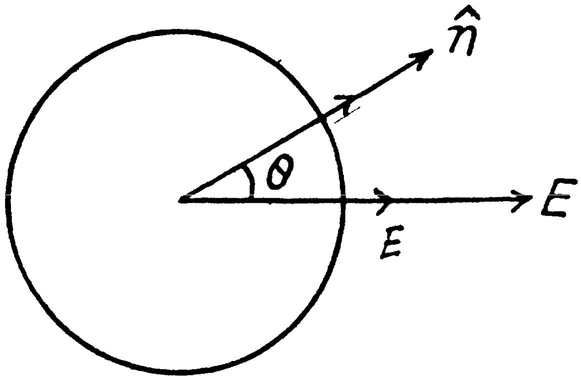
or, $\quad \dfrac{\rho_0}{\rho} = 1 - \dfrac{1}{\varepsilon} = \dfrac{\varepsilon-1}{\varepsilon}$

or, $\quad \rho = \rho_0\dfrac{\varepsilon}{\varepsilon-1}$

where ρ_0 is the density of K–oil and ρ that of the material of which the balls are made.

3.88 Within the ball the electric field can be resolved into normal and tangential components.

$$E_n = E\cos\theta,\ E_t = E\sin\theta$$



Then, $\quad D_n = \varepsilon\,\varepsilon_0 E\cos\theta$

and $\quad P_n = (\varepsilon-1)\ \varepsilon_0 E\cos\theta$

or, $\quad \sigma' = (\varepsilon-1)\,\varepsilon_0 E\cos\theta$

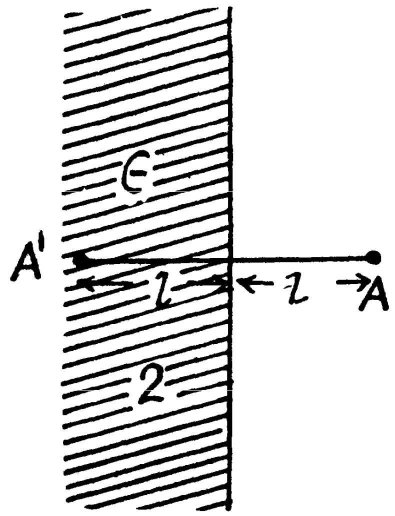
so, $\quad \sigma_{max} = (\varepsilon-1)\,\varepsilon_0 E,$

and total charge of one sign,

$$q' = \int_0^1 (\varepsilon - 1)\, \varepsilon_0 E \cos\theta\, 2\pi R^2\, d(\cos\theta) = \pi R^2 \varepsilon_0 (\varepsilon - 1) E$$

(Since we are interested in the total charge of one sign we must intergrate $\cos\theta$ from 0 to 1 only).

3.89 The charge is at A in the medium 1 and has an image point at A' in the medium 2. The electric field in the medium 1 is due to the actual charge q at A and the image charge q' at A'. The electric field in 2 is due to a corrected charge q'' at A. Thus on the boundary between 1 and 2,



$$E_{1n} = \frac{q'}{4\pi\varepsilon_0 r^2}\cos\theta - \frac{q}{4\pi\varepsilon_0 r^2}\cos\theta$$

$$E_{2n} = \frac{-q''}{4\pi\varepsilon_0 r^2}\cos\theta$$

$$E_{1t} = \frac{q'}{4\pi\varepsilon_0 r^2}\sin\theta + \frac{q}{4\pi\varepsilon_0 r^2}\sin\theta$$

$$E_{2t} = \frac{q''}{4\pi\varepsilon_0 r^2}\sin\theta$$

The boundary conditions are

$$D_{1n} = D_{2n} \text{ and } E_{1t} = E_{2t}$$

$$\varepsilon q'' = q - q'$$

$$q'' = q + q'$$

So,
$$q'' = \frac{2q}{\varepsilon + 1}, \quad q' = -\frac{\varepsilon - 1}{\varepsilon + 1} q$$

(a) The surface density of the bound charge on the surface of the dielectric

$$\sigma' = P_{2n} = D_{2n} - \varepsilon_0 E_{2n} = (\varepsilon - 1)\varepsilon_0 E_{2n}$$

$$= -\frac{\varepsilon - 1}{\varepsilon + 1}\frac{q}{2\pi r^2}\cos\theta = -\frac{\varepsilon - 1}{\varepsilon + 1}\frac{ql}{2\pi r^3}$$

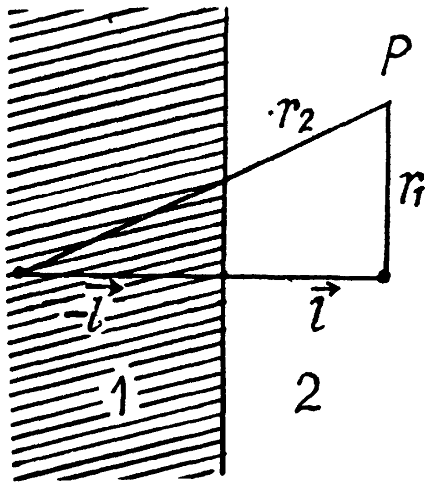
(b) Total bound charge is, $-\frac{\varepsilon - 1}{\varepsilon + 1} q \int_0^\infty \frac{l}{2\pi (l^2 + x^2)^{3/2}} 2\pi x\, dx = -\frac{\varepsilon - 1}{\varepsilon + 1} q$

3.90 The force on the point charge q is due to the bound charges. This can be calculated from the field at this charge after extracting out the self field. This image field is

$$E_{\text{image}} = \frac{\varepsilon - 1}{\varepsilon + 1}\frac{q}{4\pi\varepsilon_0 (2l)^2}$$

Thus,
$$F = \frac{\varepsilon - 1}{\varepsilon + 1}\frac{q^2}{16\pi\varepsilon_0 l^2}$$

3.91 $E_p = \dfrac{q\vec{r_1}}{4\pi\varepsilon_0 r_1^3} + \dfrac{q'\vec{r_2}}{4\pi r_2^3\varepsilon_0}$, P in 1



$E_p = \dfrac{q''\vec{r_1}}{4\pi\varepsilon_0 r_1^3}$, P in 2

where $q'' = \dfrac{2q}{\varepsilon+1}$, $q' = q'' - q$

In the limit $\vec{l} \to 0$

$$\vec{E}_p = \frac{(q+q')\vec{r}}{4\pi\varepsilon_0 r^3} = \frac{q\vec{r}}{2\pi\varepsilon_0(1+\varepsilon)r^3}, \text{ in either part.}$$

Thus,
$$E_p = \frac{q}{2\pi\varepsilon_0(1+\varepsilon)r^2}$$

$$\varphi = \frac{q}{2\pi\varepsilon_0(1+\varepsilon)r}$$

$$D = \frac{q}{2\pi\varepsilon_0(1+\varepsilon)r^2} \times \begin{cases} 1 & \text{in vacuum} \\ \varepsilon & \text{in dielectric} \end{cases}$$

3.92 $\vec{E}_p = \dfrac{q\vec{r_2}}{4\pi\varepsilon_0\varepsilon r_2^3} + \dfrac{q'\vec{r_1}}{4\pi\varepsilon_0 r_1^3}$; P in 2

$\vec{E}_p = \dfrac{q''\vec{r_2}}{4\pi\varepsilon_0 r_2^3}$; P in 1

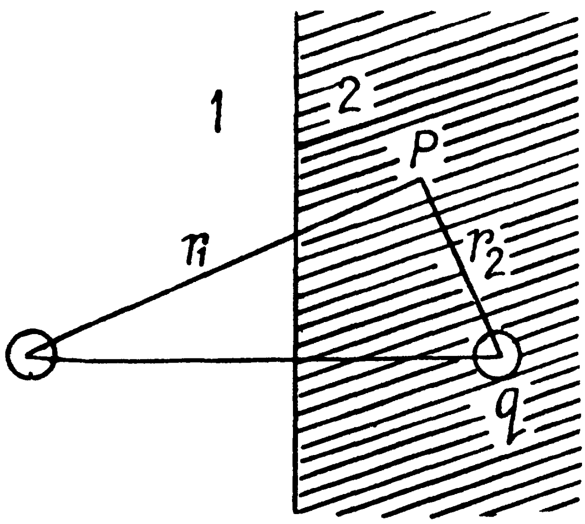
Using the boundary conditions,

$$E_{1n} = \varepsilon E_{2n},\ E_{1t} = E_{2t}$$

This implies

$q - \varepsilon q' = q''$ and $q + \varepsilon q' = \varepsilon q''$

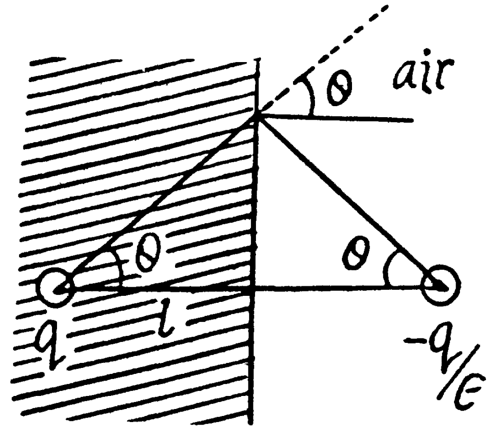
So, $q'' = \dfrac{2q}{\varepsilon+1}$, $q' = \dfrac{\varepsilon-1}{\varepsilon+1}\dfrac{q}{\varepsilon}$



Then, as earlier,

$$\sigma' = \frac{ql}{2\pi r^3}\cdot\left(\frac{\varepsilon-1}{\varepsilon+1}\right)\cdot\frac{1}{\varepsilon}$$

3.93 To calculate the electric field, first we note that an image charge will be needed to ensure that the electric field on the metal boundary is normal to the surface.



The image charge must have magnitude $-\frac{q}{\varepsilon}$ so that the tangential component of the electric field may vanish. Now,

$$E_n = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\varepsilon r^2}\right)2\cos\theta = \frac{ql}{2\pi\varepsilon_0\varepsilon r^3}$$

Then $P_n = D_n - \varepsilon_0 E_n = \frac{(\varepsilon-1)ql}{2\pi\varepsilon r^3} = \sigma'$

This is the density of bound charge on the surface.

3.94 Since the condenser plates are connected,

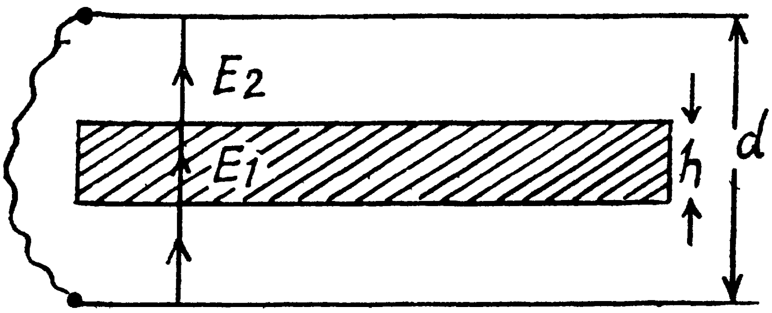
$$E_1 h + E_2(d-h) = 0$$

and $\quad P + \varepsilon_0 E_1 = \varepsilon_0 E_2$

or, $\quad E_1 + \frac{P}{\varepsilon_0} = E_2$

Thus, $E_2 d - \frac{Ph}{\varepsilon_0} = 0$, or, $E_2 = \frac{Ph}{\varepsilon_0 d}$

$$E_1 = -\frac{P}{\varepsilon_0}\left(1-\frac{h}{d}\right)$$



3.95 Given $\vec{P} = \alpha\vec{r}$, where $\vec{r}$ = distance from the axis. The space density of charges is given by, $\rho' = -\operatorname{div}\vec{P} = -2\alpha$

On using. $\quad \operatorname{div}\vec{r} = \frac{1}{r}\frac{\partial}{\partial r}(\vec{r}\cdot\vec{r}) = 2$

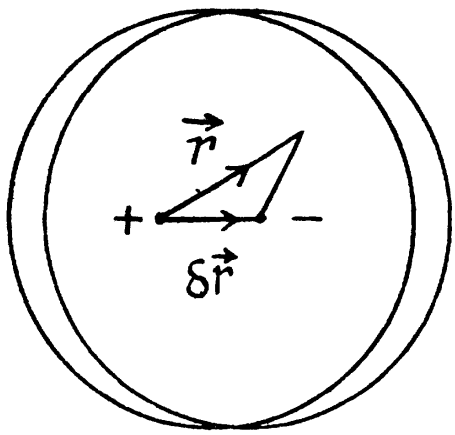
3.96 In a uniformly charged sphere,

$$E_r = \frac{\rho_0 r}{3\varepsilon_0} \quad \text{or,} \quad \vec{E} = \frac{\rho_0}{3\varepsilon_0}\vec{r}$$

The total electric field is

$$\vec{E} = \frac{1}{3\varepsilon_0}\rho_0\vec{r} - \frac{1}{3\varepsilon_0}(\vec{r}-\delta\vec{r})\,\rho_0$$

$$= \frac{1}{3\varepsilon_0}\rho_0\,\delta\vec{r} = -\frac{\vec{P}}{3\varepsilon_0}$$



where $\rho\,\delta\vec{r} = -\vec{P}$ (dipole moment is defined with its direction being from the *-ve* charge to *+ve* charge.)

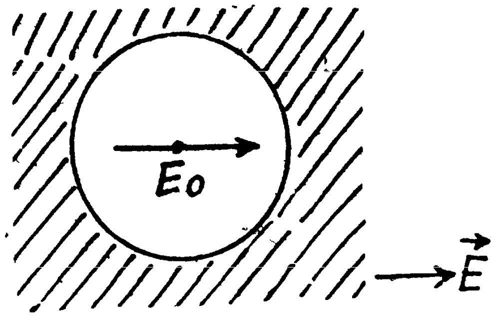
The potential outside is

$$\varphi = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} - \frac{Q}{|\vec{r}-\delta\vec{r}|}\right), = \frac{\vec{p_0}\cdot\vec{r}}{4\pi\varepsilon_0 r^3},\ r > R$$

where $\vec{p_0} = -\frac{4\pi}{3}R^3\rho_0\,\partial\vec{r}$ is the total dipole moment.

3.97 The electric field $\vec{E}_0$ in a spherical cavity in a uniform dielectric of permittivity ε is related to the far away field $\vec{E}$, in the following manner. Imagine the cavity to be filled up with the dielectric. Then there will be a uniform field $\vec{E}$ everywhere and a polarization $\vec{P}$, given by,

$$\vec{P} = (\varepsilon - 1)\,\varepsilon_0 \vec{E}$$



Now take out the sphere making the cavity, the electric field inside the sphere will be $-\frac{\vec{P}}{3\varepsilon_0}$

By superposition. $\vec{E}_0 - \frac{\vec{P}}{3\varepsilon_0} = \vec{E}$

or, $\vec{E}_0 = \vec{E} + \frac{1}{3}(\varepsilon - 1)\vec{E} = \frac{1}{3}(\varepsilon + 2)\vec{E}$

3.98 By superposition the field $\vec{E}$ inside the ball is given by

$$\vec{E} = \vec{E}_0 - \frac{\vec{P}}{3\varepsilon_0}$$

On the other hand, if the sphere is not too small, the macroscopic equation $\vec{P} = (\varepsilon - 1)\,\varepsilon_0 \vec{E}$ must hold. Thus,

$$\vec{E}\left(1 + \frac{1}{3}(\varepsilon - 1)\right) = \vec{E}_0 \quad \text{or,} \quad \vec{E} = \frac{3\vec{E}_0}{\varepsilon + 2}$$

Also $$\vec{P} = 3\,\varepsilon_0 \frac{\varepsilon - 1}{\varepsilon + 2}\vec{E}_0$$

3.99 This is to be handled by the same trick as in 3.96. We have effectively a two dimensional situation. For a uniform cylinder full of charge with charge density ρ_0 (charge per unit volume), the electric field E at an inside point is along the (cylindrical) radius vector $\vec{r}$ and equal to,

$$\vec{E} = \frac{1}{2\varepsilon_0}\rho\,\vec{r}$$

$$\left(\operatorname{div}\vec{E} = \frac{l}{r}\frac{\partial}{\partial r}(rE_r) = \frac{\rho}{\varepsilon_0}, \quad \text{hence,} \quad E_r = \frac{\rho}{2\varepsilon_0}r\right)$$

Therefore the polarized cylinder can be thought of as two equal and opposite charge distributions displaced with respect to each other

$$\vec{E} = \frac{1}{2\varepsilon_0}\rho\,\vec{r} - \frac{1}{2\varepsilon_0}\rho\,(\vec{r} - \delta\vec{r}) = \frac{1}{2\varepsilon_0}\rho\,\delta\,\vec{r} = -\frac{\vec{P}}{2\varepsilon_0}$$

Since $\vec{P} = -\rho\,\delta\,\vec{r}$. (direction of electric dipole moment vector being from the negative charge to positive charge .)

3.100 As in 3.98, we write $\vec{E} = \vec{E}_0 - \frac{\vec{P}}{2\varepsilon_0}$

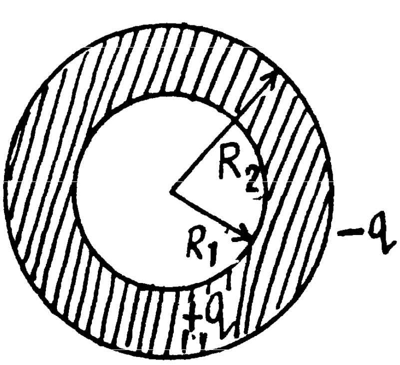
using here the result of the foregoing problem.

Also $$\vec{P} = (\varepsilon - 1)\,\varepsilon_0 \vec{E}$$

So, $$\vec{E}\left(\frac{\varepsilon + 1}{2}\right) = \vec{E}_0, \quad \text{or,} \quad \vec{E} = \frac{2\vec{E}_0}{\varepsilon + 1} \quad \text{and} \quad \vec{P} = 2\varepsilon_0 \frac{\varepsilon - 1}{\varepsilon + 1}\vec{E}_0$$

3.3 ELECTRIC CAPACITANCE ENERGY OF AN ELECTRIC FIELD

3.101 Let us mentally impart a charge q on the conductor, then



$$\varphi_+ - \varphi_- = \int_{R_1}^{R_2} \frac{q}{4\pi\varepsilon_0 \varepsilon r^2} dr + \int_{R_2}^{\infty} \frac{q}{4\pi\varepsilon_0 r^2} dr$$

$$= \frac{q}{4\pi\varepsilon_0\varepsilon}\left[\frac{1}{R_1} - \frac{1}{R_2}\right] + \frac{q}{4\pi\varepsilon_0}\frac{1}{R_2}$$

$$= \frac{q}{4\pi\varepsilon_0\varepsilon}\left[\frac{(\varepsilon-1)}{R_2} + \frac{1}{R_1}\right]$$

Hence the sought capacitance,

$$C = \frac{q}{\varphi_+ - \varphi_-} = \frac{q\,4\pi\varepsilon_0\varepsilon}{q\left[\frac{(\varepsilon-1)}{R_2} + \frac{1}{R_1}\right]} = \frac{4\pi\varepsilon_0\varepsilon R_1}{(\varepsilon-1)\frac{R_1}{R_2} + 1}$$

3.102 From the symmetry of the problem, the voltage across each capacitor, $\Delta\varphi = \xi/2$ and charge on each capacitor $q = C\xi/2$ in the absence of dielectric.

Now when the dielectric is filled up in one of the capacitors, the equivalent capacitance of the system,

$$C'_0 = \frac{C\varepsilon}{1+\varepsilon}$$

and the potential difference across the capacitor, which is filled with dielectric,

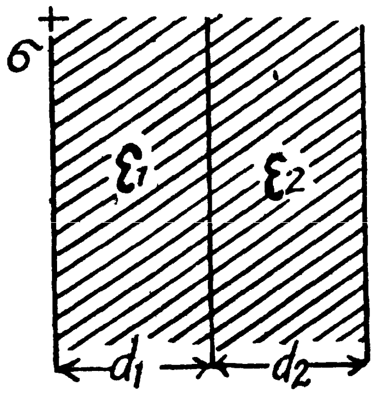
$$\Delta\varphi' = \frac{q'}{\varepsilon C} = \frac{C\varepsilon}{(1+\varepsilon)}\frac{\xi}{C\varepsilon} = \frac{\xi}{(1+\varepsilon)}$$

But $\qquad \varphi \,\alpha\, E$

So, as φ decreases $\frac{1}{2}(1+\varepsilon)$ times, the field strength also decreases by the same factor and flow of charge, $\Delta q = q' - q$

$$= \frac{C\varepsilon}{(1+\varepsilon)}\xi - \frac{C}{2}\xi = \frac{1}{2}C\xi\frac{(\varepsilon-1)}{(\varepsilon+1)}$$

3.103 (a) As it is series combination of two capacitors,



$$\frac{1}{C} = \frac{d_1}{\varepsilon_0\varepsilon_1 S} + \frac{d_2}{\varepsilon_0\varepsilon_2 S} \quad \text{or,} \quad C = \frac{\varepsilon_0 S}{(d_1/\varepsilon_1) + (d_2/\varepsilon_2)}$$

(b) Let, σ be the initial surface charge density, then density of bound charge on the boundary plane.

$$\sigma' = \sigma\left(1 - \frac{1}{\varepsilon_1}\right) - \sigma\left(1 - \frac{1}{\varepsilon_2}\right) = \sigma\left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)$$

But, $$\sigma = \frac{q}{S} = \frac{CV}{S} = \frac{\varepsilon_0 S\, \varepsilon_1 \varepsilon_2}{\varepsilon_2 d_1 + \varepsilon_1 d_2} \frac{V}{S}$$

So, $$\sigma' = \frac{\varepsilon_0 V (\varepsilon_1 - \varepsilon_2)}{\varepsilon_2 d_1 + \varepsilon_1 d_2}$$

3.104 (a) We point the x-axis lowards right and place the origin on the left hand side plate. The left plate is assumed to be positively charged.

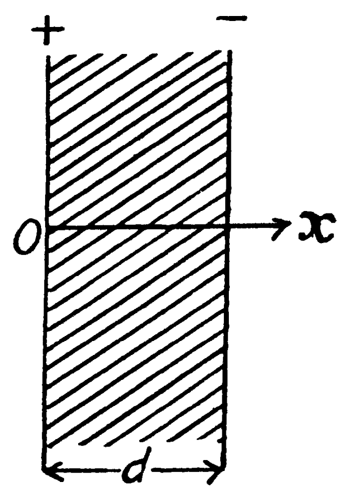
Since ε varies linearly, we can write,

$$\varepsilon(x) = a + bx$$

where a and b can be determined from the boundary condition. We have $\varepsilon = \varepsilon_1$ at $x = 0$ and $\varepsilon = \varepsilon_2$ at $x = d$,

Thus, $$\varepsilon(x) = \varepsilon_1 + \left(\frac{\varepsilon_2 - \varepsilon_1}{d}\right) x$$

Now potential difference between the plates



$$\varphi_+ - \varphi_- = \int_0^d \vec{E} \cdot d\vec{r} = \int_0^d \frac{\sigma}{\varepsilon_0\, \varepsilon(x)} dx$$

$$= \int_0^d \frac{\sigma}{\varepsilon_0 \left(\varepsilon_1 + \frac{\varepsilon_2 - \varepsilon_1}{d} x\right)} dx = \frac{\sigma d}{(\varepsilon_2 - \varepsilon_1)\, \varepsilon_0} \ln \frac{\varepsilon_2}{\varepsilon_1}$$

Hence, the sought capacitance, $C = \dfrac{\sigma S}{\varphi_+ - \varphi_-} = \dfrac{(\varepsilon_2 - \varepsilon_1)\, \varepsilon_0 S}{(\ln \varepsilon_2/\varepsilon_1)\, d}$

(b) $D = \frac{q}{S}$ and $P = \frac{q}{S} - \frac{q}{S\varepsilon(x)}$

and the space density of bound charges is

$$\rho' = -\operatorname{div} P = -\frac{q(\varepsilon_2 - \varepsilon_1)}{Sd\varepsilon^2(x)}$$

3.105 Let, us mentally impart a charge q to the conductor. Now potential difference between the plates,

$$\varphi_+ - \varphi_- = \int_{R_1}^{R_2} \vec{E} \cdot d\vec{r}$$

$$= \int_{R_1}^{R_2} \frac{q}{4\pi\varepsilon_0\, a/r} \frac{1}{r^2} dr = \frac{q}{4\pi\varepsilon_0 a} \ln R_2/R_1$$

Hence, the sought capacitance,

$$C = \frac{q}{\varphi_+ - \varphi_-} = \frac{q\, 4\pi\varepsilon_0 a}{q \ln R_2/R_1} = \frac{4\pi\varepsilon_0 a}{\ln R_2/R_1}$$

3.106 Let λ be the linear charge density then,

$$E_{1m} = \frac{\lambda}{2\pi\varepsilon_0 R_1 \varepsilon_1} \quad (1)$$

and,
$$E_{2m} = \frac{\lambda}{2\pi\varepsilon_0 R_2 \varepsilon_2} \quad (2)$$

The breakdown in either case will occur at the smaller value of r for a simultaneous breakdown of both dielectrics.

From (1) and (2)

$E_{1m} R_1 \varepsilon_1 = E_{2m} R_2 \varepsilon_2$, which is the sought relationship.

3.107 Let, λ be the linear charge density then, the sought potential difference,

$$\varphi_+ - \varphi_- = \int_{R_1}^{R_2} \frac{\lambda}{2\pi\varepsilon_0 \varepsilon_1 r} dr + \int_{R_2}^{R_3} \frac{\lambda}{2\pi\varepsilon_0 \varepsilon_2 r} dr$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{1}{\varepsilon_1} \ln R_2/R_1 + \frac{1}{\varepsilon_2} \ln R_3/R_2\right]$$

Now, as, $E_1 R_1 \varepsilon < E_2 R_2 \varepsilon_2$, so

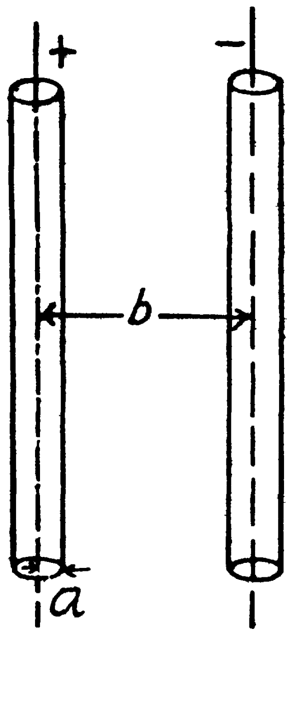
$$\frac{\lambda}{2\pi\varepsilon_0} = E_1 R_1 \varepsilon_1$$

is the maximum acceptable value, and for values greater than $E_1 R_1 \varepsilon_1$, dielectric breakdown will take place,

Hence, the maximum potential difference between the plates,

$$\varphi_+ - \varphi_- = E_1 R_1 \varepsilon_1 \left[\frac{1}{\varepsilon_1} \ln R_2/R_1 + \frac{1}{\varepsilon_2} \ln R_3/R_2\right] = E_1 R_1 \left[\ln R_2/R_1 + \frac{\varepsilon_1}{\varepsilon_2} \ln R_3/R_2\right]$$

3.108 Let us suppose that linear charge density of the wires be λ then, the potential difference, $\varphi_+ - \varphi_- = \varphi - (-\varphi) = 2\varphi$. The intensity of the electric field created by one of the wires at a distance x from its axis can be easily found with the help of the Gauss's theorem,



$$E = \frac{\lambda}{2\pi\varepsilon_0 x}$$

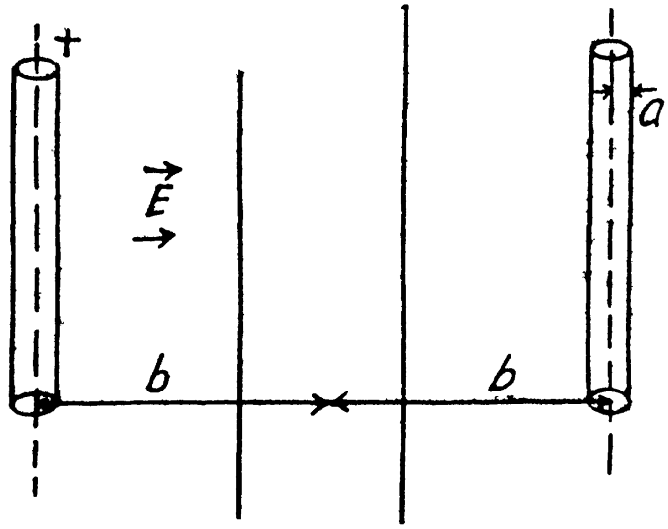
Then, $\varphi = \int_a^{b-a} E\,dx = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{b-a}{a}$

Hence, capacitance, per unit length,

$$\frac{\lambda}{\varphi_+ - \varphi_-} = \frac{2\pi\varepsilon_0}{\ln b/a}$$

3.109 The field in the region between the conducting plane and the wire can be obtained by using an oppositely charged wire as an image on the other side.

Then the potential difference between the wire and the plane,



$$\Delta\varphi = \int_b \vec{E}\cdot d\vec{r}$$

$$= \int_a^b \left[\frac{\lambda}{2\pi\varepsilon_0 r} + \frac{\lambda}{2\pi\varepsilon_0 (2b-r)}\right] dr$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b}{a} - \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b}{2b-a}$$

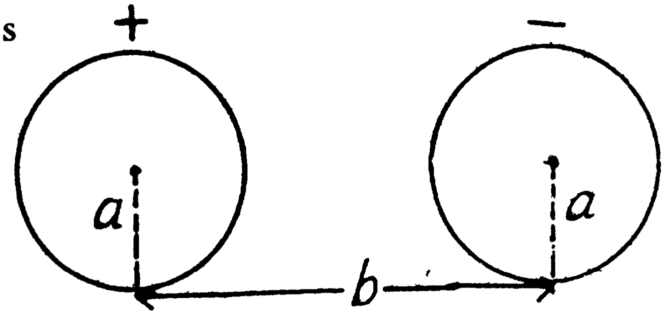
$$= \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{2b-a}{a}$$

$$\approx \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{2b}{a}, \text{ as } b >> a$$

Hence, the sought mutual capacitance of the system per unit length of the wire

$$= \frac{\lambda}{\Delta\varphi} = \frac{2\pi\varepsilon_0}{\ln 2b/a}$$

3.110 When $b >> a$, the charge distribution on each spherical conductor is practically unaffected by the presence of the other conductor. Then, the potential φ_+ (φ_-) on the positive (respectively negative) charged conductor is $+\frac{q}{4\pi\varepsilon_0\varepsilon a}\left(-\frac{q}{4\pi\varepsilon_0\varepsilon a}\right)$

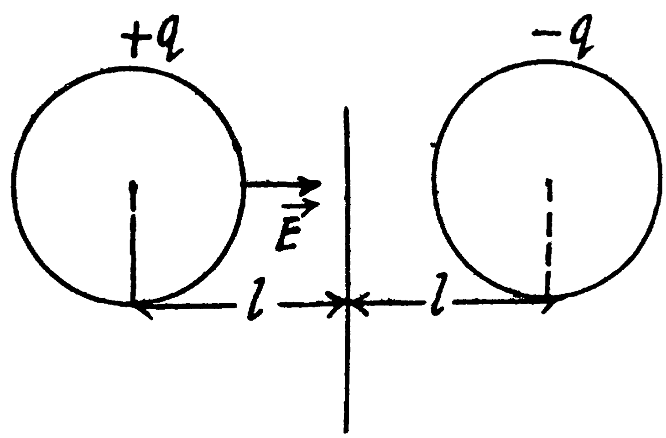


Thus $\varphi_+ - \varphi_- = \frac{q}{2\pi\varepsilon\varepsilon_0 a}$

and $C = \frac{q}{\varphi_+ - \varphi_-} = 2\pi\varepsilon_0\varepsilon a.$

Note : if we require terms which depend on $\frac{a}{b}$, we have to take account of distribution of charge on the conductors.

3.111 As in 3.109 we apply the method of image. Then the potentical difference between the +vely charged sphere and the conducting plane is one half the nominal potential difference between the sphere and its image and is

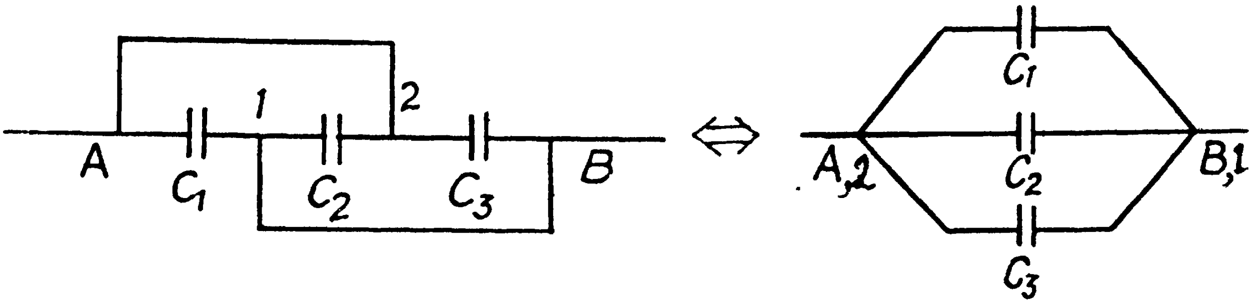


$$\Delta\varphi = \frac{1}{2}(\varphi_+ - \varphi_-) = \frac{q}{4\pi\varepsilon_0 a}$$

Thus

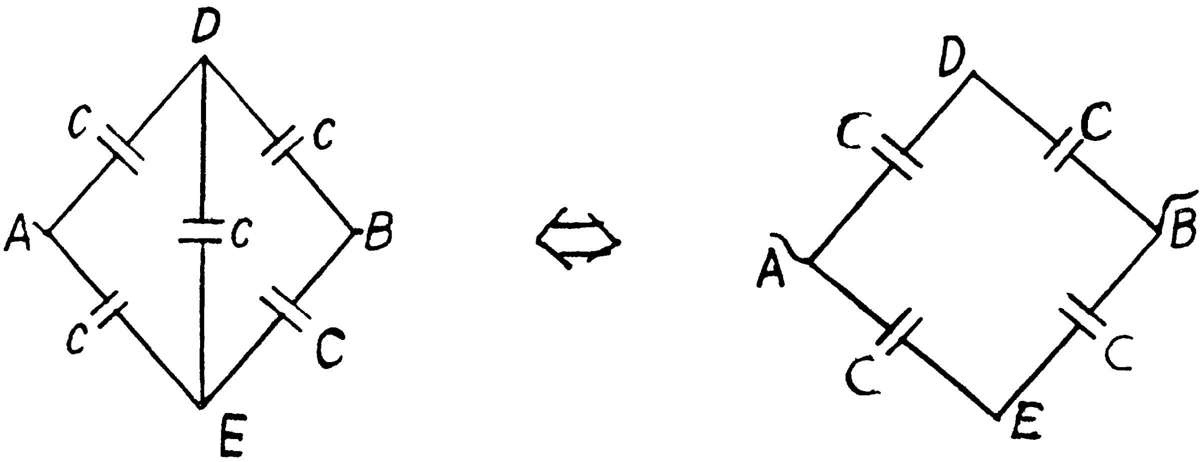
$$C = \frac{q}{\Delta\varphi} = 4\pi\varepsilon_0 a. \text{ for } l >> a.$$

3.112



(a) Since $\varphi_1 = \varphi_B$ and $\varphi_2 = \varphi_A$

The arrangement of capacitors shown in the problem is equivalent to the arrangement shown in the Fig.



and hence the capacitance between A and B is,

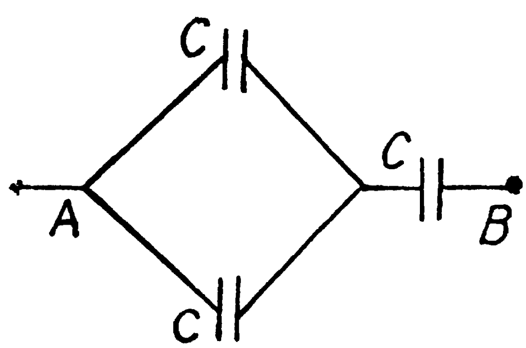
$$C = C_1 + C_2 + C_3$$

(B) From the symmetry of the problem, there is no P.d. between D and E.. So, the combination reduces to a simple arrangement shown in the Fig and hence the net capacitance,

$$C_0 = \frac{C}{2} + \frac{C}{2} = C$$

3.113 (a) In the given arrangement, we have three capacitors of equal capacitance $C = \frac{\varepsilon_0 S}{d}$ and the first and third plates are at the same potential.

Hence, we can resolve the network into a simple form using series and parallel grouping of capacitors, as shown in the figure.



Thus the equivalent capacitance

$$C_0 = \frac{(C + C)\,C}{(C + C) + C} = \frac{2}{3}C$$

(b) Let us mentally impart the charges $+q$ and $-q$ to the plates 1 and 2 and then distribute them to other plates using charge conservation and electric induction. (Fig.).

As the potential difference between the plates 1 and 2 is zero,

$$-\frac{q_1}{C}+\frac{q_2}{C}-\frac{q_1}{C}=0, \quad \left(\text{where } C=\frac{\varepsilon_0 S}{d}\right)$$

or, $\qquad q_2 = 2\,q_1,$

The potential difference between A and B,

$$\varphi = \varphi_A - \varphi_B = q_2/C$$

Hence the sought capacitane,

$$C_0 = \frac{q}{\varphi} = \frac{q_1+q_2}{q_2/C} = \frac{3\,q_1}{2q_1/C} = \frac{3}{2}C = \frac{3\,\varepsilon_0 S}{2d}$$

3.114 Amount of charge, that the capacitor of capacitance C_1 can withstand, $q_1 = C_1 V_1$ and similarly the charge, that the capacitor of capacitance C_2 can withstand, $q_2 = C_2 V_2$. But in series combination, charge on both the capacitors will be same, so, q_{max}, that the combination can withstand $= C_1 V_1$,

as $C_1 V_1 < C_2 V_2$, from the numerical data, given.

Now, net capacitance of the system,

$$C_0 = \frac{C_1 C_2}{C_1 + C_2}$$

and hence, $V_{max} = \dfrac{q_{max}}{C_0} = \dfrac{C_1 V_1}{C_1 C_2/C_1 + C_2} = V_1\left(1+\dfrac{C_1}{C_2}\right) = 9\text{ kV}$

3.115 Let us distribute the charges, as shown in the figure.

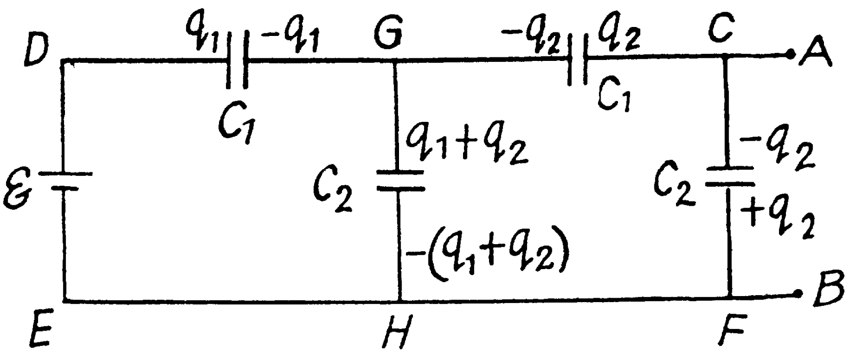
Now, we know that in a closed circuit, $-\Delta\varphi = 0$

So, in the loop, *DCFED*,

$$\frac{q_1}{C_1}-\frac{q_2}{C_1}-\frac{q_2}{C_2}=\xi \quad \text{or,} \quad q_1 = C_1\left[\xi + q_2\left(\frac{1}{C_1}+\frac{1}{C_2}\right)\right] \tag{1}$$

Again in the loop *DGHED*,

$$\frac{q_1}{C_1}+\frac{q_1+q_2}{C_2}=\xi \tag{2}$$



Using Eqs. (1) and (2), we get

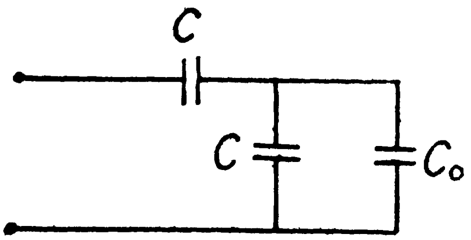
$$q_2\left[\frac{1}{C_1}+\frac{3}{C_2}+\frac{C_1}{C_2}\right]=-\frac{\xi C_1}{C_2}$$

Now, $$\varphi_A-\varphi_B=\frac{-q_2}{C_2}=\frac{\xi}{C_2^2/C_1}\,\frac{1}{\left[\frac{1}{C_1}+\frac{3}{C_2}+\frac{C_1}{C_2^2}\right]}$$

or, $$\varphi_A-\varphi_B=\frac{\xi}{\left[\frac{C_2^2}{C_1^2}+\frac{3\,C_2}{C_1}+1\right]}=\frac{\xi}{\eta^2+3\,\eta+1}=10\text{ V}$$

3.116 The infinite circuit, may be reduced to the circuit, shown in the Fig. where, C_0 is the net capacitance of the combination.

So, $$\frac{1}{C+C_0}+\frac{1}{C}=\frac{1}{C_0}$$



Solving the quadratic,

$$C\,C_0+C_0^2-C^2=0,$$

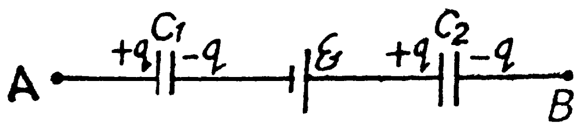
we get,

$$C_0=\frac{\left(\sqrt{5}-1\right)}{2}C,$$ taking only $+ve$ value as C_0 can not be negative.

3.117 Let, us make the charge distribution, as shown in the figure.

Now, $$\varphi_A-\varphi_B=\frac{q}{C_1}-\xi+\frac{q}{C_2}$$

or, $$q=\frac{(\varphi_A-\varphi_B)+\xi}{C_1+C_2}\,C_1C_2$$



Hence, voltage across the capacitor C_1

$$=\frac{q}{C_1}=\frac{(\varphi_A-\varphi_B)+\xi}{C_1+C_2}C_2=10\text{ V}$$

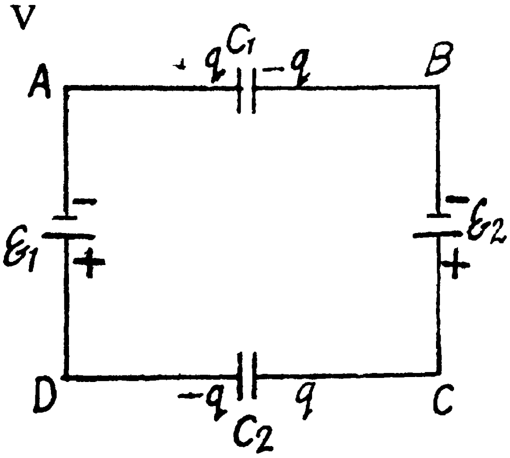
and voltage across the capacitor, C_2

$$=\frac{q}{C_2}=\frac{(\varphi_A-\varphi_B)+\xi}{C_1+C_2}C_1=5\text{ V}$$

3.118 Let $\xi_2>\xi_1$, then using $-\Delta\varphi=0$ in the closed circuit, (Fig.)

$$\frac{-q}{C_1}+\xi_2-\frac{q}{C_2}-\xi_1=0$$

or, $$q=\frac{(\xi_2-\xi_1)\,C_1C_2}{(C_1+C_2)}$$



Hence the P.D. accross the left and right plates of capacitors,

$$\varphi_1 = \frac{q}{C_1} = \frac{(\xi_2 - \xi_1)\, C_2}{C_1 + C_2}$$

and similarly

$$\varphi_2 = \frac{-q}{C_2} = \frac{(\xi_1 - \xi_2)\, C_1}{C_1 + C_2}$$

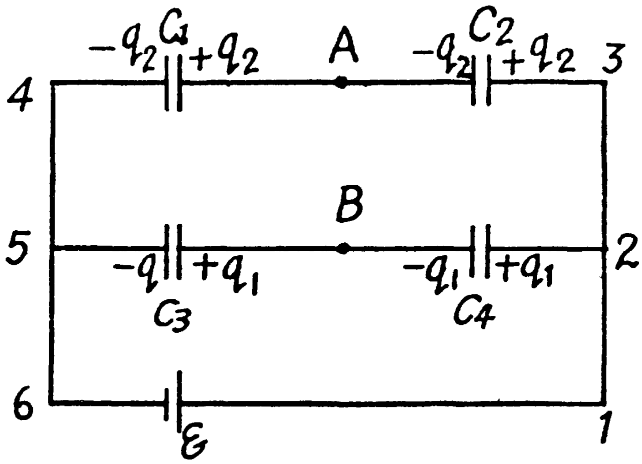
3.119 Taking benefit of the foregoing problem, the amount of charge on each capacitor

$$|q| = \frac{|\xi_2 - \xi_1|\, C_1 C_2}{C_1 + C_2}$$

3.120 Make the charge distribution, as shown in the figure. In the circuit, 12561.

$-\Delta\varphi = 0$ yields

$$\frac{q_1}{C_4} + \frac{q_1}{C_3} - \xi = 0 \quad \text{or,} \quad q_1 = \frac{\xi C_3 C_4}{C_3 + C_4}$$



and in the circuit 13461,

$$\frac{q_2}{C_2} + \frac{q_2}{C_1} - \xi = 0 \quad \text{or,} \quad q_2 = \frac{\xi C_1 C_2}{C_1 + C_2}$$

Now $\qquad \varphi_A - \varphi_B = \dfrac{q_2}{C_1} - \dfrac{q_1}{C_3}$

$$= \xi \left[\frac{C_2}{C_1 + C_2} - \frac{C_4}{C_3 + C_4} \right] = \xi \left[\frac{C_2 C_3 - C_1 C_4}{(C_1 + C_2)(C_3 + C_4)} \right]$$

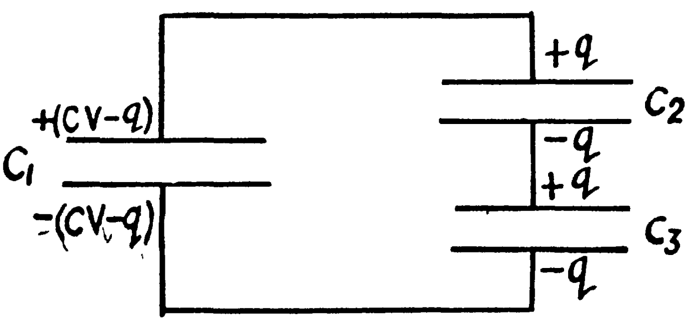
It becomes zero, when

$$(C_2 C_3 - C_1 C_4) = 0. \quad \text{or} \quad \frac{C_1}{C_2} = \frac{C_3}{C_4}$$

3.121 Let, the charge q flows through the connecting wires, then at the state of equilibrium, charge distribution will be as shown in the Fig. In the closed circuit 12341, using $-\Delta\varphi = 0$

$$-\frac{(C_1 V - q)}{C_1} + \frac{q}{C_2} + \frac{q}{C_3} = 0$$

$$\text{or, } q = \frac{V}{(1/C_1 + 1/C_2 + 1/C_3)} = 0{\cdot}06 \text{ mC}$$



3.122 Initially, charge on the capacitor C_1 or C_2,

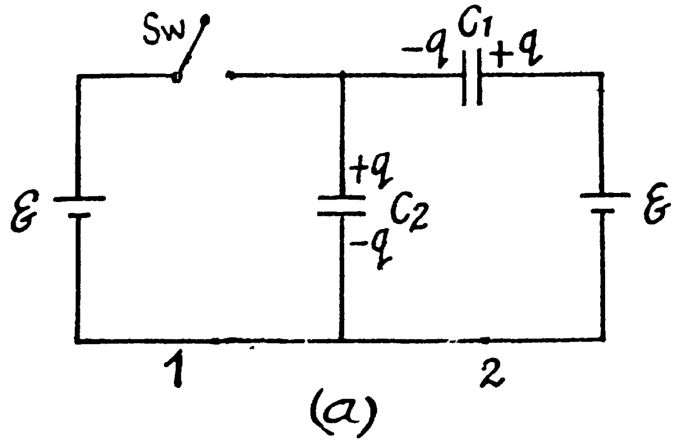
$$q = \frac{\xi C_1 C_2}{C_1 + C_2}, \text{ as they are in series combination (Fig.-a)}$$

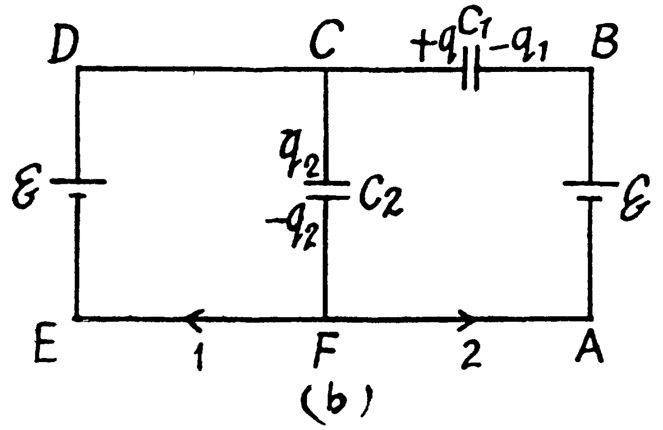
when the switch is closed, in the circuit CDEFC from $-\Delta\varphi = 0$, (Fig. b)

$$\xi - \frac{q_2}{C_2} = 0 \quad \text{or} \quad q_2 = C_2 \xi \tag{1}$$

And in the closed loop BCFAB from $-\Delta\varphi = 0$

$$\frac{-q_1}{C_1} + \frac{q_2}{C_2} - \xi = 0 \tag{2}$$





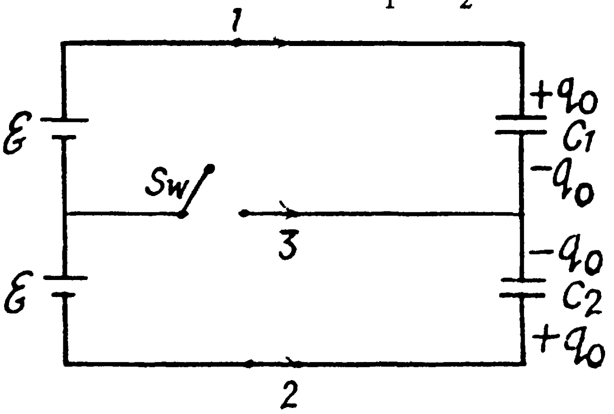
From (1) and (2) $q_1 = 0$

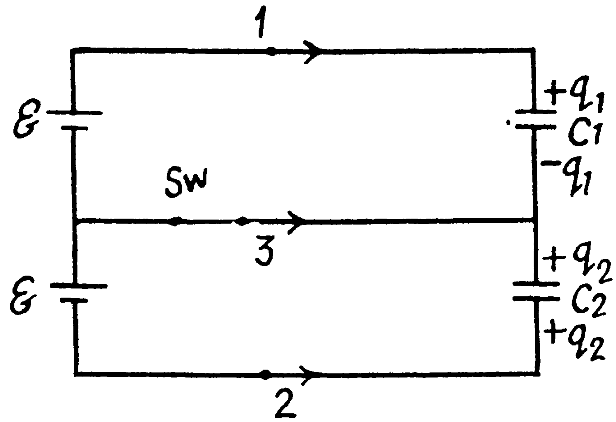
Now, charge flown through section 1 $= (q_1 + q_2) - 0 = C_2 \xi$

and charge flown through section 2 $= -q_1 - q = -\dfrac{\xi C_1 C_2}{C_1 + C_2}$

3.123 When the switch is open, (Fig-a)

$$q_0 = \frac{2 \xi C_1 C_2}{C_1 + C_2}$$





and when the switch is closed,

$$q_1 = \xi C_1 \text{ and } q_2 = \xi C_2$$

Hence, the flow of charge, due to the shortening of switch,

through section 1 $= q_1 - q_0 = \xi C_1 \left[\dfrac{C_1 - C_2}{C_1 + C_2}\right] = -24\ \mu C$

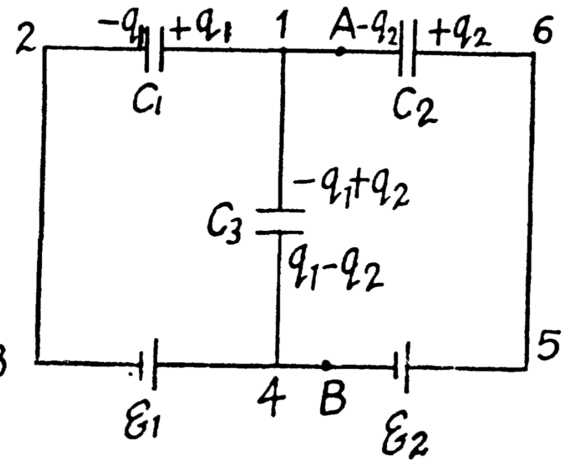
through the section 2 $= -q_2 - (q_0) = \xi C_2 \left[\dfrac{C_1 - C_2}{C_1 + C_2}\right] = -36\ \mu C$

and through the section 3 $= q_2 - (q_2 - q_1) - 0 = \xi (C_2 - C_1) = -60\ \mu C$

3.124 First of all, make the charge distribution, as shown in the figure.

In the loop 12341, using $-\Delta\varphi = 0$

$$\frac{q_1}{C_1} - \xi_1 + \frac{q_1 - q_2}{C_3} = 0 \qquad (1)$$



Similarly, in the loop 61456, using $-\Delta\varphi = 0$

$$\frac{q_2}{C_2} + \frac{q_2 - q_1}{C_3} - \xi_2 = 0 \qquad (2)$$

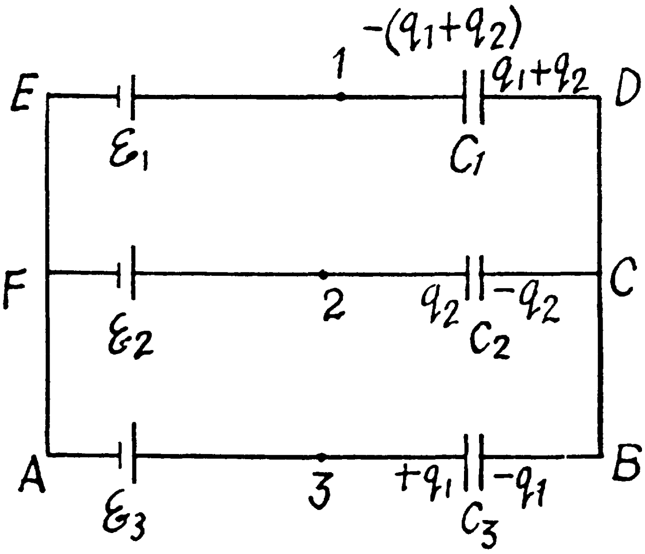
From Eqs. (1) and (2) we have

$$q_2 - q_1 = \frac{\xi_2 C_2 - \xi_1 C_1}{\frac{C_2}{C_3} + \frac{C_1}{C_3} + 1}$$

Hence, $$\varphi_A - \varphi_B = \frac{q_2 - q_1}{C_3} = \frac{\xi_2 C_2 - \xi_1 C_1}{C_1 + C_2 + C_3}$$

3.125 In the loop ABDEA, using $-\Delta\varphi = 0$

$$-\xi_3 + \frac{q_1}{C_3} + \frac{q_1 + q_2}{C_1} + \xi_1 = 0 \qquad (1)$$



Similarly in the loop ODEF, O

$$\frac{q_1 + q_2}{C_1} + \xi_1 - \xi_2 + \frac{q_2}{C_2} = 0 \qquad (2)$$

Solving Eqs. (1) and (2), we get,

$$q_1 + q_2 = \frac{\xi_2 C_2 - \xi_1 C_2 - \xi_1 C_3 + \xi_3 C_3}{\frac{C_3}{C_1} + \frac{C_2}{C_1} + 1}$$

Now, $\varphi_1 - \varphi_0 = \varphi_1 = -\dfrac{(q_1 + q_2)}{C_1}$, as $(\varphi_0 = 0)$

$$= \frac{\xi_1 (C_2 + C_3) - \xi_2 C_2 - \xi_3 C_3}{C_1 + C_2 + C_3}$$

And using the symmetry, $\varphi_2 = \dfrac{\xi_2 (C_1 + C_3) - \xi_1 C_1 - \xi_3 C_3}{C_1 + C_2 + C_3}$

and $$\varphi_3 = \frac{\xi_3 (C_1 + C_2) - \xi_1 C_1 - \xi_2 C_2}{C_1 + C_2 + C_3}$$

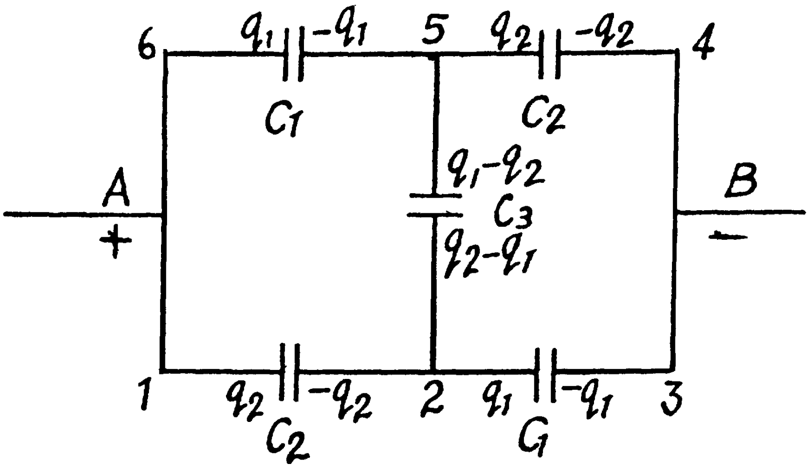
The answers have wrong sign in the book.

3.126 Taking the advantage of symmetry of the problem charge distribution may be made, as shown in the figure.

In the loop, 12561, $-\Delta\varphi = 0$

or $$\frac{q_2}{C_2} + \frac{q_2 - q_1}{C_3} - \frac{q_1}{C_1} = 0$$

or $$\frac{q_1}{q_2} = \frac{C_1 (C_3 + C_2)}{C_2 (C_1 + C_3)} \qquad (1)$$



Now, capacitance of the network,

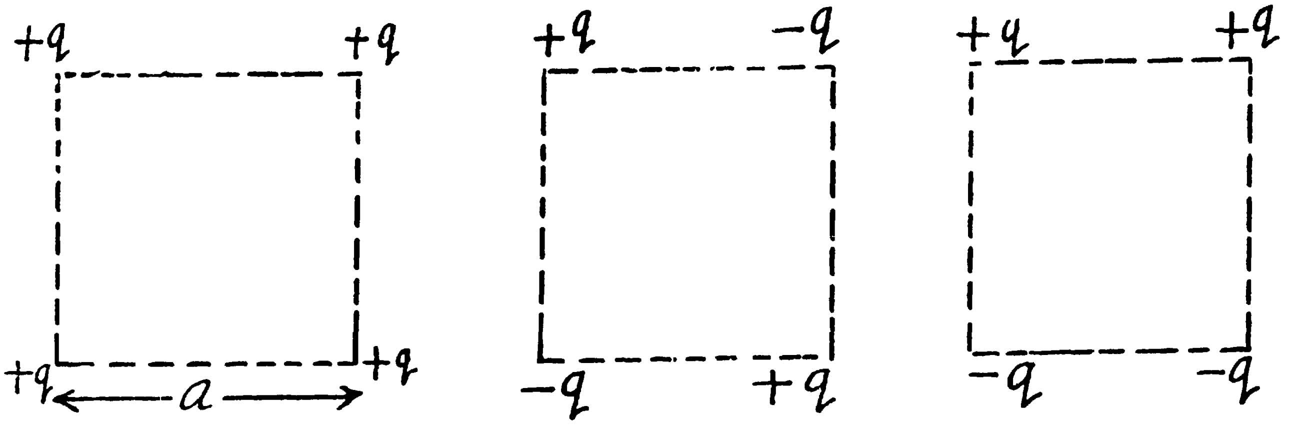
$$C_0 = \frac{q_1 + q_2}{\varphi_A - \varphi_B} = \frac{q_1 + q_2}{q_2/C_2 + q_1/C_1}$$

$$= \frac{(1 + q_1/q_2)}{\left(\frac{1}{C_2} + \frac{q_1}{q_2 C_1}\right)} \qquad (2)$$

From Eqs. (1) and (2)

$$C_0 = \frac{2\, C_1 C_2 + C_3 (C_1 + C_2)}{C_1 + C_2 + 2\, C_3}$$

3.127 (a) Interaction energy of any two point charges q_1 and q_2 is given by $\frac{q_1 q_2}{4 \pi \varepsilon_0 r}$ where r is the separation between the charges.



Hence, interaction energy of the system,

$$U_a = 4 \frac{q^2}{4 \pi \varepsilon_0 a} + 2 \frac{q^2}{4 \pi \varepsilon_0 (\sqrt{2}\, a)}$$

$$U_b = 4 \frac{-q^2}{4 \pi \varepsilon_0 a} + 2 \frac{q^2}{4 \pi \varepsilon_0 (\sqrt{2}\, a)}$$

and $$U_c = 2 \frac{q^2}{4 \pi \varepsilon_0 a} - \frac{2\, q^2}{4 \pi \varepsilon_0 a} - \frac{2\, q^2}{4 \pi \varepsilon_0 (\sqrt{2}\, a)} = - \frac{\sqrt{2}\, q^2}{4 \pi \varepsilon_0 a}$$

3.128 As the chain is of infinite length any two charge of same sign will occur symmetrically to any other charge of opposite sign.

So, interaction energy of each charge with all the others,

$$U = -2\frac{q^2}{4\pi\varepsilon_0 a}\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\ldots \text{ up to } \infty\right] \quad (1)$$

But $\qquad \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \ldots\ldots\ldots \text{ up to } \infty$

and putting $x = 1$ we get $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots \text{ up to } \infty \quad (2)$

From Eqs. (1) and (2),

$$U = \frac{-2q^2 \ln 2}{4\pi\varepsilon_0 a}$$

3.129 Using electrical image method, interaction energy of the charge q with those induced on the plane.

$$U = \frac{-q^2}{4\pi\varepsilon_0(2l)} = -\frac{q^2}{8\pi\varepsilon_0 l}$$

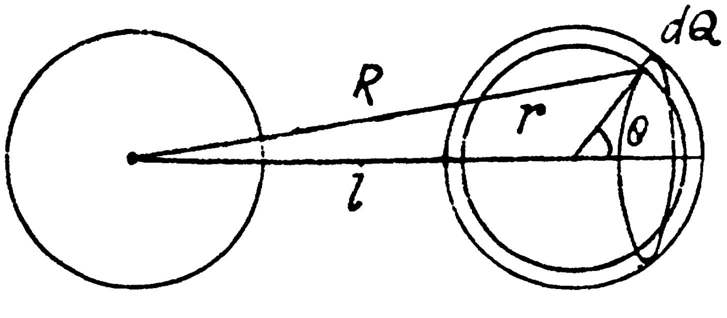
3.130 Consider the interaction energy of one of the balls (say 1) and thin spherical shell of the other. This interaction energy can be written as $\int d\varphi q$

$$= \int \frac{q_1}{4\pi\varepsilon_0 R}\rho_2(r)\,2\pi r^2 \sin\theta\, d\theta\, dr = \int_0^{\pi} \frac{\rho_2(r)\, q_1 r^2 \sin\theta\, d\theta\, dr}{2\varepsilon_0(l^2+r^2+2lr\cos\theta)^{1/2}}$$

$$= \frac{q_1 r}{2\varepsilon_0 l}dr \int_{l-r}^{l+r} dx\, \rho_2(r)$$

$$= \frac{q_1 r}{2\varepsilon_0 l}dr \cdot 2r\,\rho_2(r)\cdot 2$$

$$= \frac{q_1}{4\pi\varepsilon_0 l}4\pi r^2\, dr\, \rho_2(r)$$



Hence finally integrating

$$U_{\text{int}} = \frac{q_1 q_2}{4\pi\varepsilon_0 l} \quad \text{where, } q_2 = \int_0^{\infty} 4\pi r^2 \rho_2(r)\, dr$$

3.131 Charge contained in the capacitor of capacitance C_1 is $q = C_1\varphi$ and the energy, stored in it :

$$U_i = \frac{q^2}{2C_1} = \frac{1}{2}C_1\varphi^2$$

Now, when the capacitors are connected in parallel, equivalent capacitance of the system, $C = C_1 + C_2$ and hence, energy stored in the system :

$$U_f = \frac{C_1^2 \varphi^2}{2(C_1 + C_2)}$$, as charge remains conserved during the process.

So, increment in the energy,

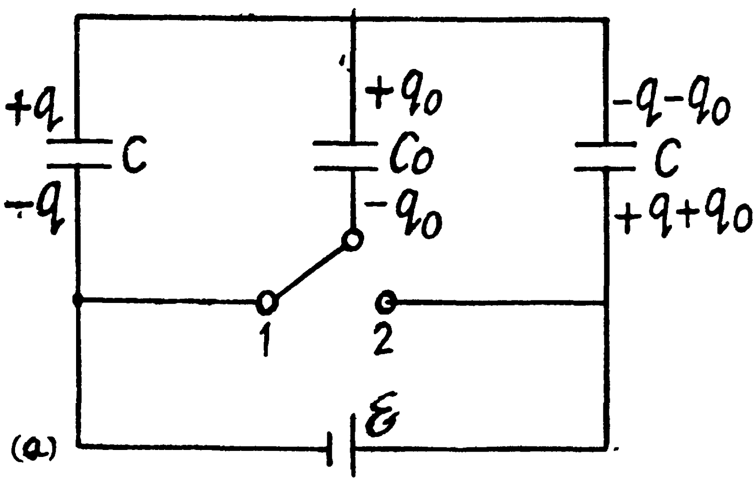
$$\Delta U = \frac{C_1^2 \varphi^2}{2}\left(\frac{1}{C_1 + C_2} - \frac{1}{C_1}\right) = \frac{-C_2 C_1 \varphi^2}{2(C_1 + C_2)} = -0{\cdot}03 \text{ mJ}$$

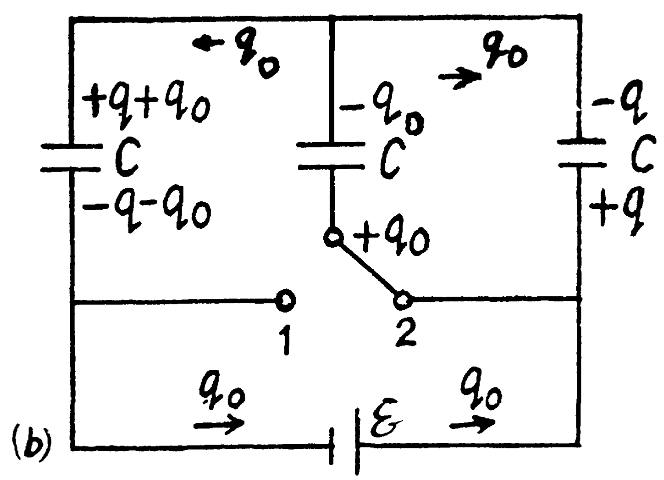
3.132 The charge on the condensers in position 1 are as shown. Here

$$\frac{q}{C} = \frac{q_0}{C_0} = \frac{q + q_0}{C + C_0}$$

and $$(q + q_0)\left(\frac{1}{C + C_0} + \frac{1}{C}\right) = \xi \text{ or, } q + q_0 = \frac{C(C + C_0)\xi}{C_0 + 2C}$$

Hence, $$q = \frac{C^2 \xi}{C_0 + 2C} \text{ and } q_0 = \frac{C C_0 \xi}{C_0 + 2C}$$





After the switch is thrown to position 2, the charges change as shown in (Fig-b).

A charge q_0 has flown in the right loop through the two condensers and a charge q_0 through the cell, Because of the symmetry of the problem there is no change in the energy stored in the condensers. Thus

H (Heat produced) = Energy delivered by the cell

$$= \Delta q \xi = q_0 \xi = \frac{C C_0 \xi^2}{C_0 + 2C}$$

3.133 Initially, the charge on the right plate of the capacitor, $q = C(\xi_1 - \xi_2)$ and finally, when switched to the position, 2. charge on the same plate of capacitor ;

$$q' = C \xi_1$$

So, $$\Delta q = q' = C \xi_2$$

Now, from energy conservation,

ΔU + Heat liberated = A_{cell}, where ΔU is the electrical energy.

$$\frac{1}{2} C \xi_1^2 - \frac{1}{2} C (\xi_1 - \xi_2)^2 + \text{ Heat liberated } = \Delta q \, \xi_1$$

as only the cell with e.m.f. ξ_1 is responsible for redistribution of the charge. So,

$$C \xi_1 \xi_2 - \frac{1}{2} C \xi_2^2 + \text{Heat liberated} = C \xi_2 \xi_1.$$

Hence heat liberated $= \frac{1}{2} C \xi_2^2$

3.134 Self energy of each shell is given by $\frac{q \varphi}{2}$, where φ is the potential of the shell, created only by the charge q, on it.

Hence, self energy of the shells 1 and 2 are :

$$W_1 = \frac{q_1^2}{8 \pi \varepsilon_0 R_1} \text{ and } W_2 = \frac{q_2^2}{8 \pi \varepsilon_0 R_2}$$

The interaction energy between the charged shells equals charge q of one shell, multiplied by the potential φ, created by other shell, at the point of location of charge q.

So,
$$W_{12} = q_1 \frac{q_2}{4 \pi \varepsilon_0 R_2} = \frac{q_1 q_2}{4 \pi \varepsilon_0 R_2}$$

Hence, total enegy of the system,

$$U = W_1 + W_2 + W_{12}$$

$$= \frac{1}{4 \pi \varepsilon_0} \left[\frac{q_1^2}{2 R_1} + \frac{q_2^2}{2 R_2} + \frac{q_1 q_2}{R_2} \right]$$

3.135 Electric fields inside and outside the sphere with the help of Gauss theorem :

$$E_1 = \frac{q r}{4 \pi \varepsilon_0 R^2} (r \le R), E_2 = \frac{q}{4 \pi \varepsilon_0} \frac{1}{r^2} (r > R)$$

Sought self energy of the ball

$$U = W_1 + W_2$$

$$= \int_0^R \frac{\varepsilon_0 E_1^2}{2} 4 \pi r^2 \, dr + \int_R^\infty \frac{\varepsilon_0 E_2^2}{2} 4 \pi r^2 \, dr = \frac{q^2}{8 \pi \varepsilon_0 R} \left(\frac{1}{5} + 1 \right)$$

Hence,
$$U = \frac{3 q^2}{4 \pi \varepsilon_0 5 R} \text{ and } \frac{W_1}{W_2} = \frac{1}{5}$$

3.136 (a) By the expression $\int \frac{1}{2} \varepsilon_0 \varepsilon E^2 \, dV = \int \frac{1}{2} \varepsilon \varepsilon_0 E^2 \, 4 \pi r^2 \, dr$, for a spherical layer.

To find the electrostatic energy inside the dielectric layer, we have to integrate the upper expression in the limit $[a, b]$

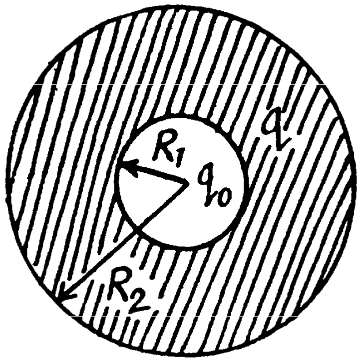
$$U = \frac{1}{2} \varepsilon_0 \varepsilon \int_a^b \left(\frac{q}{4 \pi \varepsilon_0 \varepsilon r^2} \right)^2 4 \pi r^2 \, dr = \frac{q^2}{8 \pi \varepsilon_0 \varepsilon} \left[\frac{1}{a} - \frac{1}{b} \right] = 27 \text{ mJ}$$

3.137 As the field is conservative total work done by the field force,

$$A_{fd} = U_i - U_f = \frac{1}{2} q (\varphi_1 - \varphi_2)$$

$$= \frac{1}{2} \frac{q^2}{4 \pi \varepsilon_0} \left[\frac{1}{R_1} - \frac{1}{R_2}\right] = \frac{q^2}{8 \pi \varepsilon_0} \left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

3.138 Initially, energy of the system, $U_i = W_1 + W_{12}$ where, W_1 is the self energy and W_{12} is the mutual energy.



So, $$U_i = \frac{1}{2} \frac{q^2}{4 \pi \varepsilon_0 R_1} + \frac{q q_0}{4 \pi \varepsilon_0 R_1}$$

and on expansion, energy of the system,

$$U_f = W'_1 + W'_{12}$$

$$= \frac{1}{2} \frac{q^2}{4 \pi \varepsilon_0 R_2} + \frac{q q_0}{4 \pi \varepsilon_0 R_2}$$

Now, work done by the field force, A equals the decrement in the electrical energy,

i.e. $$A = (U_i - U_f) = \frac{q (q_0 + q/2)}{4 \pi \varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Alternate : The work of electric forces is equal to the decrease in electric energy of the system,

$$A = U_i - U_f$$

In order to find the difference $U_i - U_f$, we note that upon expansion of the shell, the electric field and hence the energy localized in it, changed only in the hatched spherical layer consequently (Fig.).

$$U_i - U_f = \int_{R_1}^{R_2} \frac{\varepsilon_0}{2} (E_1^2 - E_2^2) \cdot 4 \pi r^2 \, dr$$

where E_1 and E_2 are the field intensities (in the hatched region at a distance r from the centre of the system) before and after the expansion of the shell. By using Gauss' theorem, we find

$$E_1 = \frac{1}{4 \pi \varepsilon_0} \frac{q + q_0}{r^2} \text{ and } E_2 = \frac{1}{4 \pi \varepsilon_0} \frac{q_0}{r^2}$$

As a result of integration, we obtain

$$A = \frac{q (q_0 + q/2)}{4 \pi \varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

3.139 Energy of the charged sphere of radius r, from the equation

$$U = \frac{1}{2} q \varphi = \frac{1}{2} q \frac{q}{4\pi\varepsilon_0 r} = \frac{q^2}{8\pi\varepsilon_0 r}$$

If the radius of the shell changes by dr then work done is

$$4\pi r^2 F_u \, dr = -dU = q^2/8\pi\varepsilon_0 r^2$$

Thus sought force per unit area,

$$F_u = \frac{q^2}{4\pi r^2 (8\pi\varepsilon_0 r^2)} = \frac{(4\pi r^2 \sigma)^2}{4\pi r^2 \times 8\pi\varepsilon_0 r^2} = \frac{\sigma^2}{2\varepsilon_0}$$

3.140 Initially, there will be induced charges of magnitude $-q$ and $+q$ on the inner and outer surface of the spherical layer respectively. Hence, the total electrical energy of the system is the sum of self energies of spherical shells, having radii a and b, and their mutual energies including the point charge q.

$$U_i = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0 b} + \frac{1}{2}\frac{(-q)^2}{4\pi\varepsilon_0 a} + \frac{-q\,q}{4\pi\varepsilon_0 a} + \frac{qq}{4\pi\varepsilon_0 b} + \frac{-qq}{4\pi\varepsilon_0 b}$$

or

$$U_i = \frac{q^2}{8\pi\varepsilon_0}\left[\frac{1}{b} - \frac{1}{a}\right]$$

Finally, charge q is at infinity hence, $U_f = 0$

Now, work done by the agent = increment in the energy

$$= U_f - U_i = \frac{q^2}{8\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

3.141 (a) Sought work is equivalent to the work performed against the electric field created by one plate, holding at rest and to bring the other plate away. Therefore the required work,

$$A_{\text{agent}} = qE\,(x_2 - x_1),$$

where $E = \dfrac{\sigma}{2\varepsilon_0}$ is the intensity of the field created by one plate at the location of other.

So, $$A_{agent} = q\frac{\sigma}{2\varepsilon_0}(x_2 - x_1) = \frac{q^2}{2\varepsilon_0 S}(x_2 - x_1)$$

Alternate : $A_{\text{ext}} = \Delta U$ (as field is potential)

$$= \frac{q^2}{2\varepsilon_0 S} x_2 - \frac{q^2}{2\varepsilon_0 S} x = \frac{q^2}{2\varepsilon_0 S}(x_2 - x_1)$$

(b) When voltage is kept const., the force acing on each plate of capacitor will depend on the distance between the plates.

So, elementary work done by agent, in its displacement over a distance dx, relative to the other,

$$dA = -F_x\,dx$$

But, $$F_x = -\left(\frac{\sigma(x)}{2\varepsilon_0}\right) S\,\sigma(x) \quad \text{and} \quad \sigma(x) = \frac{\varepsilon_0 V}{x}$$

Hence, $$A = \int dA = \int_{x_1}^{x_2} \frac{1}{2}\varepsilon_0 \frac{S V^2}{x^2}\,dx = \frac{\varepsilon_0 S V^2}{2}\left[\frac{1}{x_1} - \frac{1}{x_2}\right]$$

Alternate : From energy Conservation,

$$U_f - U_i = A_{\text{cell}} + A_{\text{agent}}$$

or $$\frac{1}{2}\frac{\varepsilon_0 S}{x_2}V^2 - \frac{1}{2}\frac{\varepsilon_0 S}{x_1}V^2 = \left[\frac{\varepsilon_0 S}{x_2} - \frac{\varepsilon_0 S}{x_1}\right]V^2 + A_{\text{agent}}$$

(as $A_{\text{cell}} = (q_f - q_i)\,V = (C_f - C_i)\,V^2$)

So $$A_{\text{agent}} = \frac{\varepsilon_0 S V^2}{2}\left[\frac{1}{x_1} - \frac{1}{x_2}\right]$$

3.142 (a) When metal plate of thickness ηd is inserted inside the capacitor, capacitance of the system becomes $C_0 = \frac{\varepsilon_0 S}{d(1-\eta)}$

Now, initially, charge on the capacitor, $q_0 = C_0 V = \frac{\varepsilon_0 S V}{d(1-\eta)}$

Finally, capacitance of the capacitor, $C = \frac{\varepsilon_0 S}{d}$

As the source is disconnected, charge on the plates will remain same during the process. Now, from energy conservation,

$$U_f - U_i = A_{\text{agent}} \text{ (as cell does no work)}$$

or, $$\frac{1}{2}\frac{q_0^2}{C} - \frac{1}{2}\frac{q_0^2}{C_0} = A_{\text{agent}}$$

Hence $$A_{\text{agent}} = \frac{1}{2}\left[\frac{\varepsilon_0 S V}{d(1-\eta)}\right]^2\left[\frac{1}{C} - \frac{(1-\eta)}{C}\right] = \frac{1}{2}\frac{C V^2 \eta}{(1-\eta)^2} = 1{\cdot}5 \text{ mJ}$$

(b) Initially, capacitance of the system is given by,

$C_0 = \frac{C\varepsilon}{\eta(1-\varepsilon)+\varepsilon}$ (this is the capacitance of two capacitors in series)

So, charge on the plate, $q_0 = C_0 V$

Capacitance of the capacitor, after the glass plate has been removed equals C

From energy conservation,

$$A_{\text{agent}} = U_f - U_i$$

$$= \frac{1}{2}q_0^2\left[\frac{1}{C} - \frac{1}{C_0}\right] = \frac{1}{2}\frac{C V^2 \varepsilon \eta(\varepsilon-1)}{[\varepsilon - \eta(\varepsilon-1)]^2} = 0{\cdot}8 \text{ mJ}$$

3.143 When the capactior which is immersed in water is connected to a constant voltage source, it gets charged. Suppose σ_0 is the free charge density on the condenser plates. Because water is a dielectric, bound charges also appear in it. Let σ' be the surface density of bound charges. (Because of homogeneity of the medium and uniformity of the field when we ignore edge effects no volume density of bound charges exists.) The electric field due to free charges only $\frac{\sigma_0}{\varepsilon_0}$; that due to bound charges is $\frac{\sigma'}{\varepsilon_0}$ and the total electric field is $\frac{\sigma_0}{\varepsilon\varepsilon_0}$. Recalling that the sign of bound charges is opposite of the free charges, we have

$$\frac{\sigma_0}{\varepsilon\varepsilon_0} = \frac{\sigma_0}{\varepsilon_0} - \frac{\sigma'}{\varepsilon_0} \quad \text{or,} \quad \sigma' = \left(\frac{\varepsilon - 1}{\varepsilon}\right)\sigma_0$$

Because of the field that exists due to the free charges (not the total field; the field due to the bound charges must be excluded for this purpose as they only give rise to self energy effects), there is a force attracting the bound charges to the near by plates. This force is

$$\frac{1}{2}\sigma' \frac{\sigma_o}{\varepsilon_0} = \frac{(\varepsilon - 1)\,\sigma_0^2}{2\varepsilon\varepsilon_0} \quad \text{per unit area.}$$

The factor $\frac{1}{2}$ needs an explanation. Normally the force on a test charge is qE in an electric field E. But if the charge itself is produced by the electric filed then the force must be constructed bit by bit and is

$$F = \int_0^E q(E')\, dE'$$

if $\quad q(E') \propto E'$ then we get

$$F = \frac{1}{2} q(E)\, E$$

This factor of $\frac{1}{2}$ is well known. For example the energy of a dipole of moment $\vec{p}$ in an electric field $\vec{E}_0$ is $-\vec{p}\cdot\vec{E}_0$ while the energy per unit volume of a linear dielectric in an electric field is $-\frac{1}{2}\vec{P}\cdot\vec{E}_0$ where $\vec{P}$ is the polarization vector (i.e. dipole moment per unit volume). Now the force per unit area manifests ifself as excess pressure of the liquid.

Noting that
$$\frac{V}{d} = \frac{\sigma_0}{\varepsilon\varepsilon_0}$$

We get
$$\Delta p = \frac{\varepsilon_0\varepsilon(\varepsilon - 1)V^2}{2d^2}$$

Substitution, using $\varepsilon = 81$ for water, gives $\Delta p = 7{\cdot}17$ k Pa $= 0{\cdot}07$ atm.

3.144 One way of doing this problem will be exactly as in the previous case so let us try an alternative method based on energy. Suppose the liquid rises by a distance h. Then let us calculate the extra energy of the liquid as a sum of polarization energy and the ordinary gravitiational energy. The latter is

$$\frac{1}{2} h \cdot \rho g \cdot Sh = \frac{1}{2}\rho g S h^2$$

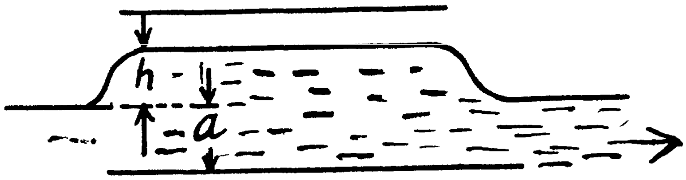
If σ is the free charge surface density on the plate, the bound charge density is, from the previous problem,

$$\sigma' = \frac{\varepsilon - 1}{\varepsilon}\sigma$$

This is also the volume density of induced dipole moment i.e. Polarization. Then the energy is, as before

$$-\frac{1}{2}\cdot\sigma' E_0 = \frac{-1}{2}\cdot\sigma' \frac{\sigma}{\varepsilon_0} = \frac{-(\varepsilon - 1)\,\sigma^2}{2\varepsilon_0\varepsilon}$$

and the total polarization energy is



$$-S(a+h)\frac{(\varepsilon-1)\sigma^2}{2\varepsilon_0\varepsilon}$$

Then, total energy is

$$U(h)=-S(a+h)\frac{(\varepsilon-1)\sigma^2}{2\varepsilon_0\varepsilon}+\frac{1}{2}\rho g S h^2$$

The actual height to which the liquid rises is determined from the formula

$$\frac{dU}{dh}=U'(h)=0$$

This gives
$$h=\frac{(\varepsilon-1)\sigma^2}{2\varepsilon_0\,\varepsilon\,\rho g}.$$

3.145 We know that energy of a capacitor, $U=\dfrac{q^2}{2C}$.

Hence, from $F_x=\left.\dfrac{\partial U}{\partial x}\right|_{q=\text{Const.}}$ we have, $F_x=\dfrac{q^2}{2}\dfrac{\partial c}{\partial x}\Big/C^2$ (1)

Now, since $d \ll R$, the capacitance of the given capacitor can be calculated by the formula of a parallel plate capacitor. Therefore, if the dielectric is introduced upto a depth x and the length of the capacitor is l, we have,

$$C=\frac{2\pi\varepsilon_0\varepsilon R x}{d}+\frac{2\pi R\varepsilon_0(l-x)}{d} \tag{2}$$

From (1) and (2), we get,

$$F_x=\varepsilon_0(\varepsilon-1)\frac{\pi R V^2}{d}$$

3.146 When the capacitor is kept at a constant potential difference V, the work performed by the moment of electrostatic forces between the plates when the inner moveable plate is rotated by an angle $d\varphi$ equals the increase in the potential energy of the system. This comes about because when charges are made, charges flow from the battery to keep the potential constant and the amount of the work done by these charges is twice in magnitude and opposite in sign to the change in the energy of the capacitor Thus

$$N_z=\frac{\partial U}{\partial\varphi}=\frac{1}{2}V^2\frac{\partial C}{\partial\varphi}$$

Now the capacitor can be thought of as made up two parts (with and without the dielectric) in parallel.

Thus $C=\dfrac{\varepsilon_0 R^2\varphi}{2d}+\dfrac{\varepsilon_0\varepsilon(\pi-\varphi)R^2}{2d}$

as the area of a sector of angle φ is $\frac{1}{2}R^2\varphi$. Differentiation then gives

$$N_z=-\frac{(\varepsilon-1)\varepsilon_0 R^2 V^2}{4d}$$

The negative sign of N_z indicates that the moment of the force is acting clockwise (i.e. trying to suck in the dielectric).

3.4 ELECTRIC CURRENT

3.147 The convection current is

$$I = \frac{dq}{dt} \tag{1}$$

here, $dq = \lambda\, dx$, where λ is the linear charge density.

But, from the Gauss' theorem, electric field at the surface of the cylinder,

$$E = \frac{\lambda}{2\pi\varepsilon_0 a}$$

Hence, substituting the value of λ and subsequently of dq in Eqs. (1), we get

$$I = \frac{2E\pi\varepsilon_0\, a dx}{dt}$$

$$= 2\pi\varepsilon_0 E\, av, \quad \text{as } \frac{dx}{dt} = v$$

3.148 Since $d << r$, the capacitance of the given capacitor can be calculated using the formula for a parallel plate capacitor. Therefore if the water (permittivity ε) is introduced up to a height x and the capacitor is of length l, we have,

$$C = \frac{\varepsilon\varepsilon_0 \cdot 2\pi r x}{d} + \frac{\varepsilon_0 (l - x)\, 2\pi r}{d} = \frac{\varepsilon_0\, 2\pi r}{d}(\varepsilon x + l - x)$$

Hence charge on the plate at that instant, $q = CV$

Again we know that the electric current intensity,

$$I = \frac{dq}{dt} = \frac{d(CV)}{dt}$$

$$= \frac{V\varepsilon_0\, 2\pi r}{d}\frac{d(\varepsilon x + l - x)}{dt} = \frac{V\, 2\pi r\varepsilon_0}{d}(\varepsilon - 1)\frac{dx}{dt}$$

But, $$\frac{dx}{dt} = v,$$

So, $$I = \frac{2\pi r\varepsilon_0(\varepsilon - 1)V}{d}v = 0{\cdot}11\ \mu\text{A}$$

3.149 We have, $R_t = R_0(1 + \alpha t)$,(1)

where R_t and R_0 are resistances at $t°$ C and $0°$ C respectively and α is the mean temperature coefficient of resistance.

So, $$R_1 = R_0(1 + \alpha_1 t) \text{ and } R_2 = \eta R_0(1 + \alpha_2 t)$$

(a) In case of series combination, $R = \Sigma R_i$

so $$R = R_1 + R_2 = R_0[(1 + \eta) + (\alpha_1 + \eta\alpha_2)t] \tag{1}$$

$$= R_0(1 + \eta)\left[1 + 2\frac{\alpha_1 + \eta\alpha_2}{1 + \eta}t\right] \tag{2}$$

Comparing Eqs. (1) and (2), we conclude that temperature co-efficient of resistance of the circuit,

$$\alpha = \frac{\alpha_1 + \eta\,\alpha_2}{1 + \eta}$$

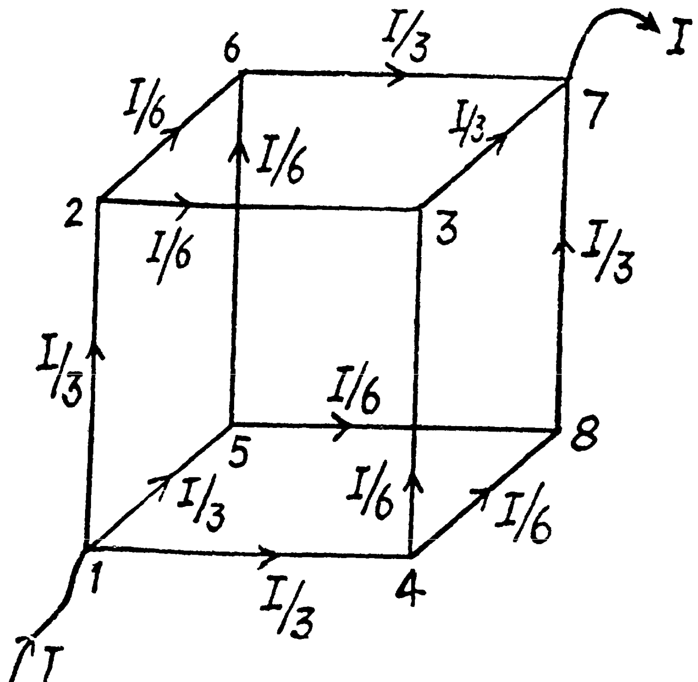
(b) In parallel combination

$$R = \frac{R_0\,(1 + \alpha_1\, t)\, R_0\, \eta\,(1 + \alpha_2\, t)}{R_0\,(1 + \alpha_1\, t) + \eta\, R_0\,(1 + \alpha_2\, t)} = R'\,(1 + \alpha'\, t), \text{ where } R' = \frac{\eta R_0}{1 + \eta}$$

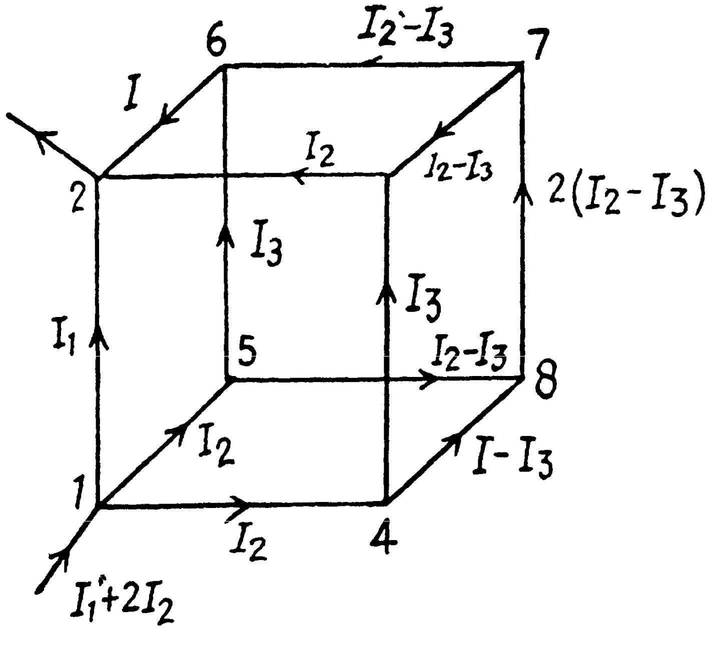
Now, neglecting the terms, proportional to the product of temperature coefficients, as being very small, we get,

$$\alpha' \approx \frac{\eta\,\alpha_1 + \alpha_2}{1 + \eta}$$

3.150 (a) The currents are as shown. From Ohm's law applied between 1 and 7 via 1487 (say)



(a)



(b)

$$IR_{eq} = \frac{I}{3}R + \frac{I}{6}R + \frac{I}{3}R = \frac{5}{6}\,RI$$

Thus, $$R_{eq} = \frac{5R}{6}$$

(b) Between 1 and 2 from the loop 14321,

$I_1 R = 2I_2 R + I_3 R$ or, $I_1 = I_3 + 2I_2$

From the loop 48734,

$$(I_2 - I_3)\, R + 2\,(I_2 - I_3)\, R + (I_2 - I_3)\, R = I_3\, R.$$

or, $$4\,(I_2 - I_3) = I_3 \text{ or } I_3 = \frac{4}{5} I_2$$

so $$I_1 = \frac{14}{5} I_2$$

Then, $(I_1 + 2I_2)\, R_{eq} = \dfrac{24}{5} I_2\, R_{eq} = I_1 R = \dfrac{14}{5} I_2\, R$

or $$R_{eq} = \frac{7}{12} R$$

(c) Between 1 and 3

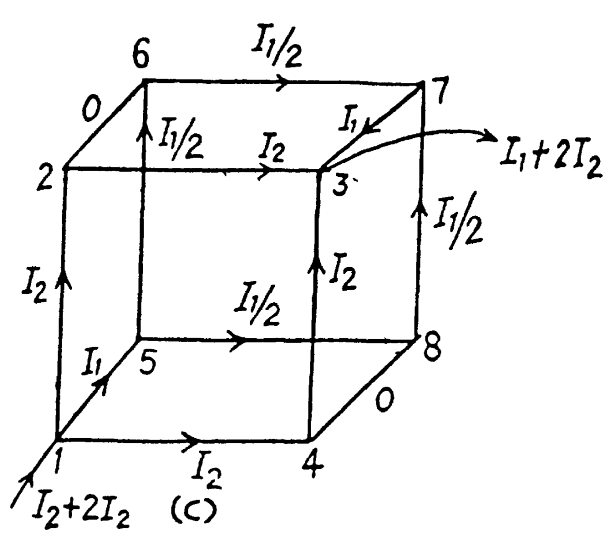
From the loop 15621

$$I_2 R = I_1 R + \frac{I_1}{2} R \quad \text{or,} \quad I_2 = 3 \frac{I_1}{2}$$

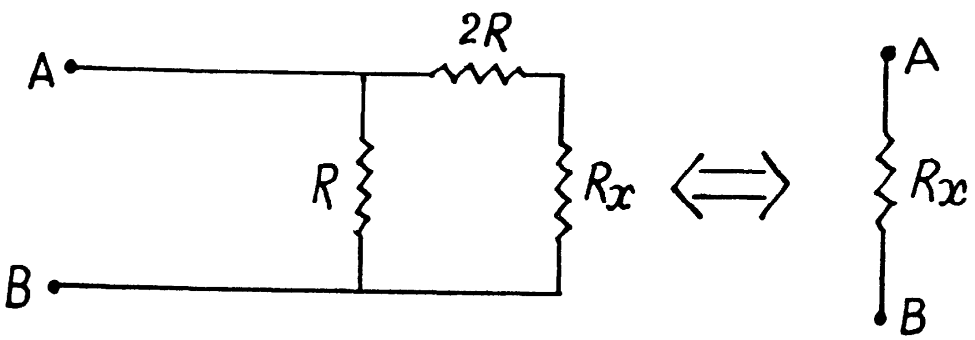
Then, $(I_1 + 2I_2) R_{eq} = 4 I_1 R_{eq}$

$$= I_2 R + I_2 R = 3 I_1 R$$

Hence, $$R_{eq} = \frac{3}{4} R .$$



3.151 Total resistance of the circuit will be independent of the number of cells,



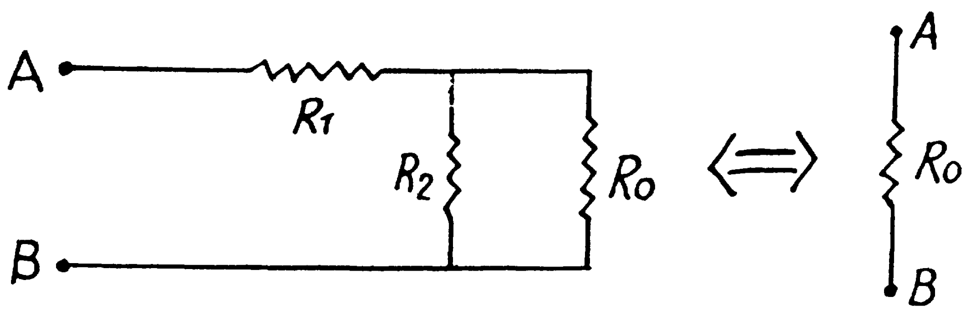
if $$R_x = \frac{(R_x + 2R) R}{R_x + 2R + R}$$

or, $$R_x^2 + 2R R_x - 2R^2 = 0$$

On solving and rejecting the negative root of the quadratic equation, we have,

$$R_x = R(\sqrt{3} - 1)$$

3.152 Let R_0 be the resistance of the network,



then, $R_0 = \dfrac{R_0 R_2}{R_0 + R_2}$ or $R_0^2 - R_0 R_1 - R_1 R_2 = 0$

On solving we get,

$$R_0 = \frac{R_1}{2}\left(1 + \sqrt{1 + 4\frac{R_2}{R_1}}\right) = 6\,\Omega$$

3.153 Suppose that the voltage V is applied between the points A and B then

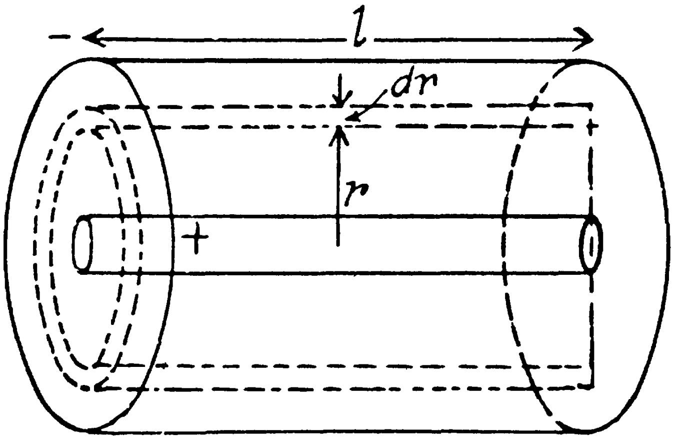
$$V = IR = I_0 R_0,$$

where R is resistance of whole the grid, I, the current through the grid and I_0, the current through the segment AB. Now from symmetry, $I/4$ is the part of the current, flowing through all the four wire segments, meeting at the point A and similarly the amount of current flowing through the wires, meeting at B is also $I/4$. Thus a current $I/2$ flows through the conductor AB, i.e.

$$I_0 = \frac{I}{2}$$

Hence, $$R = \frac{R_0}{2}$$

3.154 Let us mentally isolate a thin cylindrical layer of inner and outer radii r and $r + dr$ respectively. As lines of current at all the points of this layer are perpendicular to it, such a layer can be treated as a cylindrical conductor of thickness dr and cross-sectional area $2\pi rl$. So, we have,



$$dR = \rho \frac{dr}{S(r)} = \rho \frac{dr}{2\pi rl}$$

and integrating between the limits, we get,

$$R = \frac{\rho}{2\pi l} \ln \frac{b}{a}$$

3.155 Let us mentally isolate a thin spherical layer of inner and outer radii r and $r + dr$. Lines of current at all the points of the this layer are perpendicular to it and therefore such a layer can be treated as a spherical conductor of thickness dr and cross sectional area $4\pi r^2$. So

$$dR = \rho \frac{dr}{4\pi r^2} \tag{1}$$

And integrating (1) between the limits $[a, b]$, we get,

$$R = \frac{\rho}{4\pi}\left[\frac{1}{a} - \frac{1}{b}\right]$$

Now, for $b \rightarrow \infty$, we have

$$R = \frac{\rho}{4\pi a}$$

3.156 In our system, resistance of the medium $R = \frac{\rho}{4\pi}\left[\frac{1}{a} - \frac{1}{b}\right]$,

where ρ is the resistivity of the medium

The current $$i = \frac{\varphi}{R} = \frac{\varphi}{\frac{\rho}{4\pi}\left[\frac{1}{a} - \frac{1}{b}\right]}$$

Also, $i = \frac{-dq}{dt} = -\frac{d(C\varphi)}{dt} = -C\frac{d\varphi}{dt}$, as capacitance is constant. (2)

So, equating (1) and (2) we get,

$$\frac{\varphi}{\frac{\rho}{4\pi}\left[\frac{1}{a}-\frac{1}{b}\right]} = -C\frac{d\varphi}{dt}$$

or,
$$-\int\frac{d\varphi}{\varphi} = \frac{\Delta t}{\frac{C\rho}{4\pi}\left[\frac{1}{a}-\frac{1}{b}\right]}$$

or,
$$\ln\eta = \frac{\Delta t\, 4\pi ab}{C\rho(b-a)}$$

Hence, resistivity of the medium,

$$\rho = \frac{4\pi\Delta t\, ab}{C(b-a)\ln\eta}$$

3.157 Let us mentally impart the charge $+q$ and $-q$ to the balls respectively. The electric field strength at the surface of a ball will be determined only by its own charge and the charge can be considered to be uniformly distributed over the surface, because the other ball is at infinite distance. Magnitude of the field strength is given by,

$$E = \frac{q}{4\pi\varepsilon_0 a^2}$$

So, current density $j = \frac{1}{\rho}\frac{q}{4\pi\varepsilon_0 a^2}$ and electric current

$$I = \int \vec{j}\cdot d\vec{S} = jS = \frac{q}{\rho\, 4\pi\varepsilon_0 a^2}\cdot 4\pi a^2 = \frac{q}{\rho\,\varepsilon_0}$$

But, potential difference between the balls,

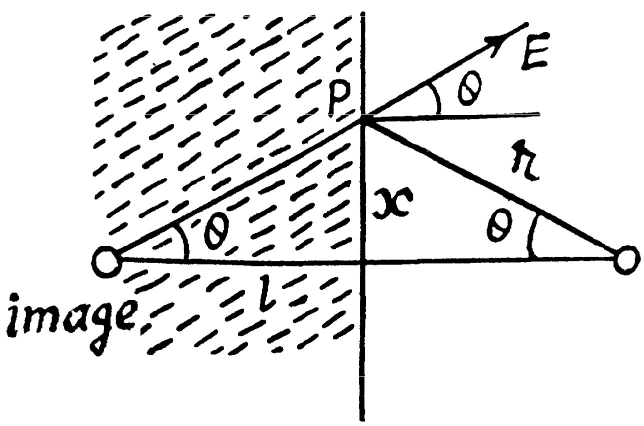
$$\varphi_+ - \varphi_- = 2\frac{q}{4\pi\varepsilon_0 a}$$

Hence, the sought resistance,

$$R = \frac{\varphi_+ - \varphi_-}{I} = \frac{2q/4\pi\varepsilon_0 a}{q/\rho\,\varepsilon_0} = \frac{\rho}{2\pi a}$$

3.158 (a) The potential in the unshaded region beyond the conductor as the potential of the given charge and its image and has the form

$$\varphi = A\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$



where r_1, r_2 are the distances of the point from the charge and its image. The potential has been taken to be zero on the conducting plane and on the ball

$$\varphi = A\left(\frac{1}{a}-\frac{1}{2l}\right) = V$$

So $A \approx Va$. In this calculation the conditions $a << l$ is used to ignore the variation of φ over the ball.

The electric field at P can be calculated similarly. The charge on the ball is

$$Q = 4\pi\varepsilon_0 Va$$

and
$$E_P = \frac{Va}{r^2}\ 2\cos\theta = \frac{2alV}{r^3}$$

Then $j = \dfrac{1}{\rho}E = \dfrac{2alV}{\rho r^3}$ normal to the plane.

(b) The total current flowing into the conducting plane is

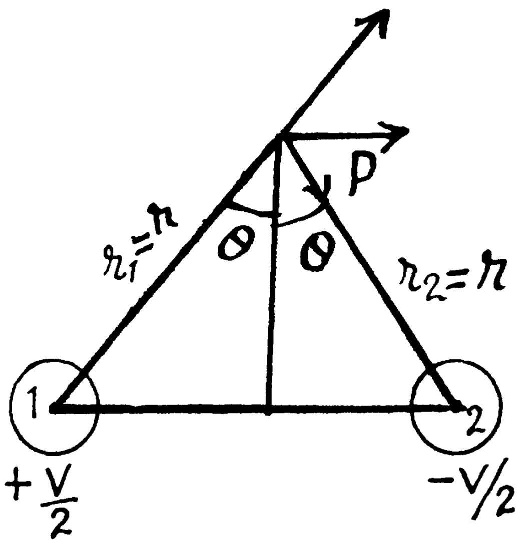
$$I = \int_0^\infty 2\pi x\, dx\, j = \int_0^\infty 2\pi x\, dx \frac{2alV}{\rho\left(\pi^2 + l^2\right)^{3/2}}$$

(On putting $y = x^2 + l^2$)

$$I = \frac{2\pi alV}{\rho}\int_{l^2}^\infty \frac{dy}{y^{3/2}} = \frac{4\pi aV}{\rho}$$

Hence
$$R = \frac{V}{I} = \frac{\rho}{2\pi a}$$

3.159 (a) The wires themselves will be assumed to be perfect conductors so the resistance is entirely due to the medium. If the wire is of length L, the resistance R of the medium is $\alpha \dfrac{1}{L}$ because different sections of the wire are connected in parallel (by the medium) rather than in series. Thus if R_1 is the resistance per unit length of the wire then $R = R_1/L$. Unit of R_1 is ohm-meter.



The potential at a point P is by symmetry and superposition
(for $l >> a$)

$$\varphi = \frac{A}{2}\ln\frac{r_1}{a} - \frac{A}{2}\ln\frac{r_2}{a}$$

$$= \frac{A}{2}\ln\frac{r_1}{r_2}$$

Then $\varphi_1 = \dfrac{V}{2} = \dfrac{A}{2}\ln\dfrac{a}{l}$ (for the potential of 1)

or, $$A = -V/\ln\frac{l}{a}$$

and $$\varphi = -\frac{V}{2\ln l/a}\ln r_1/r_2$$

We then calculate the field at a point P which is equidistant from 1 & 2 and at a distance r from both :

Then $$E = \frac{V}{2\ln l/a}\left(\frac{1}{r}\right)\times 2\sin\theta$$

$$= \frac{Vl}{2\ln l/a}\frac{1}{r^2}$$

and $$J = \sigma E = \frac{1}{\rho}\;\frac{V}{2\ln l/a}\;\frac{1}{r^2}$$

(b) Near either wire $$E = \frac{V}{2\ln l/a}\;\frac{1}{a}$$

and $$J = \sigma E = \frac{1}{\rho}\;\frac{V}{2\ln l/a}$$

Then $$I = \frac{V}{R} = L\frac{V}{R_1} = J\; 2\pi a L$$

Which gives $$R_1 = \frac{\rho}{\pi}\ln l/a$$

3.160 Let us mentally impart the charges $+q$ and $-q$ to the plates of the capacitor.

Then capacitance of the network,

$$C = \frac{q}{\varphi} = \frac{\varepsilon\varepsilon_0\int E_n\,dS}{\varphi} \tag{1}$$

Now, electric current,

$$i = \int \vec{j}\cdot d\vec{S} = \int \sigma E_n\,dS \text{ as } \vec{j}\uparrow\uparrow\vec{E}. \tag{2}$$

Hence, using (1) in (2), we get,

$$i = \frac{C\varphi}{\varepsilon\varepsilon_0}\sigma = \frac{C\varphi}{\rho\,\varepsilon\varepsilon_0} = 1{\cdot}5\,\mu\,\text{A}$$

3.161 Let us mentally impart charges $+q$ and $-q$ to the conductors. As the medium is poorly conducting, the surfaces of the conductors are equipotential and the field configuration is same as in the absence of the medium.

Let us surround, for example, the positively charged conductor, by a closed surface S, just containing the conductor,

then, $$R = \frac{\varphi}{i} = \frac{\varphi}{\int\vec{j}\cdot d\vec{S}} = \frac{\varphi}{\int\sigma E_n\,dS};\text{ as } \vec{j}\uparrow\uparrow\vec{E}$$

and $$C = \frac{q}{\varphi} = \frac{\varepsilon\varepsilon_0\int E_n\,dS}{\varphi}$$

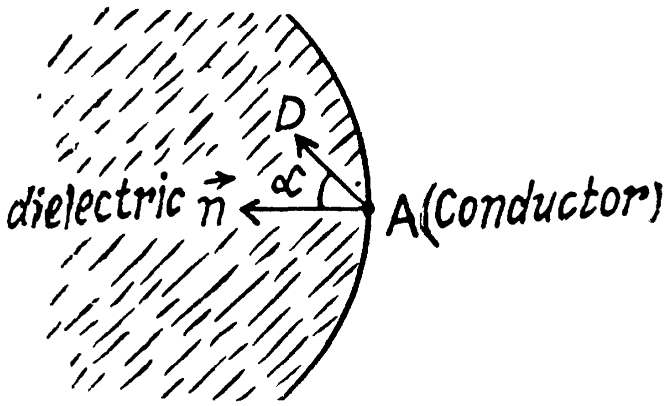
So, $$RC = \frac{\varepsilon\varepsilon_0}{\sigma} = \rho\,\varepsilon\varepsilon_0$$

3.162 The dielectric ends in a conductor. It is given that on one side (the dielectric side) the electric displacement D is as shown. Within the conductor, at any point A, there can be no normal component of electric field. For if there were such a field, a current will flow towards depositing charge there which in turn will set up countering electric field causing the normal component to vanish. Then by Gauss theorem, we easily derive

$\sigma = D_n = D \cos\alpha$ where σ is the surface charge density at A.

The tangential component is determined from the circulation theorem

$$\oint \vec{E} \cdot d\vec{r} = 0$$



It must be continuous across the surface of the conductor. Thus, inside the conductor there is a tangential electric field of magnitude,

$$\frac{D \sin\alpha}{\varepsilon_0 \varepsilon} \text{ at } A\,.$$

This implies a current, by Ohm's law, of

$$j = \frac{D \sin\alpha}{\varepsilon_0 \varepsilon\, \rho}$$

3.163 The resistance of a layer of the medium, of thickness dx and at a distance x from the first plate of the capacitor is given by,

$$dR = \frac{1}{\sigma(x)} \frac{dx}{S} \tag{1}$$

Now, since σ varies linearly with the distance from the plate. It may be represented as,

$\sigma = \sigma_1 + \left(\dfrac{\sigma_2 - \sigma_1}{d}\right) x$, at a distance x from any one of the plate.

From Eq. (1)

$$dR = \frac{1}{\sigma_1 + \left(\dfrac{\sigma_2 - \sigma_1}{d}\right) x} \frac{dx}{S}$$

or,
$$R = \frac{1}{S} \int_0^d \frac{dx}{\sigma_1 + \left(\dfrac{\sigma_2 - \sigma_1}{d}\right) x} = \frac{d}{S(\sigma_2 - \sigma_1)} \ln \frac{\sigma_2}{\sigma_1}$$

Hence,
$$i = \frac{V}{R} = \frac{S V (\sigma_2 - \sigma_1)}{d \ln \dfrac{\sigma_2}{\sigma_1}} = 5 \text{ nA}$$

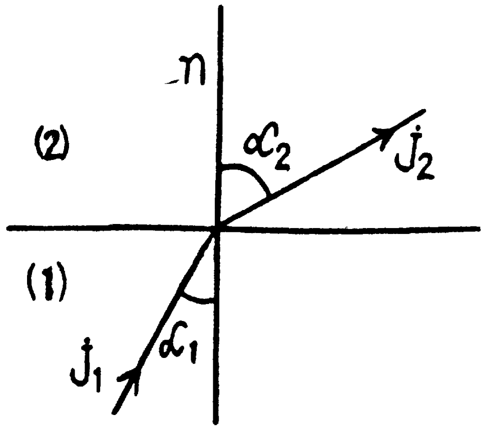
3.164 By charge conservation, current j, leaving the medium (1) must enter the medium (2). Thus

$$j_1 \cos\alpha_1 = j_2 \cos\alpha_2$$

Another relation follows from

$$E_{1t} = E_{2t}\,,$$

which is a consequence of $\oint \vec{E} \cdot d\vec{r} = 0$



Thus $\frac{1}{\sigma_1} j_1 \sin \alpha_1 = \frac{1}{\sigma_2} j_2 \ \sin \alpha_2$

or, $$\frac{\tan \alpha_1}{\sigma_1} = \frac{\tan \alpha_2}{\sigma_2}$$

or, $$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\sigma_1}{\sigma_2}$$

3.165 The electric field in conductor 1 is

$$E_1 = \frac{\rho_1 I}{\pi R^2}$$

and that in 2 is $E_2 = \dfrac{\rho_2 I}{\pi R^2}$

Applying Gauss' theorem to a small cylindrical pill-box at the boundary.

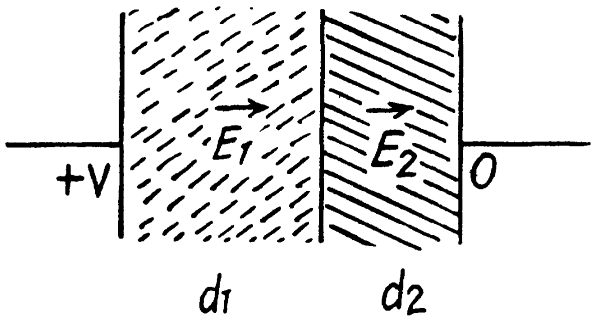
$$-\frac{\rho_1 I}{\pi R^2} dS + \frac{\rho_2 I}{\pi R^2} dS = \frac{\sigma \, dS}{\varepsilon_0}$$

Thus, $$\sigma = \varepsilon_0 \ (\rho_2 - \rho_1) \frac{1}{\pi R^2}$$

and charge at the boundary $= \varepsilon_0 \, (\rho_2 - \rho_1) I$

3.166 We have, $E_1 d_1 + E_2 d_2 = V$

and by current conservation



$$\frac{1}{\rho_1} E_1 = \frac{1}{\rho_2} E_2$$

Thus, $$E_1 = \frac{\rho_1 V}{\rho_1 d_1 + \rho_2 d_2},$$

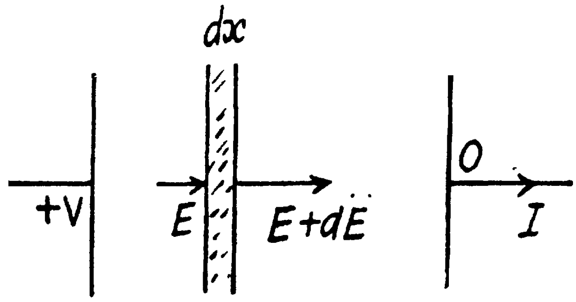
$$E_2 = \frac{\rho_2 V}{\rho_1 d_1 + \rho_2 d_2}$$

At the boundary between the two dielectrics,

$$\sigma = D_2 - D_1 = \varepsilon_0 \, \varepsilon_2 E_2 - \varepsilon_0 \, \varepsilon_1 E_1$$

$$\frac{\varepsilon_0 V}{\rho_1 d_1 + \rho_2 d_2} (\varepsilon_2 \rho_2 - \varepsilon_1 \rho_1)$$

3.167 By current conservation



$$\frac{E(x)}{\rho(x)} = \frac{E(x) + dE(x)}{\rho(x) + d\rho(x)} = \frac{dE(x)}{d\rho(x)}$$

This has the solution,

$$E(x) = C \rho(x) = \frac{I \rho(x)}{A}$$

Hence charge induced in the slice per unit area

$$d\sigma = \varepsilon_0 \frac{I}{A}[\{\varepsilon(x) + d\varepsilon(x)\}\{\rho(x) + d\rho(x)\} - \varepsilon(x)\rho(x)] = \varepsilon_0 \frac{I}{A} d[\varepsilon(x)\rho(x)]$$

Thus, $$dQ = \varepsilon_0 I d[\varepsilon(x)\rho(x)]$$

Hence total charge induced, is by integration,

$$Q = \varepsilon_0 I(\varepsilon_2 \rho_2 - \varepsilon_1 \rho_1)$$

3.168 As in the previous problem

$$E(x) = C\rho(x) = C(\rho_0 + \rho_1 x)$$

where $$\rho_0 + \rho_1 d = \eta \rho_0 \quad \text{or,} \quad \rho_1 = \frac{(\eta - 1)\rho_0}{d}$$

By integration $$V = \int_0^d C\rho(x)\,dx = C\rho_0 d\left(1 + \frac{\eta - 1}{2}\right) = \frac{1}{2}C\rho_0 d(\eta + 1)$$

Thus $$C = \frac{2V}{\rho_0 d(\eta + 1)}$$

Thus volume density of charge present in the medium

$$= \frac{dQ}{Sdx} = \varepsilon_0\, dE(x)/dx$$

$$= \frac{2\varepsilon_0 V}{\rho_0 d(\eta + 1)} \times \frac{(\eta - 1)\rho_0}{d} = \frac{2\varepsilon_0 V(\eta - 1)}{(\eta + 1)d^2}$$

3.169 (a) Consider a cylinder of unit length and divide it into shells of radius r and thickness dr Different sections are in parallel. For a typical section,

$$d\left(\frac{1}{R_1}\right) = \frac{2\pi r\,dr}{(\alpha/r^2)} = \frac{2\pi r^3\,dr}{\alpha}$$

Integrating, $$\frac{1}{R_1} = \frac{\pi R^4}{2\alpha} = \frac{S^2}{2\pi\alpha}$$

or, $$R_1 = \frac{2\pi\alpha}{S^2}, \quad \text{where } S = \pi R^2$$

(b) Suppose the electric filed inside is $E_z = E_o$ (Z axis is along the axiz of the conductor). This electric field cannot depend on r in steady conditions when other components of E are absent, otherwise one violates the circulation theorem

$$\oint \vec{E} \cdot \vec{dr} = 0$$

The current through a section between radii $(r + dr,\ r)$ is

$$2\pi r dr \frac{1}{\alpha/r^2} E = 2\pi r^3 dr \frac{E}{\alpha}$$

Thus $$I = \int_0^R 2\pi r^3 dr \frac{E}{\alpha} = \frac{\pi R^4 E}{2\alpha}$$

Hence $$E = \frac{2\alpha\pi I}{S^2} \text{ when } S = \pi R^2$$

3.170 The formula is,

$$q = C V_0 (1 - e^{-t/RC})$$

or, $$V = \frac{q}{C} = V_0 (1 - e^{-t/RC}) \quad \text{or,} \quad \frac{V}{V_0} = 1 - e^{-t/RC}$$

or, $$e^{-t/RC} = 1 - \frac{V}{V_0} = \frac{V_0 - V}{V_0}$$

Hence, $t = RC \ \ln \dfrac{V_0}{V_0 - V} = RC \ \ln 10$, if $V = 0{\cdot}9\, V_0$.

Thus $t = 0.6\ \mu S$.

3.171 The charge decays according to the foumula

$$q = q_0 e^{-t/RC}$$

Here, RC = mean life = Half-life/ln 2

So, half life= $T = RC \ \ln 2$

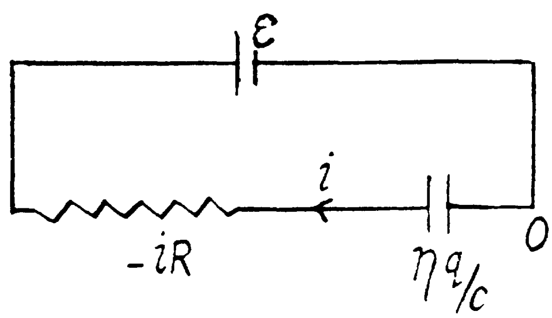
But, $$C = \frac{\varepsilon\varepsilon_0 A}{d},\ R = \frac{\rho d}{A}$$

Hence, $$\rho = \frac{T}{\varepsilon\varepsilon_0 \ln 2} = 1.4 \times 10^{13}\ \Omega \cdot \text{m}$$

3.172 Suppose q is the charge at time t. Initially $q = C\xi$, at $t = 0$.

Then at time t,

$$\frac{\eta q}{C} - iR - \xi = 0$$



But, $i = -\dfrac{dq}{dt}$ (- sign because charge decreases)

So $$\frac{\eta q}{C} + R\frac{dq}{dt} = \xi$$

$$\frac{dq}{dt} + \frac{\eta}{RC}\, q = \frac{\xi}{R}$$

or, $$\frac{d}{dt}\, q\, e^{t\eta/RC} = \frac{\xi}{R} e^{t\eta/RC}$$

or, $$q = \frac{C\xi}{\eta} + A\, e^{-t\eta/RC}$$

$$A = C\xi\left(1 - \frac{1}{\eta}\right), \text{ from } q = C\xi \text{ at } t = 0$$

Hence, $$q = C\xi\left(\frac{1}{\eta} + \left(1 - \frac{1}{\eta}\right) e^{-\eta t/RC}\right)$$

Finally, $$i = -\frac{dq}{dt} = \frac{\xi(\eta - 1)}{R} e^{-t\eta/RC}$$

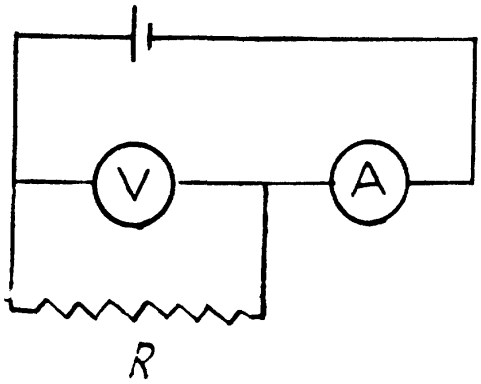
3.173 Let r = internal resistance of the battery. We shall take the resistance of the ammeter to be = 0 and that of voltmeter to be G.

Initially, $V = \xi - Ir$, $I = \dfrac{\xi}{r + G}$

So, $$V = \xi \frac{G}{r+G} \qquad (1)$$

After the voltmeter is shunted

$$\frac{V}{\eta} = \xi - \frac{\xi r}{r + \frac{RG}{R+G}} \text{ (Voltmeter)} \qquad (2)$$



and $$\frac{\xi}{r + \frac{RG}{R+G}} = \eta \frac{\xi}{r+G} \text{ (Ammeter)} \qquad (3)$$

From (2) and (3) we have

$$\frac{V}{\eta} = \xi - \frac{\eta r \xi}{r+G} \qquad (4)$$

From (1) and (4)

$$\frac{G}{\eta} = r + G - \eta r \text{ or } G = \eta r$$

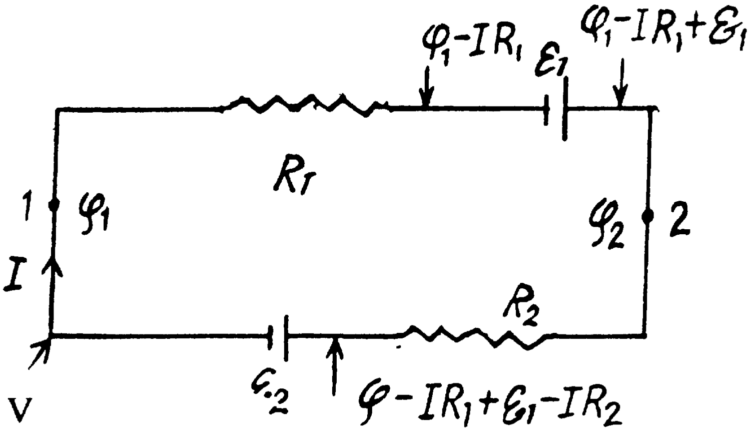
Then (1) gives the required reading

$$\frac{V}{\eta} = \frac{\xi}{\eta + 1}$$

3.174 Assume the current flow, as shown. Then potentials are as shown. Thus,

$$\varphi_1 = \varphi_1 - IR_1 + \xi_1 - IR_2 - \xi_2$$

or, $$I = \frac{\xi_1 - \xi_2}{R_1 + R_2}$$



And $$\varphi_2 = \varphi_1 - IR_1 + \xi_1$$

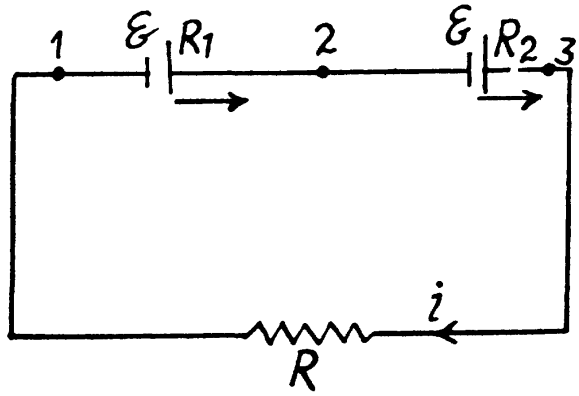
So, $$\varphi_1 - \varphi_2 = -\xi_1 + \frac{\xi_1 - \xi_2}{R_1 + R_2} R_1$$

$$= -(\xi_1 R_2 + \xi_2 R_1)/(R_1 + R_2) = -4 \text{ V}$$

3.175 Let, us consider the current i, flowing through the circuit, as shown in the figure. Applying loop rule for the circuit, $-\Delta\varphi = 0$

$$-2\xi + iR_1 + iR_2 + iR = 0$$

or, $$i(R_1 + R_2 + R) = 2\xi$$



or, $$i = \frac{2\xi}{R + R_1 + R_2}$$

Now, if $$\varphi_1 - \varphi_2 = 0$$

$$-\xi + iR_1 = 0$$

or, $\dfrac{2\xi R_1}{R + R_1 + R_2} = \xi$ and $2R_1 = R_2 + R + R_1$

or, $R = R_1 - R_2$, which is not possible as $R_2 > R_1$

Thus, $$\varphi_2 - \varphi_3 = -\xi + iR_2 = 0$$

or, $$\frac{2\xi R_2}{R+R_1+R_2} = \xi$$

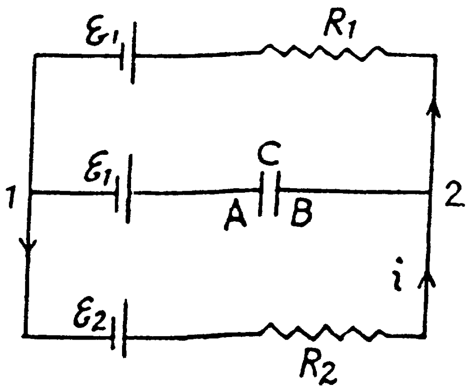
So, $R = R_2 - R_1$, which is the required resistance.

3.176 (a) Current, $i = \dfrac{N\xi}{NR} = \dfrac{N\alpha R}{NR} = \alpha$, as $\xi = \alpha R$ (given)

(b) $\varphi_A - \varphi_B = n\xi - niR = n\alpha R - n\alpha R = 0$

3.177 As the capacitor is fully charged, no current flows through it. So, current

$$i = \frac{\xi_2 - \xi_1}{R_1 + R_2} \text{ (as } \xi_2 > \xi_1)$$



And hence, $\varphi_A - \varphi_B = \xi_1 - \xi_2 + iR_2$

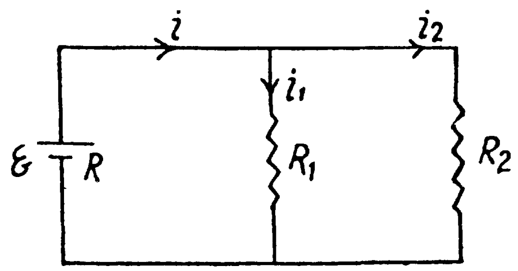
$$= \xi_1 - \xi_2 + \frac{\xi_2 - \xi_1}{R_1 + R_2} R_2$$

$$= \frac{(\xi_1 - \xi_2) R_1}{R_1 + R_2} = -0{\cdot}5 \text{ V}$$

3.178 Let us make the current distribution, as shown in the figure.

Current $i = \dfrac{\xi}{R + \dfrac{R_1 R_2}{R_1 + R_2}}$ (using loop rule)

So, current through the resistor R_1,



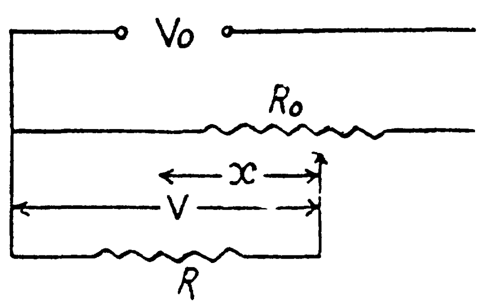
$$i_1 = \frac{\xi}{R + \dfrac{R_1 R_2}{R_1 + R_2}} \frac{R_2}{R_1 + R_2}$$

$$= \frac{\xi R_2}{R R_1 + R R_2 + R_1 R_2} = 1{\cdot}2 \text{ A}$$

and similary, current through the resistor R_2,

$$i_2 = \frac{\xi}{R + \dfrac{R_1 R_2}{R_1 + R_2}} \frac{R_1}{R_1 + R_2} = \frac{\xi R_1}{R R_1 + R_1 R_2 + R R_2} = 0{\cdot}8 \text{ A}$$

3.179 Total resistance $= \dfrac{l-x}{l} R_0 + \dfrac{R \cdot \dfrac{x R_0}{l}}{R + \dfrac{x R_0}{l}}$



$$= \frac{l-x}{l} R_0 + \frac{x R R_0}{lR + xR_0}$$

$$= R_0 \left[\frac{l-x}{l} + \frac{xR}{lR + xR_0} \right]$$

Then $V = V_0 \dfrac{xR}{lR + xR_0} \Big/ \left(1 - \dfrac{x}{l} + \dfrac{xR}{xR_0 + lR}\right) = V_0 R x \Big/ \left\{lR + R_0 x\left(1 - \dfrac{x}{l}\right)\right\}$

For $R >> R_0,\ V \approx V_0 \dfrac{x}{l}$

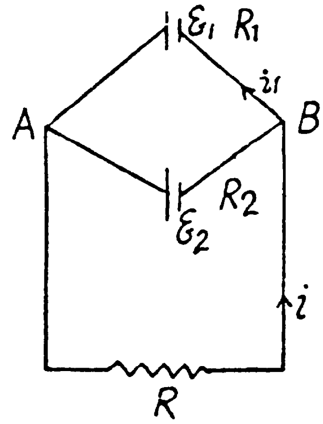
3.180 Let us connect a load of resistance R between the points A and B (Fig.)

From the loop rule, $\Delta\varphi = 0$, we obtain

$iR = \xi_1 - i_1 R_1$ (1)

and $iR = \xi_2 - (i - i_1) R_2$

or $i(R + R_2) = \xi + i_1 R_2$ (2)



Solving Eqs. (1) and (2), we get

$$i = \frac{\xi_1 R_1 + \xi_2 R_2}{R_1 + R_2} \Big/ R + \frac{R_1 R_2}{R_1 + R_2} = \frac{\xi_0}{R + R_0} \quad (3)$$

where $\xi_0 = \dfrac{\xi_1 R_1 + \xi_2 R_2}{R_1 + R_2}$ and $R_0 = \dfrac{R_1 R_2}{R_1 + R_2}$

Thus one can replace the given arrangement of the cells by a single cell having the emf ξ_0 and internal resistance R_0.

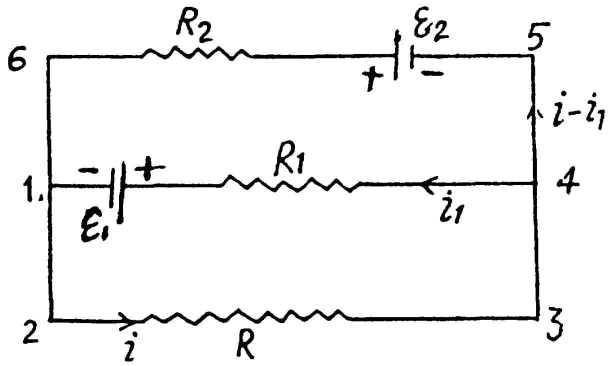
3.181 Make the current distribution, as shown in the diagram.

Now, in the loop 12341, applying $-\Delta\varphi = 0$

$$iR + i_1 R_1 + \xi_1 = 0 \quad (1)$$

and in the loop 23562,

$$iR - \xi_2 + (i - i_1) R_2 = 0 \quad (2)$$



Solving (1) and (2), we obtain current through the resistance R,

$$i = \frac{(\xi_2 R_1 - \xi_1 R_2)}{R R_1 + R R_2 + R_1 R_2} = 0{\cdot}02\ \text{A}$$

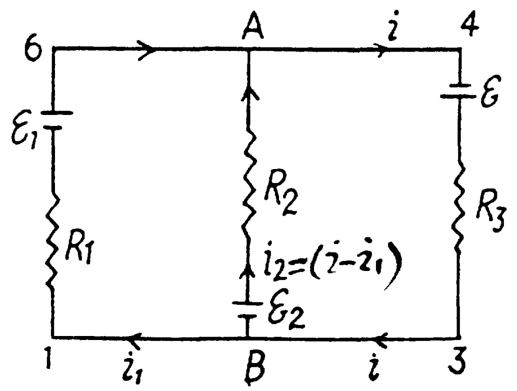
and it is directed from left to the right

3.182 At first indicate the currents in the branches using charge conservation (which also includes the point rule).

In the loops 1 BA 61 and B34AB from the loop rule, $-\Delta\varphi = 0$, we get, respectively

$$-\xi_2 + (i - i_1)R_2 + \xi_1 - i_1 R_1 = 0 \quad (1)$$

$$iR_3 + \xi_3 - (i - i_1) R_2 + \xi_2 = 0 \quad (2)$$



On solving Eqs (1) and (2), we obtain

$$i_1 = \frac{(\xi_1 - \xi_2) R_3 + R_2 (\xi_1 + \xi_3)}{R_1 R_2 + R_2 R_3 + R_3 R_1} \approx 0{\cdot}06\ \text{A}$$

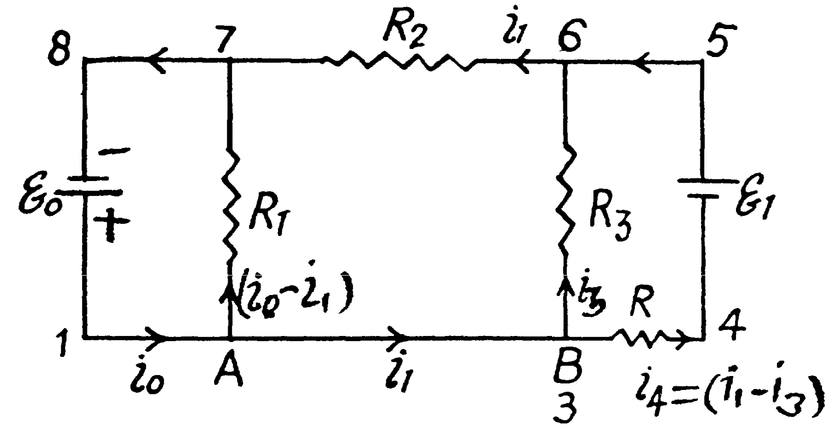
Thus $\varphi_A - \varphi_B = \xi_2 - i_2 R_2 \approx 0{\cdot}9\ \text{V}$

3.183 Indicate the currents in all the branches using charge conservation as shown in the figure. Applying loop rule, $-\Delta\varphi = 0$ in the loops 1A781, 1B681 and B456B, respectively, we get

$$\xi_0 = (i_0 - i_1)\,R_1 \qquad (1)$$

$$i_3 R_3 + i_1 R_2 - \xi_0 = 0 \qquad (2) \text{ and}$$

$$(i_1 - i_3)\,R - \xi - i_3 R_3 = 0 \qquad (3)$$



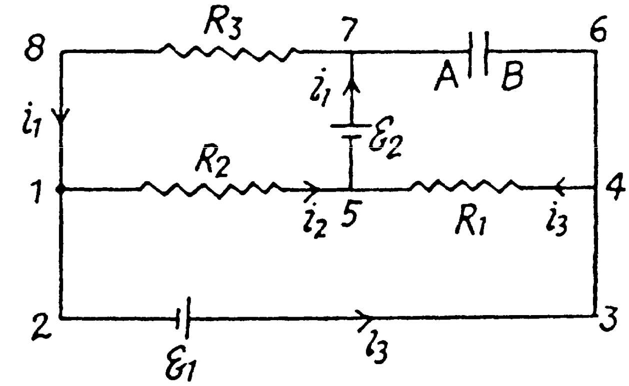
Solving Eqs. (1), (2) and (3), we get the sought current

$$(i_1 - i_3) = \frac{\xi\,(R_2 + R_3) + \xi_0 R_3}{R\,(R_2 + R_3) + R_2 R_3}$$

3.184 Indicate the currents in all the branches using charge conservation as shown in the figure. Applying the loop rule $(-\Delta\varphi = 0)$ in the loops 12341 and 15781, we get

$$-\xi_1 + i_3 R_1 - (i_1 - i_3)\,R_2 = 0 \qquad (1)$$

$$\text{and } (i_1 - i_3)\,R_2 - \xi_2 + i_1 R_3 = 0 \qquad (2)$$



Solving Eqs. (1) and (2), we get

$$i_3 = \frac{\xi_1\,(R_2 + R_3) + \xi_2 R_2}{R_1 R_2 + R_2 R_3 + R_3 R_1}$$

Hence, the sought p.d.

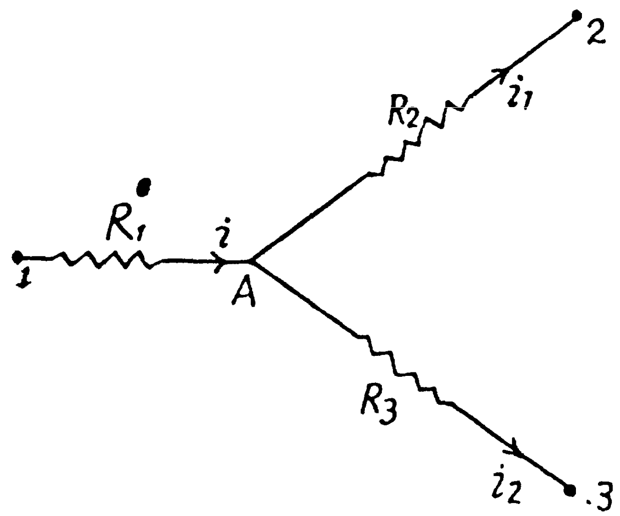
$$\varphi_A - \varphi_B = \xi_2 - i_3 R_1$$

$$= \frac{\xi_2 R_3 (R_1 + R_2) - \xi_1 R_1 (R_2 + R_3)}{R_1 R_2 + R_2 R_3 + R_3 R_1} = -1\text{ V}$$

3.185 Let us distribute the currents in the paths as shown in the figure.

Now, $\varphi_1 - \varphi_2 = i R_1 + i_1 R_2 \qquad (1)$

and $\varphi_1 - \varphi_3 = i R_1 + (i - i_1)\,R_3 \qquad (2)$



Simplifying Eqs. (1) and (2) we get

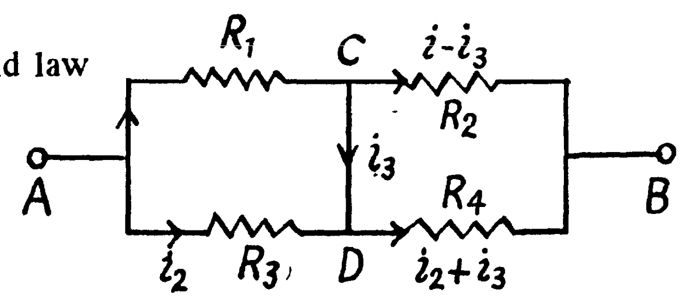
$$i = \frac{R_3\,(\varphi_1 - \varphi_2) + R_2\,(\varphi_1 - \varphi_3)}{R_1 R_2 + R_2 R_3 + R_3 R_1} = 0{\cdot}2\text{ A}$$

3.186 Current is as shown. From Kirchhoff's Second law

$$i_1 R_1 = i_2 R_3\,,$$

$$i_1 R_1 + (i_1 - i_3)\,R_2 = V\,,$$

$$i_2 R_3 + (i_3 + i_2)\,R_4 = V$$



Eliminating i_2

$$i_1(R_1+R_2) - i_3 R_2 = V$$

$$i_1 \frac{R_1}{R_3}(R_3+R_4) + i_3 R_4 = V$$

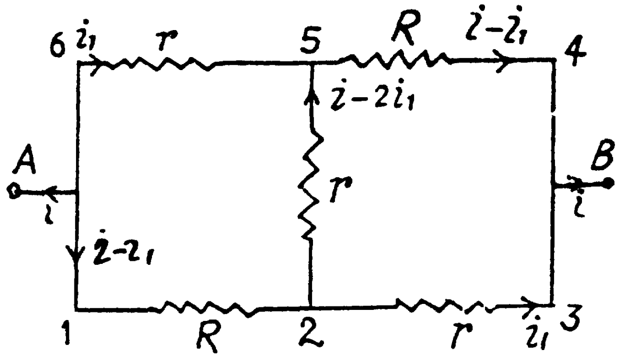
Hence
$$i_3\left[R_4(R_1+R_2) + \frac{R_1 R_2}{R_3}(R_3+R_4)\right]$$

$$= V\left[(R_1+R_2) - \frac{R_1}{R_3}(R_3+R_4)\right]$$

or,
$$i_3 = \frac{R_3(R_1+R_2) - R_1(R_3+R_4)}{R_3 R_4(R_1+R_2) + R_1 R_2(R_3+R_4)}$$

On substitution we get $i_3 = 1{\cdot}0$ A from C to D

3.187 From the symmetry of the problem, current flow is indicated, as shown in the figure.



Now, $\varphi_A - \varphi_B = i_1 r + (i - i_1) R$ (1)

In the loop 12561, from $-\Delta\varphi = 0$

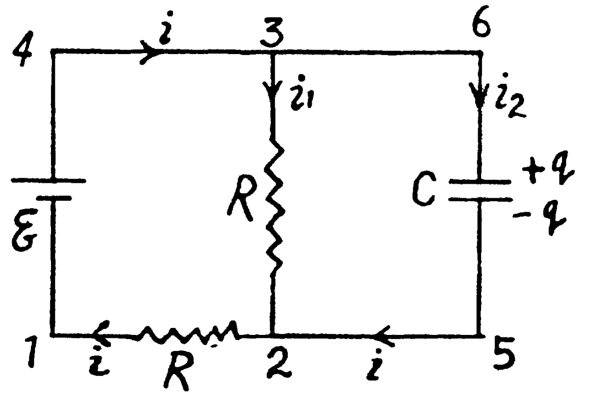
$$(i - i_1)R + (i - 2i_1) r - i_1 r = 0$$

or, $$i_1 = \frac{(R+r)}{3r+R} i \quad (2)$$

Equivalent resistance between the terminals A and B using (1) and (2),

$$R_0 = \frac{\varphi_A - \varphi_B}{i} = \frac{\left[\frac{R+r}{3r+R}(r-R)+R\right] i}{i} = \frac{r(3R+r)}{3r+R}$$

3.188 Let, at any moment of time, charge on the plates be $+q$ and $-q$ respectively, then voltage across the capacitor, $\varphi = q/C$ (1)



Now, from charge conservation,

$$i = i_1 + i_2, \text{ where } i_2 = \frac{dq}{dt} \quad (2)$$

In the loop 65146, using $-\Delta\varphi = 0$.

$$\frac{q}{C} + \left(i_1 + \frac{dq}{dt}\right) R - \xi = 0 \quad (3)$$

[using (1) and (2)]

In the loop 25632, using $-\Delta\varphi = 0$

$$-\frac{q}{C} + i_1 R = 0 \quad \text{or,} \quad i_1 R = \frac{q}{C} \quad (4)$$

From (1) and (2),

$$\frac{dq}{dt} R = \xi_1 - \frac{2q}{C} \quad \text{or,} \quad \frac{dq}{\xi - \frac{2q}{C}} = \frac{dt}{R} \tag{5}$$

On integrating the expression (5) between suitable limits,

$$\int_0^q \frac{dq}{\xi - \frac{2q}{C}} = \frac{1}{R}\int_0^t dt \quad \text{or,} \quad -\frac{C}{2}\ln\frac{\xi - \frac{2q}{C}}{\xi} = \frac{t}{R}$$

Thus $$\frac{q}{C} = V = \frac{1}{2}\xi\left(1 - e^{-2t/RC}\right)$$

3.189 (a) As current i is linear function of time, and at $t = 0$ and Δt, it equals i_0 and zero respectively, it may be represented as,

$$i = i_0\left(1 - \frac{t}{\Delta t}\right)$$

Thus $$q = \int_0^{\Delta t} i\,dt = \int_0^{\Delta t} i_0\left(1 - \frac{t}{\Delta t}\right)dt = \frac{i_0\,\Delta t}{2}$$

So, $$i_0 = \frac{2q}{\Delta t}$$

Hence, $$i = \frac{2q}{\Delta t}\left(1 - \frac{t}{\Delta t}\right)$$

The heat generated.

$$H = \int_0^{\Delta t} i^2 R\,dt = \int_0^{\Delta t}\left[\frac{2q}{\Delta t}\left(1 - \frac{t}{\Delta t}\right)\right]^2 R\,dt = \frac{4q^2 R}{3\,\Delta t}$$

(b) Obviously the current through the coil is given by

$$i = i_0\left(\frac{1}{2}\right)^{t/\Delta t}$$

Then charge $$q = \int_0^{\infty} i\,dt = \int_0^{\infty} i_0\,2^{-t/\Delta t}\,dt = \frac{i_0\,\Delta t}{\ln 2}$$

So, $$i_0 = \frac{q\ln 2}{\Delta t}$$

And hence, heat generated in the circuit in the time interval $t\,[0, \infty]$,

$$H = \int_0^{\infty} i^2 R\,dt = \int_0^{\infty}\left[\frac{q\ln 2}{\Delta t}2^{-t/\Delta t}\right]^2 R\,dt = -\frac{q^2\ln 2}{2\,\Delta t}R$$

3.190 The equivalent circuit may be drawn as in the figure.

Resistance of the network = $R_0 + (R/3)$

Let, us assume that e.m.f. of the cell is ξ, then current

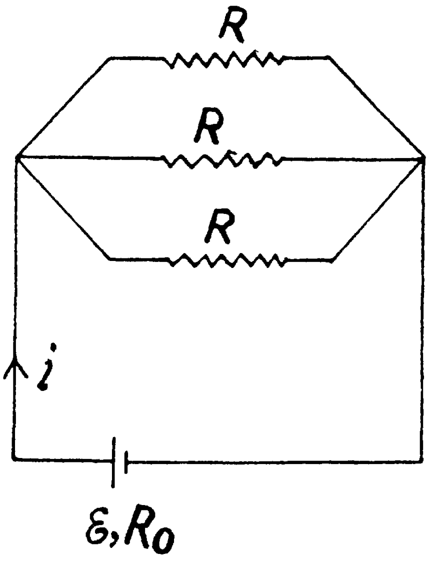
$$i = \frac{\xi}{R_0 + (R/3)}$$

Now, thermal power, generated in the circuit

$$P = i^2 R/3 = \frac{\xi^2}{(R_0 + (R/3))^2}(R/3)$$

For P to be maximum, $\frac{dP}{dR} = 0$, which yields

$$R = 3R_0$$

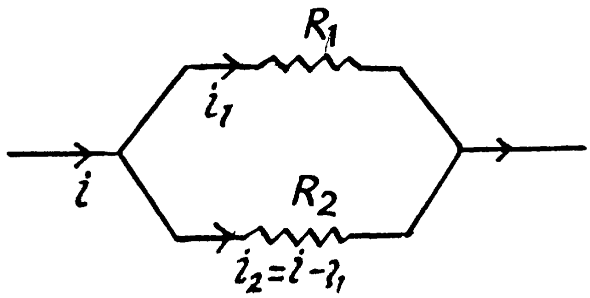


3.191 We assume current conservation but not Kirchhoff's second law. Then thermal power dissipated is

$$P(i_1) = i_1^2 R_1 + (i - i_1)^2 R_2$$

$$= i_1^2 (R_1 + R_2) - 2ii_1R_2 + i^2R_2$$

$$= [R_1 + R_2]\left[i_1 - \frac{R_2}{R_1+R_2}i\right]^2 + i^2\frac{R_1R_2}{R_1+R_2}$$



The resistances being positive we see that the power dissipated is minimum when

$$i_1 = i\frac{R_2}{R_1+R_2}$$

This corresponds to usual distribution of currents over resistance joined is parallel.

3.192 Let, internal resistance of the cell be r, then

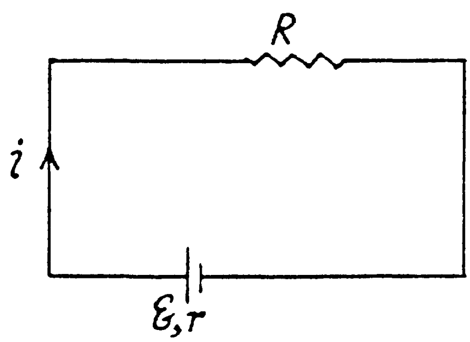
$$V = \xi - ir \quad (1)$$

where i is the current in the circuit. We know that thermal power generated in the battery.

$$Q = i^2 r \quad (2)$$

Putting r from (1) in (2), we obtain,

$$Q = (\xi - V)\, i = 0{\cdot}6 \text{ W}$$

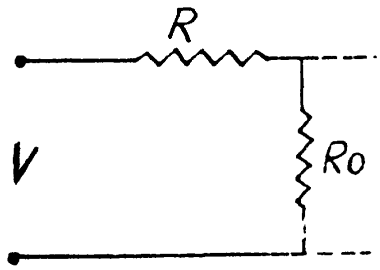


In a battery work is done by electric forces (whose origin lies in the chemical processes going on inside the cell). The work so done is stored and used in the electric circuit outside. Its magnitude just equals the power used in the electric circuit. We can say that net power developed by the electric forces is

$$P = -IV = -2 \cdot 0\text{W}$$

Minus sign means that this is generated not consumed.

3.193 As far as motor is concerned the power delivered is dissipated and can be represented by a load, R_0. Thus



$$I = \frac{V}{R + R_0}$$

and $$P = I^2 R_0 = \frac{V^2 R_0}{(R_0 + R)^2}$$

This is maximum when $R_0 = R$ and the current I is then

$$I = \frac{V}{2R}$$

The maximum power delivered is

$$\frac{V^2}{4R} = P_{max}$$

The power input is $\frac{V^2}{R + R_0}$ and its value when P is maximum is $\frac{V^2}{2R}$

The efficiency then is $\frac{1}{2} = 50\%$

3.194 If the wire diameter decreases by δ then by the information given

$P =$ Power input $= \frac{V^2}{R} =$ heat lost through the surface, H.

Now, $H \propto (1 - \delta)$ like the surface area and

$$R \propto (1 - \delta)^{-2}$$

So, $$\frac{V^2}{R_0}(1 - \delta)^2 = A\,(1 - \delta) \quad \text{or,} \quad V^2 (1 - \delta) = \text{constant}$$

But $$V \propto 1 + \eta \quad \text{so} \quad (1 + \eta)^2 (1 - \delta) = \text{Const} = 1$$

Thus $$\delta = 2\eta = 2\%$$

3.195 The equation of heat balance is

$$\frac{V^2}{R} - k\,(T - T_0) = C\,\frac{dT}{dt}$$

Put $$T - T_0 = \xi$$

So, $$C\dot{\xi} + k\,\xi = \frac{V^2}{R} \quad \text{or,} \quad \dot{\xi} + \frac{k}{C}\xi = \frac{V^2}{CR}$$

or, $$\frac{d}{dt}(\xi e^{kt/c}) = \frac{V^2}{CR} e^{kt/c}$$

or, $$\xi\, e^{kt/c} = \frac{V^2}{kR} e^{kt/c} + A$$

where A is a constant. Clearly

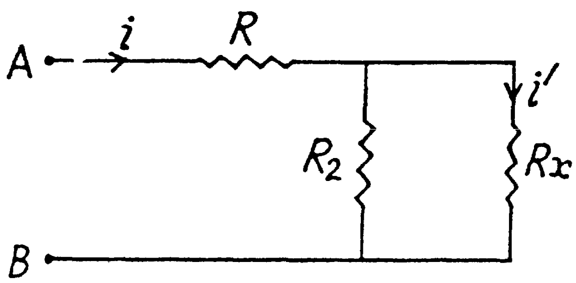
$$\xi = 0 \text{ at } t = 0, \text{ so } A = -\frac{V^2}{kR} \text{ and hence,}$$

$$T = T_0 + \frac{V^2}{kR}(1 - e^{-kt/C})$$

3.196 Let, $\varphi_A - \varphi_B = \varphi$

Now, thermal power generated in the resistance R_x,

$$P = i'^2 R_x = \left[\frac{\varphi}{R_1 + \dfrac{R_2 R_x}{R_2 + R_x}} \frac{R_2}{R_2 + R_x}\right]^2 R_x,$$



For P to be independent of R_x,

$$\frac{dP}{dR_x} = 0, \text{ which yeilds}$$

$$R_x = \frac{R_1 R_2}{R_1 + R_2} = 12\ \Omega$$

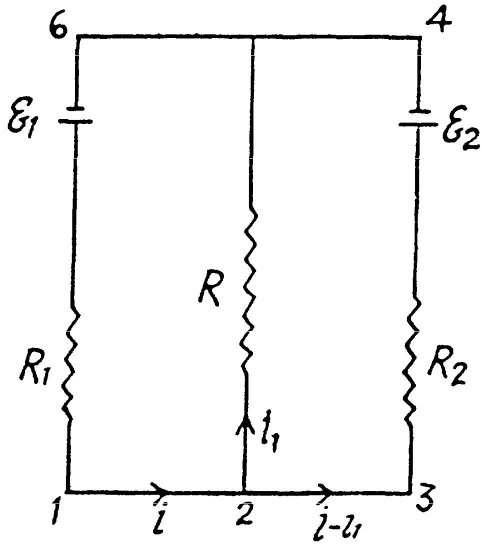
3.197 Indicate the currents in the circuit as shown in the figure.

Appying loop rule in the closed loop 12561, $-\Delta\varphi = 0$ we get

$$i_1 R - \xi_1 + i R_1 = 0 \qquad (1)$$

and in the loop 23452,

$$(i - i_1) R_2 + \xi_2 - i_1 R = 0 \qquad (2)$$



Solving (1) and (2), we get,

$$i_1 = \frac{\xi_1 R_2 + \xi_2 R_1}{R R_1 + R_1 R_2 + R R_2}$$

So, thermal power, generated in the resistance R,

$$P = i_1^2 R = \left[\frac{\xi_1 R_2 + \xi_2 R_1}{R R_1 + R_1 R_2 + R R_2}\right]^2 R$$

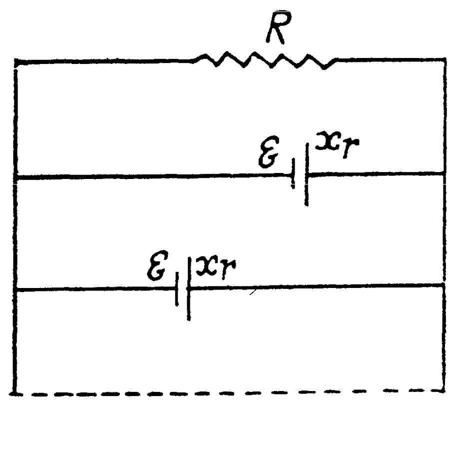
For P to be maximum, $\dfrac{dP}{dR} = 0$, which fields

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

Hence,
$$P_{\max} = \frac{(\xi_1 R_2 + \xi_2 R_1)^2}{4 R_1 R_2 (R_1 + R_2)}$$

3.198 Let, there are x number of cells, connected in series in each of the n parallel groups

then, $n x = N$ or, $x = \dfrac{N}{n}$ (1)



Now, for any one of the loop, consisting of x cells and the resistor R, from loop rule

$$i R + \frac{i}{n} x r - x \xi = 0$$

So, $i = \dfrac{x\xi}{R + \dfrac{xr}{n}} = \dfrac{\dfrac{N}{n}\xi}{R + \dfrac{Nr}{n^2}}$, using (1)

Heat generated in the resistor R,

$$Q = i^2 R = \left(\frac{N n \xi}{n^2 R + N R}\right)^2 R \qquad (2)$$

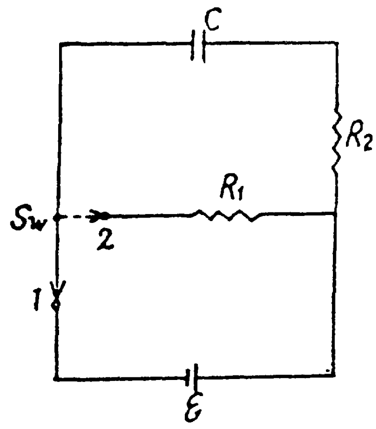
and for Q to be maximum, $\frac{dQ}{dn} = 0$, which yields

$$n = \sqrt{\frac{Nr}{R}} = 3$$

3.199 When switch 1 is closed, maximum charge accumulated on the capacitor,

$$q_{max} = C\xi, \qquad (1)$$

and when switch 2 is closed, at any arbitrary instant of time,



$$(R_1 + R_2)\left(-\frac{dq}{dt}\right) = q/C,$$

because capacitor is discharging.

$$\text{or,} \int_{q_{max}}^{q} \frac{1}{q} dq = -\frac{1}{(R_1 + R_2)C} \int_0^t dt$$

Integrating, we get

$$\ln \frac{q}{q_{max}} = \frac{-t}{(R_1 + R_2)C} \quad \text{or,} \quad q = q_{max}\, e^{\frac{-t}{(R_1 + R_2)C}} \qquad (2)$$

Differentiating with respect to time,

$$i(t) = \frac{dq}{dt} = q_{max}\, e^{\frac{-t}{(R_1 + R_2)C}} \left(-\frac{1}{(R_1 + R_2)C}\right)$$

$$\text{or,} \qquad i(t) = \frac{C\xi}{(R_1 + R_2)C}\, e^{\frac{-t}{(R_1 + R_2)C}}$$

Negative sign is ignored, as we are not interested in the direction of the current.

$$\text{thus,} \qquad i(t) = \frac{\xi}{(R_1 + R_2)}\, e^{\frac{-1}{(R_1 + R_2)C}} \qquad (3)$$

When the switch (Sw) is at the position 1, the charge (maximum) accumalated on the capacitor is,

$$q = C\xi$$

When the Sw is thrown to position 2, the capacitor starts discharging and as a result the electric energy stored in the capacitor totally turns into heat energy tho' the resistors R_1 and R_2 (during a very long interval of time). Thus from the energy conservation, the total heat liberated tho' the resistors.

$$H = U_i = \frac{q^2}{2C} = \frac{1}{2} C\xi^2$$

During the process of discharging of the capacitor, the current tho' the resistors R_1 and R_2 is the same at all the moments of time, thus

$$H_1 \propto R_1 \text{ and } H_2 \propto R_2$$

So, $$H_1 = \frac{H\,R_1}{(R_1 + R_2)} \quad (\text{as } H = H_1 + H_2)$$

Hence $$H_1 = \frac{1}{2}\frac{CR_1}{R_1 + R_2}\,\xi^2$$

3.200 When the plate is absent the capacity of the condenser is

$$C = \frac{\varepsilon_0 S}{d}$$

When it is present, the capacity is

$$C' = \frac{\varepsilon_0 S}{d\,(1-\eta)} = \frac{C}{1-\eta}$$

(a) The energy increment is clearly.

$$\Delta U = \frac{1}{2}CV^2 - \frac{1}{2}C'V^2 = \frac{C\eta}{2\,(1-\eta)}\,V^2$$

(b) The charge on the plate is

$$q_i = \frac{CV}{1-\eta} \text{ initially and } q_f = CV \text{ finally.}$$

A charge $\frac{CV\eta}{1-\eta}$ has flown through the battery charging it and withdrawing $\frac{CV^2\eta}{1-\eta}$ units of energy from the system into the battery. The energy of the capacitor has decreased by just half of this. The remaining half i.e. $\frac{1}{2}\frac{CV^2\eta}{1-\eta}$ must be the work done by the external agent in withdrawing the plate. This ensures conservation of energy.

3.201 Initially, capacitance of the system $= C\,\varepsilon$.

So, initial energy of the system : $U_i = \frac{1}{2}(C\,\varepsilon)\,V^2$

and finally, energy of the capacitor : $U_f = \frac{1}{2}C\,V^2$

Hence capacitance energy increment,

$$\Delta U = \frac{1}{2}C\,V^2 - \frac{1}{2}(C\,\varepsilon)\,V^2 = -\frac{1}{2}\,CV^2\,(\varepsilon - 1) = -0{\cdot}5 \text{ mJ}$$

From energy conservation

$$\Delta\,U = A_{\text{cell}} + A_{\text{agent}}$$

(as there is no heat liberation)

But $A_{\text{cell}} = (C_f - C_i)\,V^2 = (C - C\varepsilon)V^2$

Hence $A_{\text{agent}} = \Delta U - A_{\text{cell}}$

$$= \frac{1}{2}C(1-\varepsilon)V^2 = 0{\cdot}5 \text{ m J}$$

3.202 If C_0 is the initial capacitance of the condenser before water rises in it then

$$U_i = \frac{1}{2}C_0 V^2, \quad \text{where } C_0 = \frac{\varepsilon_0\, 2l\pi R}{d}$$

(R is the mean radius and l is the length of the capacitor plates.)

Suppose the liquid rises to a height h in it. Then the capacitance of the condenser is

$$C = \frac{\varepsilon\varepsilon_0 h 2\pi R}{d} + \frac{\varepsilon(l-h)\, 2\pi R}{d} = \frac{\varepsilon_0 2\pi R}{d}(l + (\varepsilon - 1)h)$$

and energy of the capacitor and the liquid (including both gravitational and electrosatic contributions) is

$$\frac{1}{2}\frac{\varepsilon_0\, 2\pi R}{d}(l + (\varepsilon-1)h)V^2 + \rho g\,(2\pi R\, hd)\,\frac{h}{2}$$

If the capacitor were not connected to a battery this energy would have to be minimized. But the capacitor is connected to the battery and, in effect, the potential energy of the whole system has to be minimized. Suppose we increase h by δh. Then the energy of the capacitor and the liquid increases by

$$\delta h\left(\frac{\varepsilon_0\, 2\pi R}{2d}(\varepsilon-1)V^2 + \rho g\,(2\pi R d)\,h\right)$$

and that of the cell diminishes by the quantity A_{cell} which is the product of charge flown and V

$$\delta h\,\frac{\varepsilon_0\,(2\pi R)}{d}(\varepsilon-1)V^2$$

In equilibrium, the two must balance; so

$$\rho g d h = \frac{\varepsilon_0(\varepsilon-1)V^2}{2d}$$

Hence
$$h = \frac{\varepsilon_0(\varepsilon-1)V^2}{2\rho g d^2}$$

3.203 (a) Let us mentally islolate a thin spherical layer with inner and outer radii r and $r + dr$ respectively. Lines of current at all the points of this layer are perpendicular to it and therefore such a layer can be treated as a spherical conductor of thickness dr and cross sectional area $4\pi r^2$. Now, we know that resistance,

$$dR = \rho\frac{dr}{S(r)} = \rho\frac{dr}{4\pi r^2} \tag{1}$$

Integrating expression (1) between the limits,

$$\int_0^R dR = \int_a^b \rho \frac{dr}{4\pi r^2} \quad \text{or,} \quad R = \frac{\rho}{4\pi}\left[\frac{1}{a} - \frac{1}{b}\right] \tag{2}$$

Capacitance of the network, $C = \dfrac{4\pi\varepsilon_0\varepsilon}{\left[\dfrac{1}{a} - \dfrac{1}{b}\right]}$ (3)

and $$q = C\varphi \left[\begin{array}{l}\text{where } q \text{ is the charge} \\ \text{at any arbitrary moment}\end{array}\right] \tag{4}$$

also, $$\varphi = \left(\frac{-dq}{dt}\right)R, \text{ as capacitor is discharging.} \tag{5}$$

From Eqs. (2), (3), (4) and (5) we get,

$$q = \frac{4\pi\varepsilon_0\varepsilon}{\left[\dfrac{1}{a} - \dfrac{1}{b}\right]} \frac{\left[-\dfrac{dq}{dt}\right]\rho\left[\dfrac{1}{a} - \dfrac{1}{b}\right]}{4\pi} \quad \text{or,} \quad \frac{dq}{q} = \frac{dt}{\rho\varepsilon\varepsilon_0}$$

Integrating $$\int_{q_0}^{q} -\frac{dq}{q} = \frac{1}{\rho\varepsilon_0\varepsilon}\int_0^t dt \quad = \quad \frac{dt}{\rho\,\varepsilon\varepsilon_0}$$

Hence $$q = q_0\, e^{\frac{-t}{\rho\,\varepsilon_0\varepsilon}}$$

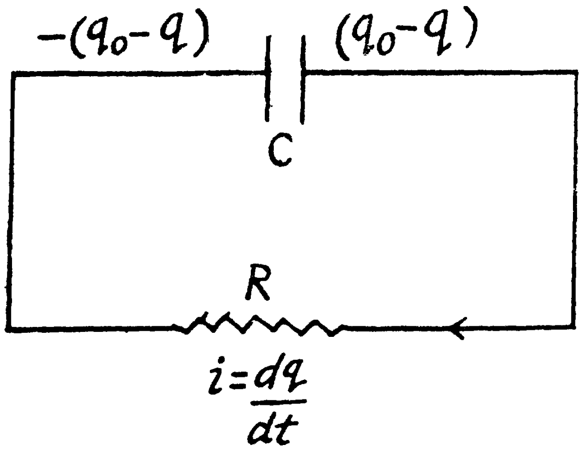
(b) From energy conservation heat generated, during the spreading of the charge,

$$H = U_i - U_f \text{ [because } A_{\text{cell}} = 0]$$

$$= \frac{1}{2}\frac{q_0^2}{4\pi\varepsilon_0\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right] - 0 = \frac{q_0^2}{8\pi\varepsilon_0\varepsilon}\frac{b-a}{ab}$$

3.204 (a) Let, at any moment of time, charge on the plates be $(q_0 - q)$ then current through the resistor, $i = -\dfrac{d(q_0 - q)}{dt}$, because the capacitor is discharging.

or, $$i = \frac{dq}{dt}$$



Now, applying loop rule in the circuit,

$$iR - \frac{q_0 - q}{C} = 0$$

or, $$\frac{dq}{dt}R = \frac{q_0 - q}{C}$$

or, $$\frac{dq}{q_0 - q} = \frac{1}{RC}dt$$

At $t = 0$, $q = 0$ and at $t = \tau$, $q = q$

So, $$\ln \frac{q_0 - q}{q_0} = \frac{-\tau}{RC}$$

Thus $$q = q_0\left(1 - e^{-\tau/RC}\right) = 0{\cdot}18\text{ mC}$$

(b) Amount of heat generated = decrement in capacitance energy

$$= \frac{1}{2}\frac{q_0^2}{C} - \frac{1}{2}\frac{\left[q_0 - q_0\left(1 - e^{-\tau/RC}\right)\right]^2}{C}$$

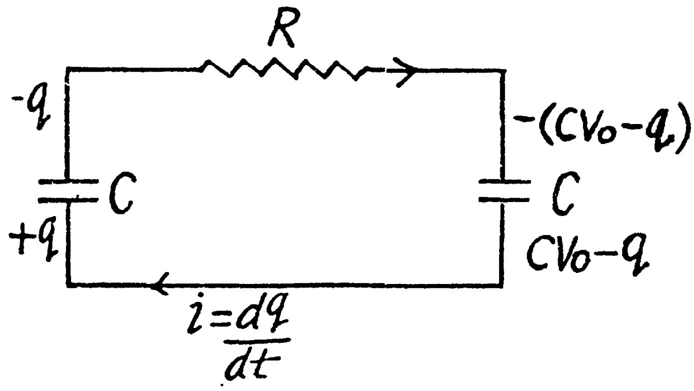
$$= \frac{1}{2}\frac{q_0^2}{C}\left[1 - e^{-\frac{2\tau}{RC}}\right] = 82\text{ mJ}$$

3.205 Let, at any moment of time, charge flown be q then current $i = \frac{dq}{dt}$

Applying loop rule in the circuit, $-\Delta\varphi = 0$, we get :

$$\frac{dq}{dt}IR - \frac{(C V_0 - q)}{C} + \frac{q}{C} = 0$$

or, $$\frac{dq}{C V_0 - 2q} = \frac{1}{RC}\,dt$$



So, $\ln \frac{C V_0 - 2q}{CV_0} = -2\frac{t}{RC}$ for $0 \le t \le t$

or, $$q = \frac{C V_0}{2}\left(1 - e^{\frac{-2t}{RC}}\right)$$

Hence, $$i = \frac{dq}{dt} = \frac{C V_0}{2}\frac{2}{RC}\,e^{-2t/RC} = \frac{V_0}{R}e^{-2t/RC}$$

Now, heat liberaled,

$$Q = \int_0^\infty i^2 R\,dt = \frac{V_0^2}{R^2}R\int_0^\infty e^{\frac{-4t}{RC}}\,dt = \frac{1}{4}C V_0^2$$

3.206 In a rotating frame, to first order in ω, the main effect is a coriolis force $2m\vec{v}' \times \vec{\omega}$.

This unbalanced force will cause electrons to react by setting up a magnetic field $\vec{B}$ so that the magnetic force $e\vec{v} \times \vec{B}$ balances the coriolis force.

Thus $$-\frac{e}{2m}\vec{B} = \vec{\omega} \quad \text{or,} \quad \vec{B} = -\frac{2m}{e}\vec{\omega}$$

The flux associated with this is

$$\Phi = N\pi r^2 B = N\pi r^2\frac{2m}{e}\omega$$

where $N = \dfrac{l}{2\pi r}$ is the number of turns of the ring. If ω changes (and there is time for the electron to rearrange) then B also changes and so does Φ. An emf will be induced and a current will flow. This is

$$I = N \pi r^2 \frac{2m}{e} \omega / R$$

The total charge flowing through the ballastic galvanometer, as the ring is stopped, is

$$q = N \pi r^2 \Big/ \frac{2m}{e} \omega / R$$

So,
$$\frac{e}{m} = \frac{2 N \pi r^2 \omega}{qR} = \frac{l \omega r}{qR}$$

3.207 Let, n_0 be the total number of electorns then, total momentum of electorns,

$$p = n_0 m_e v_d \tag{1}$$

Now,
$$I = \rho S_x v_d = \frac{n_0 e}{V} S_x v_d = \frac{ne}{l} v_d \tag{2}$$

Here S_x = Cross sectional area, ρ = electron charge density, V = volume of sample
From (1) and (2)

$$p = \frac{m_e}{e} Il = 0{\cdot}40 \ \mu \text{ Ns}$$

3.208 By definition
$n e v_d = j$ (where v_d is the drift velocity, n is number density of electrons.)

Then
$$\tau = \frac{l}{v_d} = \frac{nel}{j}$$

So distance actually travelled

$$S = \langle v \rangle \tau = \frac{nel \langle v \rangle}{j}$$

($\langle v \rangle$ = mean velocity of thermal motion of an electron)

3.209 Let, n be the volume density of electrons, then from $I = \rho S_x v_d$,

$$I = n e S_x \, |\langle \vec{v} \rangle| = n e S_x \frac{l}{t}$$

So,
$$t = \frac{n e S_x l}{I} = 3 \ \mu\text{s}.$$

(b) Sum of electric forces

$$= |(nv) e \vec{E}| = |n S l e \rho \vec{j}|, \text{ where } \rho \text{ is resistivity of the material.}$$

$$= n S l e \rho \frac{I}{S} = n e l \rho I = 1{\cdot}0 \ \mu\text{N}$$

3.210 From Gauss theorem field strength at a surface of a cylindrical shape equals, $\frac{\lambda}{2\pi\varepsilon_0 r}$, where λ is the linear charge density.

Now, $$eV = \frac{1}{2} m_e v^2 \quad \text{or,} \quad v = \sqrt{\frac{2eV}{m_e}} \tag{1}$$

Also, $$dq = \lambda\, dx \quad \text{so,} \quad \frac{dq}{dt} = \lambda \frac{dx}{dt}$$

or, $$I = \lambda v \quad \text{or,} \quad \lambda = \frac{I}{v} = \frac{I}{\sqrt{\frac{2eV}{m_e}}}, \text{ using (1)}$$

Hence $$E = \frac{I}{2\pi\varepsilon_0 r}\sqrt{\frac{m_e}{2eV}} = 32\ \text{V/m}$$

(b) For the point, inside the solid charged cylinder, applying Gauss' theorem,

$$2\pi r h E = \pi r^2 h \frac{q}{\varepsilon_0 \pi R^2 l}$$

or, $$E = \frac{q/l}{2\pi\varepsilon_0 R^2} r = \frac{\lambda r}{2\pi\varepsilon_0 R^2}$$

So,from $$E = -\frac{d\varphi}{dr},$$

$$\int_{\varphi_1}^{\varphi_2} -d\varphi = \int_0^R \frac{\lambda}{2\pi\varepsilon_0 R^2} r\, dr$$

or, $$\varphi_1 - \varphi_2 = \frac{\lambda}{2\pi\varepsilon_0 R^2}\left[\frac{r^2}{2}\right]_0^R = \frac{\lambda}{4\pi\varepsilon_0}$$

Hence, $$\varphi_1 - \varphi_2 = \frac{VI}{4\pi\varepsilon_0}\sqrt{\frac{m_e}{2eV}} = 0{\cdot}80\ \text{V}$$

3.211 Between the plates $\varphi = a x^{4/3}$

or, $$\frac{\partial\varphi}{\partial x} = a \times \frac{4}{3} x^{1/3}$$

$$\frac{d^2\varphi}{\partial x^2} = \frac{4}{9} a x^{-2/3} = -\rho/\varepsilon_0$$

or, $$\rho = -\frac{4\varepsilon_0 a}{9} x^{-2/3}$$

Let the charge on the electron be $-e$,

then $$\frac{1}{2} m v^2 - e\varphi = \text{Const.} = 0,$$

as the electron is initially emitted with neligible energy.

$$v^2 = \frac{2e\varphi}{m}, \quad v = \sqrt{\frac{2e\varphi}{m}}$$

So, $$j = -\rho v = \frac{4\varepsilon_0 a}{9}\sqrt{\frac{2\varphi}{m}}\, x^{-2/3}.$$

(j is measued from the anode to cathode, so the - ve sign.)

3.212 $E = \frac{V}{d}$

So by the definition of the mobility

$$v^+ = u^+_0 \frac{V}{d}, \quad v^- = u^-_0 - \frac{V}{d}$$

and $$j = (n_+ u_0^+ + n_- u_0^-)\frac{eV}{d}$$

(The negative ions move towards the anode and the positive ion towards the cathode and the total current is the sum of the currents due to them.)

On the other hand, in equilibrium $n_+ = n_-$

So, $$n_+ = n_- = \frac{I}{S} \Big/ (u_0^+ + u_0^-)\frac{eV}{d}$$

$$= \frac{I\,d}{e\,V S\,(u_0^+ + u_0^-)} = 2{\cdot}3 \times 10^8 \text{ cm}^{-3}$$

3.213 Velocity = mobility × field

or, $v = u\frac{V_0}{l}\sin\omega t$, which is positive for $0 \le \omega t \le \pi$

So, maximum displacement in one direction is

$$x_{max} = \int_0^{\pi} u\frac{V_0}{l}\sin\omega t\, dt = \frac{2uV_0}{l\omega}$$

At $\omega = \omega_0$, $x_{max} = l$, so, $\frac{2uV_0}{l\omega} = l$

Thus $$u = \frac{\omega l^2}{2V_0}$$

3.214 When the current is saturated, all the ions, produced, reach the plate.

Then, $$\dot{n}_i = \frac{I_{sat}}{eV} = 6 \times 10^9 \text{ cm}^{-3}\text{ s}^{-1}$$

(Both positive ions and negative ions are counted here)

The equation of balance is, $\frac{dn}{dt} = \dot{n}_i - rn^2$

The first term on the right is the production rate and the second term is the recombination rate which by the usual statistical arguments is proportional to n^2 (= no of positive ions × no. of $-ve$ ion). In equilibrium,

$$\frac{dn}{dt} = 0$$

so,
$$n_{eq} = \sqrt{\frac{\dot{n}_i}{r}} = 6 \times 10^7 \text{ cm}^{-3}$$

3.215 Initially $n = n_0 = \sqrt{\dot{n}_i / r}$

Since we can assume that the long exposure to the ionizer has caused equilibrium to be set up. Afer the ionizer is switched off,

$$\frac{dn}{dt} = -rn^2$$

or
$$r\,dt = -\frac{dn}{n^2}, \quad \text{or,} \quad rt = \frac{1}{n} + \text{constant}$$

But
$$n = n_0 \text{ at } t = 0, \text{ so, } rt = \frac{1}{n} - \frac{1}{n_0}$$

The concentration will decrease by a factor η when

$$r\,t_0 = \frac{1}{n_0/\eta} - \frac{1}{n_0} = \frac{\eta - 1}{n_0}$$

or,
$$t_0 = \frac{\eta - 1}{\sqrt{r\,\dot{n}_i}} = 13 \text{ ms}$$

3.216 Ions produced will cause charge to decay. Clearly,

$$\eta\,CV = \text{decrease of charge} = \dot{n}_i\,e A\,dt = \frac{\varepsilon_0 A}{d} V \eta$$

or,
$$t = \frac{\varepsilon_0 V \eta}{\dot{n}_i\,e\,d^2} = 4{\cdot}6 \textit{ days}$$

Note, that n_i, here, is the number of ion pairs produced.

3.217 If ν = number of electrons moving to the anode at distance x, then

$$\frac{d\nu}{dx} = \alpha\,\nu \quad \text{or} \quad \nu = \nu_0\,e^{\alpha x}$$

Assuming saturation, $I = e\,\nu_0\,e^{\alpha d}$

3.218 Since the electrons are produced uniformly through the volume, the total current attaining saturation is clearly,

$$I = \int_0^d e\,(\dot{n}_i A\,dx)\,e^{\alpha x} = e\,\dot{n}_i A \left(\frac{e^{\alpha d} - 1}{\alpha}\right)$$

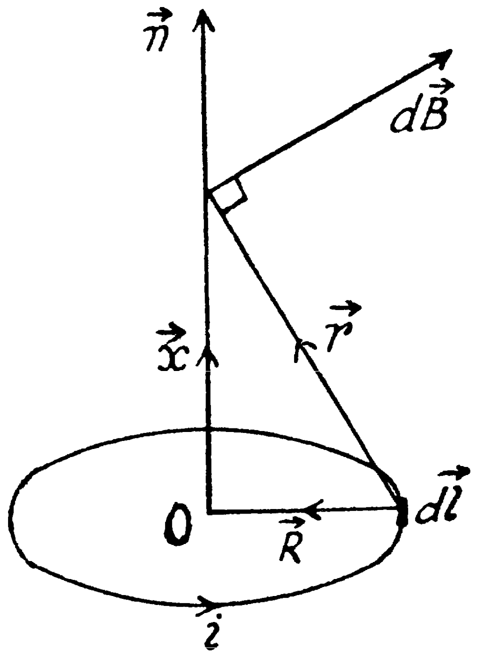
Thus,
$$j = \frac{I}{A} = e\,\dot{n}_i \left(\frac{e^{\alpha d} - 1}{\alpha}\right)$$

3.5 CONSTANT MAGNETIC FIELD. MAGNETICS

3.219 (a) From the Biot - Savart law,

$$d\vec{B} = \frac{\mu_0}{4\pi} i \frac{d\vec{l} \times \vec{r}}{r^3}, \text{ so}$$

$$dB = \frac{\mu_0}{4\pi} i \frac{(R\, d\theta)\, R}{R^3} \text{ (as } d\vec{l} \perp \vec{r}\text{)}$$



From the symmetry

$$B = \int dB = \int_0^{2\pi} \frac{\mu_0}{4\pi} \frac{i}{R} d\theta = \frac{\mu_0}{2} \frac{i}{R} = 6{\cdot}3\ \mu\text{T}$$

(b) From Biot-Savart's law :

$$\vec{B} = \frac{\mu_0}{4\pi} i \int \frac{d\vec{l} \times \vec{r}}{r^3} \text{ (here } \vec{r} = \vec{R} + \vec{x}\text{)}$$

So, $$\vec{B} = \frac{\mu_0}{4\pi} i \left[\oint d\vec{l} \times \vec{R} + \oint d\vec{l} \times \vec{x}\right]$$

Since $\vec{x}$ is a constant vector and $|\vec{R}|$ is also constant

So, $$\oint d\vec{l} \times \vec{x} = \left(\oint d\vec{l}\right) \times \vec{x} = 0 \left(\text{because } \oint d\vec{l} = 0\right)$$

and $$\oint d\vec{l} \times \vec{R} = \oint R\, dl\, \vec{n}$$

$$= \vec{n} R \oint dl = 2\pi R^2 \vec{n}$$

Here $\vec{n}$ is a unit vector perpendicular to the plane containing the current loop (Fig.) and in the direction of $\vec{x}$

Thus we get $$\vec{B} = \frac{\mu_0}{4\pi} \frac{2\pi R^2 i}{(x^2 + R^2)^{3/2}} \vec{n}$$

3.220 As $\angle AOB = \frac{2\pi}{n}$, OC or perpendicular distance of any segment from centre equals $R \cos \frac{\pi}{n}$. Now magnetic induction at O, due to the right current carrying element AB

$$= \frac{\mu_0}{4\pi} \frac{i}{R \cos \frac{\pi}{n}} 2 \sin \frac{\pi}{n}$$

(From Biot–Savart's law, the magnetic field at O due to any section such as AB is perpendicular to the plane of the figure and has the magnitude.)

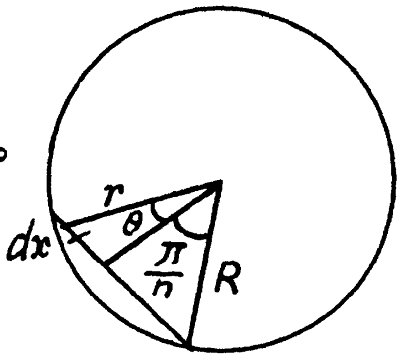
$$B = \int \frac{\mu_0}{4\pi} i \frac{dx}{r^2} \cos\theta = \int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} \frac{\mu_0 i}{4\pi} \frac{R\cos\frac{\pi}{n}\sec^2\theta d\theta}{R^2\cos\frac{2\pi}{n}\sec^2\theta} \cos\theta = \frac{\mu_0 i}{4\pi} \frac{1}{R\cos\frac{\pi}{n}} 2\sin\frac{\pi}{n}$$

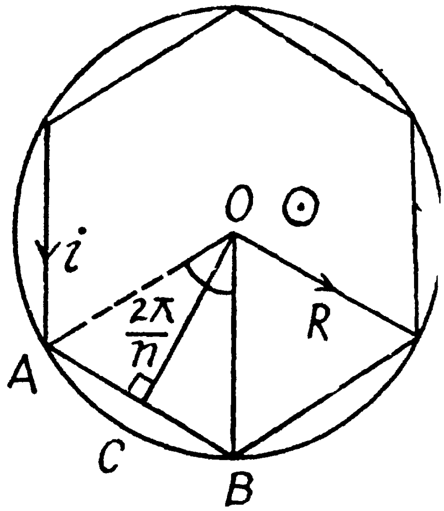
As there are n number of sides and magnetic induction vectors, due to each side at O, are equal in magnitude and direction. So,

$$B_0 = \frac{\mu_0}{4\pi} \frac{ni}{R\cos\frac{\pi}{n}} 2\sin\frac{\pi}{n} \cdot n$$

$$= \frac{\mu_0}{2\pi} \frac{ni}{R} \tan\frac{\pi}{n} \quad \text{and for } n \to \infty$$

$$B_0 = \frac{\mu_0}{2} \frac{i}{R} \underset{n\to\infty}{Lt} \left(\frac{\tan\frac{\pi}{n}}{\pi/n} \right) = \frac{\mu_0}{2} \frac{i}{R}$$





3.221 We know that magnetic induction due to a straight current carrying wire at any point, at a perpendicular distance from it is given by :

$$B = \frac{\mu_0}{4\pi} \frac{i}{r} (\sin\theta_1 + \sin\theta_2),$$

where r is the perpendicular distance of the wire from the point, considered, and θ_1 is the angle between the line, joining the upper point of straight wire to the considered point and the perpendicular drawn to the wire and θ_2 that from the lower point of the straight wire.

Here, $$B_1 = B_3 = \frac{\mu_0}{4\pi} \frac{i}{(d/2)\sin\frac{\varphi}{2}} \left\{ \cos\frac{\varphi}{2} + \cos\frac{\varphi}{2} \right\}$$

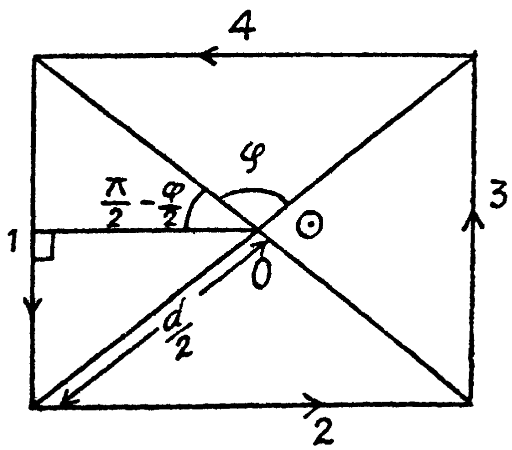
and $$B_2 = B_4 = \frac{\mu_0}{4\pi} \frac{i}{(d/2)\cos\frac{\varphi}{2}} \left(\sin\frac{\varphi}{2} + \sin\frac{\varphi}{2} \right)$$

Hence, the magnitude of total magnetic induction at O,

$$B_0 = B_1 + B_2 + B_3 + B_4$$

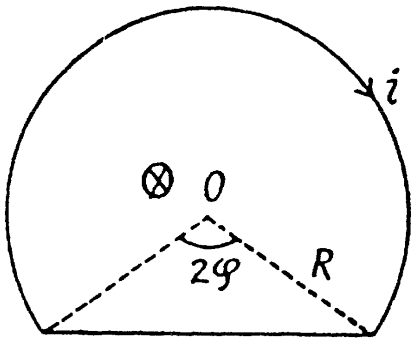
$$= \frac{\mu_0}{4\pi} \frac{4i}{d/2} \left[\frac{\cos\frac{\varphi}{2}}{\sin\frac{\varphi}{2}} + \frac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}} \right]$$

$$= \frac{4\mu_0 i}{\pi d \sin\varphi} = 0{\cdot}10 \text{ mT}$$



3.222 Magnetic induction due to the arc segment at O,

$$B_{arc} = \frac{\mu_0}{4\pi}\frac{i}{R}(2\pi - 2\varphi)$$



and magnetic induction due to the line segment at O,

$$B_{line} = \frac{\mu_0}{4\pi}\frac{i}{R\cos\varphi}[2\sin\varphi]$$

So, total magnetic induction at O,

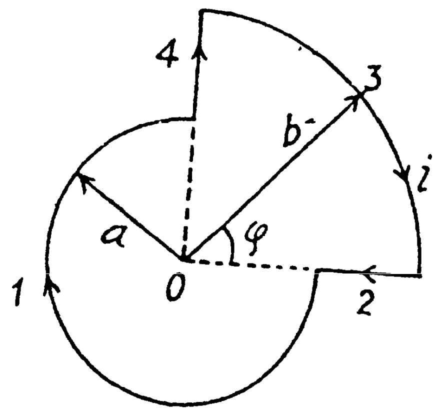
$$B_0 = B_{arc} + B_{line} = \frac{\mu_0}{2\pi}\frac{i}{R}[\pi - \varphi + \tan\varphi] = 28\ \mu\text{T}$$

3.223 (a) From the Biot-Savart law,

$$dB = \frac{\mu_0}{4\pi}i\frac{(\vec{dl}\times\vec{r})}{r^3}$$

So, magnetic field induction due to the segment 1 at O,

$$B_1 = \frac{\mu_0}{4\pi}\frac{i}{a}(2\pi - \varphi)$$



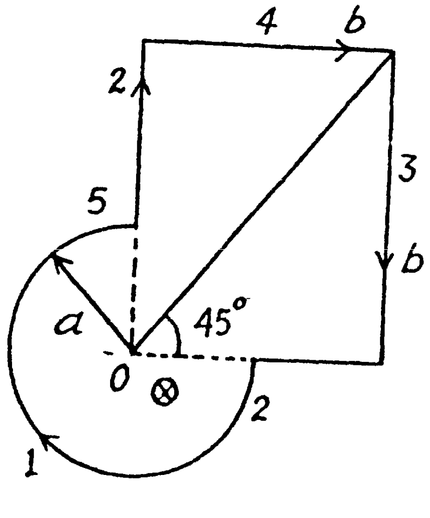
also $\quad B_2 = B_4 = 0$, as $\vec{dl}\uparrow\uparrow\vec{r}$

and $\qquad B_3 = \frac{\mu_0}{4\pi}\frac{i}{b}\varphi$

Hence, $B_0 = B_1 + B_2 + B_3 + B_4$

$$= \frac{\mu_0}{4\pi}i\left[\frac{2\pi - \varphi}{a} + \frac{\varphi}{b}\right],$$

(b) Here, $B_1 = \frac{\mu_0}{4\pi}\frac{i}{a}\frac{3\pi}{a}$, $\vec{B_2} = 0$,



$$B_3 = \frac{\mu_0}{4\pi}\frac{i}{b}\sin 45°,$$

$$B_4 = \frac{\mu_0}{4\pi}\frac{i}{b}\sin 45°,$$

and $\qquad B_5 = 0$

So, $\quad B_0 = B_1 + B_2 + B_3 + B_4 + B_5$

$$= \frac{\mu_0}{4\pi}\frac{i}{a}\frac{3\pi}{2} + 0 + \frac{\mu_0}{4\pi}\frac{i}{b}\sin 45° + \frac{\mu_0}{4\pi}\frac{i}{b}\sin 45° + 0$$

$$= \frac{\mu_0}{4\pi}i\left[\frac{3\pi}{2a} + \frac{\sqrt{2}}{b}\right]$$

3.224 The thin walled tube with a longitudinal slit can be considered equivalent to a full tube and a strip carrying the same current density in the opposite direction. Inside the tube, the former does not contribute so the total magnetic field is simply that due to the strip. It is

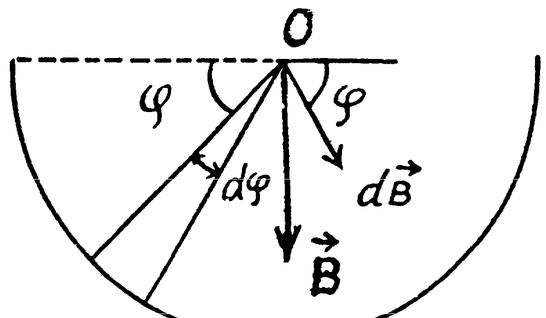
$$B = \frac{\mu_0}{2\pi}\frac{(I/2\pi R)\,h}{r} = \frac{\mu_0 I h}{4\pi^2 R r}$$

where r is the distance of the field point from the strip.

3.225 First of all let us find out the direction of vector $\vec{B}$ at point O. For this purpose, we divide the entire conductor into elementary fragments with current di. It is obvious that the sum of any two symmetric fragments gives a resultant along $\vec{B}$ shown in the figure and consequently, vector $\vec{B}$ will also be directed as shown

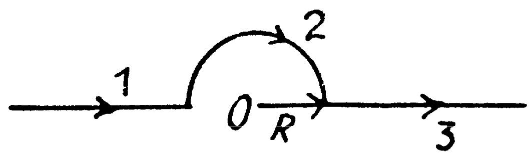
So, $\quad |\vec{B}| = \int dB \sin\varphi \qquad (1)$

$$= \int \frac{\mu_0}{2\pi R}\,di\,\sin\varphi$$



$$= \int_0^{\pi} \frac{\mu_0}{2\pi^2 R}\,i\sin\varphi\,d\varphi, \quad \left(\text{as } di = \frac{i}{\pi}\,d\varphi\right)$$

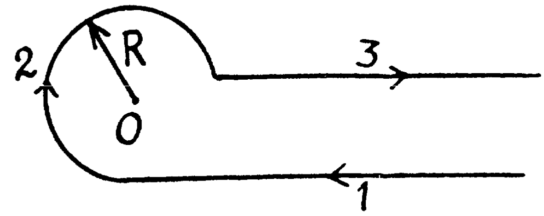
Hence $\quad B = \mu_0 i/\pi^2 R$



3.226 (a) From symmetry

$$B_0 = B_1 + B_2 + B_3$$

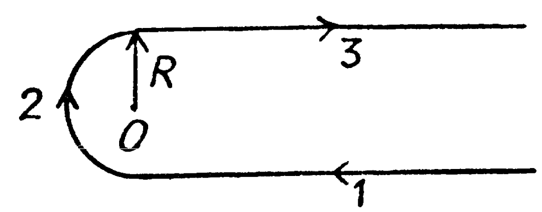
$$= 0 + \frac{\mu_0}{4\pi}\frac{i}{R}\,\pi + 0 = \frac{\mu_0}{4}\frac{i}{R}$$



(b) From symmetry

$$B_0 = B_1 + B_2 + B_3$$

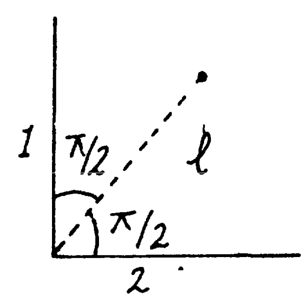
$$= \frac{\mu_0}{4\pi}\frac{i}{R} + \frac{\mu_0}{2\pi}\frac{i}{R}\frac{3\pi}{2} + 0 = \frac{\mu_0}{4\pi}\frac{i}{R}\left[1 + \frac{3\pi}{2}\right]$$



(c) From symmetry

$$B_0 = B_1 + B_2 + B_3$$

$$= \frac{\mu_0}{4\pi}\frac{i}{R} + \frac{\mu_0}{4\pi}\frac{i}{R}\,\pi + \frac{\mu_0}{4\pi}\frac{i}{R} = \frac{\mu_0}{4\pi}\frac{i}{R}(2+\pi)$$



3.227 $\vec{B}_0 = \vec{B}_1 + \vec{B}_2$

or, $|\vec{B}_0| = \dfrac{\mu_0 i}{4\pi l}\sqrt{2} = 2{\cdot}0\,\mu\text{T}$, (using 3.221)

3.228 (a)
$$\vec{B}_0 = \vec{B}_1 + \vec{B}_2 + \vec{B}_3$$
$$= \frac{\mu_0}{4\pi}\frac{i}{R}(-\vec{k}) + \frac{\mu_0}{4\pi}\frac{i}{R}\pi(-\vec{i}) + \frac{\mu_0}{4\pi}\frac{i}{R}(-\vec{k})$$
$$= -\frac{\mu_0}{4\pi}\frac{i}{R}[2\vec{k} + \pi\vec{i}]$$

So,
$$|\vec{B}_0| = \frac{\mu_0}{4\pi}\frac{i}{R}\sqrt{\pi^2+4} = 0{\cdot}30\ \mu\text{T}$$

(b) $\vec{B}_0 = \vec{B}_1 + \vec{B}_2 + \vec{B}_3$
$$= \frac{\mu_0}{4\pi}\frac{i}{R}(-\vec{k}) + \frac{\mu_0}{4\pi}\frac{i}{R}\pi(-\vec{i}) + \frac{\mu_0}{4\pi}\frac{i}{R}(-\vec{i})$$
$$= -\frac{\mu_0}{4\pi}\frac{i}{R}[\vec{k} + (\pi+1)\vec{i}]$$

So,
$$|\vec{B}_0| = \frac{\mu_0}{4\pi}\frac{i}{R}\sqrt{1+(\pi+1)^2} = 0{\cdot}34\ \mu\text{T}$$

(c) Here using the law of parallel resistances
$$i_1 + i_2 = i \text{ and } \frac{i_1}{i_2} = \frac{1}{3},$$

So,
$$\frac{i_1+i_2}{i_2} = \frac{4}{3}$$

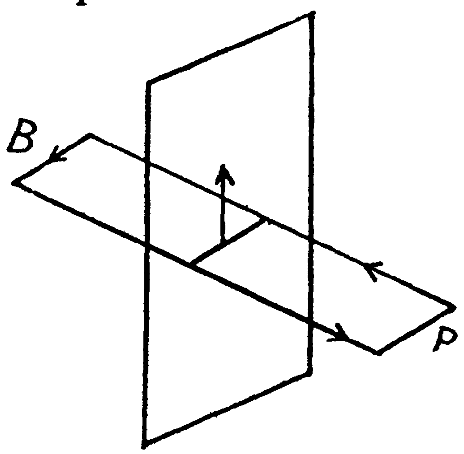
Hence
$$i_2 = \frac{3}{4}i, \text{ and } i_1 = \frac{1}{4}i$$

Thus $\vec{B}_0 = \frac{\mu_0}{4\pi}\frac{i}{R}(-\vec{k}) + \frac{\mu_0}{4\pi}\frac{i}{R}(-\vec{j}) + \left[\frac{\mu_0}{4\pi}\left(\frac{3\pi}{2}\right)\frac{i_1}{R}(-\vec{i}) + \frac{\mu_0}{4\pi}\frac{(\pi/2)\,i_2}{R}\vec{i}\right]$
$$= \frac{-\mu_0}{4\pi}\frac{i}{R}(\vec{j}+\vec{k}) + 0$$

Thus,
$$|\vec{B}_0| = \frac{\mu_0}{4\pi}\frac{\sqrt{2}\,i}{R} = 0{\cdot}11\ \mu\text{T}$$

3.229 (a) We apply circulation theorem as shown. The current is vertically upwards in the plane and the magnetic field is horizontal and parallel to the plane.

$$\oint \vec{B}\cdot d\vec{l} = 2Bl = \mu_0 il \quad \text{or,} \quad B = \frac{\mu_0 i}{2}$$



(b) Each plane contributes $\mu_0\frac{i}{2}$ between the planes and outside the plane that cancel.

Thus
$$B = \begin{cases} \mu_0 i \text{ between the plane} \\ 0 \text{ outside.} \end{cases}$$

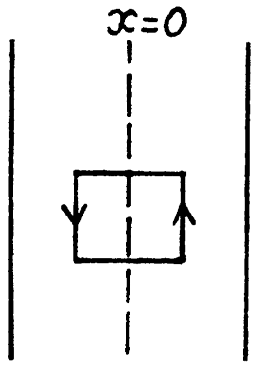
3.230 We assume that the current flows perpendicular to the plane of the paper, by circulation theorem,

$$2B\,dl = \mu_0\,(2\,x\,dl\,)\,j$$

or, $\quad B = \mu_0\,x\,j,\ |x| \le d$

Outside, $2\,B\,dl = \mu_0\,(2d\,dl\,)\,j$

or, $\quad B = \mu_0\,d\,j\ \ |x| \ge d.$

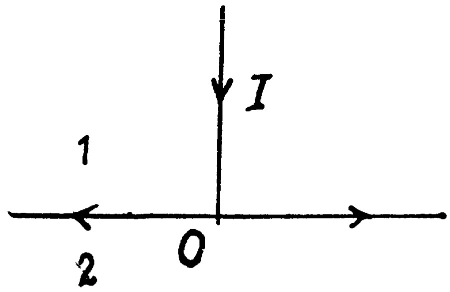


3.231 It is easy to convince oneself that both in the regions. 1 and 2, there can only be a circuital magnetic field (i.e. the component B_φ). Any radial field in region 1 or any B_Z away from the current plane will imply a violation of Gauss' law of magnetostatics, B_φ must obviously be symmetrical about the straight wire. Then in 1,

$$B_\varphi\,2\,\pi\,r = \mu_0\,I$$

or, $$B_\varphi = \frac{\mu_0\,I}{2\,\pi\,r}$$

In 2, $\quad B_\varphi \cdot 2\,\pi\,r = 0,\quad \text{or}\quad B_\varphi = 0$



3.232 On the axis, $B = \dfrac{\mu_0\,I\,R^2}{2\,(R^2+x^2)^{3/2}} = B_x$, along the axis.

Thus, $$\int \vec{B}\cdot d\vec{r} = \int_{-\infty}^{\infty} B_x\,dx = \frac{\mu_0\,I\,R^2}{2}\int_{-\infty}^{\infty}\frac{dx}{(R^2+x^2)^{3/2}}$$

$$= \frac{\mu_0\,I\,R^2}{2}\int_{-\pi/2}^{\pi/2}\frac{R\sec^2\theta\,d\theta}{R^3\sec^3\theta}, \text{ on putting } x = R\tan\theta$$

$$= \mu_0\,I\,\frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta = \mu_0\,I$$

The physical interpretation of this result is that $\int_{-\infty}^{\infty} B_x\,dx$ can be thought of as the circulation of B over a closed loop by imaging that the two ends of the axis are connected, by a line at infinity (e.g. a semicircle of infinite radius).

3.233 By circulation theorem inside the conductor

$$B_\varphi\,2\,\pi\,r = \mu_0\,j_z\,\pi\,r^2 \quad \text{or,}\quad B_\varphi = \mu_0\,j_z\,r/2$$

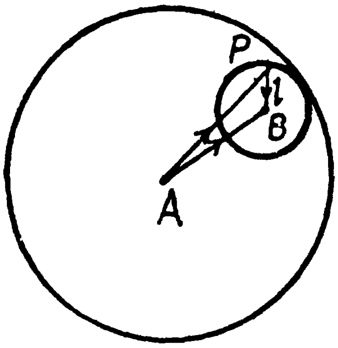
i.e., $$\vec{B} = \frac{1}{2}\,\mu_0\,\vec{j}\times\vec{r}$$

Similarly outside the conductor,

$$B_\varphi 2\pi r = \mu_0 j_z \pi R^2 \quad \text{or,} \quad B_\varphi = \frac{1}{2}\mu_0 j_z \frac{R^2}{r}$$

So, $$\vec{B} = \frac{1}{2}\mu_0 (\vec{j} \times \vec{r}) \frac{R^2}{r^2}$$

2.234 We can think of the given current which will be assumed uniform, as arising due to a negative current, flowing in the cavity, superimposed on the true current, everywhere including the cavity. Then from the previous problem, by superposition.



$$\vec{B} = \frac{1}{2}\mu_0 \vec{j} \times (A\vec{P} - B\vec{P}) = \frac{1}{2}\mu_0 \vec{j} \times \vec{l}$$

If $\vec{l}$ vanishes so that the cavity is concentric with the conductor, there is no magnetic field in the cavity.

3.235 By Circulation theorem,

$$B_\varphi \cdot 2\pi r = \mu_0 \int_0^r j(r') \times 2\pi r' \, dr'$$

or using $B_\varphi = br^\alpha$ inside the stream,

$$br^{\alpha+1} = \mu_0 \int_0^r j(r') \, r' \, dr'$$

So by differentiation,

$$(\alpha + 1)\, br^\alpha = \mu_0 j(r)\, r$$

Hence, $$j(r) = \frac{b(\alpha+1)}{\mu_0} r^{\alpha-1}$$

3.236 On the surface of the solenoid there is a surface current density

$$\vec{j_s} = n I \hat{e}_\varphi$$

Then, $$\vec{B} = -\frac{\mu_0}{4\pi} n I \int R \, d\varphi \, dz \frac{\hat{e}_\varphi \times \vec{r_0}}{r_0^3}$$

where $\vec{r_0}$ is the vector from the current element to the field point, which is the centre of the solenoid,

Now, $$-\hat{e}_\varphi \times \vec{r_0} = R\hat{e}_z$$

$$r_0 = (z^2 + R^2)^{1/2}$$

Thus, $$B = B_z = \frac{\mu_0 n I}{4\pi} \times 2\pi R^2 \int_{-l/2}^{-l/2} \frac{dz}{(R^2 + z^2)^{3/2}}$$

$$= \frac{1}{2}\mu_0 n I \int_{-\tan^{-1}\frac{l}{2R}}^{+\tan^{-1}\frac{l}{2R}} \cos\alpha \, d\alpha \qquad \text{(on putting } z = R\tan\alpha\text{)}$$

$$= \mu_0 n I \sin\alpha = \mu_0 nI \frac{l/2}{\sqrt{(l/2)^2 + R^2}} = \mu_0 n I \Big/ \sqrt{1 + \left(\frac{2R}{l}\right)^2}$$

3.237 We proceed exactly as in the previous problem. Then (a) the magnetic induction on the axis at a distance x from one end is clearly,

$$B = \frac{\mu_0 nI}{4\pi} \times 2\pi R^2 \int_0^{\infty} \frac{dz}{[R^2 + (z-x)^2]^{3/2}} = \frac{1}{2}\mu_0 n I R^2 \int_x^{\infty} \frac{dz}{(z^2 + R^2)^{3/2}}$$

$$= \frac{1}{2}\mu_0 n I \int_{\tan^{-1}\frac{x}{R}}^{\pi/2} \cos\theta \, d\theta = \frac{1}{2}\mu_0 n I \left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)$$

$x > 0$ menas that the field point is outside the solenoid. B then falls with x. $x < 0$ means that the field point gets more and more inside the solenoid. B then increases with (x) and eventually becomes constant, equal to $\mu_0 n I$. The $B - x$ graph is as given in the answer script.

(b) We have, $\dfrac{B_0 - \delta B}{B_0} = \dfrac{1}{2}\left[1 - \dfrac{x_0}{\sqrt{R^2 + x_0^2}}\right] = 1 - \eta$

or, $$-\frac{x_0}{\sqrt{R^2 + x_0^2}} = 1 - 2\eta$$

Since η is small ($\sim 1\%$), x_0 must be negative. Thus $x_0 = -|x_0|$

and $$\frac{|x_0|}{\sqrt{R^2 + |x_0|^2}} = 1 - 2\eta$$

$$|x_0|^2 = (1 - 4\eta + 4\eta^2)(R^2 + |x_0|^2)$$

$$0 = (1 - 2\eta)^2 R^2 - 4\eta(1-\eta)|x_0|^2$$

or, $$|x_0| = \frac{(1 - 2\eta)R}{2\sqrt{\eta(1-\eta)}}$$

3.238 If the strip is tightly wound, it must have a pitch of h. This means that the current will flow obliquely, partly along $\hat{e}_\varphi$ and partly along $\hat{e}_z$. Obviously, the surface current density is,

$$\vec{J}_s = \frac{I}{h}\left[\sqrt{1 - (h/2\pi R)^2}\,\hat{e}_\varphi + \frac{h}{2\pi R}\hat{e}_z\right].$$

By comparision with the case of a solenoid and a hollow straight conductor, we see that field inside the coil

$$= \mu_0 \frac{I}{h} \sqrt{1 - (h/2\pi R)^2}$$

(Cf. $B = \mu_0 nI$).

Outside, only the other term contributes, so

$$B_\varphi \times 2\pi r = \mu_0 \frac{I}{h} \times \frac{h}{2\pi R} \times 2\pi R$$

or,
$$B_\varphi = \frac{\mu_0}{4\pi} \cdot \frac{2I}{r} .$$

Note - *Surface current density is defined as current flowing normally acrǫss a unit length over a surface.*

3.239 Suppose a is the radius of cross section of the core. The winding has a pitch $2\pi R/N$, so the surface current density is

$$\vec{J_s} = \frac{I}{2\pi R/N} \vec{e_1} + \frac{I}{2\pi a} \vec{e_2}$$

where $\vec{e_1}$ is a unit vector along the cross section of the core and $\vec{e_2}$ is a unit vector along its length.

The magnetic field inside the cross section of the core is due to first term above, and is given by

$$B_\varphi \cdot 2\pi R = \mu_0 NI$$

(NI is total current due to the above surface current (first term.))

Thus,
$$B_\varphi = \mu_0 NI/2\pi R.$$

The magnetic field at the centre of the core can be obtained from the basic formula.

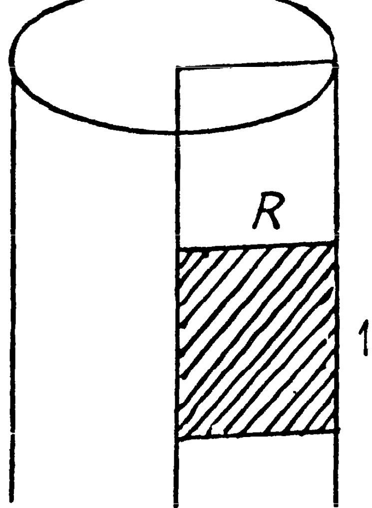
$$\vec{dB} = \frac{\mu_0}{4\pi} \frac{\vec{J_s} \times \vec{r_0}}{r_0^3} dS \text{ and is due to the second term.}$$

So,
$$\vec{B} = B_z \vec{e_z} = \vec{e_z} \frac{\mu_0}{4\pi} \frac{I}{2\pi a} \int \frac{1}{R^3} R d\varphi \times 2\pi a$$

or,
$$B_z = \frac{\mu_0 I}{2R}$$

The ratio of the two magnetic field, is $= \frac{N}{\pi}$

3.240 We need the flux through the shaded area.



Now by Ampere's theorem,

$$B_\varphi 2\pi r = \mu_0 \frac{I}{\pi R^2} \cdot \pi r^2$$

or,
$$B_\varphi = \frac{\mu_0}{2\pi} I \frac{r}{R^2}$$

The flux through the shaded region is,

$$\varphi_1 = \int_0^R 1\cdot dr \cdot B_\varphi(r)$$

$$= \int_0^R dr \frac{\mu_0}{2\pi} I \frac{r}{R^2} = \frac{\mu_0}{4\pi} I.$$

3.241 Using 3.237, the magnetic field is given by,

$$B = \frac{1}{2}\mu_0 nI\left(1 - \frac{x}{\sqrt{x^2+R^2}}\right)$$

At the end, $B = \frac{1}{2}\mu_0 n I = \frac{1}{2}B_0$, where $B_0 = \mu_0 nI$,

is the field deep inside the solenoid. Thus,

$$\Phi = \frac{1}{2}\mu_0 nIS = \Phi_0/2, \text{ where } \Phi = \mu_0 nI S$$

is the flux of the vector B through the cross section deep inside the solenid.

3.242
$$B_\varphi 2\pi r = \mu_0 N I$$

or,
$$B_\varphi = \frac{\mu_0 N I}{2\pi r}$$

Then, $\Phi = \int_a^b B_\varphi h dr, a \le r \le b = \frac{\mu_0}{4\pi} 2N I h \ln \eta$, where $\eta = b/a$

3.243 Magnetic moment of a current loop is given by $p_m = n i S$ (where n is the number of turns and S, the cross sectional area.) In our problem, $n = 1$, $S = \pi R^2$ and $B = \frac{\mu_0}{2}\frac{i}{R}$

So,
$$p_m = \frac{2BR}{\mu_0}\pi R^2 = \frac{2\pi B R^3}{\mu_0}$$

3.244 Take an element of length $r d\theta$ containing $\frac{N}{\pi r}\cdot r d\theta$ turns. Its magnetic moment is

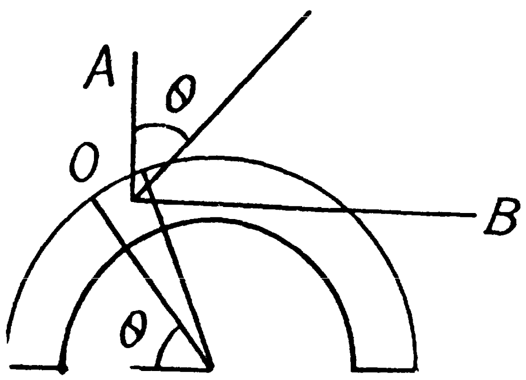
$$\frac{N}{\pi} d\theta \cdot \frac{\pi}{4} d^2 I$$

normal to the plane of cross section. We resolve it along OA and OB. The moment along OA integrates to

$$\int_0^\pi \frac{N}{4} d^2 I d\theta \cos\theta = 0$$

while that along OB gives

$$p_m = \int_0^\pi \frac{N d^2 I}{4} \sin\theta \, d\theta = \frac{1}{2} N d^2 I$$



3.245 (a) From Biot-Savart's law, the magnetic induction due to a circular current carrying wire loop at its centre is given by,

$$B_r = \frac{\mu_0}{2r} i$$

The plane spiral is made up of concentric circular loops, having different radii, varying from a to b. Therefore, the total magnetic induction at the centre,

$$B_0 = \int \frac{\mu_0}{2r} dN \qquad (1)$$

where $\frac{\mu_0}{2r} i$ is the contribution of one turn of radius r and dN is the number of turns in the interval $(r, r + dr)$

i.e. $$dN = \frac{N}{b-a} dr$$

Substituting in equation (1) and integrating the result over r between a and b, we obtain,

$$B_0 = \int_a^b \frac{\mu_0 i}{2r} \frac{N}{(b-a)} dr = \frac{\mu_0 \, i N}{2\,(b-a)} \ln \frac{b}{a}$$

(b) The magnetic moment of a turn of radius r is $p_m = i \pi r^2$ and of all turns,

$$p = \int p_m \, dN = \int_a^b i \pi r^2 \frac{N}{b-a} dr = \frac{\pi \, i N (b^3 - a^2)}{3\,(b-a)}$$

3.246 (a) Let us take a ring element of radius r and thickness dr, then charge on the ring element., $d\,q = \sigma \, 2 \pi \, r \, dr$

and current, due to this element, $di = \frac{(\sigma \, 2 \pi \, r \, dr)\, \omega}{2\pi} = \sigma \, \omega \, r \, dr$

So, magnetic induction at the centre, due to this element : $dB = \frac{\mu_0}{2} \frac{di}{r}$

and hence, from symmetry : $B = \int dB = \int_0^R \frac{\mu_0 \, \sigma \, \omega \, r \, dr}{r} = \frac{\mu_0}{2} \sigma \, \omega R$

(b) Magnetic moment of the element, considered,

$$dp_m = (di)\, \pi r^2 = \sigma \, \omega \, dr \, \pi \, r^2 = \sigma \, \pi \, \omega \, r^3 \, dr$$

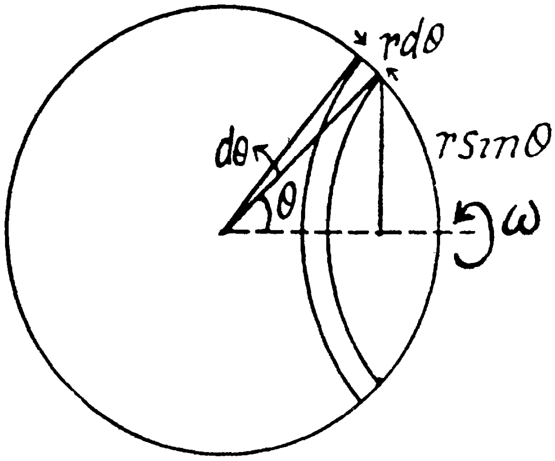
Hence, the sought magnetic moment,

$$p_m = \int dp_m = \int_0^R \sigma \, \pi \, \omega \, r^3 \, dr = \sigma \, \omega \, \pi \frac{R^4}{4}$$

3.247 As only the outer surface of the sphere is charged, consider the element as a ring, as shown in the figure.

The equivalent current due to the ring element,

$$di = \frac{\omega}{2\pi}(2\,\pi r \sin\theta\, r d\,\theta)\,\sigma \qquad (1)$$



and magnetic induction due to this loop element at the centre of the sphere, O,

$$dB = \frac{\mu_0}{4\pi} di \frac{2\,\pi\, r \sin\theta\, r \sin\theta}{r^3} = \frac{\mu_0}{4\pi} di \frac{\sin^2\theta}{r}$$

[Using 3.219 (b)]

Hence, the total magnetic induction due to the sphere at the centre, O,

$$B = \int dB = \int_0^{\pi/2} \frac{\mu_0}{4\pi}\frac{\omega}{2\pi}\frac{2\,\pi\, r^2 \sin\theta\, d\,\theta \sin^2\theta\,\sigma}{r} \quad \text{[using (1)]}$$

Hence, $$B = \int_0^{\pi/2} \frac{\mu_0\,\sigma\,\omega r}{4\pi}\sin^3\theta\, d\,\theta = \frac{2}{3}\mu_0\,\sigma\,\omega r = 29\text{ pT}$$

3.248 The magnetic moment must clearly be along the axis of rotation. Consider a volume element dV. It contains a charge $\frac{q}{4\,\pi/3\,R^3}dV$. The rotation of the sphere causes this charge to revolve around the axis and constitute a current.

$$\frac{3q}{4\,\pi R^3}dV \times \frac{\omega}{2\pi}$$

Its magnetic moment will be

$$\frac{3q}{4\,\pi R^3}dV \times \frac{\omega}{2\pi} \times \pi r^2 \sin^2\theta$$

So the total magnetic moment is

$$p_m = \int_0^R \int_0^\pi \frac{3q}{2R^3} r^2 \sin\theta d\theta \times \frac{\omega\, r^2 \sin^2\theta}{2} dr = \frac{3q}{2R^3} \times \frac{\omega}{2} \times \frac{R^5}{5} \times \frac{4}{3} = \frac{1}{5} qR^2\,\omega$$

The mechanical moment is

$$M = \frac{2}{5} mR^2\,\omega, \text{ So, } \frac{p_m}{M} = \frac{q}{2m}.$$

3.249 Because of polarization a space charge is present within the cylinder. It's density is

$$\rho_p = -\text{div}\,\vec{P} = -2\alpha$$

Since the cylinder as a whole is neutral a surface charge density σ_p must be present on the surface of the cylinder also. This has the magnitude (algebraically)

$$\sigma_p \times 2\pi R = 2\,\alpha\,\pi R^2 \quad \text{or,} \quad \sigma_p = \alpha R$$

When the cylinder rotates, currents are set up which give rise to magnetic fields. The contribution of ρ_p and σ_p can be calculated separately and then added.

For the surface charge the current is (for a particular element)

$$\alpha R \times 2\pi R\, dx \times \frac{\omega}{2\pi} = \alpha R^2 \omega\, dx$$

Its contribution to the magnetic field at the centre is

$$\frac{\mu_0 R^2 (\alpha R^2 \omega\, dx)}{2\,(x^2 + R^2)^{3/2}}$$

and the total magnetic field is

$$B_s = \int_{\infty}^{-\infty} \frac{\mu_0 R^2 (\alpha R^2 \omega dx)}{2\,(x^2+R^2)^{3/2}} = \frac{\mu_0 \alpha R^4 \omega}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+R^2)^{3/2}} = \frac{\mu_0 \alpha R^4 \omega}{2} \times \frac{2}{R^2} = \mu_0 \alpha R^2 \omega$$

As for the volume charge density consider a circle of radius r, radial thickness dr and length dx.

The current is $-\,2\alpha \times 2\pi r\, dr\, dx \times \dfrac{\omega}{2\pi} = -\,2\,\alpha r\, dr\, \omega\, dx$

The total magnetic field due to the volume charge distribution is

$$B_v = -\int_0^R dr \int_{-\infty}^{\infty} dx\, 2\pi r \omega \frac{\mu_0 r^2}{2\,(x^2+r^2)^{3/2}} = -\int_0^R \alpha \mu_0 \omega r^3\, dr \int_{-\infty}^{\infty} dx\, (x^2+r^2)^{3/2}$$

$$= -\int_0^R \alpha \mu_0 \omega r\, dr \times 2 = -\,\mu_0 \alpha \omega R^2 \quad \text{so,} \quad B = B_s + B_v = 0$$

3.250 Force of magnetic interaction, $\vec{F}_{mag} = e\,(\vec{v} \times \vec{B})$

Where, $$\vec{B} = \frac{\mu_0}{4\pi} \frac{e\,(\vec{v} \times \vec{r})}{r^3}$$

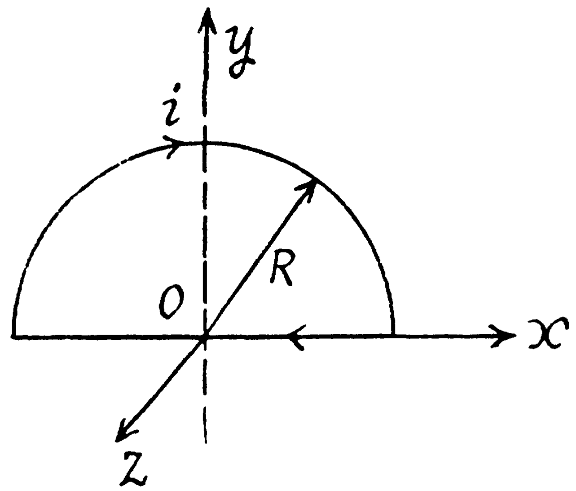
So, $$\vec{F}_{mag} = \frac{\mu_0}{4\pi} \frac{e^2}{r^3} [\vec{v} \times (\vec{v} \times \vec{r})]$$

$$= \frac{\mu_0}{4\pi} \frac{e^2}{r^3} [(\vec{v}\cdot\vec{r}) \times \vec{v} - (\vec{v}\cdot\vec{v}) \times \vec{r}] = \frac{\mu_0}{4\pi} \frac{e^2}{r^3} (-\,v^2 \vec{r})$$

And $$\vec{F}_{ele} = e\vec{E} = e \frac{1}{4\pi\varepsilon_0} \frac{e\vec{r}}{|\vec{r}|^3}$$

Hence, $$\frac{|\vec{F}_{mag}|}{|\vec{F}_{electic}|} = -\,v^2 \mu_0 \varepsilon_0 = \left(\frac{v}{c}\right)^2 = 1{\cdot}00 \times 10^{-6}$$

3.251 (a) The magnetic field at O is only due to the curved path, as for the line element, $d\vec{l} \uparrow\uparrow \vec{r}$.

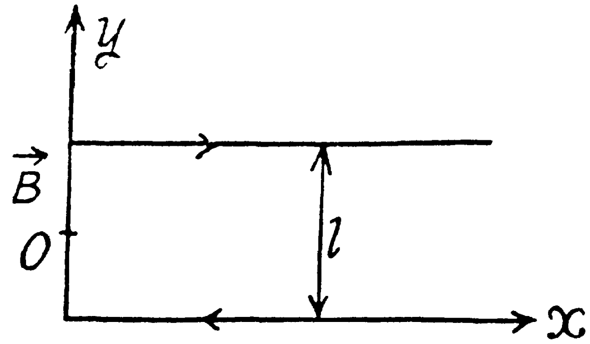


Hence, $\vec{B} = \frac{\mu_0 i}{4\pi R}\pi(-\vec{k}) = \frac{\mu_0 i}{4R}(-\vec{k})$

Thus $\vec{F}_u = iB(-\vec{j}) = \frac{\mu_0 i^2}{4R}(-\vec{j})$

So, $F_u = \frac{\pi_0 i^2}{4R} = 0{\cdot}20$ N/m

(b) In this part, magnetic induction $\vec{B}$ at O will be effective only due to the two semi infinite segments of wire. Hence



$$\vec{B} = 2\cdot\frac{\mu_0 i}{4\pi\left(\frac{l}{2}\right)}\sin\frac{\pi}{2}(-\vec{k})$$

$$= \frac{\mu_0 i}{\pi l}(-\vec{k})$$

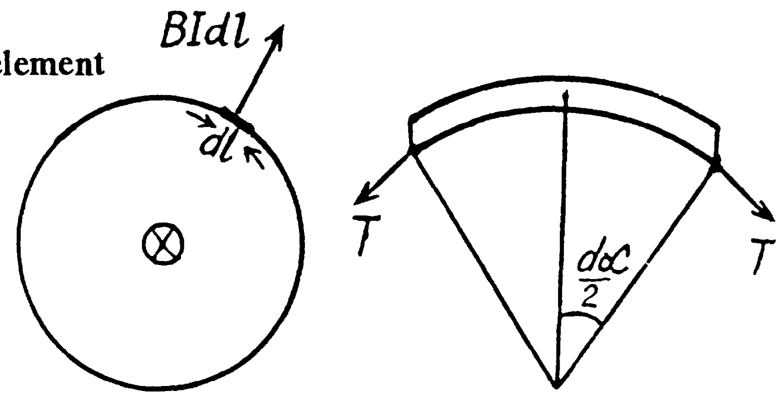
Thus force per unit length,

$$\vec{F}_u = \frac{\mu_0 l^2}{\pi l}(-\vec{i})$$

3.252 Each element of length dl experiences a force $BI\,dl$. This causes a tension T in the wire. For equilibrium,

$$T\,d\alpha = BI\,dl,$$

where $d\alpha$ is the angle subtended by the element at the centre.



Then, $$T = BI\frac{dl}{d\alpha} = BIR$$

The wire experiences a stress

$$\frac{BIR}{\pi d^2/4}$$

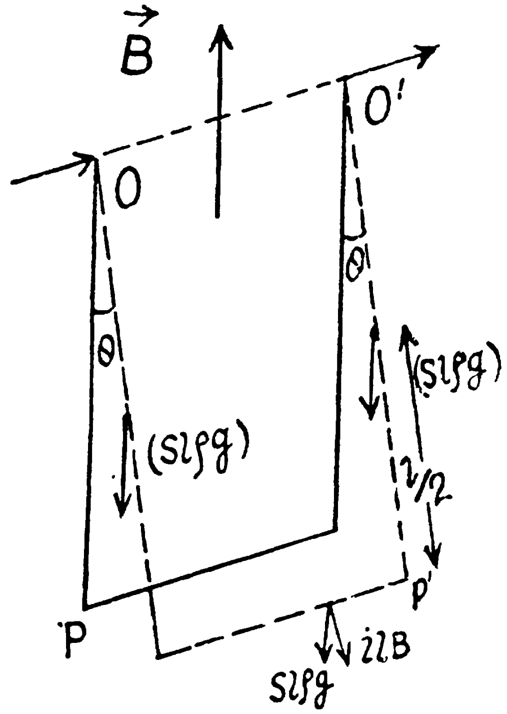
This must equals the breaking stress σ_m for rupture. Thus,

$$B_{max} = \frac{\pi d^2 \sigma_m}{4IR}$$

3.253 The Ampere forces on the sides OP and $O'P'$ are directed along the same line, in opposite directions and have equal values, hence the net force as well as the net torque of these forces about the axis OO' is zero. The Ampere-force on the segment PP' and the corresponding moment of this force about the axis OO' is effective and is deflecting in nature.

In equilibrium (in the dotted position) the deflecting torque must be equal to the restoring torque, developed due to the weight of the shape.

Let, the length of each side be l and ρ be the density of the material then,



$$ilB\,(l\cos\theta) = (S\,l\,\rho)\,g\frac{l}{2}\sin\theta + (S\,l\,\rho)\,g\frac{l}{2}\sin\theta + (S\,l\,\rho)\,gl\sin\theta$$

or, $$il^2 B\cos\theta = 2S\rho\,gl^2\sin\theta$$

Hence, $$B = \frac{2S\rho g}{i}\tan\theta$$

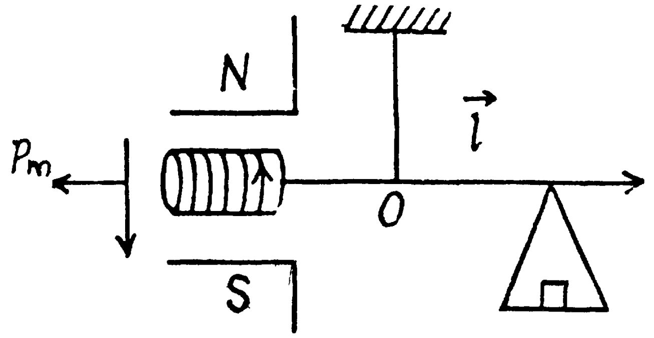
3.254 We know that the torque acting on a magnetic dipole.

$$\vec{N} = \vec{p}_m \times \vec{B}$$

But, $\vec{p}_m = iS\hat{n}$, where $\hat{n}$ is the normal on the plane of the loop and is directed in the direction of advancement of a right handed screw, if we rotate the screw in the sense of current in the loop.

On passing a current through the coil, this torque acting on the magnetic dipol, is counterbalanced by the moment of additional weight, about O.

Hence, the direction of current in the loop must be in the direction, shown in the figure.



$$\vec{p}_m \times \vec{B} = -\vec{l} \times \Delta m\vec{g}$$

or, $$N\,i\,S\,B = \Delta\,mgl$$

So, $$B = \frac{\Delta mgl}{N\,i\,S} = 0{\cdot}4\ \text{T on putting the values.}$$

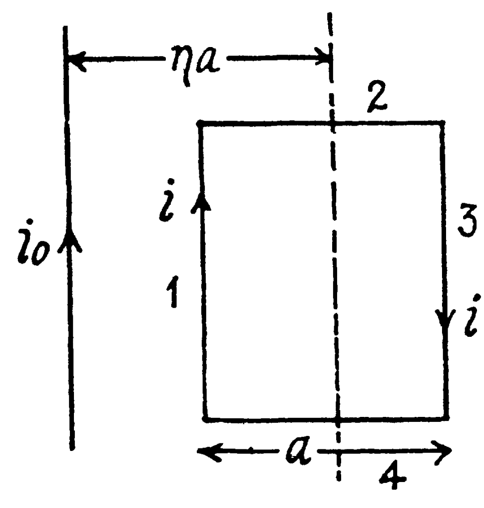
3.255 (a) As is clear from the condition, Ampere's forces on the sides (2) and (4) are equal in magnitude but opposite in direction. Hence the net effective force on the frame is the resultant of the forces, experienced by the sides (1) and (3).

Now, the Ampere force on (1),

$$F_1 = \frac{\mu_0}{2\pi}\frac{ii_0}{\left(\eta - \frac{1}{2}\right)}$$

and that on (3),

$$F_3 = \frac{\mu_0}{2\pi}\frac{i_0\,i}{\left(\eta + \frac{1}{2}\right)}$$



So, the resultant force on the frame $= F_1 - F_3$, (as they are opposite in nature.)

$$= \frac{2\mu_0 i i_0}{\pi(4\eta^2 - 1)} = 0{\cdot}40\ \mu\,\text{N}.$$

(b) Work done in turning the frame through some angle, $A = \int i\, d\Phi = i(\Phi_f - \Phi_i)$, where Φ_f is the flux through the frame in final position, and Φ_i, that in the the initial position.

Here, $$|\Phi_f| = |\Phi_i| = \Phi \text{ and } \Phi_i = -\Phi_f$$

so, $$\Delta\Phi = 2\Phi \text{ and } A = i\,2\Phi$$

Hence, $$A = 2i\int \vec{B}\cdot d\vec{S}$$

$$= 2i\int_{a\left(\eta - \frac{1}{2}\right)}^{a\left(\eta + \frac{1}{2}\right)} \frac{\mu_0}{2\pi}\frac{i_0 a}{r}\,dr = \frac{\mu_0 i i_0 a}{\pi}\ln\left(\frac{2\eta + 1}{2\eta - 1}\right)$$

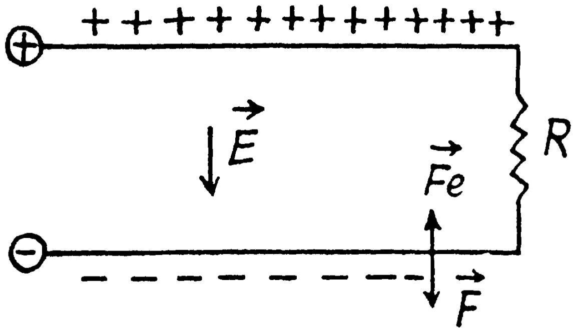
3.256 There are excess surface charges on each wire (irrespective of whether the current is flowing through them or not). Hence in addition to the magnetic force $\vec{F}_m$, we must take into account the electric force $\vec{F}_e$. Suppose that an excess charge λ corresponds to a unit length of the wire, then electric force exerted per unit length of the wire by other wire can be found with the help of Gauss's theorem.

$$F_e = \lambda E = \lambda\frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{l} = \frac{2\lambda^2}{4\pi\varepsilon_0 l}, \quad (1)$$

where l is the distance between the axes of the wires. The magnetic force acting per unit length of the wire can be found with the help of the theorem on circulation of vector $\vec{B}$

$$F_m = \frac{\mu_0}{4\pi}\frac{2i^2}{l},$$

where i is the current in the wire. (2)



Now, from the relation,

$\lambda = C\varphi$, where C is the capacitance of the wires per unit lengths and is given in problem 3.108 and $\varphi = iR$

$$\lambda = \frac{\pi\varepsilon_0}{\ln\eta}iR \quad \text{or,} \quad \frac{i}{\lambda} = \frac{\ln\eta}{\pi\varepsilon_0 R} \quad (3)$$

Dividing (2) by (1) and then substuting the value of $\frac{i}{\lambda}$ from (3), we get,

$$\frac{F_m}{F_e} = \frac{\mu_0}{\varepsilon_0}\frac{(\ln\eta)^2}{\pi^2 R^2}$$

The resultant force of interaction vanishes when this ratio equals unity. This is possible when $R = R_0$, where

$$R_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{\ln \eta}{\pi} = 0{\cdot}36\ \text{k}\Omega$$

3.257 Use 3.225

The magnetic field due to the conductor with semicircular cross section is

$$B = \frac{\mu_0 I}{\pi^2 R}$$

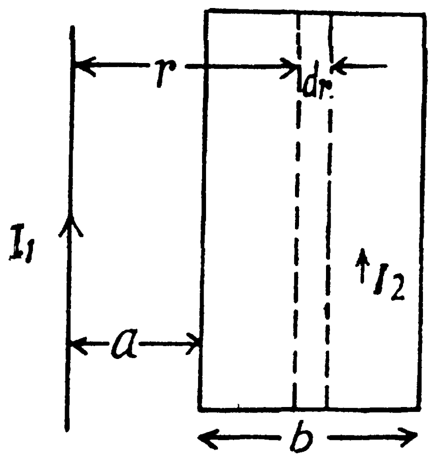
Then $$\frac{\partial F}{\partial l} = BI = \frac{\mu_0 I^2}{\pi^2 R}$$

3.258 We know that Ampere's force per unit length on a wire element in a magnetic field is given by.

$d\vec{F}_n = i\,(\hat{n} \times \vec{B})$ where $\hat{n}$ is the unit vector along the direction of current. (1)

Now, let us take an element of the conductor i_2, as shown in the figure. This wire element is in the magnetic field, produced by the current i_1, which is directed normally into the sheet of the paper and its magnitude is given by,

$$|\vec{B}| = \frac{\mu_0 I_1}{2\pi r} \qquad (2)$$



From Eqs. (1) and (2)

$d\vec{F}_n = \frac{I_2}{b}\,dr\,(\hat{n} \times \vec{B})$, (because the current through the element equals $\frac{I_2}{b}\,dr$)

So, $$d\vec{F}_n = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{b}\frac{dr}{r}, \text{ towards left (as } \hat{n} \perp^r \vec{B}).$$

Hence the magnetic force on the conductor :

$$\vec{F}_n = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{b}\int_a^{a+b}\frac{dr}{r} \text{ (towards left)} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{b}\ln\frac{a+b}{a} \text{ (towards left).}$$

Then according to the Newton's third law the magnitude of sought magnetic interaction force

$$= \frac{\mu_0 I_1 I_2}{2\pi b}\ln\frac{a+b}{a}$$

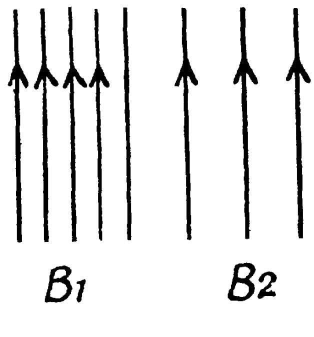
3.259 By the circulation theorem $B = \mu_0 i$,

where $i =$ current per unit length flowing along the plane perpendicular to the paper. Currents flow in the opposite sense in the two planes and produce the given field B by superposition. The field due to one of the plates is just $\frac{1}{2}B$. The force on the plate is,

$$\frac{1}{2}B \times i \times \text{Length} \times \text{Breadth} = \frac{B^2}{2\mu_0} \text{ per unit area.}$$

(Recall the formula $F = BIl$ on a straight wire)

3.260 (a) The external field must be $\frac{B_1 + B_2}{2}$, which when superposed with the internal field $\frac{B_1 - B_2}{2}$ (of opposite sign on the two sides of the plate) must give actual field. Now



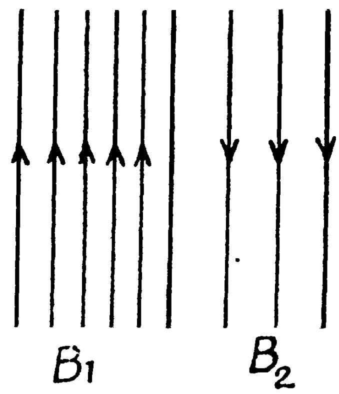
$$\frac{B_1 - B_2}{2} = \frac{1}{2}\mu_0 i$$

or, $$i = \frac{B_1 - B_2}{\mu_0}$$

Thus, $$F = \frac{B_1^2 - B_2^2}{2\mu_0}$$

(b) Here, the external field must be $\frac{B_1 - B_2}{2}$ upward with an internal field, $\frac{B_1 + B_2}{2}$, upward on the left and downward on the right. Thus,

$$i = \frac{B_1 + B_2}{\mu_0} \text{ and } F = \frac{B_1^2 - B_2^2}{2\mu_0}.$$



(c) Our boundary condition following from Gauss' law is, $B_1 \cos\theta_1 = B_2 \cos\theta_2$.

Also,$(B_1 \sin\theta_1 + B_2 \sin\theta_2) = \mu_0 i$ where i = current per unit length.

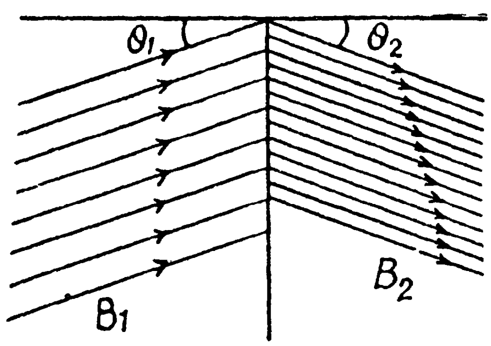
The external field parallel to the plate must be $\frac{B_1 \sin\theta_1 - B_2 \sin\theta_2}{2}$

(The perpendicular component $B_1 \cos\theta_1$, does not matter since the corresponding force is tangential)

Thus, $$F = \frac{B_1^2 \sin^2\theta_1 - B_2^2 \sin^2\theta_2}{2\mu_0} \text{ per unit area}$$

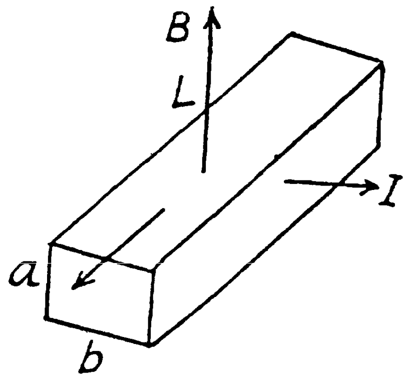
$$= \frac{B_1^2 - B_2^2}{2\mu_0} \text{ per unit area.}$$

The direction of the current in the plane conductor is perpendicular to the paper and beyond the drawing.



3.261 The Current density is $\frac{I}{aL}$, where L is the length of the section. The difference in pressure produced must be,

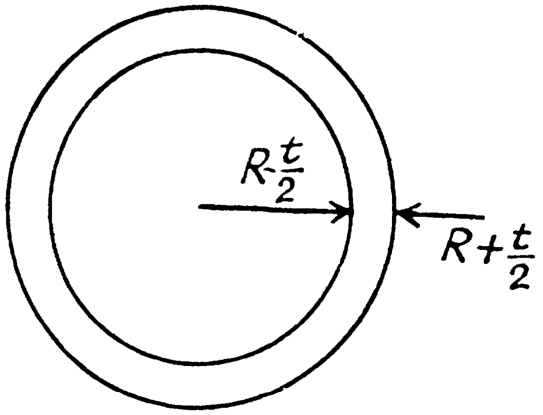
$$\Delta p = \frac{1}{aL} \times B \times (abL)/ab = \frac{IB}{a}$$



3.262 Let $t =$ thickness of the wall of the cylinder. Then,

$J = I/2\pi R t$ along z axis. The magnetic field due to this at a distance r

$\left(R - \frac{t}{2} < r < R + \frac{t}{2}\right)$, is given by,



$$B_\varphi (2\pi r) = \mu_0 \frac{I}{2\pi R t} \pi \left\{ r^2 - \left(R - \frac{t}{2}\right)^2 \right\}$$

or, $$B_\varphi = \frac{\mu_0 I}{4\pi R r t} \left\{ r^2 - (R - t/2)^2 \right\}$$

Now, $\vec{F} = \int \vec{J} \times \vec{B}\, dV$

and $$p = \frac{F_r}{2\pi R L} = \frac{1}{2\pi R L} \int_{R - \frac{t}{2}}^{R + \frac{t}{2}} \frac{\mu_0 I^2}{8\pi^2 R^2 t^2 r} \left\{ r^2 - \left(R - \frac{t}{2}\right)^2 \right\} \times 2\pi r L\, dr$$

$$= \frac{\mu_0 I^2}{8\pi^2 R^3 t^2} \int_{R - \frac{t}{2}}^{R + \frac{t}{2}} \left\{ r^2 - \left(R - \frac{t}{2}\right)^2 \right\} dr = \frac{\mu_0 I^2}{8\pi^2 R^3 t^2} \left[\frac{\left(R + \frac{t}{2}\right)^3 - \left(R - \frac{t}{2}\right)^3}{3} - \left(R - \frac{t}{2}\right)^2 t \right]$$

$$= \frac{\mu_0 I^2}{8\pi^2 R^3 t} \left[R t + 0 (t^2) \right] \approx \frac{\mu_0 I^2}{8\pi^2 R^2}$$

3.263 When self-forces are involved, a typical factor of $\frac{1}{2}$ comes into play. For example, the force on a current carrying straight wire in a magnetic induction B is BIl. If the magnetic induction B is due to the current itself then the force can be written as,

$$F = \int_0^I B(I')\, dI'\, l$$

If $B(I') \propto I'$, then this becomes, $F = \frac{1}{2} B(I)\, I\, l$.

In the present case, $B(I) = \mu_0 n I$ and this acts on nI ampere turns per unit length, so,

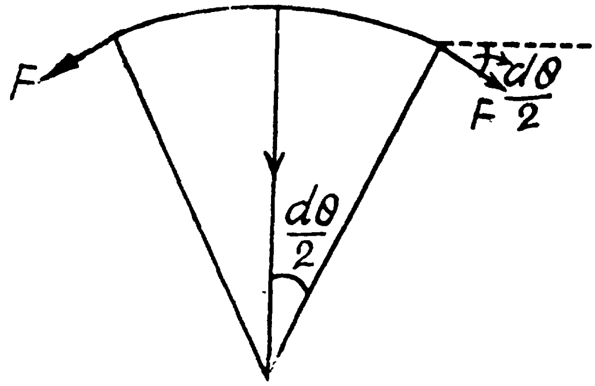
pressure $$p = \frac{F}{\text{Area}} = \frac{1}{2} \mu_0 n \frac{I \times nI \times 1 \times l}{1 \times l} = \frac{1}{2} \mu_0 n^2 I^2$$

3.264 The magnetic induction B in the solenoid is given by $B = \mu_0 nI$. The force on an element dl of the current carrying conductor is,

$$dF = \frac{1}{2} \mu_0 n I I\, dl = \frac{1}{2} \mu_0 n I^2\, dl$$

This is radially outwards. The factor $\frac{1}{2}$ is explained above.

To relate dF to the tensile strength $F_{\lim}$ we proceed as follows. Consider the equilibrium of the element dl. The longitudinal forces F have a radial component equal to,



$$dF = 2F \sin\frac{d\theta}{2} = F\,d\theta$$

Thus using $dl = R\,d\theta$, $F = \frac{1}{2}\mu_0 nI^2 R$

This equals $F_{\lim}$ when, $I = I_{\lim} = \sqrt{\frac{2F_{\lim}}{\mu_0 n R}}$

Note that $F_{\lim}$, here, is actually a force and not a stress.

3.265 Resistance of the liquid between the plates= $\frac{\rho d}{S}$

Voltage between the plates = $Ed = v\,Bd$,

Current through the plates = $\dfrac{vBd}{R + \frac{\rho d}{S}}$

Power, generated, in the external resistance R,

$$P = \frac{v^2B^2d^2R}{\left(R+\frac{\rho d}{S}\right)^2} = \frac{v^2B^2d^2}{\left(\sqrt{R}+\frac{\rho d}{S\sqrt{R}}\right)^2} = \frac{v^2B^2d^2}{\left[\left\{R^{1/4}-\left(\frac{\rho d}{S\sqrt{R}}\right)^{1/2}\right\}^2+2\sqrt{\frac{\rho d}{S}}\right]^2}$$

This is maximum when $R = \frac{\rho D}{S}$ and $P_{max} = \frac{v^2B^2Sd}{4\rho}$

3.266 The electrons in the conductor are drifting with a speed of,

$$v_d = \frac{J}{ne} = \frac{I}{\pi R^2 ne},$$

where e = magnitude of the charge on the electron, n = concentration of the conduction electorns.

The magnetic field inside the conductor due to this current is given by,

$$B_\varphi(2\pi r) = \pi r^2 \frac{I}{\pi R^2}\mu_0 \quad \text{or,} \quad B_\varphi = \frac{\mu_0}{2\pi}\frac{Ir}{R^2}$$

A radial electric field vB_φ must come into being in equilibrium. Its P.D. is,

$$\Delta\varphi = \int_0^R \frac{I}{\pi R^2 ne}\frac{\mu_0}{2\pi}\frac{Ir}{R^2}\,dr = \frac{I}{\pi R^2 ne}\left(\frac{\mu_0}{4\pi}I\right) = \frac{\mu_0 I^2}{4\pi R^2 ne}$$

3.267 Here, $v_d = \frac{E}{B}$ and $j = ne\, v_d$

so, $$n = \frac{jB}{eE} = \frac{200 \times 10^4 \frac{\text{A}}{\text{m}^2} \times 1\text{ T}}{1{\cdot}6 \times 10^{-19}\text{ C} \times 5 \times 10^{-4}\text{ V/m}}$$

$$= 2{\cdot}5 \times 10^{28} \text{ per m}^3 = 2{\cdot}5 \times 10^{22} \text{ per c.c.}$$

Atomic weight of Na being 23 and its density $\approx$ 1, molar volume is 23 c.c. Thus number of atoms per unit volume is $\frac{6 \times 10^{23}}{23} = 2{\cdot}6 \times 10^{22}$ per c.c.

Thus there is almost one conduction electron per atom.

3.268 By definition, mobility $= \frac{\text{dirft velocity}}{\text{Electric field component causing this drift}}$ or $\mu = \frac{v}{E_L}$

On other hand,

$$E_T = vB = \frac{E_L}{\eta}, \text{ as given} \quad \text{so, } \mu = \frac{1}{\eta B} = 3{\cdot}2 \times 10^{-3}\ \text{m}^2/(\text{V}\cdot\text{s})$$

3.269 Due to the straight conductor, $B_\varphi = \frac{\mu_0 I}{2\pi r}$

We use the formula, $\vec{F} = (\vec{p}_m \cdot \vec{\nabla})\vec{B}$

(a) The vector $\vec{p}_m$ is parallel to the straight conductor.

$$\vec{F} = p_m \frac{\partial}{\partial Z}\vec{B} = 0,$$

because neither the direction nor the magnitude of $\vec{B}$ depends on z

(b) The vector $\vec{p}_m$ is oriented along the radius vector $\vec{r}$

$$\vec{F} = p_m \frac{\partial}{\partial r}\vec{B}$$

The direction of $\vec{B}$ at $r + dr$ is parallel to the direction at r. Thus only the φ component of $\vec{F}$ will survive.

$$F_\varphi = p_m \frac{\partial}{\partial r}\frac{\mu_0 I}{2\pi r} = -\frac{\mu_0 I p_m}{2\pi r^2}$$

(c) The vector $\vec{p}_m$ coincides in direction with the magnetic field, produced by the conductor carrying current I

$$\vec{F} = p_m \frac{\partial}{r\partial\varphi}\frac{\mu_0 I}{2\pi}\vec{e}_\varphi = \frac{\mu_0 I p_m}{2\pi r^2}\frac{\partial \vec{e}_\varphi}{\partial\varphi}$$

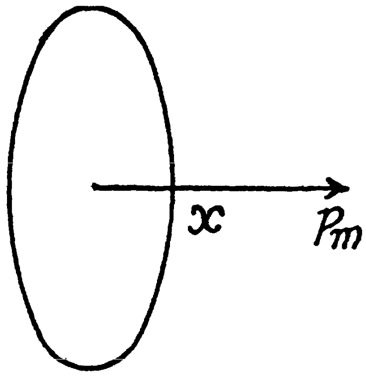
So, $$\vec{F} = -\frac{\mu_0 I p_m}{2\pi r^2}\vec{e}_r \quad \text{As, } \frac{\partial \vec{e}_\varphi}{\partial_\varphi} = -\vec{e}_r$$

3.270 $F_x = p_m \frac{\partial}{\partial x} B_x$

But, $B_x = \frac{\mu_0 I}{4\pi} \int \frac{R\,dl}{(x^2 + R^2)^{3/2}} = \frac{\mu_0 I R^2}{2\,(x^2 + R^2)^{3/2}}$

So, $F = \frac{\mu_0}{4\pi} \frac{I \cdot 2\pi R^2}{(x^2 + R^2)^{5/2}} \frac{3}{2} \cdot 2x \cdot p_m$

$= \frac{\mu_0}{4\pi} \frac{6\pi R^2 I p_m x}{(x^2 + R^2)^{5/2}}$



3.271

$$F = P_{2m} \frac{\partial}{\partial l} \left[\frac{\mu_0}{4\pi} \frac{3\,\vec{p}_{1m} \cdot \vec{r}\,\vec{r} - \vec{p}_{1m} r^2}{r^5} \right]$$

$$= P_{2m} \frac{\partial}{\partial l} \left[\frac{\mu_0}{2\pi} \frac{p_{1m}}{l^3} \right] = \frac{-3}{2} \frac{\mu_0 p_{1m} p_{2m}}{\pi l^4} = 9 \text{ nN}$$

3.272 From 3.270, for $x >> R$,

$$B_x \approx \frac{\mu_0 I' R^2}{2x^3}$$

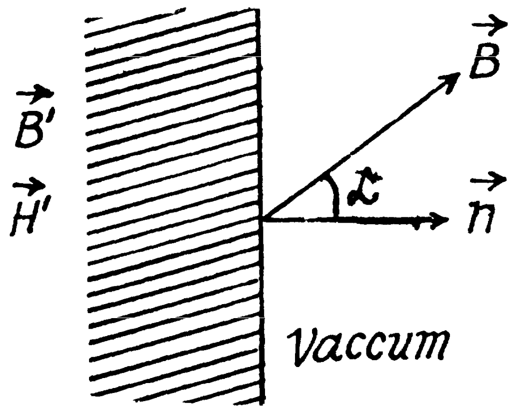
or, $I' \approx \frac{2B_x x^3}{\mu_0 R^2} = \frac{2 \times 3 \times 10^{-5}\,T \times (10^{-1}\text{ m})^3}{1{\cdot}26 \times 10^{-6} \times (10^{-2}\text{ m})^4} \approx 0{\cdot}5$ kA

3.273

$$B'_n = B \cos \alpha,$$

$$H'_t = \frac{1}{\mu_0} B \sin \alpha,$$

$$B'_t = \mu B \sin \alpha$$

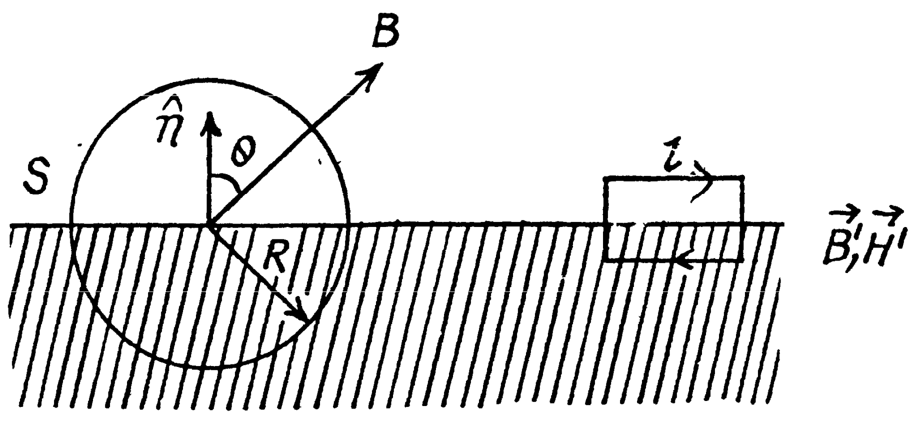


so,

$$B' = B \sqrt{\mu^2 \sin^2 \alpha + \cos^2 \alpha}$$

3.274 (a) $\oint \vec{H} \cdot d\vec{S} = \oint \left(\frac{\vec{B}}{\mu_0} - \vec{J} \right) \cdot d\vec{S} = -\oint \vec{J} \cdot d\vec{S}$, since $\oint \vec{B} \cdot d\vec{S} = 0$

Now $\vec{J}$ is nonvanishing only in the bottom half of the sphere.



Here, $B'_n = B\cos\theta$, $H'_t = \dfrac{1}{\mu_0}B\sin\theta$, $B'_t = \mu B\sin\theta$, $H'_n = \dfrac{B}{\mu\mu_0}\cos\theta$

$$J_n = \frac{B\cos\theta}{\mu_0}\left(1 - \frac{1}{\mu}\right) \text{ and } J_t = \frac{\mu - 1}{\mu_0}B\sin\theta\ .$$

Only J_n contributes the surface integral and

$$-\oint \vec{J}\cdot d\vec{S} = -\oint_{lower} \vec{J}\cdot d\vec{S} = \oint_{lower} J_n\, dS = \frac{\pi R^2 B\cos\theta}{\mu_0}\left(1 - \frac{1}{\mu}\right)$$

(b) $\displaystyle\oint_{\Gamma} \vec{B}\cdot d\vec{r} = (B_t - B'_t)\, l = (1-\mu)\, B\, l\sin\theta$

3.275 Inside the cylindrical wire there is an external current of density $\dfrac{I}{\pi R^2}$. This gives a magnetic field H_φ with

$$H_\varphi\, 2\pi r = I\frac{r^2}{R^2} \quad \text{or,} \quad H_\varphi = \frac{Ir}{2\pi R^2}$$

From this $\displaystyle B_\varphi = \frac{\mu\mu_0 Ir}{2\pi R^2}$ and $\displaystyle J_\varphi = \frac{\mu-1}{2\pi}\frac{Ir}{R^2} = \frac{\chi Ir}{2\pi R^2}$ = Magnetization.

Hence total volume molecular current is,

$$\oint_{r=R} \vec{J_\varphi}\cdot d\vec{r} = \int \frac{\chi I}{2\pi R}\, dl = \chi I$$

The surface current is obtained by using the equivalence of the surface current density to $\vec{J}\times\vec{n}$, this gives rise to a surface current density in the z-direction of $-\dfrac{\chi I}{2\pi R}$

The total molecular surface current is,

$$I'_s = -\frac{\chi I}{2\pi R}(2\pi R) = -\chi I.$$

The two currents have opposite signs.

3.276 We can obtain the form of the curves, required here, by qualitative arguments.

From
$$\oint \vec{H}\cdot d\vec{l} = I,$$

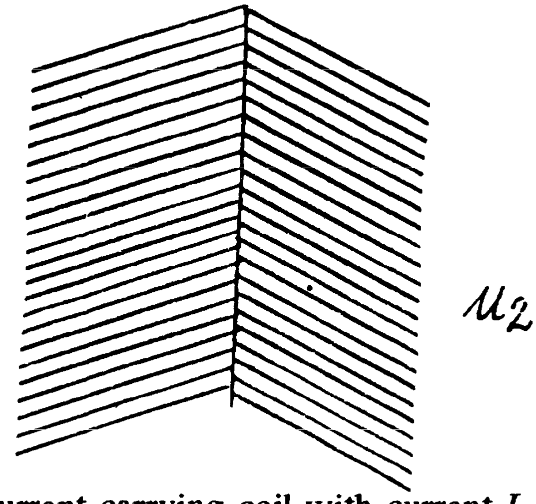
we get
$$H(x \gg 0) = H(x \ll 0) = nI$$

Then
$$B(x \gg 0) = \mu\mu_0\, nI$$
$$B(x < 0) = \mu_0\, nI$$

Also,
$$B(x<0) = \mu_0 H(x<0)$$
$$J(x<0) = 0$$

B is continuous at $x = 0$, H is not. These give the required curves as shown in the answer-sheet.

3.277 The lines of the B as well as H field are circles around the wire. Thus



$$H_1 \pi r + H_2 \pi r = I \quad \text{or,} \quad H_1 + H_2 = \frac{I}{\pi r}$$

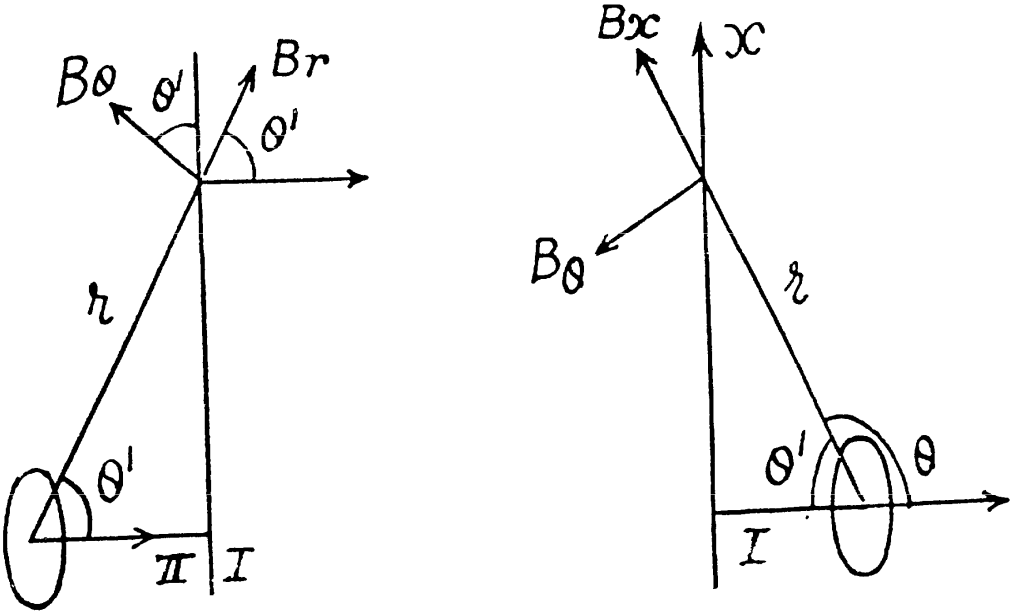
Also $\mu_0 \mu_1 H_1 = \mu_2 H_2 \mu_0 = B_1 = B_2 = B$

Thus $$H_1 = \frac{\mu_2}{\mu_1 + \mu_2} \frac{I}{\pi r},$$

$$H_2 = \frac{\mu_1}{\mu_1 + \mu_2} \frac{I}{\pi r}$$

and $$B = \mu_0 \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \frac{I}{\pi r}.$$

3.278 The medium I is vacuum and contains a circular current carrying coil with current I. The medium II is a magnetic with permeability μ. The boundary is the plane $z = 0$ and the coil is in the plane $z = l$. To find the magnetic induction, we note that the effect of the magnetic medium can be written as due to an image coil in II as far as the medium I is concerned. On the other hand, the induction in II can be written as due to the coil in I, carrying a different current. It is sufficient to consider the far away fields and ensure that the boundary conditions are satisfied there. Now for actual coil in medium I,



$$B_r = -\frac{2p_m \cos\theta'}{r^3} \cdot \left(\frac{\mu_0}{4\pi}\right), \quad B_\theta = \frac{p_m \sin\theta'}{r^3}\left(\frac{\mu_0}{4\pi}\right)$$

so, $$B_z = \frac{\mu_0 p_m}{4\pi}(2\cos^2\theta' - \sin^2\theta') \text{ and } B_x = \frac{\mu_0 p_m}{4\pi}(-3\sin\theta'\cos\theta')$$

where $$p_m = I(\pi a^2), \quad a = \text{radius of the coil.}$$

Similarly due to the image coil,

$$B_z = \frac{\mu_0 p'_m}{4\pi}(2\cos^2\theta' - \sin^2\theta'), \; B_x = \frac{\mu_0 p'_m}{4\pi}(3\sin\theta'\cos\theta'), \; p'_m = I'(\pi a^2)$$

As far as the medium II is concerned, we write similarly

$$B_z = \frac{\mu_0 p''_m}{4\pi}(2\cos^2\theta' - \sin^2\theta'), \; B_x = \frac{\mu_0 p''_m}{4\pi}(-3\sin\theta'\cos\theta'), p''_m = I''(\pi a^2)$$

The boundary conditions are, $p_m + p'_m = p''_m$ (from $B_{1n} = B_{2n}$)

$-p_m + p'_m = -\frac{1}{\mu} p''_m$ (from $H_{1t} = H_{2t}$)

Thus, $$I'' = \frac{2\mu}{\mu+1} I,\ I' = \frac{\mu-1}{\mu+1} I$$

In the limit, when the coil is on the boundary, the magnetic field enverywhere can be obtained by taking the current to be $\frac{2\mu}{\mu+1} I$. Thus, $\vec{B} = \frac{2\mu}{\mu+1} \vec{B}_0$

3.279 We use the fact that within an isolated uniformly magnetized ball, $\vec{H}' = -\vec{J}/3$, $\vec{B}' = \frac{2\mu_0 \vec{J}}{3}$, where $\vec{J}$ is the magnetization vector. Then in a uniform magnetic field with induction $\vec{B}_0$, we have by superposition,

$$\vec{B}_{in} = \vec{B}_0 + \frac{2\mu_0 \vec{J}}{3},\quad \vec{H}_{in} = \frac{\vec{B}_0}{\mu_0} - \vec{J}/3$$

or, $$\vec{B}_{in} + 2\,\mu_0 \vec{H}_{in} = 3\,\vec{B}_0$$

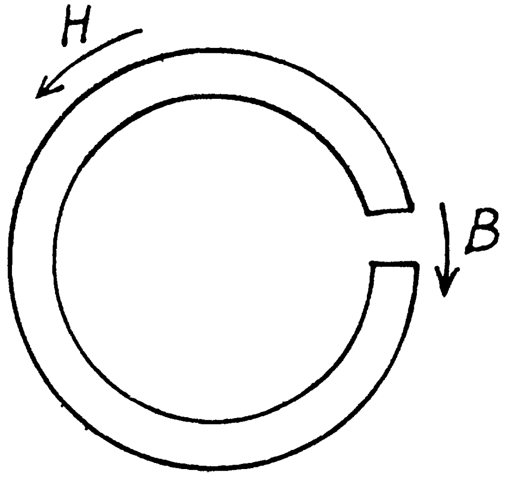
also, $$\vec{B}_{in} = \mu\,\mu_0 \vec{H}_{in}$$

Thus, $$\vec{H}_{in} = \frac{3\,\vec{B}_0}{\mu_0(\mu+2)} \text{ and } \vec{B}_{in} = \frac{3\,\mu\,\vec{B}_0}{\mu+2}$$

3.280 The coercive force H_c is just the magnetic field within the cylinder. This is by circulation theorem, $H_c = \frac{NI}{l} = 6$ kA/m

(from $\oint \vec{H}\cdot d\vec{r} = I$, total current, considering a rectangular contour.)

3.281 We use, $\oint \vec{H}\cdot d\vec{l} = 0$



Neglecting the fringing of the lines of force, we write this as

$$H(\pi d - b) + \frac{B}{\mu_0} b = 0$$

or, $$H \approx \frac{-Bb}{\mu_0 \pi d} = 101 \text{ A/m}$$

The sense of H is opposite to B

3.282 Here, $\oint \vec{H}\cdot d\vec{l} = NI$ or, $H(2\pi R) + \frac{Bb}{\mu_0} = NI$, so, $H = \frac{NI\mu_0 - Bb}{2\pi R\mu_0}$

Hence, $$\mu = \frac{B}{\mu_0 H} = \frac{2\pi RB}{\mu_0 NI - Bb} = 3700$$

3.283 One has to draw the graph of $\mu = \frac{B}{\mu_0 H}$ versus H from the given graph. The $\mu - H$ graph starts out horizontally, and then rises steeply at about $H = 0{\cdot}04$ k A/m before falling agian. It is easy to check that $\mu_{max} \approx 10{,}000$.

3.284 From the theorem on circulation of vector $\vec{H}$.

$$H\pi d+\frac{Bb}{\mu_0}=NI \quad \text{or,} \quad B=\frac{\mu_0 NI}{b}-\frac{\mu_0 \pi d}{b}H=(1\cdot 51-0\cdot 987)\,H,$$

where B is in Tesla and H in kA/m. Besides, B and H are interrelated as in the Fig. 3.76 of the text. Thus we have to solve for B, H graphically by simultaneously drawing the two curves (the hysterisis curve and the straight line, given above) and find the point of intersection. It is at

$$H \approx 0\cdot 26\ \text{kA/m},\ B=1\cdot 25\ \text{T}$$

Then,
$$\mu=\frac{B}{\mu_0 H}\approx 4000.$$

3.285 From the formula,

$$\vec{F}=(\vec{p}_m\cdot\vec{\nabla})\vec{B}\to\vec{F}-P\int(\vec{J}\cdot\vec{\nabla})\vec{B}\,dV,$$

Thus
$$\vec{F}=\frac{\chi}{\mu\,\mu_0}\int(\vec{B}\cdot\vec{\nabla})\vec{B}\,dV$$

or since $\vec{B}$ is predominantly along the x–axis,

$$F_x=\frac{\chi}{\mu\,\mu_0}\int B_x\frac{\partial B_x}{\partial x}\,Sdx=\frac{\chi S}{2\mu\,\mu_0}\int_{x=0}^{x=L}dB_x^2=-\frac{\chi SB^2}{2\mu\,\mu_0}\approx\frac{\chi SB^2}{2\,\mu_0}$$

3.286 The force in question is,

$$\vec{F}=\left(\vec{p}_m\cdot\vec{\nabla}\right)\vec{B}=\frac{\chi BV}{\mu\,\mu_0}\frac{dB}{dx}$$

since B is essentiatlly in the x-direction.

So,
$$F_x\approx\frac{\chi V}{2\,\mu_0}\frac{dB^2}{dx}=\frac{\chi B_0^2 V}{2\,\mu_0}\frac{d}{dx}\left(e^{-2ax^2}\right)=-4\,ax\,e^{-2ax^2}\frac{\chi B_0^2}{2\mu_0}V$$

This is maximum when its derivative vanishes

i.e.
$$16\,a^2x^2-4\,a=0,\quad \text{or,}\quad x_m=\frac{1}{\sqrt{4a}}$$

The maximum force is,

$$F_{\max}=4a\frac{1}{\sqrt{4a}}e^{-1/2}\frac{\chi B_0^2 V}{2\,\mu_0}=\frac{\chi B_0^2 V}{\mu_0}\sqrt{\frac{a}{e}}$$

So,
$$\chi\approx\left(\mu_0 F_{\max}\sqrt{\frac{e}{a}}\right)\Big/\ VB_0^2=3\cdot 6\times 10^{-4}$$

3.287 $F_x=(\vec{p}_m\cdot\vec{\nabla})B_x=\dfrac{\chi BV}{\mu\,\mu_0}\dfrac{dB}{dx}\approx\dfrac{\chi V}{2\,\mu_0}\dfrac{dB^2}{dx}$

This force is attractive and an equal force must be applied for balance. The work done by applied forces is,

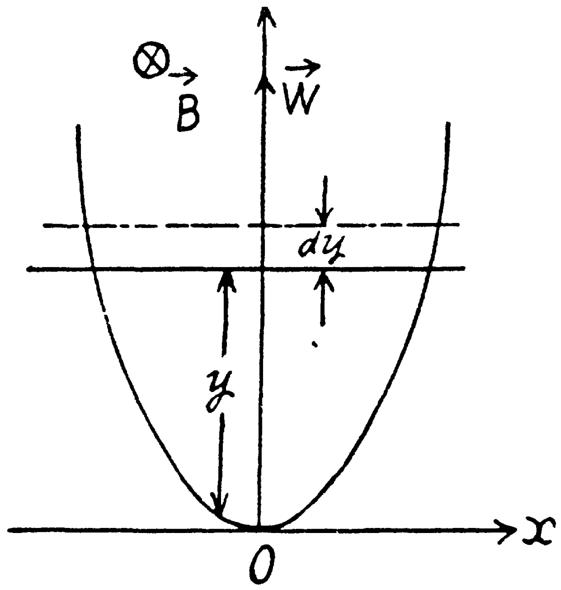
$$A=\int_{x=0}^{x=L}-F_x\,dx=\frac{\chi V}{2\,\mu_0}\left(-B^2\right)_{x=0}^{x=L}\approx\frac{\chi VB^2}{2\,\mu_0}$$

3.6 ELECTROMAGNETIC INDUCTION. MAXWELL'S EQUATIONS

3.288 Obviously, from Lenz's law, the induced current and hence the induced e.m.f. in the loop is anticlockwise.

From Faraday's law of electromagnetic indcution,

$$\xi_{in} = \left|\frac{d\Phi}{dt}\right|$$



Here, $d\Phi = \vec{B}\cdot d\vec{S} = -2B\,x\,dy,$

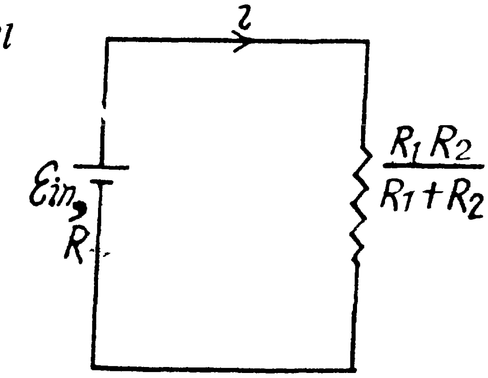
and from $y = ax^2, x = \sqrt{\frac{y}{a}}$

Hence, $\xi_{in} = 2B\sqrt{\frac{y}{a}}\frac{dy}{dt}$

$$= By\sqrt{\frac{8w}{a}}, \text{ using } \frac{dy}{dt} = \sqrt{2\,wy}$$

3.289 Let us assume, $\vec{B}$ is directed into the plane of the loop. Then the motional e.m.f.

$$\xi_{in} = \left|\int -(\vec{v}\times\vec{B})\cdot d\vec{l}\right| = v\,Bl$$



and directed in the same of $(\vec{v}\times\vec{B})$ (Fig.)

So, $i = \dfrac{\xi_{in}}{R + \dfrac{R_1 R_2}{R_1 + R_2}} = \dfrac{B\,v\,l}{R + R_u}$

As R_1 and R_2 are in parallel connections.

3.290 (a) As the metal disc rotates, any free electron also rotates with it with same angular velocity ω, and that's why an electron must have an acceleration $\omega^2 r$ directed towards the disc's centre, where r is separation of the electron from the centre of the disc. We know from Newton's second law that if a particle has some acceleration then there must be a net effecetive force on it in the direction of acceleration. We also know that a charged particle can be influenced by two fields electric and magnetic. In our problem magnetic field is absent hence we reach at the conslusion that there is an electric field near any electron and is directed opposite to the acceleration of the electron.

If E be the electric field strength at a distance r from the centre of the disc, we have from Newton's second law.

$$F_n = m\,w_n$$

$$eE = m\,r\,\omega^2, \text{ or, } E = \frac{m\,\omega^2\,r}{e},$$

and the potential difference,

$$\varphi_{cen} - \varphi_{rim} = \int_0^a \vec{E}\cdot d\vec{r} = \int_0^a \frac{m\,\omega^2\,r}{e}\,dr, \text{ as } \vec{E}\uparrow\downarrow d\vec{r}$$

Thus $$\varphi_{cen} - \varphi_{rim} = \Delta\varphi = \frac{m\,\omega^2}{e}\frac{a^2}{2} = 3{\cdot}0\ \text{nV}$$

(b) When field $\vec{B}$ is present, by definition, of motional e.m.f. :

$$\varphi_1 - \varphi_2 = \int_1^2 -(\vec{v}\times\vec{B})\cdot d\vec{r}$$

Hence the sought potential difference,

$$\varphi_{cen} - \varphi_{rim} = \int_0^a -v\,B\,dr = \int_0^a -\omega\,r\,B\,dr\,,\ \text{(as } v = \omega\,r)$$

Thus $$\varphi_{rim} - \varphi_{cen} = \varphi = \frac{1}{2}\omega\,B\,a^2 = 20\ \text{mV}$$

(In general $\omega < \frac{eB}{m}$ so we can neglect the effect discussed in (1) here).

3.291 By definition,

$$\vec{E} = -(\vec{v}\times\vec{B})$$

So, $$\int_A^C \vec{E}\cdot d\vec{r} = \int_A^C -(\vec{v}\times\vec{B})\cdot d\vec{r} = \int_0^d -v\,B\,dr$$

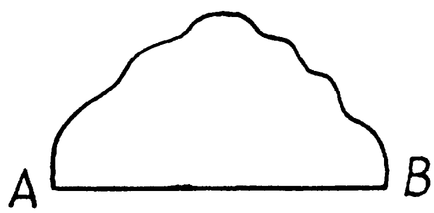
But, $v = \omega r$, where r is the perpendicular distance of the point from A.

Hence, $$\int_A^C \vec{E}\cdot d\vec{r} = \int_0^d -\omega\,B\,r\,dr = -\frac{1}{2}\omega\,B\,d^2 = -10\ \text{mV}$$

This result can be generalized to a wire AC of arbitary planar shape. We have

$$\int_A^C \vec{E}\cdot d\vec{r} = -\int_A^C (\vec{v}\times\vec{B})\cdot d\vec{r} = -\int_A^C ((\vec{\omega}\times\vec{r})\times\vec{B})\cdot d\vec{r}$$

$$= -\int_A^C (\vec{B}\cdot\vec{r}\,\vec{\omega} - \vec{B}\cdot\vec{\omega}\,\vec{r})\cdot d\vec{r}$$



$$= -\frac{1}{2}B\omega d^2,$$

d being AC and $\vec{r}$ being measured from A.

3.292 Flux at any moment of time,

$$|\Phi_t| = |\vec{B}\cdot d\vec{S}| = B\left(\frac{1}{2}R^2\varphi\right)$$

where φ is the sector angle, enclosed by the field.

Now, magnitude of induced e.m.f. is given by,

$$\xi_{in} = \left|\frac{d\,\Phi_t}{dt}\right| = \left|\frac{BR^2}{2}\frac{d\varphi}{dt}\right| = \frac{BR^2}{2}\omega,$$

where ω is the angular velocity of the disc. But as it starts rotating from rest at $t = 0$ with an angular acceleration β its angular velocity $\omega(t) = \beta t$. So,

$$\xi_{in} = \frac{BR^2}{2}\beta t.$$

According to Lenz law the first half cycle current in the loop is in anticlockwise sense, and in subsequent half cycle it is in clockwise sense.

Thus in general, $\xi_{in} = (-1)^n \dfrac{BR^2}{2}\beta\, t$, where n in number of half revolutions.

The plot $\xi_{in}(t)$, where $t_n = \sqrt{2\pi n/\beta}$ is shown in the answer sheet.

3.293 Field, due to the current carrying wire in the region, right to it, is directed into the plane of the paper and its magnitude is given by,

$B = \dfrac{\mu_0}{2\pi}\dfrac{i}{r}$ where r is the perpendicular distance from the wire.

As B is same along the length of the rod thus motional e.m.f.

$$\xi_{in} = \left|-\int_1^2 (\vec{v}\times\vec{B})\cdot d\vec{l}\right| = vBl$$

and it is directed in the sense of $(\vec{v}\times\vec{B})$

So, current (induced) in the loop,

$$i_{in} = \frac{\xi_{in}}{R} = \frac{1}{2}\frac{\mu_0 I vi}{\pi R r}$$

3.294 Field, due to the current carrying wire, at a perpendicular distance x from it is given by,

$$B(x) = \frac{\mu_0}{2\pi}\frac{i}{x}$$

Motional e.m.f is given by $\left|\int -(\vec{v}\times\vec{B})\cdot d\vec{l}\right|$

There will be no induced e.m.f. in the segments (2) and (4) as, $\vec{v}\uparrow\uparrow d\vec{l}$ and magnitude of e.m.f. induced in 1 and 3, will be

$$\xi_1 = v\left(\frac{\mu_0}{2\pi}\frac{i}{x}\right)a \text{ and } \xi_2 = v\left(\frac{\mu_0}{2\pi}\frac{i}{(a+x)}\right)a,$$

respectively, and their sense will be in the direction of $(\vec{v}\times\vec{B})$.

So, e.m.f., induced in the network $= \xi_1 - \xi_2$ [as $\xi_1 > \xi_2$]

$$= \frac{a v \mu_0 i}{2\pi}\left[\frac{1}{x} - \frac{1}{a+x}\right] = \frac{v a^2 \mu_0 i}{2\pi x(a+x)}$$

3.295 As the rod rotates, an emf.

$$\frac{d}{dt}\frac{1}{2}a^2\,\theta \cdot B = \frac{1}{2}a^2 B\,\omega$$

is induced in it. The net current in the conductor is then $\dfrac{\xi(t) - \frac{1}{2}a^2 B\,\omega}{R}$

A magnetic force will then act on the conductor of magnitude BI per unit length. Its direction will be normal to B and the rod and its torque will be

$$\int_0^a \left(\frac{\xi(t) - \frac{1}{2}a^2 B\,\omega}{R}\right) dx\, B\, x$$

Obviously both magnetic and mechanical torque acting on the C.M. of the rod must be equal but opposite in sense. Then

for equilibrium at constant ω

$$\frac{\xi(t) - \frac{1}{2}a^2 B\,\omega}{R} \cdot \frac{B\,a^2}{2} = \frac{1}{2}mga\sin\omega t$$

or, $\quad \xi(t) = \dfrac{1}{2}a^2 B\,\omega + \dfrac{mg\,R}{aB}\sin\omega\, t = \dfrac{1}{2\,aB}(a^3 B^2\,\omega + 2\,mg\,R\sin\omega\, t)$

(The answer given in the book is incorrect dimensionally.)

3.296 From Lenz's law, the current through the connector is directed form A to B. Here $\xi_{in} = vBl$ between A and B

where v is the velocity of the rod at any moment.

For the rod, from $F_x = mw_x$

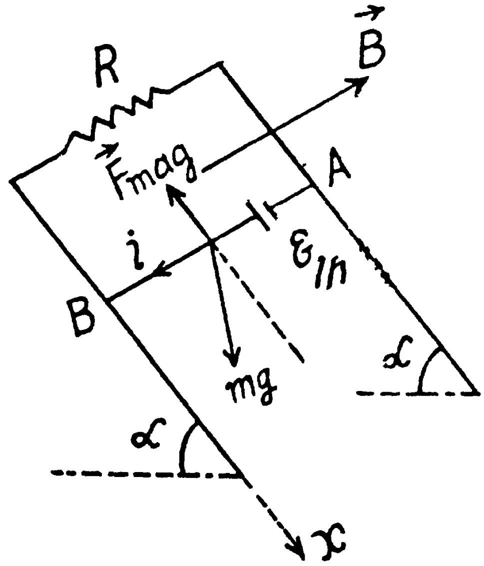
or, $\quad mg\sin\alpha - i\,l\,B = mw$

For steady state, acceleration of the rod must be equal to zero.

Hence, $\quad mg\sin\alpha = i\,l\,B \qquad (1)$

But, $\quad i = \dfrac{\xi_{in}}{R} = \dfrac{v\,B\,l}{R}$

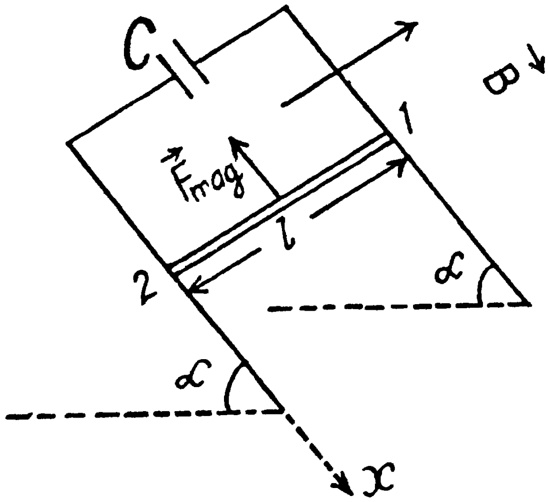
From (1) and (2) $\quad v = \dfrac{mg\sin\alpha\, R}{B^2\, l^2}$



3.297 From Lenz's law, the current through the copper bar is directed from 1 to 2 or in other words, the induced crrrent in the circuit is in clockwise sense.

Potential difference across the capacitor plates,

$$\frac{q}{C} = \xi_{in} \quad \text{or,} \quad q = C\,\xi_{in}$$



Hence, the induced current in the loop,

$$i = \frac{dq}{dt} = C\frac{d\xi_{in}}{dt}$$

But the variation of magnetic flux through the loop is caused by the movement of the bar. So, the induced e.m.f. $\xi_{in} = B\,lv$

and, $$\frac{d\xi_{in}}{dt} = Bl\frac{dv}{dt} = Blw$$

Hence, $$i = C\frac{d\xi}{dt} = C\,B\,l\,w$$

Now, the forces acting on the bars are the weight and the Ampere's force, where $F_{amp} = i\,lB\,(CB\,l\,w)\;B = C\,l^2 B^2 w.$

From Newton's second law, for the rod, $F_x = mw_x$

or, $$mg\sin\alpha - C\,l^2 B^2\,w = mw$$

Hence $$w = \frac{mg\sin\alpha}{C\,l^2 B^2 + m} = \frac{g\sin\alpha}{1 + \dfrac{l^2 B^2 C}{m}}$$

3.298 Flux of $\vec{B}$, at an arbitrary moment of time t :

$$\Phi_t = \vec{B}\cdot\vec{S} = B\frac{\pi a^2}{2}\cos\omega t,$$

From Faraday's law, induced e.m.f., $\xi_{in} = -\dfrac{d\Phi}{dt}$

$$= -\frac{d\left(B\pi\dfrac{a^2}{2}\cos\omega t\right)}{dt} = \frac{B\pi a^2\omega}{2}\sin\omega t.$$

and induced current, $$i_{in} = \frac{\xi_{in}}{R} = \frac{B\pi a^2}{2R}\,\omega\sin\omega t.$$

Now thermal power, generated in the circuit, at the moment $t = t$:

$$P(t) = \xi_{in}\times i_{in} = \left(\frac{B\pi a^2\omega}{2}\right)^2\frac{1}{R}\sin^2\omega t$$

and mean thermal power generated,

$$<P> = \frac{\left[\dfrac{B\pi a^2\omega}{2}\right]^2\dfrac{1}{R}\displaystyle\int_0^T \sin^2\omega t\,dt}{\displaystyle\int_0^T dt} = \frac{1}{2R}\left(\frac{\pi\omega a^2 B}{2}\right)^2$$

Note : The claculation of ξ_{in} which can also be checked by using motional emf is correct even though the conductor is not a closed semicircle , for the flux linked to the rectangular part containing the resistance R is not changing. The answer given in the book is off by a factor 1/4.

3.299 The flux through the coil changes sign. Initially it is BS per turn.

Finally it is $-BS$ per turn. Now if flux is Φ at an intermediate state then the current at that moment will be

$$i = \frac{-N \dfrac{d\Phi}{dt}}{R}$$

So charge that flows during a sudden turning of the coil is

$$q = \int i\,dt = -\frac{N}{R}[\Phi - (-\Phi)] = 2NBS/R$$

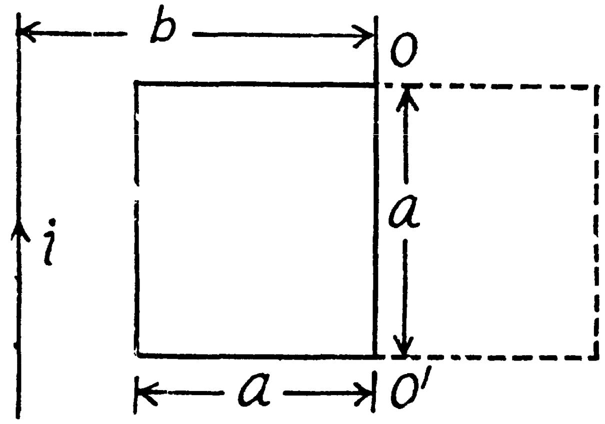
Hence, $$B = \frac{1}{2}\frac{qR}{NS} = 0{\cdot}5\text{ T on putting the values.}$$

3.300 According to Ohm's law and Faraday's law of induction, the current i_0 appearing in the frame, during its rotation, is determined by the formula,

$$i_0 = -\frac{d\Phi}{dt} = -\frac{L\,d\,i_0}{dt}$$

Hence, the required amount of electricity (charge) is,

$$q = \int i_0\,dt = -\frac{1}{R}\int (d\Phi + L\,di_0) = -\frac{1}{R}(\Delta\Phi + L\,\Delta i_0)$$



Since the frame has been stopped after rotation, the current in it vanishes, and hence $\Delta i_0 = 0$. It remains for us to find the increment of the flux $\Delta\Phi$ through the frame ($\Delta\Phi = \Phi_2 - \Phi_1$).

Let us choose the normal $\vec{n}$ to the plane of the frame, for instance, so that in the final position, $\vec{n}$ is directed behind the plane of the figure (along $\vec{B}$).

Then it can be easily seen that in the final position, $\Phi_2 > 0$, while in the initial position, $\Phi_1 < 0$ (the normal is opposite to $\vec{B}$), and $\Delta\Phi$ turns out to be simply equal to the fulx through the surface bounded by the final and initial positions of the frame :

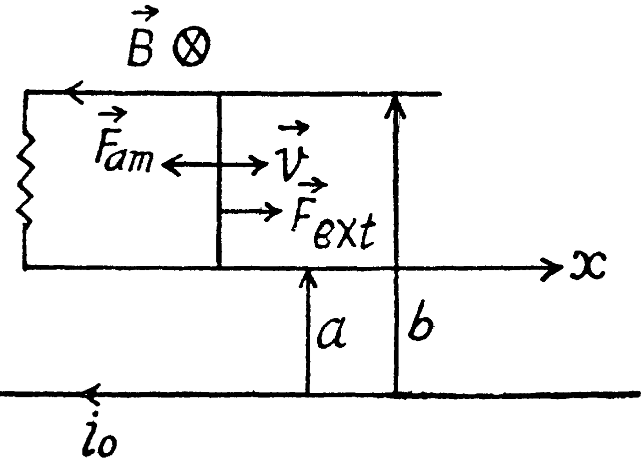
$$\Delta\Phi = \Phi_2 + |\Phi_1| = \int_{b-a}^{b+a} B\,a\,dr,$$

where B is a function of r, whose form can be easily found with the help of the theorem of circulation. Finally omitting the minus sign, we obtain,

$$q = \frac{\Delta\Phi}{R} = \frac{\mu_0 a\,i}{2\pi R}\ln\frac{b+a}{b-a}$$

3.301 As $\vec{B}$, due to the straight current carrying wire, varies along the rod (connector) and enters linerarly so, to make the calculations simple, $\vec{B}$ is made constant by taking its average value in the range $[a, b]$.

$$\langle B \rangle = \frac{\int_a^b B\,dr}{\int_a^b dr} = \frac{\int_a^b \frac{\mu_0}{2\pi}\frac{i_0}{r}\,dr}{\int_a^b dr}$$



or, $$\langle B \rangle = \frac{\mu_0}{2\pi}\frac{i_0}{(b-a)}\ln\frac{b}{a}$$

(a) The flux of $\vec{B}$ changes through the loop due to the movement of the connector. According to Lenz's law, the current in the loop will be anticlockwise. The magnitude of motional e.m.f.,

$$\xi_{in} = v \langle B \rangle (b-a)$$

$$= \frac{\mu_0}{2\pi}\frac{i_0}{(b-a)}\ln\frac{b}{a}(b-a)\frac{dx}{dt} = \frac{\mu_0}{2\pi}i_0 \ln\frac{b}{a}v$$

So, induced current

$$i_{in} = \frac{\xi_{in}}{R} = \frac{\mu_0}{2\pi}\frac{i_0 v}{R}\ln\frac{b}{a}$$

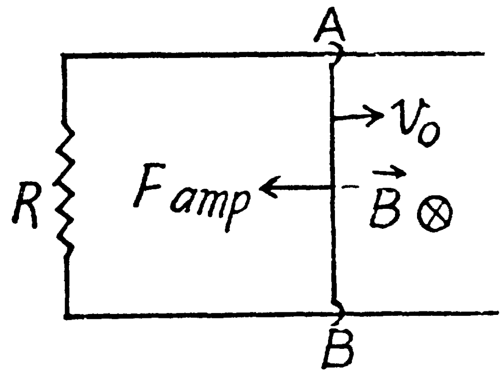
(b) The force required to maintain the constant velocity of the connector must be the magnitude equal to that of Ampere's acting on the connector, but in opposite direction.

So, $$F_{ext} = i_{in}\, l \langle B \rangle = \left(\frac{\mu_0}{2\pi}\frac{i_0}{R}v\ln\frac{b}{a}\right)(b-a)\left(\frac{\mu_0}{2\pi}\frac{i_0}{(b-a)}\ln\frac{b}{a}\right)$$

$$= \frac{v}{R}\left(\frac{\mu_0}{2\pi}i_0\ln\frac{b}{a}\right)^2$$, and will be directed as shown in the (Fig.)

3.302 (a) The flux through the loop changes due to the movement of the rod AB. According to Lenz's law current should be anticlockwise in sense as we have assumed $\vec{B}$ is directed into the plane of the loop. The motion e.m.f $\xi_{in}(t) = B\,l\,v$

and induced current $i_{in} = \frac{vBl}{R}$



From Newton's law in projection form $F_x = mw_x$

$$-F_{amp} = m\frac{v\,dv}{dx}$$

But $$F_{amp} = i_{in}\,l\,B = \frac{vB^2l^2}{R}$$

So, $$-\frac{vB^2l^2}{R} = mv\frac{dv}{dx}$$

or, $$\int_0^x dx = -\frac{mR}{B^2 l^2}\int_{v_0}^0 dv \quad \text{or,} \quad x = \frac{mRv_0}{B^2 l^2}$$

(b) From equation of energy conservation; $E_f - E_i +$ Heat liberated $= A_{cell} + A_{ext}$

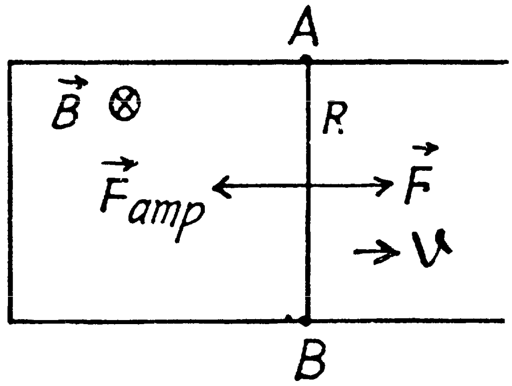
$$\left[0 - \frac{1}{2}m v_0^2\right] + \text{Heat liberated} = 0 + 0$$

So, heat liberated $= \frac{1}{2}m v_0^2$

3.303 With the help of the calculation, done in the previous problem, Ampere's force on the connector,

$\vec{F}_{amp} = \frac{vB^2l^2}{R}$ directed towards left.

Now from Newton's second law,



$$F - F_{amp} = m\frac{dv}{dt}$$

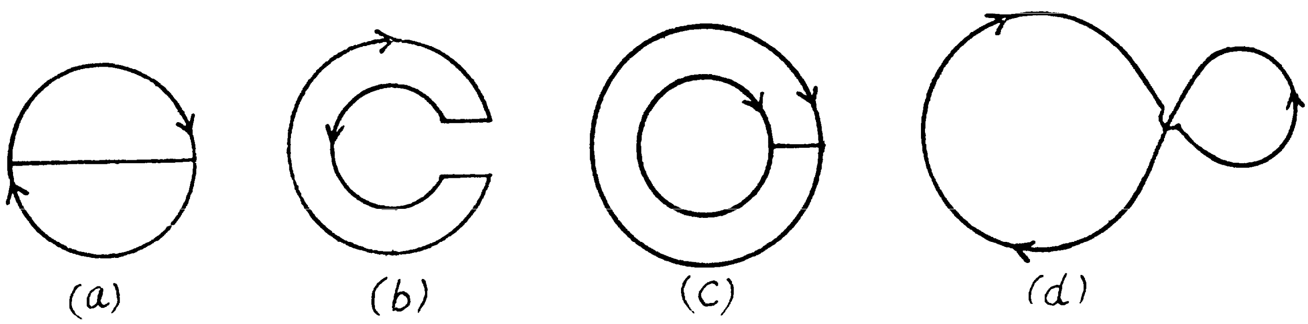
So, $$F = \frac{vB^2l^2}{R} + m\frac{dv}{dt}$$

or, $$\int_0^t dt = m\int_0^v \frac{dv}{F - \frac{vB^2l^2}{R}}$$

or, $$\frac{t}{m} = -\frac{R}{B^2l^2}\ln\left(\frac{F - \frac{vB^2l^2}{R}}{F}\right)$$

Thus $$v = \left(1 - e^{\frac{-tB^2l^2}{Rm}}\right)\frac{RF}{B^2l^2}$$

3.304 According to Lenz, the sense of induced e.m.f. is such that it opposes the cause of change of flux. In our problem, magnetic field is directed away from the reader and is diminishing.



So, in figure (a), in the round conductor, it is clockwise and there is no current in the connector

In figure (b) in the outside conductor, clockwise.

In figure (c) in both the conductor, clockwise; and there is no current in the connector to obey the charge conservation.

In figure (d) in the left side of the figure, clockwise.

3.305 The loops are connected in such a way that if the current is clockwise in one, it is anticlockwise in the other. Hence the e.m.f. in loop b opposes the e.m.f. in loop a.

e.m.f. in loop $a = \frac{d}{dt}(a^2 B) = a^2 \frac{d}{dt}(B_0 \sin \omega t)$

Similarly, e.m.f. in loop $b = b^2 B_0 \omega \cos \omega t$.

Hence, net e.m.f. in the circuit $= (a^2 - b^2) B_0 \omega \cos \omega t$, as both the e.m.f's are in opposite sense, and resistance of the circuit $= 4(a+b)\rho$

Therefore, the amplitude of the current

$$= \frac{(a^2 - b^2) B_0 \omega}{4(a+b)\rho} = 0{\cdot}5 \text{ A}.$$

3.306 The flat shape is made up of concentric loops, having different radii, varying from 0 to a.

Let us consider an elementary loop of radius r, then e.m.f. induced due to this loop

$$= \frac{-d(\vec{B} \cdot \vec{S})}{dt} = \pi r^2 B_0 \omega \cos \omega t.$$

and the total induced e.m.f.,

$$\xi_{ind} = \int_0^a (\pi r^2 B_0 \omega \cos \omega t)\, dN, \tag{1}$$

where $\pi r^2 \omega \cos \omega t$ is the contribution of one turn of radius r and dN is the number of turns in the interval $[r, r + dr]$.

So, $$dN = \left(\frac{N}{a}\right) dr \tag{2}$$

From (1) and (2), $\xi = \int_0^a -(\pi r^2 B_0 \omega \cos \omega t)\frac{N}{a} dr = \frac{\pi B_0 \omega \cos \omega t N a^2}{3}$

Maximum value of e.m.f. amplitude $\xi_{max} = \frac{1}{3}\pi B_0 \omega N a^2$

3.307 The flux through the loop changes due to the variation in $\vec{B}$ with time and also due to the movement of the connector.

So, $\xi_{in} = \left|\frac{d(\vec{B} \cdot \vec{S})}{dt}\right| = \left|\frac{d(BS)}{dt}\right|$ as $\vec{S}$ and $\vec{B}$ are colliniear

But, B, after t sec. of beginning of motion $= Bt$, and S becomes $= l\frac{1}{2}wt^2$, as connector starts moving from rest with a constant acceleration w.

So, $$\xi_{ind} = \frac{3}{2} B l w t^2$$

3.308 We use $B = \mu_0 n I$

Then, from the law of electromagnetic induction

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

So, for $r < a$

$$E_\varphi 2\pi r = -\pi r^2 \mu_0 n \dot{I} \text{ or, } E_\varphi = -\frac{1}{2}\mu_0 n r \dot{I}. \text{ (where } \dot{I} = dI/dt)$$

For $r>a$

$$E_\varphi 2\pi r = -\pi a^2 \mu_0 n \dot{I} \text{ or, } E_\varphi = -\mu_0 n \dot{I} a^2/2r$$

The meaning of minus sign can be deduced from Lenz's law.

3.309 The e.m.f. induced in the turn is $\mu_0 n \dot{I} \pi \frac{d^2}{4}$

The resistance is $\frac{\pi d}{S}\rho$.

So, the current is $\frac{\mu_0 n \dot{I} S d}{4\rho} = 2\text{ mA}$, where ρ is the resistivity of copper.

3.310 The changing magnetic field will induce an e.m.f. in the ring, which is obviously equal, in the two parts by symmetry (the e.m.f. induced by electromagnetic induction does not depend on resistance). The current, that will flow due to this, will be different in the two parts. This will cause an acceleration of charge, leading to the setting up of an electric field E which has opposite sign in the two parts. Thus,

$$\frac{\xi}{2} - \pi a E = r I \text{ and, } \frac{\xi}{2} + \pi a E = \eta r I,$$

where ξ is the total induced e.m.f. From this,

$$\xi = (\eta + 1)\, rI,$$

and $$E = \frac{1}{2\pi a}(\eta - 1)\, rI = \frac{1}{2\pi a}\frac{\eta - 1}{\eta + 1}\xi$$

But by Faraday's law, $\xi = \pi a^2 b$

so, $$E = \frac{1}{2}ab\frac{\eta - 1}{\eta + 1}$$

3.311 Go to the rotating frame with an instantaneous angular velocity $\vec{\omega}(t)$. In this frame, a Coriolis force, $2m\vec{v}' \times \vec{\omega}(t)$

acts which must be balanced by the magnetic force, $e\vec{v} \times \vec{B}(t)$.

Thus, $$\vec{\omega}(t) = -\frac{e}{2m}\vec{B}(t).$$

(It is assumed that $\vec{\omega}$ is small and varies slowly, so ω^2 and $\dot{\omega}$ can be neglected.)

3.312 The solenoid has an inductance,

$$L = \mu_0 n^2 \pi b^2 l,$$

where n = number of turns of the solenoid per unit length. When the solenoid is connected to the source an e.m.f. is set up, which, because of the inductance and resistance, rises slowly, according to the equation,

$$RI + L\dot{I} = V$$

This has the well known solution,

$$I = \frac{V}{R}(1 - e^{-tR/L}).$$

Corresponding to this current, an e.m.f. is induced in the ring. Its magnetic field $B = \mu_0 n I$ in the solenoid, produces a force per unit length, $\frac{dF}{dl} = B\,i = \mu_0^2 n^2 \pi a^2 II/r$

$$= \frac{\mu_0^2 \pi a^2 V^2}{r}\left(\frac{n^2}{RL}\right) e^{-tR/L}(1 - e^{-tR/L}),$$

acting on each segment of the ring. This force is zero initially and zero for large t. Its maximum value is for some finite t. The maximum value of

$$e^{-tR/L}(1 - e^{-tR/L}) = \frac{1}{4} - \left(\frac{1}{2} - e^{-tR/L}\right)^2 \text{ is } \frac{1}{4}.$$

So $$\frac{dF_{max}}{dl} = \frac{\mu_0^2 \pi a^2 V^2}{r}\frac{n^2}{4RL} = \frac{\mu_0 a^2 V^2}{4rRlb^2}$$

3.313 The amount of heat generated in the loop during a small time interval dt,

$$dQ = \xi^2/R\,dt, \text{ but, } \xi = -\frac{d\Phi}{dt} = 2at - a\tau,$$

So, $$dQ = \frac{(2at - a\tau)^2}{R}dt$$

and hence, the amount of heat, generated in the loop during the time interval 0 to τ.

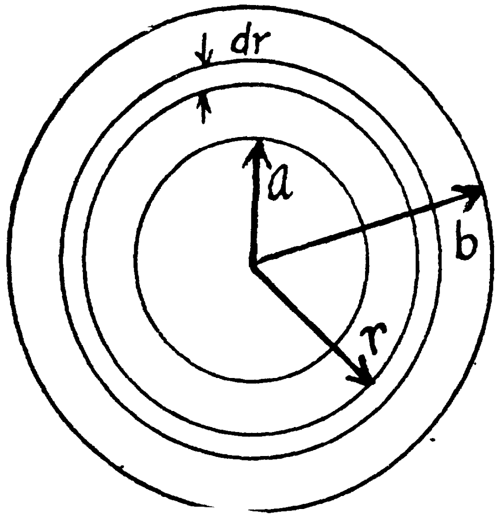
$$Q = \int_0^\tau \frac{(2at - a\tau)^2}{R}dt = \frac{1}{3}\frac{a^2\tau^3}{R}$$

3.314 Take an elementary ring of radius r and width dr.

The e.m.f. induced in this elementary ring is $\pi r^2 \beta$.

Now the conductance of this ring is.

$$d\left(\frac{1}{R}\right) = \frac{h\,dr}{\rho\,2\pi r} \text{ so } dI = \frac{h\,r\,dr}{2\rho}\beta$$



Integrating we get the total current,

$$I = \int_a^b \frac{hr\,dr}{2\rho}\beta = \frac{h\beta(b^2 - a^2)}{4\rho}$$

3.315 Given $L = \mu_0 n^2 V = \mu_0 n^2 l_0 \pi R^2$, where R is the radius of the solenoid.

Thus, $$n = \sqrt{\frac{L}{\mu_0 l_0 \pi}}\frac{1}{R}.$$

So, length of the wire required is,

$$l = n\, l_0\, 2\pi R = \sqrt{\frac{4\pi L\, l_0}{\mu_0}} = 0{\cdot}10\ \text{km}.$$

3.316 From the previous problem, we know that,

l' = length of the wire needed $= \sqrt{\dfrac{L\, l\, 4\pi}{\mu_0}}$, where l = length of solenoid here.

Now, $R = \dfrac{\rho_0 l'}{S}$, (where S = area of crossection of the wire. Also $m = \rho S l'$)

Thus, $$l' = \frac{RS}{\rho_0} = \frac{Rm}{\rho\rho_0 l'} \quad or \quad l' = \sqrt{\frac{Rm}{\rho\rho_0}}$$

where ρ_0 = resistivity of copper and ρ = its density.

Equating, $$\frac{Rm}{\rho\rho_0} = \frac{Ll}{\mu_0/4\pi}$$

or, $$L = \frac{\mu_0}{4\pi}\frac{mR}{\rho\rho_0 l}$$

3.317 The current at time t is given by,

$$I(t) = \frac{V}{R}\left(1 - e^{-tR/L}\right)$$

The steady state value is, $I_0 = \dfrac{V}{R}$

and $$\frac{I(t)}{I_0} = \eta = 1 - e^{-tR/L} \text{ or }, \ e^{-tR/L} = 1 - \eta$$

or, $$t_o\frac{R}{L} = \ln\frac{1}{1-\eta} \quad \text{or,} \quad t_o = \frac{L}{R}\ln\frac{1}{1-\eta} = 1{\cdot}49\ \text{s}$$

3.318 The time constant τ is given by

$$\tau = \frac{L}{R} = \frac{L}{\rho_0 \dfrac{l_0}{S}},$$

where, ρ_0 = resistivity, l_0 = length of the winding wire, S = cross section of the wire.

But $$m = l\,\rho_0\, S$$

So eliminating S, $\tau = \dfrac{L}{\dfrac{\rho_0 l_0}{m/\rho l_0}} = \dfrac{mL}{\rho\rho_0 l_0^2}$

From problem 3.315 $l_0 = \sqrt{\dfrac{4\pi l L}{\mu_0}}$

(note the interchange of l and l_0 because of difference in notation here.)

Thus, $$\tau = \frac{mL}{\rho\rho_0 \frac{4\pi}{\mu_0} L\, l} = \mu_0 4\pi \frac{m}{\rho\, \rho_0\, l} = 0{\cdot}7 \text{ ms},$$

3.319 Between the cables, where $a < r < b$, the magnetic field $\vec{H}$ satisfies

$$H_\varphi 2\pi r = I \quad \text{or,} \quad H_\varphi = \frac{I}{2\pi r}$$

So $$B_\varphi = \frac{\mu\, \mu_0 I}{2\pi r}$$

The associated flux per unit length is, $\Phi = \displaystyle\int_{r=a}^{r=b} \frac{\mu\, \mu_0 I}{2\pi r} \times 1 \times dr = \frac{\mu\, \mu_0 I}{2\pi} \ln \frac{b}{a}$

Hence, the inductance per unit length $L_1 = \dfrac{\Phi}{I} = \dfrac{\mu\mu_0}{2\pi} \ln \eta$, where $\eta = \dfrac{b}{a}$

We get $L_1 = 0{\cdot}26 \dfrac{\mu H}{m}$

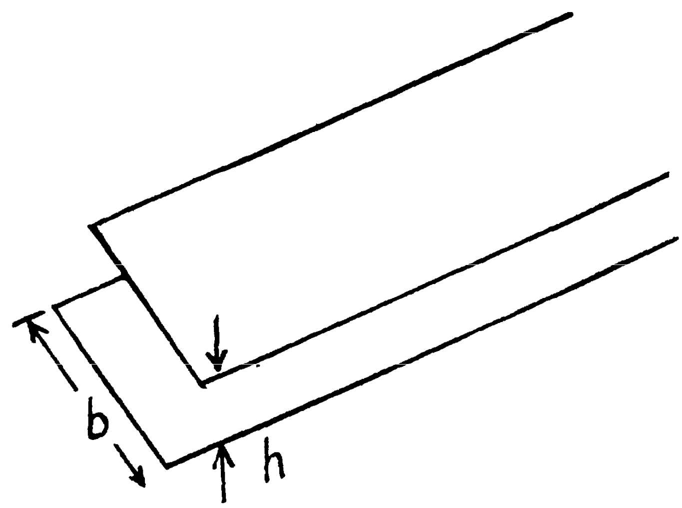
3.320 Within the solenoid, $H_\varphi \cdot 2\pi r = NI$ or $H_\varphi = \dfrac{NI}{2\pi r}$, $B_\varphi = \mu\mu_0 \dfrac{NI}{2\pi r}$

and the flux, $\Phi = N\Phi_1$, $= N \dfrac{\mu\mu_0}{2\pi} NI \displaystyle\int_b^{a+b} \frac{a\, dr}{r}$

Finally, $$L = \frac{\mu\, \mu_0}{2\pi} N^2 a \ln\left(1 + \frac{a}{b}\right)$$

3.321 Neglecting end effects the magnetic field B, between the plates, which is mainly parallel to the plates, is $B = \mu_0 \dfrac{I}{b}$

(For a derivation see 3.229 b)



Thus, the associated flux per unit length of the plates is,

$$\Phi = \mu_0 \frac{I}{b} \times h \times 1 = \left(\mu_0 \frac{h}{b}\right) \times I.$$

So, L_1 = inductance per unit length $= \mu_0 \dfrac{h}{b} = 25 \text{ nH/m}.$

3.322 For a single current carrying wire, $B_\varphi = \frac{\mu_0 I}{2\pi r}$ $(r > a)$. For the double line cable, with current, flowing in opposite directions, in the two conductors,

$B_\varphi \approx \frac{\mu_0 I}{\pi r}$, between the cables, by superposition. The associated flux is,

$$\Phi = \int_a^{d-a} \frac{\mu_0 I}{\pi} \frac{dr \times 1}{r} \approx \frac{\mu_0 I}{\pi} \ln \frac{d}{a} = \frac{\mu_0}{\pi} \ln \eta \times I, \text{ per unit length}$$

Hence, $$L_1 = \frac{\mu_0}{\pi} \ln \eta$$

is the inductance per unit length.

3.323 In a superconductor there is no resistance, Hence,

$$L \frac{dI}{dt} = +\frac{d\Phi}{dt}$$

So integrating, $$I = \frac{\Delta\Phi}{L} = \frac{\pi a^2 B}{L}$$

because $$\Delta\Phi = \Phi_f - \Phi_i, \Phi_f = \pi a^2 B, \Phi_i = 0$$

Also, the work done is, $A = \int \xi I\, dt = \int I\, dt \frac{d\Phi}{dt} = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\pi^2 a^4 B^2}{L}$

3.324 In a solenoid, the inductance $L = \mu\mu_0 n^2 V = \mu \mu_0 \frac{N^2 S}{l}$,

where S = area of cross section of the solenoid, l = its length, $V = Sl$, $N = nl$ = total number of turns.

When the length of the solenoid is increased, for example, by pulling it, its inductance will decrease. If the current remains unchanged, the flux, linked to the solenoid, will also decrease. An induced e.m.f. will then come into play, which by Lenz's law will try to oppose the decrease of flux, for example, by increasing the current. In the superconducting state the flux will not change and so,

$$\frac{I}{l} = \text{constant}$$

Hence, $$\frac{I}{l} = \frac{I_0}{l_0}, \quad \text{or,} \quad I = I_0 \frac{l}{l_0} = I_0 (1 + \eta)$$

3.325 The flux linked to the ring can not change on transition to the superconduction state, for reasons, similar to that given above. Thus a current I must be induced in the ring, where,

$$I = \frac{\Phi}{L} = \frac{\pi a^2 B}{\mu_0 a \left(\ln \frac{8a}{b} - 2\right)} = \frac{\pi a B}{\mu_0 \left(\ln \frac{8a}{b} - 2\right)}$$

3.326 We write the equation of the circuit as,

$$Ri + \frac{L}{\eta}\frac{di}{dt} = \xi,$$

for $t \geq 0$. The current at $t = 0$ just after inductance is changed, is

$i = \eta \frac{\xi}{R}$, so that the flux through the inductance is unchanged.

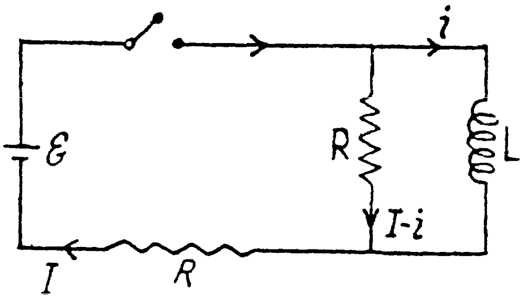
We look for a solution of the above equation in the form

$$i = A + Be^{-t/C}$$

Substituting $C = \frac{L}{\eta R}$, $B = \eta - 1, A = \frac{\xi}{R}$

Thus, $$i = \frac{\xi}{R}\left(1 + (\eta - 1)\, e^{-\eta R t/L}\right)$$

3.327 Clearly, $L\frac{di}{dt} = R\,(I - i) = \xi - RI$



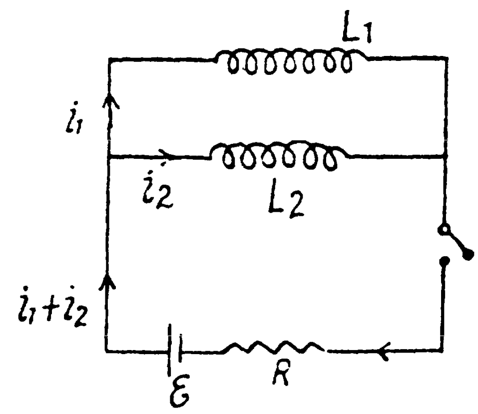
So, $2L\frac{di}{dt} = \xi - Ri$

This equation has the solution (as in 3.312)

$i = \frac{\xi}{R}\left(1 - e^{-tR/2L}\right)$

3.328 The equations are,

$$L_1\frac{di_1}{dt} = L_2\frac{di_2}{dt} = \xi - R\,(i_1 + i_2)$$



Then, $\frac{d}{dt}(L_1 i_1 - L_2 i_2) = 0$

or, $L_1 i_1 - L_2 i_2 =$ constant

But initially at $t = 0$, $i_1 = i_2 = 0$

so constant must be zero and at all times,

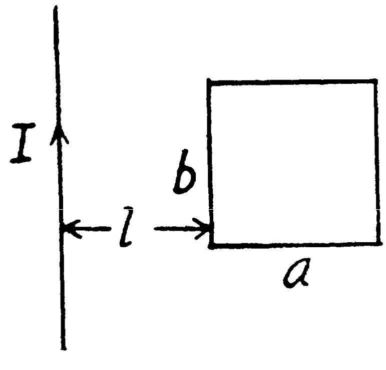
$$L_1 i_1 = L_2 i_2$$

In the final steady atate, current must obviously be $i_1 + i_2 = \frac{\xi}{R}$. Thus in steady state,

$$i_1 = \frac{\xi L_2}{R\,(L_1 + L_2)} \text{ and } i_2 = \frac{\xi L_1}{R\,(L_1 + L_2)}$$

3.329 Here, $B = \frac{\mu_0 I}{2\pi r}$ at a distance r from the wire. The flux through the frame is obtained as,

$$\Phi_{12} = \int_l^{a+l} \frac{\mu_0 I}{2\pi r}\, b\,dr = \frac{\mu_0 b}{2\pi} I \ln\left(1 + \frac{a}{l}\right)$$



Thus, $L_{12} = \frac{\Phi_{12}}{I} = \frac{\mu_0 b}{2\pi} \ln\left(1 + \frac{a}{l}\right)$

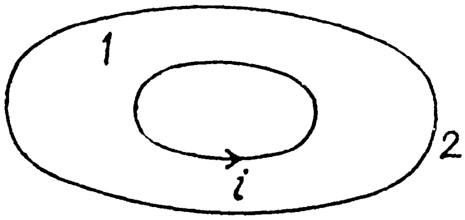
3.330 Here also, $B = \dfrac{\mu_0 I}{2\pi r}$ and $\Phi = \mu_0 \mu \dfrac{I}{2\pi} \displaystyle\int_a^b \frac{h\,dr}{r} N \cdot$

Thus, $$L_{12} = \frac{\mu \mu_0 hN}{2\pi} \ln \frac{b}{a}$$

3.331 The direct calculation of the flux Φ_2 is a rather complicated problem, since the configuration of the field itself is complicated. However, the application of the reciprocity theorem simplifies the solution of the problem. Indeed, let the same current i flow through loop 2. Then the magnetic flux created by this current through loop 1 can be easily found.

Magnetic induction at the centre of the loop, : $B = \dfrac{\mu_0 i}{2b}$

So, flux throug loop 1, : $\Phi_{12} = \pi a^2 \dfrac{\mu_0 i}{2b}$



and from reciprocity theorem,

$$\varphi_{12} = \Phi_{21}\,,\ \Phi_{21} = \frac{\mu_0 \pi a^2 i}{2b}$$

So, $L_{12} = \dfrac{\Phi_{21}}{i} = \dfrac{1}{2} \mu_0 \pi a^2 / b$

3.332 Let $\vec{p}_m$ be the magnetic moment of the magnet M. Then the magnetic field due to this magnet is,

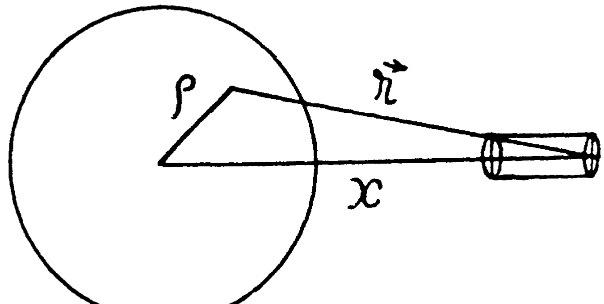
$$\frac{\mu_0}{4\pi}\left[\frac{3(\vec{p}_m \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{p}_m}{r^3}\right].$$

The flux associated with this, when the magnet is along the axis at a distance x from the centre, is

$$\Phi = \frac{\mu_0}{4\pi} \int \left[\frac{3(\vec{p}_m \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{p}_m}{r^3}\right] \cdot d\vec{S} = \Phi_1 - \Phi_2.$$

$$\text{where,}\ \Phi_2 = \frac{\mu_0}{4\pi} p_m \int_0^a \frac{2\pi \rho\, d\rho}{(x^2 + \rho^2)^{3/2}} = \frac{\mu_0 p_m}{2}\left(\frac{1}{x} - \frac{1}{\sqrt{x^2 + a^2}}\right)$$

$$\text{and}\ \ \Phi_1 = \frac{3\mu_0 p_m x^2}{4\pi} \int_0^a \frac{2\pi \rho\, d\rho}{(x^2 + \rho^2)^{5/2}}$$



$$= \frac{\mu_0 p_m x^2}{2}\left(\frac{1}{x^3} - \frac{1}{(x^2 + a^2)^{3/2}}\right)$$

So, $$\Phi = \frac{-\mu_0 p_m a^2}{2(x^2 + a^2)^{3/2}}$$

When the flux changes, an e.m.f. $-N\dfrac{d\Phi}{dt}$ is induced and a current $-\dfrac{N}{R}\dfrac{d\Phi}{dt}$ flows. The total charge q, flowing, as the magnet is removed to infinity from $x = 0$ is,

$$q = \frac{N}{R}\Phi(x=0) = \frac{N}{R}\cdot\frac{\mu_0 p_m}{2a}$$

or,
$$p_m = \frac{2aqR}{N\mu_0}$$

3.333 If a current I flows in one of the coils, the magnetic field at the centre of the other coil is,

$$B = \frac{\mu_0 a^2 I}{2(l^2+a^2)^{3/2}} = \frac{\mu_0 a^2 I}{2l^3}, \quad \text{as } l >> a.$$

The flux associated with the second coil is then approximately $\mu_0 \pi a^4 I/2l^3$

Hence,
$$L_{12} = \frac{\mu_0 \pi a^4}{2l^3}$$

3.334 When the current in one of the loop is $I_1 = \alpha t$, an e.m.f. $L_{12}\dfrac{dI_1}{dt} = L_{12}\alpha$, is induced in the other loop. Then if the current in the other loop is I_2 we must have,

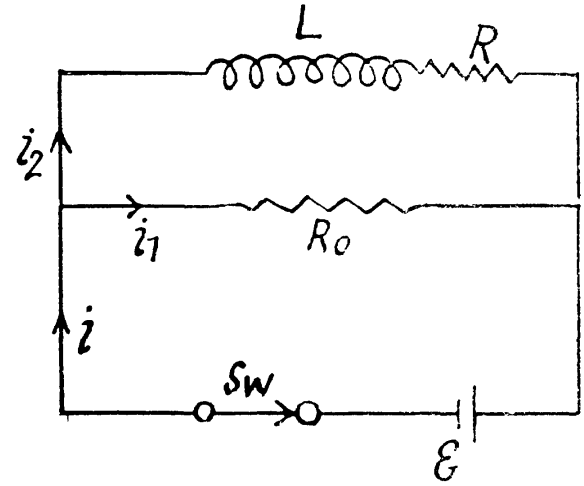
$$L_2\frac{dI_2}{dt} + RI_2 = L_{12}\alpha$$

This familiar equation has the solution,

$$I_2 = \frac{L_{12}\alpha}{R}\left(1 - e^{\frac{-tR}{L_2}}\right)$$ which is the required current

3.335 Initially, after a steady current is set up, the current is flowing as shown.

In steady condition $i_{20} = \dfrac{\xi}{R}$, $i_{10} = \dfrac{\xi}{R_0}$.



When the switch is disconnected, the current through R_0 changes from i_{10} to the right, to i_{20} to the left. (The current in the inductance cannot change suddenly.). We then have the equation,

$$L\frac{di_2}{dt} + (R+R_0)\,i_2 = 0.$$

This equation has the solution $i_2 = i_{20}\,e^{-t(R+R_0)/L}$

The heat dissipated in the coil is,

$$Q = \int_0^\infty i_2^2 R\,dt = i_{20}^2 R\int_0^\infty e^{-2t(R+R_0)/L}\,dt$$

$$= R\,i_{20}^2 \times \frac{L}{2(R+R_0)} = \frac{L\xi^2}{2R(R+R_0)} = 3\,\mu\text{J}$$

3.336 To find the magnetic field energy we recall that the flux varies linearly with current. Thus, when the flux is Φ for current i, we can write $\phi = A\, i$. The total energy inclosed in the field, when the current is I, is

$$W = \int \xi i\, dt = \int N \frac{d\Phi}{dt} i\, dt$$

$$= \int N\, d\Phi\, i = \int_0^I N A\, i\, di = \frac{1}{2} N A I^2 = \frac{1}{2} N \Phi I$$

The characteristic factor $\frac{1}{2}$ appears in this way.

3.337 We apply circulation theorem,

$$H \cdot 2\pi b = N I, \quad \text{or,} \quad H = N I / 2\pi b.$$

Thus the total energy,

$$W = \frac{1}{2} BH \cdot 2\pi b \cdot \pi a^2 = \pi^2 a^2 b\, BH.$$

Given N, I, b we know H, and can find out B from the $B - H$ curve. Then W can be calculated.

3.338 From $\oint \vec{H} \cdot d\vec{r} = N I$,

$$H \cdot \pi d + \frac{B}{\mu_0} \cdot b \approx N I, \, (d >> b)$$

Also, $\quad B = \mu \mu_0 H.$ Thus, $H = \dfrac{N I}{\pi d + \mu b}.$

Since B is continuous across the gap, B is given by,

$B = \mu \mu_0 \dfrac{N I}{\pi d + \mu b}$, both in the magnetic and the gap.

(a) $$\frac{W_{gap}}{W_{magnetic}} = \frac{\dfrac{B^2}{2 \mu_0} \times S \times b}{\dfrac{B^2}{2 \mu \mu_0} \times S \times \pi d} = \frac{\mu b}{\pi d}.$$

(b) The flux is $N \int \vec{B} \cdot d\vec{S} = N \mu \mu_0 \dfrac{N I}{\pi d + \mu b} \cdot S = \mu_0 \dfrac{S N^2 I}{b + \dfrac{\pi d}{\mu}}$,

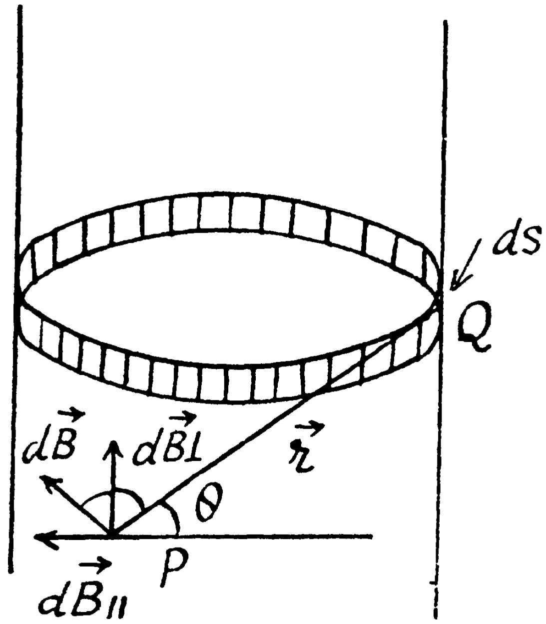
So, $$L \approx \frac{\mu_0 S N^2}{b + \dfrac{\pi d}{\mu}}.$$

Energy wise; total energy

$$= \frac{B^2}{2\mu_0} \left(\frac{\pi d}{\mu} + b \right) S = \frac{1}{2} \frac{\mu_0 N^2 S}{b + \dfrac{\pi d}{\mu}} \cdot I^2 = \frac{1}{2} L I^2$$

The L, found in the one way, agrees with that, found in the other way. Note that, in calculating the flux, we do not consider the field in the gap, since it is not linked to the winding. But the total energy includes that of the gap.

3.339 When the cylinder with a linear charge density λ rotates with a circular frequency ω, a surface current density (charge / length × time) of $i = \frac{\lambda\omega}{2\pi}$ is set up.



The direction of the surface current is normal to the plane of paper at Q and the contribution of this current to the magnetic field at P is

$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\,(\vec{e} \times \vec{r})}{r^3} dS$ where $\vec{e}$ is the direction of the current. In magnitude, $|\vec{e} \times \vec{r}| = r$, since $\vec{e}$ is normal to $\vec{r}$ and the direction of $d\vec{B}$ is as shown.

It's component, $d\vec{B}_\parallel$ cancels out by cylindrical symmetry. The component that survives is,

$$|\vec{B}_\perp| = \frac{\mu_0}{4\pi} \int \frac{i\,dS}{r^2} \cos\theta = \frac{\mu_0\, i}{4\pi} \int d\Omega = \mu_0\, i,$$

where we have used $\frac{dS\cos\theta}{r^2} = d\Omega$ and $\int d\Omega = 4\pi$, the total solid angle around any point.

The magnetic field vanishes outside the cylinder by similar argument.

The total energy per unit length of the cylinder is,

$$W_1 = \frac{1}{2\mu_0} \mu_0^2 \left(\frac{\lambda\,\omega}{2\pi}\right)^2 \times \pi a^2 = \frac{\mu_0}{8\pi} a^2 \lambda^2 \omega^2$$

3.340 $w_E = \frac{1}{2}\varepsilon_0 E^2$, for the electric field,

$w_B = \frac{1}{2\mu_0} B^2$ for the magnetic field.

Thus,
$$\frac{1}{2\mu_0} B^2 = \frac{1}{2}\varepsilon_0 E^2,$$

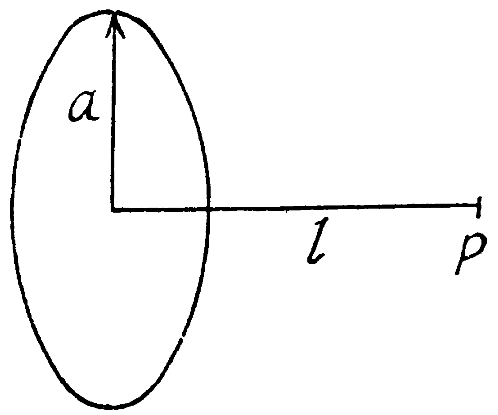
when
$$E = \frac{B}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \text{ V/m}$$

3.341 The electric field at P is,

$$E_p = \frac{ql}{4\pi\varepsilon_0 (a^2 + l^2)^{3/2}}$$

To get the magnetic field, note that the rotating ring constitutes a current $i = q\,\omega/2\,\pi$, and the corresponding magnetic field at P is,

$$B_p = \frac{\mu_0\, a^2\, i}{2\,(a^2 + l^2)^{3/2}}.$$



Thus, $$\frac{w_E}{w_M} = \frac{\varepsilon_0\,\mu_0\,E^2}{B^2} = \varepsilon_0\,\mu_0\left(\frac{ql \times 2}{4\,\pi\,\varepsilon_0\,\mu_0\,a^2 i}\right)^2$$

$$= \frac{1}{\varepsilon_0\,\mu_0}\left(\frac{l}{a^2\,\omega}\right)^2$$

or, $$\frac{w_M}{w_E} = \varepsilon_0\,\mu_0\,\omega^2\,a^4/l^2$$

3.342 The total energy of the magnetic field is,

$$\frac{1}{2}\int (\vec{B}\cdot\vec{H})\,dV = \frac{1}{2}\int \vec{B}\cdot\left(\frac{\vec{B}}{\mu_0} - \vec{J}\right)dV$$

$$= \frac{1}{2\,\mu_0}\int \vec{B}\cdot\vec{B}\,dV - \frac{1}{2}\int \vec{J}\cdot\vec{B}\,dV.$$

The second term can be interpreted as the energy of magnetization, and has the density $-\frac{1}{2}\vec{J}\cdot\vec{B}.$

3.343 (a) In series, the current I flows through both coils, and the total e.m.f. induced. when the current changes is,

$$-2L\frac{dI}{dt} = -L'\frac{dI}{dt}$$

or, $$L' = 2L$$

(b) In parallel, the current flowing through either coil is, $\frac{I}{2}$ and the e.m.f. induced is

$$-L\left(\frac{1}{2}\frac{dI}{dt}\right).$$

Equating this to $-L'\frac{dI}{dt}$, we find $L' = \frac{1}{2}L$.

3.344 We use $L_1 = \mu_0\,n_1^2\,V, L_2 = \mu_0\,n_2^2\,V$

So, $$L_{12} = \mu_0\,n_1\,n_2\,V = \sqrt{L_1\,L_2}$$

3.345 The interaction energy is

$$\frac{1}{2\,\mu_0}\int \left|\vec{B}_1 + \vec{B}_2\right|^2 dV - \frac{1}{2\,\mu_0}\int \left|\vec{B}_1\right|^2 dV - \frac{1}{2\,\mu_0}\int \left|\vec{B}_2\right|^2 dV$$

$$= \frac{1}{\mu_0}\int \vec{B}_1\cdot\vec{B}_2\,dV$$

Here, if $\vec{B}_1$ is the magnetic field produced by the first of the current carrying loops, and $\vec{B}_2$, that of the second one, then the magnetic field due to both the loops will be $\vec{B}_1 + \vec{B}_2$.

3.346 We can think of the smaller coil as constituting a magnet of dipole moment,

$$p_m = \pi a^2 I_1$$

Its direction is normal to the loop and makes an angle θ with the direction of the magnetic field, due to the bigger loop. This magnetic field is,

$$B_2 = \frac{\mu_0 I_2}{2b}$$

The interaction energy has the magnitude,

$$|W| = \frac{\mu_0 I_1 I_2}{2b} \pi a^2 \cos\theta$$

Its sign depends on the sense of the currents.

3.347 (a) There is a radial outward conduction current. Let Q be the instantaneous charge on the inner sphere, then,

$$j \times 4\pi r^2 = -\frac{dQ}{dt} \quad \text{or,} \quad \vec{j} = -\frac{1}{4\pi r^2}\frac{dQ}{dt}\hat{r}.$$

On the other hand $\vec{j}_d = \frac{\partial \vec{D}}{\partial t} = \frac{d}{dt}\left(\frac{Q}{4\pi r^2}\hat{r}\right) = -\vec{j}$

(b) At the given moment, $\vec{E} = \frac{q}{4\pi\varepsilon_0 \varepsilon r^2}\hat{r}$

and by Ohm's law, $\vec{j} = \frac{\vec{E}}{\rho} = \frac{q}{4\pi\varepsilon_0\varepsilon\rho r^2}\hat{r}$

Then,
$$\vec{j}_d = -\frac{q}{4\pi\varepsilon_0\varepsilon\rho r^2}\hat{r}$$

and
$$\oint \vec{j}_d \cdot d\vec{S} = -\frac{q}{4\pi\varepsilon_0\varepsilon\rho}\int\frac{dS\cos\theta}{r^2} = -\frac{q}{\varepsilon_0\varepsilon\rho}.$$

The surface integral must be $-ve$ because $\vec{j}_d$, being opposite of $\vec{j}$, is inward.

3.348 Here also we see that neglecting edge effects, $\vec{j}_d = -\vec{j}$. Thus Maxwell's equations reduce to, div $\vec{B} = 0$, Curl $\vec{H} = 0$, $\vec{B} = \mu\vec{H}$

A general solution of this equation is $\vec{B} =$ constant $= \vec{B}_0 \cdot \vec{B}_0$ can be thought of as an extraneous magnetic field. If it is zero, $\vec{B} = 0$.

3.349 Given $I = I_m \sin\omega t$. We see that

$$j = \frac{I_m}{S}\sin\omega t = -j_d = -\frac{\partial D}{\partial t}$$

or, $D = \frac{I_m}{\omega S}\cos\omega t$, so, $E_m = \frac{I_m}{\varepsilon_0 \omega S}$ is the amplitude of the electric field and is 7 V/cm

3.350 The electric field between the plates can be written as,

$$E = Re\frac{V_m}{d}e^{i\omega t}, \text{ instead of } \frac{V_m}{d}\cos\omega t.$$

This gives rise to a conduction current,

$$j_c = \sigma E = \text{Re}\frac{\sigma}{d}V_m e^{i\omega t}$$

and a displacement current,

$$j_d = \frac{\partial D}{\partial t} = \text{Re}\,\varepsilon_0\,\varepsilon\, i\,\omega\frac{V_m}{d}e^{i\omega t}$$

The total current is,

$$j_T = \frac{V_m}{d}\sqrt{\sigma^2 + (\varepsilon_0\,\varepsilon\,\omega)^2}\,\cos(\omega t + \alpha)$$

where, $\tan\alpha = \dfrac{\sigma}{\varepsilon_0\,\varepsilon\,\omega}$ on taking the real part of the resultant.

The corresponding magnetic field is obtained by using circulation theorem,

$$H\cdot 2\pi r = \pi r^2 j_T$$

or, $H = H_m\cos(\omega t + \alpha)$, where, $H_m = \dfrac{r V_m}{2d}\sqrt{\sigma^2 + (\varepsilon_0\,\varepsilon\,\omega)^2}$

3.351 Inside the solenoid, there is a magnetic field,

$$B = \mu_0 n I_m \sin\omega t.$$

Since this varies in time there is an associated electric field. This is obtained by using,

$$\oint_C \vec{E}\cdot d\vec{r} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S}$$

For $r < R,\ 2\pi r E = -\dot{B}\cdot\pi r^2,$ or, $E = -\dfrac{\dot{B}r}{2}$

For $r > R,\ E = -\dfrac{\dot{B}R^2}{2r}$

The associated displacement current density is,

$$j_d = \varepsilon_0\frac{\partial E}{\partial t} = \begin{bmatrix} -\varepsilon_0\ddot{B}\,r/2 \\ -\varepsilon_0\ddot{B}\,R^2/2r \end{bmatrix}$$

The answer, given in the book, is dimensionally incorrect without the factor ε_0.

3.352 In the non-relativistic limit

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r}$$

(a) On a straight line coinciding with the charge path,

$$\vec{j}_d = \varepsilon_0\frac{\partial\vec{E}}{\partial t} = \frac{q}{4\pi}\left[\frac{-\vec{V}}{r^3} - \frac{3\vec{r}\,\dot{r}}{r^4}\right], \left(\text{using, } \frac{d\vec{r}}{dt} = -\vec{v}.\right)$$

But in this case, $\dot{r} = -v$ and $v\frac{\vec{r}}{r} = \vec{v}$, so, $j_d = \frac{2q\vec{v}}{4\pi r^3}$

(b) In this case, $\dot{r} = 0$, as, $\vec{r} \perp \vec{v}$. Thus,

$$j_d = -\frac{q\vec{v}}{4\pi r^3}$$

3.353 We have, $E_p = \frac{qx}{4\pi\varepsilon_0 (a^2+x^2)^{3/2}}$

then $$j_d = \frac{\partial D}{\partial t} = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{qv}{4\pi (a^2+x^2)^{5/2}}(a^2 - 2x^2)$$

This is maximum, when $x = x_m = 0$, and minimum at some other value. The maximum displacement current density is

$$(j_d)_{\max} = \frac{qv}{4\pi a^3}$$

To check this we calculate $\frac{\partial j_d}{\partial x}$;

$$\frac{\partial j_d}{\partial x} = \frac{qv}{4\pi}[(-4x(a^2+x^2) - 5x(a^2 - 2x^2))]$$

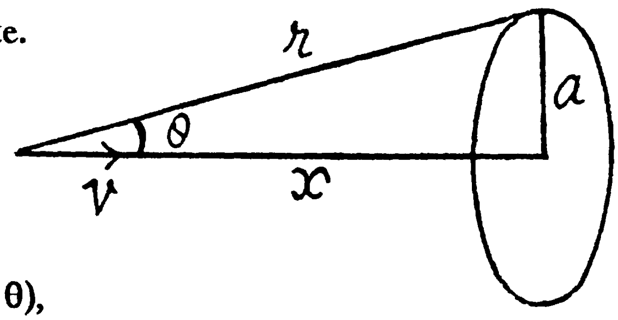
This vanishes for $x = 0$ and for $x = \sqrt{\frac{3}{2}}\,a$. The latter is easily shown to be a smaller local minimum (negative maximum).

3.354 We use Maxwell's equations in the form,

$$\oint \vec{B}\cdot d\vec{r} = \varepsilon_0 \mu_0 \frac{\partial}{\partial t}\int \vec{E}\cdot d\vec{S},$$

when the conduction current vanishes at the site.

We know that,



$$\int \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0}\int \frac{d\vec{S}\cdot\hat{r}}{r^2}$$

$$= \frac{q}{4\pi\varepsilon_0}\int d\Omega = \frac{q}{4\pi\varepsilon_0}2\pi(1-\cos\theta),$$

where, $2\pi(1-\cos\theta)$ is the solid angle, formed by the disc like surface, at the charge.

Thus, $$\oint \vec{B}\cdot d\vec{r} = 2\pi a B = \frac{1}{2}\mu_0 q\cdot\sin\theta\cdot\dot{\theta}$$

On the other hand, $x = a\cot\theta$

differentiating and using $\frac{dx}{dt} = -v$,

$$v = a\,\mathrm{cosec}^2\theta\,\dot{\theta}$$

Thus, $$B = \frac{\mu_0 q v r\sin\theta}{4\pi r^3}$$

This can be written as, $\vec{B} = \dfrac{\mu_0 q (\vec{v} \times \vec{r})}{4 \pi r^3}$

and $\vec{H} = \dfrac{q}{4\pi} \dfrac{\vec{v} \times \vec{r}}{r^3}$ (The sense has to be checked independently.)

3.355 (a) If $\vec{B} = \vec{B}(t)$, then,

Curl $\vec{E} = \dfrac{-\partial \vec{B}}{\partial t} \neq 0.$

So, $\vec{E}$ cannot vanish.

(b) Here also, curl $\vec{E} \neq 0$, so $\vec{E}$ cannot be uniform.

(c) Suppose for instance, $\vec{E} = \vec{a} f(t)$

where $\vec{a}$ is spatially and temporally fixed vector. Then $-\dfrac{\partial \vec{B}}{dt} =$ curl $\vec{E} = 0$. Generally speaking this contradicts the other equation curl $\vec{H} = \dfrac{\partial \vec{D}}{\partial t} \neq 0$ for in this case the left hand side is time independent but RHS. depends on time. The only exception is when $f(t)$ is linear function. Then a uniform field $\vec{E}$ can be time dependent.

3.356 From the equation Curl $\vec{H} - \dfrac{\partial \vec{D}}{\partial t} = \vec{j}$

We get on taking divergence of both sides

$$-\frac{\partial}{\partial t} \operatorname{div} \vec{D} = \operatorname{div} \vec{j}$$

But div $\vec{D} = \rho$ and hence div $\vec{j} + \dfrac{\partial \rho}{\partial t} = 0$

3.357 From $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

we get on taking divergence

$$0 = -\frac{\partial}{\partial t} \operatorname{div} \vec{B}$$

This is compatible with div $\vec{B} = 0$

3.358 A rotating magnetic field can be represented by,

$B_x = B_0 \cos \omega t \,;\, B_y = B_0 \sin \omega t$ and $B_z = B_{zo}$

Then curl, $\quad \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}.$

So, $\quad -(\text{Curl}\,\vec{E})_x = -\omega B_0 \sin \omega t = -\omega B_y$

$\quad -(\text{Curl}\,\vec{E})_y = \omega B_0 \cos \omega t = \omega B_x$ and $-(\text{Curl}\,\vec{E})_z = 0$

Hence, $\quad$ Curl $\vec{E} = -\vec{\omega} \times \vec{B}$,

where, $\quad \vec{\omega} = \vec{e}_3 \omega$.

3.359 Consider a particle with charge e, moving with velocity $\vec{v}$, in frame K. It experiences a force $\vec{F} = e\vec{v} \times \vec{B}$

In the frame K', moving with velocity $\vec{v}$, relative to K, the particle is at rest. This means that there must be an electric field $\vec{E}$ in K', so that the particle experinces a force,

$$\vec{F'} = e\vec{E'} = \vec{F} = e\vec{v} \times \vec{B}$$

Thus, $$\vec{E'} = \vec{v} \times \vec{B}$$

3.360 Within the plate, there will appear a $(\vec{v} \times \vec{B})$ force, which will cause charges inside the plate to drift, until a countervailing electric field is set up. This electric field is related to B, by $E = eB$, since v & B are mutually perpendicular, and E is perpendicular to both. The charge density $\pm\, \sigma$, on the force of the plate, producing this electric field, is given by

$$E = \frac{\sigma}{\varepsilon_0} \quad \text{or} \quad \sigma = \varepsilon_0 v B = 0{\cdot}40\ \text{pC/m}^2$$

3.361 Choose $\vec{\omega} \uparrow\uparrow \vec{B}$ along the z–axis, and choose $\vec{r}$, as the cylindrical polar radius vector of a reference point (perpendicular distance from the axis). This point has the velocity,

$$\vec{v} = \vec{\omega} \times \vec{r},$$

and experiences a $(\vec{v} \times \vec{B})$ force, which must be counterbalanced by an electric field,

$$\vec{E} = -(\vec{\omega} \times \vec{r}) \times \vec{B} = -(\vec{\omega} \cdot \vec{B})\, \vec{r}.$$

There must appear a space charge density,

$$\rho = \varepsilon_0 \,\text{div}\, \vec{E} = -2\,\varepsilon_0 \vec{\omega} \cdot \vec{B} = -8\ \text{pC/m}^3$$

Since the cylinder, as a whole is electrically neutral, the surface of the cylinder must acquire a positive charge of surface density,

$$\sigma = +\frac{2\,\varepsilon_0 (\vec{\omega} \cdot \vec{B})\, \pi a^2}{2\pi a} = \varepsilon_0 a\, \vec{\omega} \cdot \vec{B} = +{\cdot}2\ \text{pC/m}^2$$

3.362 In the reference frame K', moving with the particle,

$$\vec{E'} = \vec{E} + \vec{v_0} \times \vec{B} = \frac{q\vec{r}}{4\pi\varepsilon_0 r^3}$$

$$\vec{B'} = \vec{B} - \vec{v_0} \times \vec{E}/c^2 = 0.$$

Here, $\vec{v_0}$ = velocity of K', relative to the K frame, in which the particle has velocity $\vec{v}$.

Clearly, $\vec{v_0} = \vec{v}$. From the second equation,

$$\vec{B} = \frac{\vec{v} \times \vec{E}}{c^2} = \varepsilon_0 \mu_0 \times \frac{q}{4\pi\varepsilon_0} \frac{\vec{v} \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi} \frac{q\,(\vec{v} \times \vec{r})}{r^3}$$

3.363 Suppose, there is only electric field $\vec{E}$, in K. Then in K', considering nonrelativistic velocity $\vec{v}$, $\vec{E}' = \vec{E}$, $\vec{B} = -\frac{\vec{v}\times\vec{E}}{c^2}$,

So, $$\vec{E}'\cdot\vec{B}' = 0$$

In the relativistic case,

$$\left.\begin{aligned}\vec{E}'_{\parallel} &= \vec{E}_{\parallel}\\ E'_{\perp} &= \frac{\vec{E}\perp}{\sqrt{1-v^2/c^2}}\end{aligned}\right\}\begin{aligned}\vec{B}'_{\parallel} &= \vec{B}_{\parallel} = 0\\ \vec{B}'_{\perp} &= \frac{-\vec{v}\times\vec{E}/c^2}{\sqrt{1-v^2/c^2}}\end{aligned}$$

Now, $$\vec{E}'\cdot\vec{B}' = \vec{E}'_{\parallel}\cdot\vec{B}'_{\parallel} + \vec{E}'_{\perp}\cdot\vec{B}'_{\perp} = 0, \text{ since}$$

$$E'_{\perp}\cdot B'_{\perp} = -\vec{E}_{\perp}\cdot(\vec{v}\times\vec{E})/(1-v^2/c^2) = -\vec{E}_{\perp}\cdot(\vec{v}\times\vec{E}_{\perp})\Big/\left(1-\frac{v^2}{c^2}\right) = 0$$

3.364 In K, $\vec{B} = b\frac{y\hat{i}-x\hat{j}}{x^2+y^2}$, $b =$ constant.

In $$K',\ \vec{E}' = \vec{v}\times\vec{B} = bv\frac{y\hat{j}-x\hat{i}}{x^2+y^2} = bv\frac{\vec{r}}{r^2}$$

The electric field is radial ($\vec{r} = x\hat{i}+y\hat{j}$).

3.365 In K, $\vec{E} = a\frac{\vec{r}}{r^2}$, $\vec{r} = (x\hat{i}+y\hat{j})$

In $$K', \vec{B}' = -\frac{\vec{v}\times\vec{E}}{c^2} = \frac{a\,\vec{r}\times\vec{v}}{c^2 r^2}$$

The magnetic lines are circular.

3.366 In the non relativistic limit, we neglect v^2/c^2 and write,

$$\left.\begin{aligned}\vec{E}'_{\parallel} &= \vec{E}_{\parallel}\\ \vec{E}'_{\perp} &\cong \vec{E}_{\perp} + \vec{v}\times\vec{B}\end{aligned}\right\}\begin{aligned}\vec{B}'_{\parallel} &= \vec{B}_{\parallel}\\ \vec{B}'_{\perp} &\cong \vec{B}_{\perp} - \vec{v}\times\vec{E}/c^2\end{aligned}$$

These two equations can be combined to give,

$$\vec{E}' = \vec{E} + \vec{v}\times\vec{B},\ \vec{B}' = \vec{B} - \vec{v}\times\vec{E}/c^2$$

3.367 Choose $\vec{E}$ in the direction of the z–axis, $\vec{E} = (0, 0, E)$. The frame K' is moving with velocity $\vec{v} = (v\sin\alpha, 0, v\cos\alpha)$, in the $x-z$ plane. Then in the frame K',

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}\ B'_{\parallel} = 0$$

$$\vec{E}'_{\perp} = \frac{\vec{E}_{\perp}}{\sqrt{1-v^2/c^2}}\ \vec{B}'_{\perp} = \frac{-\vec{v}\times\vec{E}/c^2}{\sqrt{1-v^2/c^2}}$$

The vector along $\vec{v}$ is $\vec{e} = (\sin\alpha, 0, \cos\alpha)$ and the perpendicular vector in the $x-z$ plane is,

$$\vec{f} = (-\cos\alpha, 0, \sin\alpha),$$

(a) Thus using $\vec{E} = E\cos\alpha\,\vec{e} + E\sin\alpha\,\vec{f}$,

$$E'_{\|} = E\cos\alpha \text{ and } E'_{\|} = \frac{E\sin\alpha}{\sqrt{1-v^2/c^2}},$$

So

$$E' = E\sqrt{\frac{1-\beta^2\cos^2\alpha}{1-\beta^2}} \text{ and } \tan\alpha' = \frac{\tan\alpha}{\sqrt{1-v^2/c^2}}$$

(b) $B'_{\|} = 0$, $\vec{B}'_{\perp} = \dfrac{\vec{v}\times\vec{E}/c^2}{\sqrt{1-v^2/c^2}}$

$$B' = \frac{\beta E\sin\alpha}{c\sqrt{1-\beta^2}}$$

3.368 Choose $\vec{B}$ in the z direction, and the velocity $\vec{v} = (v\sin\alpha, 0, v\cos\alpha)$ in the $x-z$ plane, then in the K' frame,

$$\vec{E}'_{\|} = \vec{E}_{\|} = 0 \qquad \vec{B}'_{\|} = \vec{B}_{\|}$$

$$\vec{E}'_{\perp} = \frac{\vec{v}\times\vec{B}}{\sqrt{1-v^2/c^2}} \qquad \vec{B}'_{\perp} = \frac{\vec{B}_{\perp}}{\sqrt{1-v^2/c^2}}$$

We find similarly, $E' = \dfrac{c\,\beta B\sin\alpha}{\sqrt{1-\beta^2}}$

$$B' = B\sqrt{\frac{1-\beta^2\cos^2\alpha}{1-\beta^2}} \quad \tan\alpha' = \frac{\tan\alpha}{\sqrt{1-\beta^2}}$$

3.369 (a) We see that, $\vec{E}'\cdot\vec{B}' = \vec{E}'_{\|}\cdot\vec{B}'_{\|} + \vec{E}'_{\perp}\cdot\vec{B}'_{\perp}$

$$= \vec{E}_{\|}\cdot\vec{B}_{\|} + \frac{(\vec{E}_{\perp}+\vec{v}\times\vec{B})\cdot\left(\vec{B}_{\perp} - \dfrac{\vec{v}\times\vec{E}}{c^2}\right)}{1-\dfrac{v^2}{c^2}}$$

$$= \vec{E}_{\|}\cdot\vec{B}_{\|} + \frac{\vec{E}_{\perp}\cdot\vec{B}_{\perp} - (\vec{v}\times\vec{B})\cdot(\vec{v}\times\vec{E})/c^2}{1-v^2/c^2}$$

$$= \vec{E}_{\|}\cdot\vec{B}_{\|} + \frac{\vec{E}_{\perp}\cdot\vec{B}_{\perp} - (\vec{v}\times\vec{B}_{\perp})\cdot(\vec{v}\times\vec{E}_{\perp})/c^2}{1-\dfrac{v^2}{c^2}}$$

But,
$$\vec{A}\times\vec{B}\cdot\vec{C}\times\vec{D} = A\cdot C\,B\cdot D - A\cdot D\,B\cdot C,$$

so,
$$\vec{E}'\cdot\vec{B}' = \vec{E}_{\|}\cdot\vec{B}_{\|} + \vec{E}_{\perp}\cdot\vec{B}_{\perp}\frac{\left(1-\dfrac{v^2}{c^2}\right)}{1-\dfrac{v^2}{c^2}} = \vec{E}\cdot\vec{B}$$

(b) $E'^2 - c^2B'^2 = E'^2_{\|} - c^2B'^2_{\|} + E'^2_{\perp} - c^2B'^2_{\perp}$

$$= E_{\|}^2 - c^2 B_{\|}^2 + \frac{1}{1 - \frac{v^2}{c^2}} \left[(\vec{E}_\perp + \vec{v} \times \vec{B})^2 - c^2 \left(\vec{B}_\perp - \frac{\vec{v} \times \vec{E}}{c^2} \right)^2 \right]$$

$$= E_{\|}^2 - c^2 B_{\|}^2 + \frac{1}{1 - \frac{v^2}{c^2}} \left[E_\perp^2 - c^2 B_\perp^2 + (\vec{v} \times \vec{B}_\perp)^2 - \frac{1}{c^2} (\vec{v} \times E_1)^2 \right]$$

$$= E_{\|}^2 - c^2 B_{\|}^2 + \frac{1}{1 - \frac{v^2}{c^2}} [E_\perp^2 - c^2 B_\perp^2] \left(1 - \frac{v^2}{c^2} \right) = E^2 - c^2 B^2,$$

since, $(\vec{v} \times \vec{A}_\perp)^2 = v^2 A_\perp^2$

3.370 In this case, $\vec{E} \cdot \vec{B} = 0$, as the fields are mutually perpendicular. Also,

$$E^2 - c^2 B^2 = -20 \times 10^8 \left(\frac{\text{V}}{\text{m}} \right)^2 \text{ is } -ve.$$

Thus, we can find a frame, in which $E' = 0$, and

$$B' = \frac{1}{c} \sqrt{c^2 B^2 - E^2} = B \sqrt{1 - \frac{E^2}{c^2 B^2}} = 0{\cdot}20 \sqrt{1 - \left(\frac{4 \times 10^4}{3 \times 10^8 \times 2 \times 10^{-4}} \right)^2} = 0{\cdot}15 \text{ mT}$$

3.371 Suppose the charge q moves in the positive direction of the x–axis of the frame K. Let us go over to the moving frame K', at whose origin the charge is at rest. We take the x and x' axes of the two frames to be coincident, and the y & y' axes, to be parallel.

In the K' frame, $\vec{E} = \frac{1}{4 \pi \varepsilon_0} \frac{q \vec{r}}{r'^3}$,

and this has the following components,

$$E'_x = \frac{1}{4 \pi \varepsilon_0} \frac{q x'}{r'^3}, \; E_y' = \frac{1}{4 \pi \varepsilon_0} \frac{q y'}{r'^3}.$$

Now let us go back to the frame K. At the moment, when the origins of the two frames coincide, we take $t = 0$. Then,

$$x = r \cos \theta = x' \sqrt{1 - \frac{v^2}{c^2}}, \; y = r \sin \theta = y'$$

Also, $$E_x = E_x', \; E_y = E_y' / \sqrt{1 - v^2/c^2}$$

From these equations, $r'^2 = \dfrac{r^2 (1 - \beta^2 \sin^2 \theta)}{1 - \beta^2}$

$$\vec{E} = \frac{q}{4 \pi \varepsilon_0} \frac{1}{r^3 (1 - \beta^2 \sin^2 \theta)^{3/2}} \left[(1 - \beta^2)^{3/2} \left(x' \hat{i} + \frac{y'}{\sqrt{1 - \beta^2}} \hat{j} \right) \right]$$

$$= \frac{q \vec{r} (1 - \beta^2)}{4 \pi \varepsilon_0 r^3 (1 - \beta^2 \sin^2 \theta)^{3/2}}$$

3.7 MOTION OF CHARGED PARTICLES IN ELECTRIC AND MAGNETIC FIELDS

3.372 Let the electron leave the negative plate of the capacitor at time $t = 0$

As, $$E_x = -\frac{d\varphi}{dx}, \; E = \frac{\varphi}{l} = \frac{at}{l},$$

and, therefore, the acceleration of the electron,

$$w = \frac{eE}{m} = \frac{eat}{ml} \quad \text{or,} \quad \frac{dv}{dt} = \frac{eat}{ml}$$

or, $$\int_0^v dv = \frac{ea}{ml}\int_0^t t\,dt, \quad \text{or,} \quad v = \frac{1}{2}\frac{ea}{ml}t^2 \tag{1}$$

But, from $s = \int v\,dt$,

$$l = \frac{1}{2}\frac{ea}{ml}\int_0^t t^2\,dt = \frac{eat^3}{6\,ml} \quad \text{or,} \quad t = \left(\frac{6\,ml^2}{ea}\right)^{\frac{1}{3}}$$

Putting the value of t in (1),

$$v = \frac{1}{2}\frac{ea}{ml}\left(\frac{6\,ml^2}{ea}\right)^{\frac{2}{3}} = \left(\frac{9}{2}\frac{ale}{m}\right)^{\frac{1}{3}} = 16\text{ km/s}.$$

3.373 The electric field inside the capacitor varies with time as,

$$E = at.$$

Hence, electric force on the proton,

$$F = eat$$

and subsequently, acceleration of the proton,

$$w = \frac{eat}{m}$$

Now, if t is the time elapsed during the motion of the proton between the plates, then $t = \frac{l}{v_{\parallel}}$, as no acceleration is effective in this direction. (Here $v_{\parallel}$ is velocity along the length of the plate.)

From kinematics, $\frac{dv_{\perp}}{dt} = w$

so, $$\int_0^{v_{\perp}} dv_{\perp} = \int_0^t w\,dt,$$

(as initially, the component of velocity in the direction, $\perp$ to plates, was zero.)

or $$v_\perp = \int_0^t \frac{ea}{m}\frac{t^2}{2m} = \frac{ea}{2m}\frac{l^2}{v_\parallel^2}$$

Now, $$\tan\alpha = \frac{v_\perp}{v_\parallel} = \frac{e\,al^2}{2\,m\,v_\parallel^3}$$

$$= \frac{eal^2}{2m\left(\frac{2\,eV}{m}\right)^{\frac{3}{2}}}, \text{ as } v_\parallel = \left(\frac{2\,eV}{m}\right)^{\frac{1}{2}}, \text{ from energy conservation.}$$

$$= \frac{al^2}{4}\sqrt{\frac{m}{2\,eV^3}}$$

3.374 The equation of motion is,

$$\frac{dv}{dt} = v\,\frac{dv}{dx} = \frac{q}{m}(E_0 - ax)$$

Integrating

$$\frac{1}{2}v^2 - \frac{q}{m}(E_0 x - \frac{1}{2}ax^2) = \text{constant.}$$

But initially $v = 0$ when $x = 0$, so "constant" $= 0$

Thus, $$v^2 = \frac{2q}{m}\left(E_0 x - \frac{1}{2}ax^2\right)$$

Thus, $v = 0$, again for $x = x_m = \dfrac{2\,E_0}{a}$

The corresponding acceleration is,

$$\left(\frac{dv}{dt}\right)_{x_m} = \frac{q}{m}(E_0 - 2\,E_0) = -\frac{qE_0}{m}$$

3.375 From the law of relativistic conservation of energy

$$\frac{m_0\,c^2}{\sqrt{1 - (v^2/c^2)}} - e\,Ex = m_0\,c^2.$$

as the electron is at rest ($v = 0$ for $x = 0$) initially.

Thus clearly $$T = e\,Ex.$$

On the other hand, $$\sqrt{1 - (v^2/c^2)} = \frac{m_0\,c^2}{m_0\,c^2 + e\,Ex}$$

or, $$\frac{v}{c} = \frac{\sqrt{(m_0\,c^2 + e\,Ex)^2 - m_0^2\,c^4}}{m_0\,c^2 + eEx}$$

or, $$ct = \int c\,dt = \int \frac{(m_0\,c^2 + eEx)\,dx}{\sqrt{(m_0\,c^2 + eEx)^2 - m_0^2\,c^4}}$$

$$= \frac{1}{2\,eE}\int \frac{dy}{\sqrt{y - m_0^2 c^4}} = \frac{1}{eE}\sqrt{(m_0 c^2 + eEx)^2 - m_0^2 c^4} + \text{constant}$$

The "constant" $= 0$, at $t = 0$, for $x = 0$,

So, $$ct = \frac{1}{eE}\sqrt{(m_0 c^2 + eEx)^2 - m_0^2 c^4}.$$

Finally, using $T = e\,E\,x$,

$$c\,e\,E\;t_0 = \sqrt{T\,(T + 2\,m_0\,c^2)} \quad \text{or,} \quad t_0 = \frac{\sqrt{T\,(T + 2\,m_0\,c^2)}}{e\,Ec}$$

3.376 As before, $T = e\,E\,x$

Now in linear motion,

$$\frac{d}{dt}\frac{m_0\,v}{\sqrt{1 - v^2/c^2}} = \frac{m_0\,w}{\sqrt{1 - v^2/c^2}} + \frac{m_0\,w}{(1 - v^2/c^2)^{3/2}}\frac{v}{c^2}\,w$$

$$= \frac{m_0}{(1 - v^2/c^2)^{3/2}}\,w = \frac{(T + m_0\,c^2)^3}{m_0^2\,c^6}\,w = e\,E,$$

So, $$w = \frac{e\,E\,m_0^2\,c^6}{(T + m_0\,c^2)^3} = \frac{eE}{m_0}\left(1 + \frac{T}{m_0\,c^2}\right)^{-3}$$

3.377 The equations are,

$$\frac{d}{dt}\left(\frac{m_0\,v_x}{\sqrt{1 - (v^2/c^2)}}\right) = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{m_0\,v_y}{\sqrt{1 - v^2/c^2}}\right) = e\,E$$

Hence, $$\frac{v_x}{\sqrt{1 - v^2/c^2}} = \text{constant} = \frac{v_0}{\sqrt{1 - (v_0^2/c^2)}}$$

Also, by energy conservation,

$$\frac{m_0\,c^2}{\sqrt{1 - (v^2/c^2)}} = \frac{m_0\,c^2}{\sqrt{1 - (v_0^2/c^2)}} + e\,E\,y$$

Dividing $$v_x = \frac{v_0\,\varepsilon_0}{\varepsilon_0 + eEy},\quad \varepsilon_0 = \frac{m_0\,c^2}{\sqrt{1 - (v_0^2/c^2)}}$$

Also, $$\frac{m_0}{\sqrt{1 - (v^2/c^2)}} = \frac{\varepsilon_0 + e\,E\,y}{c^2}$$

Thus, $$(\varepsilon_0 + e\,Ey)\,v_y = c^2\,e\,E\,t + \text{constant}.$$

"constant" $= 0$ as $v_y = 0$ at $t = 0$.

Integrating again,

$$\varepsilon_0\,y + \frac{1}{2}\,e\,E\,y^2 = \frac{1}{2}\,c^2\,E\,t^2 + \text{constant}.$$

"constant" = 0, as $y = 0$, at $t = 0$.

Thus, $(ceEt)^2 = (eyE)^2 + 2\varepsilon_0 eEy + \varepsilon_0^2 - \varepsilon_0^2$

or, $$ceEt = \sqrt{(\varepsilon_0 + eEy)^2 - \varepsilon_0^2}$$

or, $$\varepsilon_0 + eEy = \sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}$$

Hence, $$v_x = \frac{v_0 \varepsilon_0}{\sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}} \quad \text{also,} \quad v_y = \frac{c^2 eEt}{\sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}}$$

and $$\tan\theta = \frac{v_y}{v_x} = \frac{eEt}{m_0 v_0}\sqrt{1 - (v_0^2/c^2)}.$$

3.378 From the figure,

$$\sin\alpha = \frac{d}{R} = \frac{dqB}{mv},$$

As radius of the arc $R = \frac{mv}{qB}$, where v is the velocity of the particle, when it enteres into the field. From initial condition of the problem,

$$qV = \frac{1}{2}mv^2 \quad \text{or,} \quad v = \sqrt{\frac{2qV}{m}}$$

Hence, $$\sin\alpha = \frac{dqB}{m\sqrt{\frac{2qV}{m}}} = dB\sqrt{\frac{q}{2mV}}$$

and $$\alpha = \sin^{-1}\left(dB\sqrt{\frac{q}{2mV}}\right) = 30°,$$ on putting the values.

3.379 (a) For motion along a circle, the magnetic force acted on the particle, will provide the centripetal force, necessary for its circular motion.

i.e. $$\frac{mv^2}{R} = evB \quad \text{or,} \quad v = \frac{eBR}{m}$$

and the period of revolution, $$T = \frac{2\pi}{\omega} = \frac{2\pi R}{v} = \frac{2\pi m}{eB}$$

(b) Generally, $\frac{d\vec{p}}{dt} = \vec{F}$

But, $$\frac{d\vec{p}}{dt} = \frac{d}{dt}\frac{m_0\vec{v}}{\sqrt{1-(v^2/c^2)}} = \frac{m_0\dot{\vec{v}}}{\sqrt{1-(v^2/c^2)}} + \frac{m_0}{\left(1-(v^2/c^2)\right)^{3/2}}\frac{\vec{v}(\vec{v}\cdot\dot{\vec{v}})}{c^2}$$

For transverse motion, $\vec{v}\cdot\dot{\vec{v}} = 0$ so,

$$\frac{d\vec{p}}{dt} = \frac{m_0\dot{\vec{v}}}{\sqrt{1-(v^2/c^2)}} = \frac{m_0}{\sqrt{1-(v^2/c^2)}}\frac{v^2}{r},$$ here.

Thus, $$\frac{m_0 v^2}{r\sqrt{1-(v^2/c^2)}} = Bev \quad \text{or,} \quad \frac{v/c}{\sqrt{1-(v^2/c^2)}} = \frac{Ber}{m_0 c}$$

or, $$\frac{v}{c} = \frac{Ber}{\sqrt{B^2 e^2 r^2 + m_0^2 c^2}}$$

Finally, $$T = \frac{2\pi r}{v} = \frac{2\pi m_0}{eB\sqrt{1-v^2/c^2}} = \frac{2\pi}{cBe}\sqrt{B^2 e^2 r^2 + m_0^2 c^2}$$

3.380 (a) As before, $p = B\,qr$.

(b) $T = \sqrt{c^2 p^2 + m_0^2 c^4} = \sqrt{c^2 B^2 q^2 r^2 + m_0^2 c^4}$

(c) $w = \dfrac{v^2}{r} = \dfrac{c^2}{r\left[1 + (m_0 c / Bqr)^2\right]}$

using the result for v from the previous problem.

3.381 From (3.279),

$$T = \frac{2\pi\varepsilon}{c^2 eB}\,(\text{relativistic}), \qquad T_0 = \frac{2\pi m_0 c^2}{c^2 eB}\,(\text{nonrelativistic}),$$

Here, $$m_0 c^2 \big/ \sqrt{1 - v^2/c^2} = E$$

Thus, $$\delta T = \frac{2\pi T}{c^2 eB}, \quad (T = K.E.)$$

Now, $$\frac{\delta T}{T_0} = \eta = \frac{T}{m_0 c^2}, \quad \text{so,} \quad T = \eta\, m_0 c^2$$

3.382 $T = eV = \dfrac{1}{2} m v^2$

(The given potential difference is not large enough to cause significant deviations from the nonrelativistic formula).

Thus, $$v = \sqrt{\frac{2eV}{m}}$$

So, $$v_{\parallel} = \sqrt{\frac{2eV}{m}}\cos\alpha, \quad v_{\perp} = \sqrt{\frac{2eV}{m}}\sin\alpha$$

Now, $$\frac{m v_{\perp}^2}{r} = Bev_{\perp} \quad \text{or,} \quad r = \frac{mv_{\perp}}{Be},$$

and $$T = \frac{2\pi r}{v_{\perp}} = \frac{2\pi m}{Be}$$

Pitch $$p = v_{\parallel} T = \frac{2\pi m}{Be}\sqrt{\frac{2eV}{m}}\cos\alpha = 2\pi\sqrt{\frac{2mV}{eB^2}}\cos\alpha$$

3.383 The charged particles will traverse a helical trajectory and will be focussed on the axis after traversing a number of turms. Thus

$$\frac{l}{v_0} = n \cdot \frac{2\pi m}{qB_1} = (n+1)\frac{2\pi m}{qB_2}$$

So, $$\frac{n}{B_1} = \frac{n+1}{B_2} = \frac{1}{B_2 - B_1}$$

Hence, $$\frac{l}{v_0} = \frac{2\pi m}{q(B_2 - B_1)}$$

or, $$\frac{l^2}{2qV/m} = \frac{(2\pi)^2}{(B_2 - B_1)^2} \times \frac{1}{(q/m)^2}$$

or, $$\frac{q}{m} = \frac{8\pi^2 V}{l^2 (B_2 - B_1)^2}$$

3.384 Let us take the point A as the origin O and the axis of the solenoid as z-axis. At an arbitrary moment of time let us resolve the velocity of electron into its two rectangular components, $\vec{v_\parallel}$ along the axis and $\vec{v_\perp}$ to the axis of solenoid. We know the magnetic force does no work, so the kinetic energy as well as the speed of the electron $|\vec{v_\perp}|$ will remain constant in the x-y plane. Thus $\vec{v_\perp}$ can change only its direction as shown in the Fig.. $\vec{v_\parallel}$ will remain constant as it is parallel to $\vec{B}$.

Thus at $t = t$

$$v_x = v_\perp \cos \omega t = v \sin\alpha \cos \omega t,$$

$$v_y = v_\perp \sin \omega t = v \sin \alpha \sin \omega t$$

and $$v_z = v \cos \alpha, \quad \text{where } \omega = \frac{eB}{m}$$

As at $t = 0$, we have $x = y = z = 0$, so the motion law of the electron is.

$$\left.\begin{aligned} z &= v \cos\alpha\, t \\ x &= \frac{v \sin\alpha}{\omega} \sin \omega t \\ y &= \frac{v\sin\alpha}{\omega}(\cos\omega t - 1) \end{aligned}\right]$$

(The equation of the helix)

On the screen, $$z = l, \text{ so } t = \frac{l}{v\cos\alpha}.$$

Then, $$r^2 = x^2 + y^2 = \frac{2v^2 \sin^2\alpha}{\omega^2}\left(1 - \cos\frac{\omega l}{v\cos\alpha}\right)$$

$$r = \frac{2v\sin\alpha}{\omega} \cdot \left|\sin\frac{\omega l}{2v\cos\alpha}\right| = 2\frac{mv}{eB}\sin\alpha \left|\sin\frac{leB}{2mv\cos\alpha}\right|$$

3.385 Choose the wire along the z-axis, and the initial direction of the electron, along the x-axis. Then the magnetic field in the $x-z$ plane is along the y-axis and outside the wire it is,

$$B = B_y = \frac{\mu_0 I}{2\pi x}, \ (B_x = B_z = 0, \text{ if } y = 0)$$

The motion must be confined to the $x-z$ plane. Then the equations of motion are,

$$\frac{d}{dt} m v_x = -e v_z B_y$$

$$\frac{d(m v_z)}{dt} = +e v_x B_y$$

Multiplying the first equation by v_x and the second by v_z and then adding,

$$v_x \frac{dv_x}{dt} + v_z \frac{dv_z}{dt} = 0$$

or, $$v_x^2 + v_z^2 = v_0^2, \text{ say, or, } v_z = \sqrt{v_0^2 - v_x^2}$$

Then, $$v_x \frac{dv_x}{dx} = -\frac{e}{m}\sqrt{v_0^2 - v_x^2}\,\frac{\mu_0 I}{2\pi x}$$

or, $$-\frac{v_x\, dv_x}{\sqrt{v_0^2 - v_x^2}} = \frac{\mu_0 I e}{2\pi m}\frac{dx}{x}$$

Integrating, $$\sqrt{v_0^2 - v_x^2} = \frac{\mu_0 I e}{2\pi m} \ln \frac{x}{a}$$

on using, $v_x = v_0$, if $x = a$ (i.e. initially).

Now, $$v_x = 0, \text{ when } x = x_m,$$

so, $$x_m = a\, e^{v_0/b}, \text{ where } b = \frac{\mu_0 I e}{2\pi m}.$$

3.386 Inside the capacitor, the electric field follows a $\frac{1}{r}$ law, and so the potential can be written as

$$\varphi = \frac{V \ln r/a}{\ln b/a}, \ E = \frac{-V}{\ln b/a}\frac{1}{r}.$$

Here r is the distance from the axis of the capacitor.

Also, $$\frac{mv^2}{r} = \frac{qV}{\ln b/a}\frac{1}{r} \text{ or } mv^2 = \frac{qV}{\ln b/a}$$

On the other hand,

$$mv = qBr \text{ in the magnetic field.}$$

Thus, $$v = \frac{V}{Br \ln b/a} \text{ and } \frac{q}{m} = \frac{v}{Br} = \frac{V}{B^2 r^2 \ln(b/a)}$$

3.387 The equations of motion are,

$$m\frac{dv_x}{dt} = -qBv_z\,,\ m\frac{dv_y}{dt} = qE \text{ and } m\frac{dv_z}{dt} = qv_xB$$

These equations can be solved easily.

First, $$v_y = \frac{qE}{m}t,\ y = \frac{qE}{2m}t'^2$$

Then, $$v_x^2 + v_z^2 = \text{constant} = v_0^2 \text{ as before.}$$

In fact, $v_x = v_0 \cos \omega t$ and $v_z = v_0 \sin \omega t$ as one can check.

Integrating again and using $x = z = 0$, at $t = 0$

$$x = \frac{v_0}{\omega}\sin \omega t,\ z = \frac{v_0}{\omega}(1 - \cos \omega t)$$

Thus, $$x = z = 0 \text{ for } t = t_n = n\frac{2\pi}{\omega}$$

At that instant, $$y_n = \frac{qE}{2m} \times \frac{2\pi}{qB/m} \times n^2 \times \frac{2\pi}{qB/m} = \frac{2\pi^2 mEn^2}{qB^2}$$

Also, $$\tan \alpha_n = \frac{v_x}{v_y},\ (v_z = 0 \text{ at this moment})$$

$$= \frac{mv_0}{qEt_n} = \frac{mv_0}{qE} \times \frac{qB}{m} \times \frac{1}{2\pi n} = \frac{Bv_0}{2\pi En}.$$

3.388 The equation of the trajectory is,

$$x = \frac{v_0}{\omega}\sin \omega t,\ z = \frac{v_0}{\omega}(1 - \cos \omega t)\,,\ y = \frac{qE}{2m}t^2 \text{ as before see (3.384).}$$

Now on the screen $x = l$, so

$$\sin \omega t = \frac{\omega l}{v_0} \quad \text{or,} \quad \omega t = \sin^{-1}\frac{\omega l}{v_0}$$

At that moment,

$$y = \frac{qE}{2m\omega^2}\left(\sin^{-1}\frac{\omega l}{v_0}\right)^2$$

so, $$\frac{\omega l}{v_0} = \sin\sqrt{\frac{2m\omega^2 y}{qE}} = \sin\sqrt{\frac{2qB^2y}{Em}}$$

and $$z = \frac{v_0}{\omega}2\sin^2\frac{\omega t}{2} = l\tan\frac{\omega t}{2}$$

$$= l\tan\frac{1}{2}\left[\sin^{-1}\frac{\omega l}{v_0}\right] = l\tan\sqrt{\frac{qB^2y}{2mE}}$$

For small

$$z,\ \frac{qB^2y}{2mE} = \left(\tan^{-1}\frac{z}{l}\right)^2 \approx \frac{z^2}{l^2}$$

or, $$y = \frac{2mE}{qB^2l^2}\cdot z^2 \text{ is a parabola.}$$

3.389 In crossed field,

$$eE = evB, \text{ so } v = \frac{E}{B}$$

Then, F = force exerted on the plate $= \frac{I}{e} \times m\frac{E}{B} = \frac{mIE}{eB}$

3.390 When the electric field is switched off, the path followed by the particle will be helical. and pitch, $\Delta l = v_{\parallel} T$, (where $v_{\parallel}$ is the velocity of the particle, parallel to $\vec{B}$, and T, the time period of revolution.)

$$= v\cos(90 - \varphi)\, T = v\sin\varphi\, T$$

$$= v\sin\varphi \frac{2\pi m}{qB}\left(\text{as } T = \frac{2\pi}{qB}\right) \tag{1}$$

Now, when both the fields were present, $qE = qvB\sin(90 - \varphi)$, as no net force was effective on the system.

or, $$v = \frac{E}{B\cos\varphi} \tag{2}$$

From (1) and (2), $$\Delta l = \frac{E}{B}\frac{2\pi m}{qB}\tan\varphi = 6 \text{ cm.}$$

3.391 When there is no deviation,

$$-q\vec{E} = q(\vec{v} \times \vec{B})$$

or, in scalar from, $E = vB$ (as $\vec{v} \perp \vec{B}$) or, $v = \frac{E}{B}$ (1)

Now, when the magnetic field is switched on, let the deviation in the field be x. Then,

$$x = \frac{l}{2}\left(\frac{q\,vB}{m}\right)t^2,$$

where t is the time required to pass through this region.

also, $$t = \frac{a}{v}$$

Thus $$x = \frac{1}{2}\left(\frac{qvB}{m}\right)\left(\frac{a}{v}\right)^2 = \frac{1}{2}\frac{q}{m}\frac{a^2B^2}{E} \tag{2}$$

For the region where the field is absent, velocity in upward direction

$$= \left(\frac{qvB}{m}\right)t = \frac{q}{m}aB \tag{3}$$

Now, $$\Delta x - x = \frac{qaB}{m}t'$$

$$= \frac{q}{m}\frac{aB^2b}{E} \text{ when } t' = \frac{b}{v} = \frac{bB}{E} \tag{4}$$

From (2) and (4),

$$\Delta x - \frac{1}{2}\frac{q}{m}\frac{a^2B^2}{E} = \frac{q}{m}\frac{aB^2b}{E}$$

or, $$\frac{q}{m} = \frac{2E\Delta x}{aB^2(a + 2b)}$$

3.392 (a) The equation of motion is,

$$m\frac{d^2\vec{r}}{dt^2} = q(\vec{E} + \vec{v}\times\vec{B})$$

Now, $$\vec{v}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B \end{vmatrix} = \vec{i}B\dot{y} - \vec{j}B\dot{x}$$

So, the equation becomes,

$$\frac{dv_x}{dt} = \frac{qBv_y}{m}, \quad \frac{dv_y}{dt} = \frac{qE}{m} - \frac{qB}{m}v_x, \text{ and } \frac{dv_z}{dt} = 0$$

Here, $v_x = \dot{x}$, $v_y = \dot{y}$, $v_z = \dot{z}$. The last equation is easy to integrate;

$$v_z = \text{constant} = 0,$$

since v_z is zero initially. Thus integrating again,

$$z = \text{constant} = 0,$$

and motion is confined to the $x - y$ plane. We now multiply the second equation by i and add to the first equation.

$$\xi = v_x + iv_y$$

we get the equation,

$$\frac{d\xi}{dt} = i\omega\frac{E}{B} - i\omega\xi, \quad \omega = \frac{qB}{m}.$$

This equation after being multiplied by $e^{i\omega t}$can be rewritten as,

$$\frac{d}{dt}(\xi e^{i\omega t}) = i\omega e^{i\omega t}\frac{E}{B}$$

and integrated at once to give,

$$\xi = \frac{E}{B} + Ce^{-i\omega t - i\alpha},$$

where C and α are two real constants. Taking real and imaginary parts.

$$v_x = \frac{E}{B} + C\cos(\omega t + \alpha) \text{ and } v_y = -C\sin(\omega t + \alpha)$$

Since $v_y = 0$, when $t = 0$, we can take $\alpha = 0$, then $v_x = 0$ at $t = 0$ gives, $C = -\frac{E}{B}$

and we get,

$$v_x = \frac{E}{B}(1 - \cos\omega t) \text{ and } v_y = \frac{E}{B}\sin\omega t.$$

Integrating again and using $x = y = 0$, at $t = 0$, we get

$$x(t) = \frac{E}{B}\left(t - \frac{\sin\omega t}{\omega}\right), \quad y(t) = \frac{E}{\omega B}(1 - \cos\omega t).$$

This is the equation of a cycloid.

(b) The velocity is zero, when $\omega t = 2n\pi$. We see that

$$v^2 = v_x^2 + v_y^2 = \left(\frac{E}{B}\right)^2(2 - 2\cos\omega t)$$

or, $$v = \frac{ds}{dt} = \frac{2E}{B}\left|\sin\frac{\omega t}{2}\right|$$

The quantity inside the modulus is positive for $0 < \omega t < 2\pi$. Thus we can drop the modulus and write for the distance traversed between two successive zeroes of velocity.

$$S = \frac{4E}{\omega B}\left(1 - \cos\frac{\omega t}{2}\right)$$

Putting $\omega t = 2\pi$, we get

$$S = \frac{8E}{\omega B} = \frac{8\,mE}{q\,B^2}$$

(c) The drift velocity is in the x-direction and has the magnitude,

$$< v_x > = \left< \frac{E}{B}(1 - \cos\omega t) \right> = \frac{E}{B}.$$

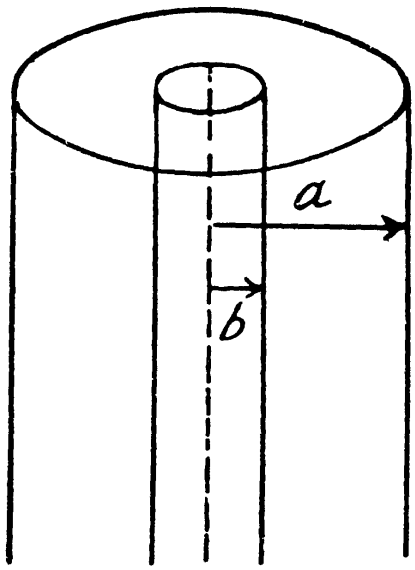
3.393 When a current I flows along the axis, a magnetic field $B_\varphi = \frac{\mu_0 I}{2\pi\rho}$ is set up where $\rho^2 = x^2 + y^2$. In terms of components,

$$B_x = -\frac{\mu_0 I y}{2\pi\rho^2},\ B_y = \frac{\mu_0 I x}{2\pi\rho^2} \text{ and } B_z = 0$$

Suppose a p.d. V is set up between the inner cathode and the outer anode. This means a potential function of the form

$$\varphi = V\frac{\ln\rho/b}{\ln a/b},\ a > \rho > b,$$

as one can check by solving Laplace equation.



The electric field corresponding to this is,

$$E_x = -\frac{Vx}{\rho^2\ln a/b},\ E_y = -\frac{Vy}{\rho^2\ln a/b},\ E_z = 0.$$

The equations of motion are,

$$\frac{d}{dt}mv_x = +\frac{|e|Vx}{\rho^2\ln a/b} + \frac{|e|\,\mu_0 I}{2\pi\rho^2}x\dot{z}$$

$$\frac{d}{dt}mv_y = +\frac{|e|Vy}{\rho^2\ln a/b} + \frac{|e|\mu_0 I}{2\pi\rho^2}y\dot{z}$$

and $$\frac{d}{dt}mv_z = -|e|\frac{\mu_0 I}{2\pi\rho^2}(x\dot{x} + y\dot{y}) = -|e|\frac{\mu_0 I}{2\pi}\frac{d}{dt}\ln\rho$$

$(-|e|)$ is the charge on the electron.

Integrating the last equation,

$$mv_z = -|e|\frac{\mu_0 I}{2\pi}\ln\rho/a = m\dot{z}.$$

since $v_z = 0$ where $\rho = a$. We now substitute this $\dot{z}$ in the other two equations to get

$$\frac{d}{dt}\left(\frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2\right)$$

$$= \left[\frac{|e|V}{\ln a/b} - \frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln\rho/b\right]\cdot\frac{x\dot{x}+y\dot{y}}{\rho^2}$$

$$= \left[\frac{|e|V}{\ln\frac{a}{b}} - \frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln\frac{\rho}{b}\right]\cdot\frac{1}{2\rho^2}\frac{d}{dt}\rho^2$$

$$= \left[\frac{|e|V}{\ln\frac{a}{b}} - \frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln\frac{\rho}{b}\right]\frac{d}{dt}\ln\frac{\rho}{b}$$

Integrating and using $v^2 = 0$, at $\rho = b$, we get,

$$\frac{1}{2}mv^2 = = \left[\frac{|e|V}{\ln\frac{a}{b}}\ln\frac{\rho}{b} - \frac{1}{2m}|e|^2\left(\frac{\mu_0 I}{2\pi}\right)^2\left(\ln\frac{\rho}{b}\right)\right]$$

The RHS must be positive, for all $a > \rho > b$. The condition for this is,

$$V \geq \frac{1}{2}\frac{|e|}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln\frac{a}{b}$$

3.394 This differs from the previous problem in $(a \leftrightarrow b)$ and the magnetic field is along the z-direction. Thus $B_x = B_y = 0$, $B_z = B$

Assuming as usual the charge of the electron to be $-|e|$, we write the equation of motion

$$\frac{d}{dt}mv_x = \frac{|e|V_x}{\rho^2\ln\frac{b}{a}} - |e|B\dot{y},\quad \frac{d}{dt}mv_y = \frac{|e|V_y}{\rho^2\ln\frac{b}{a}} + |e|B\dot{x}$$

and
$$\frac{d}{dt}mv_z = 0 \Rightarrow z = 0$$

The motion is confined to the plane $z = 0$. Eliminating B from the first two equations,

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{|e|V}{\ln b/a}\frac{x\dot{x}+y\dot{y}}{\rho^2}$$

or,
$$\frac{1}{2}mv^2 = |e|V\frac{\ln\rho/a}{\ln b/a}$$

so, as expected, since magnetic forces do not work,

$$v = \sqrt{\frac{2|e|V}{m}},\ \text{at } \rho = b.$$

On the other hand, eliminating V, we also get,

$$\frac{d}{dt} m\,(xv_y - yv_x) = |e|\,B\,(x\dot{x} + y\dot{y})$$

i.e. $$(xv_y - yv_x) = \frac{|e|B}{2m}\rho^2 + \text{constant}$$

The constant is easily evaluated, since v is zero at $\rho = a$. Thus,

$$(xv_y - yv_x) = \frac{|e|B}{2m}(\rho^2 - a^2) > 0$$

At $$\rho = b,\ (xv_y - yv_x) \le vb$$

Thus, $$vb \ge \frac{|e|B}{2m}(b^2 - a^2)$$

or, $$B \le \frac{2\,mb}{b^2 - a^2}\sqrt{\frac{2\,|e|\,V}{m}} \times \frac{1}{|e|}$$

or, $$B \le \frac{2b}{b^2 - a^2}\sqrt{\frac{2m\,B}{|e|}}$$

3.395 The equations are as in 3.392.

$$\frac{dv_x}{dt} = \frac{qB}{m}v_y,\ \frac{dv_y}{dt} = \frac{qE_m}{m}\cos\omega t - \frac{qB}{m}v_x \text{ and } \frac{dv_z}{dt}$$

with $$\omega = \frac{qB}{m},\ \xi = v_x + iv_y, \text{ we get,}$$

$$\frac{d\xi}{dt} = i\frac{E_m}{B}\omega\cos\omega t - i\,\omega\,\xi$$

or multiplying by $e^{i\omega t}$,

$$\frac{d}{dt}(\xi\,e^{i\omega t}) = i\frac{E_m}{2B}\omega\,(e^{2i\omega t} + 1)$$

or integrating, $$\xi\,e^{i\omega t} = \frac{E_m}{4B}e^{2i\omega t} + \frac{E_m}{2B}i\,\omega t$$

or, $$\xi = \frac{E_m}{4B}(e^{i\omega t} + 2\,i\,\omega t\,e^{i\omega t}) + C\,e^{i\omega t}$$

since $$\xi = 0 \text{ at } t = 0,\ C = -\frac{E_m}{4B}.$$

Thus, $$\xi = i\frac{E_m}{2B}\sin\omega t + i\frac{E_m}{2B}\omega t\,e^{i\omega t}$$

or, $$v_x = \frac{E_m}{2B}\omega t\sin\omega t \text{ and } v_y = \frac{E_m}{2B}\sin\omega t + \frac{E_m}{2B}\omega t\cos\omega t$$

Integrating again,

$$x = \frac{a}{2\,\omega^2}(\sin\omega t - \omega t\cos\omega t),\ y = \frac{a}{2\omega}t\sin\omega t.$$

where $a = \frac{qE_m}{m}$, and we have used $x = y = 0$, at $t = 0$.

The trajectory is an unwinding spiral.

3.396 We know that for a charged particle (proton) in a magnetic field,

$$\frac{mv^2}{r} = B\,ev \text{ or } mv = Ber$$

But,
$$\omega = \frac{eB}{m},$$

Thus
$$E = \frac{1}{2}mv^2 = \frac{1}{2}m\,\omega^2 r^2.$$

So,
$$\Delta E = m\omega^2 r\,\Delta r = 4\,\pi^2 \nu^2 mr\,\Delta r$$

On the other hand $\Delta E = 2\,eV$, where V is the effective acceleration voltage, across the Dees, there being two crossings per revolution. So,

$$V \geq 2\,\pi^2 \nu^2 mr\,\Delta r/e$$

3.397 (a) From $\frac{mv^2}{r} = Bev$, or, $mv = Ber$

and
$$T = \frac{(Ber)^2}{2m} = \frac{1}{2}mv^2 = 12\,\text{MeV}$$

(b) From $\frac{2\pi}{\omega} = \frac{2\pi r}{v}$

we get,
$$f_{\min} = \frac{v}{2\,\pi r} = \frac{1}{\pi r}\sqrt{\frac{T}{2m}} = 15\,\text{MHz}$$

3.398 (a) The total time of acceleration is,

$$t = \frac{1}{2\,\nu}\cdot n,$$

where n is the number of passages of the Dees.

But,
$$T = neV = \frac{B^2 e^2 r^2}{2m}$$

or,
$$n = \frac{B^2 e\,r^2}{2mV}$$

So,
$$t = \frac{\pi}{eB/m} \times \frac{B^2 er^2}{2mV} = \frac{\pi B\,r^2}{2V} = \frac{\pi^2 m\nu\,r^2}{eV} = 30\,\mu\,\text{s}$$

(b) The distance covered is, $s = \sum v_n \cdot \frac{1}{2\,\nu}$

But,
$$v_n = \sqrt{\frac{2eV}{m}}\,\sqrt{n}\,,$$

So,
$$s = \sqrt{\frac{eV}{2m\nu^2}}\sum \sqrt{n} = \sqrt{\frac{eV}{2m\nu^2}}\int \sqrt{n}\,dn = \sqrt{\frac{eV}{2m\nu^2}}\,\frac{2}{3}\,n^{3/2}$$

But, $$n = \frac{B^2 e^2 r^2}{2\,eV m} = \frac{2\,\pi^2 m\,\nu^2 r^2}{eV}$$

Thus, $$s \approx \frac{4\,\pi^3 \nu^2\, mr^2}{3eV} = 1{\cdot}24\ \text{km}$$

3.399 In the nth orbit, $\frac{2\,\pi\, r_n}{v_n} = n\ T_0 = \frac{n}{\nu}\cdot$ We ignore the rest mass of the electron and write $v_n \approx c$. Also $W \approx cp = cBer_n$.

Thus, $$\frac{2\,\pi\, W}{Bec^2} = \frac{n}{\nu}$$

or, $$n = \frac{2\,\pi\, W\nu}{Bec^2} = 9$$

3.400 The basic condition is the relativistic equation,

$$\frac{mv^2}{r} = Bqv, \quad \text{or,} \quad mv = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} = Bqr.$$

Or calling, $$\omega = \frac{Bq}{m},$$

we get, $$\omega = \frac{\omega_0}{\sqrt{1 + \frac{\omega_0^2\, r^2}{c^2}}}, \qquad \omega_0 = \frac{Bq}{m_0}\, r$$

is the radius of the instantaneous orbit.

The time of acceleration is,

$$t = \sum_{n=1}^{N} \frac{1}{2\nu_n} = \sum_{n=1}^{N} \frac{\pi}{\omega_n} = \sum_{n}^{N} \frac{\pi\, W_n}{q\,Bc^2}.$$

N is the number of crossing of either Dee.

But, $W_n = m_0\, c^2 + \frac{n\,\Delta W}{2}$, there being two crossings of the Dees per revolution.

So, $$t = \sum \frac{\pi\, m_0 c^2}{qB\, c^2} + \sum \frac{\pi\, \Delta W_n}{2q\, B\, c^2}$$

$$= N\frac{\pi}{\omega_0} + \frac{N\,(N+1)}{4}\,\frac{\pi\,\Delta\, W}{qB\, c^2} \approx N^2\,\frac{\pi\,\Delta W}{4\, q\, Bc^2}\ (N >> 1)$$

Also, $$r = r_N\,\frac{v_N}{\omega_N} \approx \frac{c}{\pi}\,\frac{\partial t}{\partial N} = \frac{\Delta W}{2q\, Bc}\, N$$

Hence finally, $$\omega = \frac{\omega_0}{\sqrt{1 + \frac{q^2 B^2}{m_0^2 c^2} \times \frac{\Delta W^2}{4 q^2 B^2 c^2} N^2}}$$

$$= \frac{\omega_0}{\sqrt{1 + \frac{(\Delta W)^2}{4 m_0^2 c^4} \times \frac{4 qBc^2}{\pi \Delta W} t}} = \frac{\omega_0}{\sqrt{1 + at}};$$

$$a = \frac{q B \Delta W}{\pi m_0^2 c^2}$$

3.401 When the magnetic field is being set up in the solenoid, and electric field will be induced in it, this will accelerate the charged particle. If $\dot{B}$ is the rate, at which the magnetic field is increasing, then.

$$\pi r^2 \dot{B} = 2 \pi r E \quad \text{or} \quad E = \frac{1}{2} r \dot{B}$$

Thus, $$m \frac{dv}{dt} = \frac{1}{2} r \dot{B} q, \quad \text{or} \quad v = \frac{qBr}{2m},$$

After the field is set up, the particle will execute a circular motion of radius ρ, where

$$mv = B q \rho, \quad \text{or} \quad \rho = \frac{1}{2} r$$

3.402 The increment in energy per revolution is $e \Phi$, so the number of revolutions is,

$$N = \frac{W}{e \Phi}$$

The distance traversed is, $s = 2 \pi r N$

3.403 On the one hand,

$$\frac{dp}{dt} = eE = \frac{e}{2\pi r} \frac{d\Phi}{dt} = \frac{e}{2\pi r} \frac{d}{dt} \int_0^r 2\pi r' B(r') \, dr'$$

On the other ,

$$p = B(r) \, er, \quad r = \text{constant.}$$

so, $$\frac{dp}{dt} = er \frac{d}{dt} B(r) = er \dot{B}(r)$$

Hence, $$er \dot{B}(r) = \frac{e}{2 \pi r} \pi r^2 \frac{d}{dt} \langle B \rangle$$

So, $$\dot{B}(r) = \frac{1}{2} \frac{d}{dt} \langle B \rangle$$

This equations is most easily satisfied by taking $B(r_0) = \frac{1}{2} \langle B \rangle$.

3.404 The condition, $B(r_0) = \frac{1}{2} \langle B \rangle = \frac{1}{2} \int_0^{r_0} B \cdot 2\pi r \, dr / \pi r_0^2$

or, $$B(r_0) = \frac{1}{r_0^2}\int_0^{r_0} Brdr$$

This gives r_0.

In the present case,

$$B_0 - ar_0^2 = \frac{1}{r_0^2}\int_0^{r_0}(B - ar^2)\, rdr = \frac{1}{2}\left(B_0 - \frac{1}{2}ar_0^2\right)$$

or, $$\frac{3}{4}ar_0^2 = \frac{1}{2}B_0 \quad \text{or} \quad r_0 = \sqrt{\frac{2B_0}{3a}}\,.$$

3.405 The induced electric field (or eddy current field) is given by,

$$E(r) = \frac{1}{2\pi r}\frac{d}{dt}\int_0^r 2\pi r'\,(r')\,B(r')\,dr'$$

Hence,

$$\frac{dE}{dr} = -\frac{1}{2\pi r^2}\frac{d}{dt}\int_0^r 2\pi r' B(r')\,dr' + \frac{dB(r)}{dt}$$

$$= -\frac{1}{2}\frac{d}{dt}< B > + \frac{dB(r)}{dt}$$

This vanishes for $r = r_0$ by the betatron condition, where r_0 is the radius of the equilibrium orbit.

3.406 From the betatron condition,

$$\frac{1}{2}\frac{d}{dt}< B > = \frac{dB}{dt}(r_0) = \frac{B}{\Delta t}$$

Thus, $$\frac{d}{dt}< B > = \frac{2B}{\Delta t}$$

and $$\frac{d\Phi}{dt} = \pi r^2\frac{d< B >}{dt} = \frac{2\pi r^2 B}{\Delta t}$$

So, energy increment per revolution is,

$$e\frac{d\Phi}{dt} = \frac{2\pi r^2 eB}{\Delta t}$$

3.407 (a) Even in the relativistic case, we know that : $p = Ber$

Thus, $W = \sqrt{c^2p^2 + m_0^2c^4} - m_0c^2 = m_0c^2\left(\sqrt{1 + (Ber/m_0c)^2} - 1\right)$

(b) The distance traversed is,

$$2\pi r\frac{W}{e\Phi} = 2\pi r\frac{W}{2\pi r^2 eB/\Delta t} = \frac{W\Delta t}{Ber},$$

on using the result of the previous problem.

PART FOUR

OSCILLATIONS AND WAVES

4.1 MECHANICAL OSCILLATIONS

4.1 (a) Given, $x = a\cos\left(\omega t - \frac{\pi}{4}\right)$

So, $v_x = \dot{x} = -a\,\omega\sin\left(\omega t - \frac{\pi}{4}\right)$ and $w_x = \ddot{x} = -a\,\omega^2\cos\left(\omega t - \frac{\pi}{4}\right)$ (1)

On-the basis of obtained expressions plots $x(t)$, $v_x(t)$ and $w_x(t)$ can be drawn as shown in the answersheet, (of the problem book).

(b) From Eqn (1)

$$v_x = -a\,\omega\sin\left(\omega t - \frac{\pi}{4}\right) \text{ So, } v_x^2 = a^2\omega^2\sin^2\left(\omega t - \frac{\pi}{4}\right) \tag{2}$$

But from the law $x = a\cos(\omega t - \pi/4)$, so, $x^2 = a^2\cos^2(\omega t - \pi/4)$

or, $$\cos^2(\omega t - \pi/4) = x^2/a^2 \quad \text{or} \quad \sin^2(\omega t - \pi/4) = 1 - \frac{x^2}{a^2} \tag{3}$$

Using (3) in (2),

$$v_x^2 = a^2\omega^2\left(1 - \frac{x^2}{a^2}\right) \quad \text{or} \quad v_x^2 = \omega^2(a^2 - x^2) \tag{4}$$

Again from Eqn (4), $w_x = -a\,\omega^2\cos(\omega t - \pi/4) = -\omega^2 x$

4.2 (a) From the motion law of the particle

$$x = a\sin^2(\omega t - \pi/4) = \frac{a}{2}\left[1 - \cos\left(2\,\omega t - \frac{\pi}{2}\right)\right]$$

or, $$x - \frac{a}{2} = -\frac{a}{2}\cos\left(2\,\omega t - \frac{\pi}{2}\right) = -\frac{a}{2}\sin 2\,\omega t = \frac{a}{2}\sin(2\,\omega t + \pi)$$

i.e. $$x - \frac{a}{2} = \frac{a}{2}\sin(2\,\omega t + \pi). \tag{1}$$

Now compairing this equation with the general equation of harmonic oscillations : $X = A\sin(\omega_0 t + \alpha)$

Amplitude, $A = \frac{a}{2}$ and angular frequency, $\omega_0 = 2\,\omega$.

Thus the period of one full oscillation, $T = \frac{2\pi}{\omega_0} = \frac{\pi}{\omega}$

(b) Differentiating Eqn (1) w.r.t. time

$v_x = a\,\omega \cos(2\,\omega\,t + \pi)$ or $v_x^2 = a^2\,\omega^2 \cos^2(2\,\omega\,t + \pi) = a^2\,\omega^2\left[1 - \sin^2(2\,\omega\,t + \pi)\right]$ (2)

From Eqn (1) $\left(x - \frac{a}{2}\right)^2 = \frac{a^2}{4}\sin^2(2\,\omega\,t + \pi)$

or, $4\frac{x^2}{a^2} + 1 - \frac{4x}{a} = \sin^2(2\,\omega\,t + \pi)$ or $1 - \sin^2(2\,\omega\,t + \pi) = \frac{4\,x}{a}\left(1 - \frac{x}{a}\right)$ (3)

From Eqns (2) and (3), $v_x = a^2\,\omega^2\,\frac{4\,x}{a}\left(1 - \frac{x}{a}\right) = 4\,\omega^2\,x\,(a - x)$

Plot of $v_x(x)$ is as shown in the answersheet.

4.3 Let the general equation of S.H.M. be

$$x = a\cos(\omega\,t + \alpha) \tag{1}$$

So,

$$v_x = -a\,\omega\sin(\omega\,t + \alpha) \tag{2}$$

Let us assume that at $t = 0$, $x = x_0$ and $v_x = v_{x_0}$.

Thus from Eqns (1) and (2) for $t = 0$, $x_0 = a\cos\alpha$, and $v_{x_0} = -\,a\,\omega\sin\alpha$

Therefore $\tan\alpha = -\frac{v_{x_0}}{\omega\,x_0}$ and $a = \sqrt{x_0^2 + \left(\frac{v_{x_0}}{\omega}\right)^2} = 35.35$ cm

Under our assumption Eqns (1) and (2) give the sought x and v_x if

$$t = t = 2{\cdot}40\text{ s},\ a = \sqrt{x_0^2 + \left(v_{x_0}/\omega\right)^2} \text{ and } \alpha = \tan^{-1}\left(-\frac{v_x}{\omega\,x_0}\right) = -\frac{\pi}{4}$$

Putting all the given numerical values, we get :

$$x = -\ 29 \text{ cm} \quad \text{and} \quad v_x = -\ 81 \text{ cm/s}$$

4.4 From the Eqn, $v_x^2 = \omega^2(a^2 - x^2)$ (see Eqn. 4 of 4.1)

$$v_1^2 = \omega^2(a^2 - x_1^2) \quad \text{and} \quad v_2^2 = \omega^2(a^2 - x_2^2)$$

Solving these Eqns simultaneously, we get

$$\omega = \sqrt{(v_1^2 - v_2^2)/(x_2^2 - x_1^2)}\ ,\quad a = \sqrt{(v_1\,x_2^2 - v_2^2\,x_1^2)/(v_1^2 - v_2^2)}$$

4.5 (a) When a particle starts from an extreme position, it is useful to write the motion law as

$$x = a\cos\omega\,t \tag{1}$$

(However x is the displacement from the equlibrium position)

It t_1 be the time to cover the distence $a/2$ then from (1)

$$a - \frac{a}{2} = \frac{a}{2} = a\cos\omega\,t_1 \quad \text{or} \quad \cos\omega\,t_1 = \frac{1}{2} = \cos\frac{\pi}{3} \ (\text{as } t_1 < T/4)$$

Thus

$$t_1 = \frac{\pi}{3\,\omega} = \frac{\pi}{3(2\,\pi/T)} = \frac{T}{6}$$

As $\qquad x = a\cos\omega t, \quad \text{so}, \; v_x = -a\omega\sin\omega t$

Thus $\qquad v = |v_x| = -v_x = a\omega\sin\omega t, \quad \text{for } t \le t_1 = T/6$

Hence sought mean velocity

$$<v> = \frac{\int v\,dt}{\int dt} = \int_0^{T/6} a\,(2\pi/T)\sin\omega t\,dt \Big/ T/6 = \frac{3a}{T} = 0{\cdot}5 \text{ m/s}$$

(b) In this case, it is easier to write the motion law in the form :

$$x = a\sin\omega t \tag{2}$$

If t_2 be the time to cover the distance $a/2$, then from Eqn (2)

$$a/2 = a\sin\frac{2\pi}{T}t_2 \quad \text{or} \quad \sin\frac{2\pi}{T}t_2 = \frac{1}{2} = \sin\frac{\pi}{6} \quad (\text{as } t_2 < T/4)$$

Thus $\qquad \frac{2\pi}{T}t_2 = \frac{\pi}{6} \quad \text{or}, \; t_2 = \frac{T}{12}$

Differentiating Eqn (2) w.r.t time, we get

$$v_x = a\omega\cos\omega t = a\frac{2\pi}{T}\cos\frac{2\pi}{T}t$$

So, $\qquad v = |v_x| = a\frac{2\pi}{T}\cos\frac{2\pi}{T}t, \quad \text{for } t \le t_2 = T/12$

Hence the sought mean velocity

$$<v> = \frac{\int v\,dt}{\int dt} = \frac{1}{(T/12)}\int_0^{T/12} a\frac{2\pi}{T}\cos\frac{2\pi}{T}t\,dt = \frac{6a}{T} = 1 \text{ m/s}$$

4.6 (a) As $x = a\sin\omega t \qquad \text{so,} \quad v_x = a\omega\cos\omega t$

Thus $\quad <v_x> = \int v_x\,dt \Big/ \int dt = \dfrac{\int_0^{\frac{3}{8}T} a\omega\cos(2\pi/T)\,t\,dt}{\frac{3}{8}T} = \dfrac{2\sqrt{2}\,a\omega}{3\pi} \left(\text{using } T = \dfrac{2\pi}{\omega}\right)$

(b) In accordance with the problem

$$\vec{v} = v_x\vec{i}, \quad \text{so}, \; |<\vec{v}>| = |<v_x>|$$

Hence, using part (a), $|<\vec{v}>| = \left|\dfrac{2\sqrt{2}\,a\omega}{3\pi}\right| = \dfrac{2\sqrt{2}\,a\omega}{3\pi}$

(c) We have got, $v_x = a\omega\cos\omega t$

$$\text{So, } v = |v_x| = \left.\begin{aligned} &a\omega\cos\omega t, \quad \text{for } t \le T/4 \\ &= -a\omega\cos\omega t, \quad \text{for } T/4 \le t \le \frac{3}{8}T \end{aligned}\right]$$

Hence, $$\langle v \rangle = \frac{\int v\,dt}{\int dt} = \frac{\int_0^{T/4} a\,\omega \cos \omega t\,dt + \int_{T/4}^{3T/8} -a\,\omega \cos \omega t\,dt}{3\,T/8}$$

Using $\omega = 2\pi/T$, and on evaluating the integral we get

$$\langle v \rangle = \frac{2\,(4-\sqrt{2}\,)\,a\,\omega}{3\,\pi}$$

4.7 From the motion law, $x = a \cos \omega t$,, it is obvious that the time taken to cover the distance equal to the amplitude (a), starting from extreme position equals $T/4$.

Now one can write

$$t = n\frac{T}{4} + t_0,\ \left(\text{where } t_0 < \frac{T}{4} \text{ and } n = 0, 1, 2, \dots\right)$$

As the particle moves according to the law, $x = a \cos \omega t$,

so at $n = 1,3,5 \dots$ or for odd n values it passes through the mean positon and for even numbers of n it comes to an extreme position (if $t_0 = 0$).

Case (1) when n is an odd number :

In this case, from the equation

$x = \pm a \sin \omega t$, if the t is counted from $nT/4$ and the distance covered in the time interval to becomes,$s_1 = a \sin \omega t_0 = a \sin \omega \left(t - n\frac{T}{4}\right) = a \sin\left(\omega t - \frac{n\pi}{2}\right)$

Thus the sought distance covered for odd n is

$$s = n\,a + s_1 = n\,a + a \sin\left(\omega t - \frac{n\pi}{2}\right) = a\left[n + \sin\left(\omega t - \frac{n\pi}{2}\right)\right]$$

Case (2), when n is even, In this case from the equation

$x = a \cos \omega t$, the distance covered (s_2) in the interval t_0, is given by

$$a - s_2 = a \cos \omega t_0 = a \cos \omega \left(t - n\frac{T}{4}\right) = a \cos\left(\omega t - n\frac{\pi}{2}\right)$$

or, $$s_2 = a\left[1 - \cos\left(\omega t - \frac{n\pi}{2}\right)\right]$$

Hence the sought distance for n is even

$$s = n\,a + s_2 = n\,a + a\left[1 - \cos\left(\omega t - \frac{n\pi}{2}\right)\right] = a\left[n + 1 - \cos\left(\omega t - \frac{n\pi}{2}\right)\right]$$

In general

$$s = \begin{cases} a\left[n + 1 - \cos\left(\omega t - \frac{n\pi}{2}\right)\right], & n \text{ is even} \\ a\left[n + \sin\left(\omega t - \frac{n\pi}{2}\right)\right], & n \text{ is odd} \end{cases}$$

4.8 Obviously the motion law is of the from, $x = a \sin \omega t$. and $v_x = \omega a \cos \omega t$.

Comparing $v_x = \omega a \cos \omega t$ with $v_x = 35 \cos \pi t$, we get

$$\omega = \pi, \; a = \frac{35}{\pi}, \text{thus } T = \frac{2\pi}{\omega} = 2 \text{ and } T/4 = 0{\cdot}5 \text{ s}$$

Now we can write

$$t = 2{\cdot}8 \text{ s} = 5 \times \frac{T}{4} + 0{\cdot}3 \quad \left(\text{where } \frac{T}{4} = 0{\cdot}5 \text{ s}\right)$$

As $n = 5$ is odd, like (4·7), we have to basically find the distance covered by the particle starting from the extreme position in the time interval 0·3 s.

Thus from the Eqn.

$$x = a \cos \omega t = \frac{35}{\pi} \cos \pi (0{\cdot}3)$$

$$\frac{35}{\pi} - s_1 = \frac{35}{\pi} \cos \pi (0{\cdot}3) \quad \text{or} \quad s_1 = \frac{35}{\pi} \{1 - \cos 0{\cdot}3\pi\}$$

Hence the sought distance

$$s = 5 \times \frac{35}{\pi} + \frac{35}{\pi} \{1 - \cos 0{\cdot}3\pi\}$$

$$= \frac{35}{\pi} \{6 - \cos 0{\cdot}3\pi\} = \frac{35}{22} \times 7 (6 - \cos 54°) = 60 \text{ cm}$$

4.9 As the motion is periodic the particle repeatedly passes through any given region in the range $-a \leq x \leq a$. The probability that it lies in the range $(x, x + dx)$ is defined as the fraction $\frac{\Delta t}{t}$ (as $t \to \infty$) where Δt is the time that the particle lies in the range $(x, x + dx)$ out of the total time t. Because of periodicity this is

$$dP = \frac{dP}{dx} dx = \frac{dt}{T} = \frac{2\,dx}{v T}$$

where the factor 2 is needed to take account of the fact that the particle is in the range $(x, x + dx)$ during both up and down phases of its motion. Now in a harmonic oscillator.

$$v = \dot{x} = \omega a \cos \omega t = \omega \sqrt{a^2 - x^2}$$

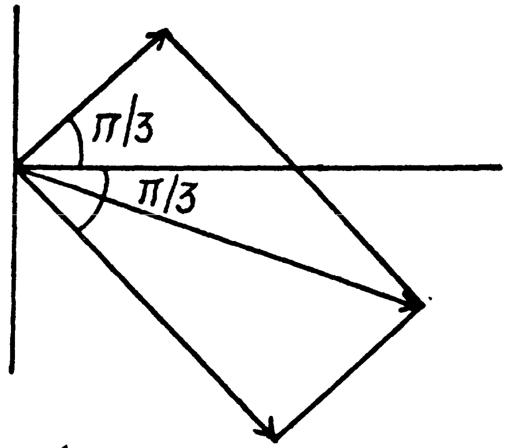
Thus since $\omega T = 2\pi$ (T is the time period)

We get
$$dP = \frac{dP}{dx} dx = \frac{1}{\pi} \frac{dx}{\sqrt{a^2 - x^2}}$$

Note that
$$\int_{-a}^{+a} \frac{dP}{dx} dx = 1$$

so
$$\frac{dP}{dx} = \frac{1}{\pi} \frac{1}{\sqrt{a^2 - x^2}} \text{ is properly normalized.}$$

4.10 (a) We take a graph paper and choose an axis (X – axis) and an origin. Draw a vector of magnitude 3 inclined at an angle $\frac{\pi}{3}$ with the X-axis. Draw another vector of magnitude 8 inclined at an angle $-\frac{\pi}{3}$ (Since $\sin(\omega t + \pi/6) = \cos(\omega t - \pi/3)$) with the X – axis. The magnitude of the resultant of both these vectors (drawn from the origin) obtained using parallelogram law is the resultant, amplitude.



Clearly $\quad R^2 = 3^2 + 8^2 + 2 \cdot 3 \cdot 8 \cdot \cos\frac{2\pi}{3} = 9 + 64 - 48 \times \frac{1}{2}$

$$= 73 - 24 = 49$$

Thus $\quad R = 7$ units

(b) One can follow the same graphical method here but the result can be obtained more quickly by breaking into sines and cosines and adding :

Resultant
$$x = \left(3 + \frac{5}{\sqrt{2}}\right)\cos\omega t + \left(6 - \frac{5}{\sqrt{2}}\right)\sin\omega t$$
$$= A\cos(\omega t + \alpha)$$

Then
$$A^2 = \left(3 + \frac{5}{\sqrt{2}}\right)^2 + \left(6 - \frac{5}{\sqrt{2}}\right)^2 = 9 + 25 + \frac{30 - 60}{\sqrt{2}} + 36$$
$$= 70 - 15\sqrt{2} = 70 - 21{\cdot}2$$

So, $\quad A = 6{\cdot}985 \approx 7$ units

Note- In using graphical method convert all oscillations to either sines or cosines but do not use both.

4.11 Given, $x_1 = a\cos\omega t$ and $x_2 = a\cos 2\omega t$

so, the net displacement,

$$x = x_1 + x_2 = a\{\cos\omega t + \cos 2\omega t\} = a\{\cos\omega t + 2\cos^2\omega t - 1\}$$

and $\quad v_x = \dot{x} = a\{-\omega\sin\omega t - 4\omega\cos\omega t\sin\omega t\}$

For $\dot{x}$ to be maximum,

$$\ddot{x} = a\omega^2\cos\omega t - 4a\omega^2\cos^2\omega t + 4a\omega^2\sin^2\omega t = 0$$

or, $\quad 8\cos^2\omega t + \cos\omega t - 4 = 0$, which is a quadratic equation for $\cos\omega t$.

Solving for acceptable value

$$\cos\omega t = 0{\cdot}644$$

thus $\quad \sin\omega t = 0{\cdot}765$

and $\quad v_{max} = |v_{x_{max}}| = +a\omega[0{\cdot}765 + 4 \times 0{\cdot}765 \times 0{\cdot}644] = +2{\cdot}73\ a\omega$

4.12 We write :

$$a \cos 2{\cdot}1\, t \cos 50{\cdot}0\, t = \frac{a}{2}\left\{\cos 52{\cdot}1\, t + \cos 47{\cdot}9\, t\right\}$$

Thus the angular frequencies of constituent oscillations are

$$52{\cdot}1\ \mathrm{s}^{-1} \text{ and } 47{\cdot}9\ \mathrm{s}^{-1}$$

To get the beat period note that the variable amplitude $a \cos 2{\cdot}1\, t$ becomes maximum (positive or negative), when

$$2{\cdot}1\, t = n\pi$$

Thus the interval between two maxima is

$$\frac{\pi}{2{\cdot}1} = 1{\cdot}5 \text{ s nearly.}$$

4.13 If the frequency of A with respect to K' is ν_0 and K' oscillates with frequency $\bar{\nu}$ with respect to K, the beat frequency of the point A in the K-frame will be ν when

$$\bar{\nu} = \nu_0 \pm \nu$$

In the present case $\bar{\nu} = 20$ or 24. This means

$$\nu_0 = 22. \ \ \& \ \ \nu = 2$$

Thus beats of $2\nu = 4$ will be heard when $\bar{\nu} = 26$ or 18.

4.14 (a) From the Eqn : $x = a \sin \omega t$

$$\sin^2 \omega t = x^2/a^2 \quad \text{or} \quad \cos^2 \omega t = 1 - \frac{x^2}{a^2} \tag{1}$$

And from the equation : $y = b \cos \omega t$

$$\cos^2 \omega t = y^2/b^2 \tag{2}$$

From Eqns (1) and (2), we get :

$$1 - \frac{x^2}{a^2} = \frac{y^2}{b^2} \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is the standard equation of the ellipse shown in the figure.
we observe that,

at $\quad t = 0,\ x = 0$ and $y = b$

and at $\quad t = \dfrac{\pi}{2\omega},\ x = +a$ and $y = 0$

Thus we observe that at $t = 0$, the point is at point 1 (Fig.) and at the following moments, the co-ordinate y diminishes and x becomes positive. Consequently the motion is clockwise.

(b) As $x = a \sin \omega t$ and $y = b \cos \omega t$

So we may write $\vec{r} = a \sin \omega t\ \vec{i} + b \cos \omega t\ \vec{j}$

Thus $\ddot{\mathbf{r}} = \vec{w} = -\omega^2 \vec{r}$

4.15 (a) From the Eqn. : $x = a \sin \omega t$, we have

$$\cos \omega t = \sqrt{1 - (x^2/a^2)}$$

and from the Eqn. :$y = a \sin 2 \omega t$

$$y = 2 a \sin \omega t \cos \omega t = 2 x \sqrt{1 - (x^2/a^2)} \quad \text{or} \quad y^2 = 4 x^2 \left(1 - \frac{x^2}{a^2}\right)$$

(b) From the Eqn. : $x = a \sin \omega t$;

$$\sin^2 \omega t = x^2/a^2$$

From$y = a \cos 2 \omega t$

$$y = a (1 - 2 \sin^2 \omega t) = a \left(1 - 2 \frac{x^2}{a^2} \right)$$

For the plots see the plots of answersheet of the problem book.

4.16 As $U(x) = U_0 (1 - \cos a x)$

So, $$F_x = -\frac{dU}{dx} = - U_0 a \sin a x \qquad (1)$$

or, $F_x = - U_0 a a x$ (because for small angle of oscillations $\sin a x \approx a x$)

or, $$F_x = - U_0 a^2 x \qquad (1)$$

But we know $F_x = - m \omega_0^2 x$, for small oscillation

Thus $$\omega_0^2 = \frac{U_0 a^2}{m} \quad \text{or} \quad \omega_0 = a \sqrt{\frac{U_0}{m}}$$

Hence the sought time period

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{a} \sqrt{\frac{m}{U_0}} = 2\pi \sqrt{\frac{m}{a^2 U_0}}$$

4.17 If $U(x) = \frac{a}{x^2} - \frac{b}{x}$

then the equilibrium position is $x = x_0$ when $U'(x_0) = 0$

or $$-\frac{2a}{x_0^3} + \frac{b}{x_0^2} = 0 \Rightarrow x_0 = \frac{2a}{b}.$$

Now write : $$x = x_0 + y$$

Then $$U(x) = \frac{a}{x_0^2} - \frac{b}{x_0} + (x - x_0) U'(x_0) + \frac{1}{2} (x - x_0)^2 U''(x_0)$$

But $$U''(x_0) = \frac{6a}{x_0^4} - \frac{2b}{x_0^3} = (2a/b)^{-3} (3b - 2b) = b^4/8a^3$$

So finally : $$U(x) = U(x_0) + \frac{1}{2} \left(\frac{b^4}{8a^3} \right) y^2 + \ldots$$

We neglect remaining terms for small oscillations and compare with the P.E. for a harmonic, oscillator :

$$\frac{1}{2} m \omega^2 y^2 = \frac{1}{2}\left(\frac{b^4}{8 a^3}\right) y^2, \quad \text{so} \quad \omega = \frac{b^2}{\sqrt{8 a^3 m}}$$

Thus $$T = 2\pi \frac{\sqrt{8 m a^3}}{b^2}$$

Note : Equilibrium position is generally a minimum of the potential energy. Then $U'(x_0) = 0$, $U''(x_0) > 0$. The equilibrium position can in principle be a maximum but then $U''(x_0) < 0$ and the frequency of oscillations about this equilibrium position will be imaginary.

The answer given in the book is incorrect both numerically and dimensionally.

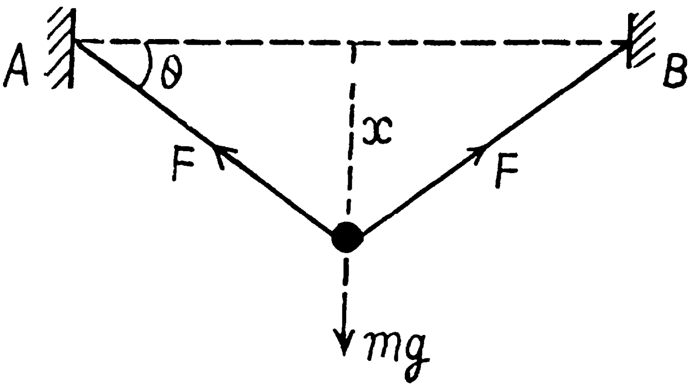
4.18 Let us locate and depict the forces acting on the ball at the position when it is at a distance x down from the undeformed position of the string.
At this position, the unbalanced downward force on the ball

$$= m g - 2 F \sin\theta$$

By Newton's law, $m\ddot{x} = m g - 2 F \sin\theta$

$$= m g - 2 F \theta \text{ (when } \theta \text{ is small)}$$

$$= m g - 2 F \frac{x}{l/2} = m g - \frac{4F}{l} x$$



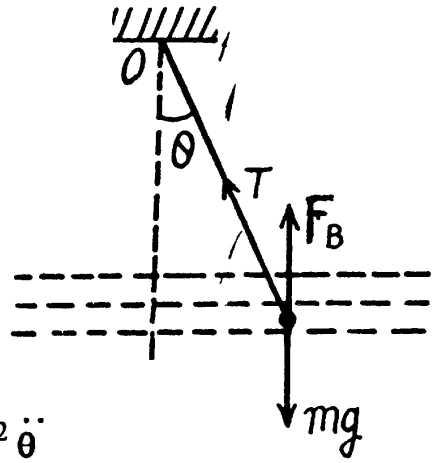
Thus $$\ddot{x} = g - \frac{4F}{m l} x = -\frac{4F}{m l}\left(x - \frac{m g l}{4F}\right)$$

putting $x' = x - \frac{m g l}{T}$, we get

$$\ddot{x}' = -\frac{4T}{m l} x'$$

Thus $T = \pi \sqrt{\frac{m l}{F}} = 0{\cdot}2$ s

4.19 Let us depict the forces acting on the oscillating ball at an arbitraty angular position θ. (Fig.), relative to equilibrium position where F_B is the force of buoyancy. For the ball from the equation :



$N_Z = I \beta_Z$, (where we have taken the positive sense of Z axis in the direction of angular velocity i.e. $\dot{\theta}$ of the ball and passes through the point of suspension of the pendulum O), we get :

$$-m g l \sin\theta + F_B l \sin\theta = m l^2 \ddot{\theta} \quad (1)$$

Using $m = \frac{4}{3}\pi r^3 \sigma$, $F_B = \frac{4}{3}\pi r^3 \rho$ and $\sin\theta \approx \theta$ for small θ, in Eqn (1), we get :

$$\ddot{\theta} = -\frac{g}{l}\left(1-\frac{\rho}{\sigma}\right)\theta$$

Thus the sought time period

$$T = 2\pi\frac{1}{\sqrt{\frac{g}{l}\left(1-\frac{\rho}{\sigma}\right)}} = 2\alpha\sqrt{\frac{l/g}{1-\frac{1}{\eta}}}$$

Hence $$T = 2\alpha\sqrt{\frac{\eta l}{g(\eta-1)}} = 1{\cdot}1\text{s}$$

4.20 Obviously for small β the ball execute part of S.H.M. Due to the perfectly elastic collision the velocity of ball simply reversed. As the ball is in S.H.M. ($|\theta| < \alpha$ on the left)its motion law in differential from can be written as

$$\ddot{\theta} = -\frac{g}{l}\theta = -\omega_0^2\theta \qquad (1)$$

If we assume that the ball is released from the extreme position, $\theta = \beta$ at $t = 0$, the solution of differential equation would be taken in the form

$$\theta = \beta\cos\omega_0 t = \beta\cos\sqrt{\frac{g}{l}}\,t \qquad (2)$$

If t' be the time taken by the ball to go from the extreme position $\theta = \beta$ to the wall i.e. $\theta = -\alpha$, then Eqn. (2) can be rewritten as

$$-\alpha = \beta\cos\sqrt{\frac{g}{l}}\,t'$$

or $$t' = \sqrt{\frac{l}{g}}\cos^{-1}\left(-\frac{\alpha}{\beta}\right) = \sqrt{\frac{l}{g}}\left(\pi - \cos^{-1}\frac{\alpha}{\beta}\right)$$

Thus the sought time $T = 2t' = 2\sqrt{\frac{l}{g}}\left(\pi - \cos^{-1}\frac{\alpha}{\beta}\right)$

$$= 2\sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + \sin^{-1}\frac{\alpha}{\beta}\right), \quad [\text{because } \sin^{-1}x + \cos^{-1}x = \pi/2]$$

4.21 Let the downward acceleration of the elevator car has continued for time t', then the sought time

$t = \sqrt{\frac{2h}{w}} + t'$, where obviously $\sqrt{\frac{2h}{w}}$ is the time of upward acceleration of the elevator.

One should note that if the point of suspension of a mathematical pendulum moves with an acceleration $\vec{w}$, then the time period of the pendulum becomes

$$2\pi\sqrt{\frac{l}{|\vec{g}-\vec{w}|}} \qquad (\text{see } 4.30)$$

In this problem the time period of the pendulum while it is moving upward with acceleration w becomes

$2\pi\sqrt{\dfrac{l}{g+w}}$ and its time period while the elevator moves downward with the same magnitude of acceleration becomes

$$2\pi\sqrt{\frac{l}{g-w}}$$

As the time of upward acceleration equals $\sqrt{\dfrac{2h}{w}}$, the total number of oscillations during this time equals

$$\frac{\sqrt{2h/w}}{2\pi\sqrt{l/(g+w)}}$$

Thus the indicated time $= \dfrac{\sqrt{2h/w}}{2\pi\sqrt{l/(g+w)}}\cdot 2\pi\sqrt{l/g} = \sqrt{2h/w}\sqrt{(g+w)/g}$

Similarly the indicated time for the time interval t'

$$= \frac{t'}{2\pi\sqrt{l/(g-w)}}\, 2\pi\sqrt{l/g} = t'\sqrt{(g-w)/g}$$

we demand that

$$\sqrt{2h/w}\sqrt{(g+w)/g} + t'\sqrt{(g-w)/g} = \sqrt{2h/w} + t'$$

or,
$$t' = \sqrt{2h/w}\,\frac{\sqrt{g+w}-\sqrt{g}}{\sqrt{g}-\sqrt{g-w}}$$

Hence the sought time

$$t = \sqrt{\frac{2h}{w}} + t' = \sqrt{\frac{2h}{w}}\,\frac{\sqrt{g+w}-\sqrt{g-w}}{\sqrt{g}-\sqrt{g-w}}$$

$$= \sqrt{\frac{2h}{w}}\,\frac{\sqrt{1+\beta}-\sqrt{1-\beta}}{1-\sqrt{1-\beta}}, \text{ where } \beta = w/g$$

4.22 If the hydromoter were in equlibrium or floating, its weight will be balanced by the buoyancy force acting on it by the fluid. During its small oscillation, let us locate the hydrometer when it is at a vertically downward distance x from its equilibrium position. Obviously the net unbalanced force on the hydrometer is the excess buoyancy force directed upward and equals $\pi r^2 x \rho g$. Hence for the hydrometer.

$$m\ddot{x} = -\pi r^2 \rho g x$$

or,
$$\ddot{x} = -\frac{\pi r^2 \rho g}{m}x$$

Hence the sought time period

$$T = 2\alpha\sqrt{\frac{m}{\pi r^2 \rho g}} = 2{\cdot}5\text{ s}.$$

4.23 At first let us calculate the stiffness κ_1 and κ_2 of both the parts of the spring. If we subject the original spring of stiffness κ having the natural length l_0 (say), under the deforming forces $F - F$ (say) to elongate the spring by the amount x, then

$$F = \kappa x \tag{1}$$

Therefore the elongation per unit length of the spring is x/l_0 . Now let us subject one of the parts of the spring of natural length $\eta\, l_0$ under the same deforming forces $F - F$. Then the elongation of the spring will be

$$\frac{x}{l_0}\,\eta\, l_0 = \eta\, x$$

Thus
$$F = \kappa_1\,(\eta\, x) \tag{2}$$

Hence from Eqns (1) and (2)

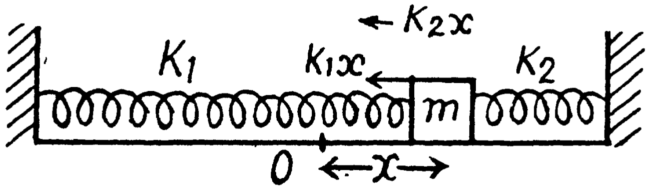
$$\kappa = \eta\,\kappa_1 \quad \text{or} \quad \kappa_1 = \kappa/\eta \tag{3}$$

Similarly
$$\kappa_2 = \frac{\kappa}{1-\eta}$$

The position of the block m when both the parts of the spring are non-deformed, is its equilibrium position O. Let us displace the block m towards right or in positive x axis by the small distance x. Let us depict the forces acting on the block when it is at a distance x from its equilibrium position (Fig.). From the second law of motion in projection form i.e. $F_x = m\, w_x$

$$-\kappa_1 x - \kappa_2 x = m\ddot{x}$$

or,
$$-\left(\frac{\kappa}{\eta} + \frac{\kappa}{1-\eta}\right)x = m\ddot{x}$$



Thus
$$\ddot{x} = -\frac{\kappa}{m}\frac{1}{\eta\,(m)}x$$

Hence the sought time period

$$T = 2\pi\sqrt{\eta\,(1-\eta)\,m/\kappa} = 0{\cdot}13\ \text{s}$$

4.24 Similar to the Soln of 4.23, the net unbalanced force on the block m when it is at a small horizontal distance x from the equilibrium position becomes $(\kappa_1 + \kappa_2)\,x$.

From $F_x = m\, w_x$ for the block :

$$-(\kappa_1 + \kappa_2)\,x = m\ddot{x}$$

Thus
$$\ddot{x} = -\left(\frac{\kappa_1 + \kappa_2}{m}\right)x$$

Hence the sought time period $T = 2\pi\sqrt{\dfrac{m}{\kappa_1 + \kappa_2}}$

Alternate : Let us set the block m in motion to perform small oscillation. Let us locate the block when it is at a distance x from its equilibrium position.

As the spring force is restoring conservative force and deformation of both the springs are same, so from the conservation of mechanical energy of oscillation of the spring-block system :

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2+\frac{1}{2}\kappa_1 x^2+\frac{1}{2}\kappa_2 x^2 = \text{Constant}$$

Differentiating with respect to time

$$\frac{1}{2}m\,2\dot{x}\ddot{x}+\frac{1}{2}(\kappa_1+\kappa_2)\,2x\dot{x} = 0$$

or,
$$\ddot{x} = -\frac{(\kappa_1+\kappa_2)}{m}x$$

Hence the sought time period $T = 2\pi\sqrt{\frac{m}{\kappa_1+\kappa_2}}$

4.25 During the vertical oscillation let us locate the block at a vertical down distance x from its equilibrium position. At this moment if x_1 and x_2 are the additional or further elongation of the upper & lower springs relative to the equilibrium position, then the net unbalanced force on the block will be $\kappa_2 x_2$ directed in upward direction. Hence

$$-\kappa_2 x_2 = m\ddot{x} \tag{1}$$

We also have
$$x = x_1+x_2 \tag{2}$$

As the springs are massless and initially the net force on the spring is also zero so for the spring

$$\kappa_1 x_1 = \kappa_2 x_2 \tag{3}$$

Solving the Eqns (1), (2) and (3) simultaneously, we get

$$-\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2}x = m\ddot{x}$$

Thus
$$\ddot{x} = -\frac{(\kappa_1\kappa_2/\kappa_1+\kappa_2)}{m}x$$

Hence the sought time period $T = 2\pi\sqrt{m\frac{(\kappa_1\kappa_2)}{\kappa_1\kappa_2}}$

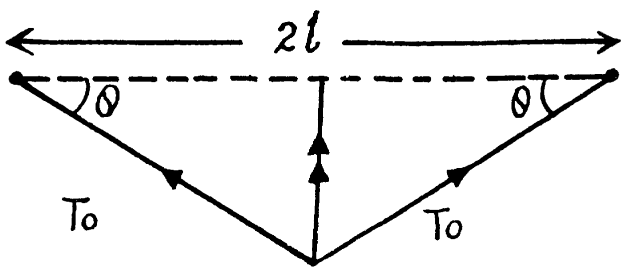
4.26 The force F, acting on the weight deflected from the position of equilibrium is $2\,T_0\sin\theta$.

Since the angle θ is small, the net restoring force, $F = 2\,T_0\frac{x}{l}$

or, $\quad F = kx$, where $k = \frac{2\,T_0}{l}$

So, by using the formula,

$$\omega_0 = \sqrt{\frac{k}{m}},\quad \omega_0 = \sqrt{\frac{2\,T_0}{m\,l}}$$



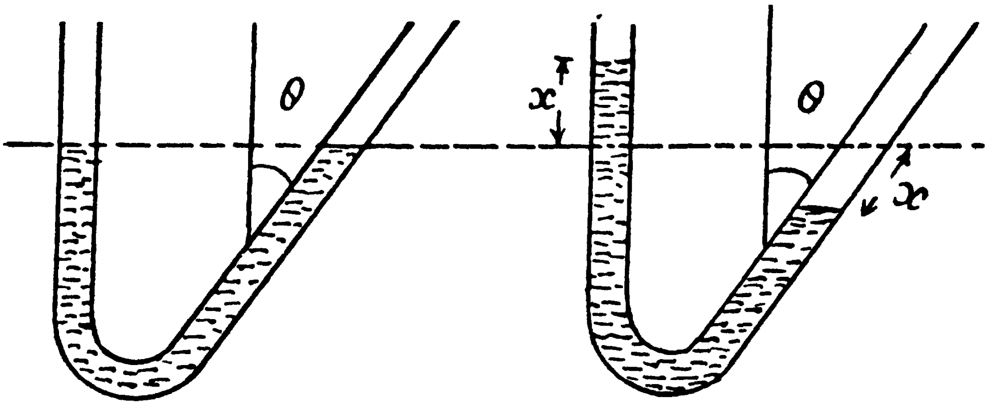
4.27 If the mercury rises in the left arm by x it must fall by a slanting length equal to x in the other arm. Total pressure difference in the two arms will then be

$$\rho g x+\rho g x\cos\theta = \rho g x(1+\cos\theta)$$

This will give rise to a restoring force

$$-\rho g S x(1+\cos\theta)$$

This must equal mass times acceleration which can be obtained from work energy principle.



The K.E. of the mercury in the tube is clearly : $\frac{1}{2} m \dot{x}^2$

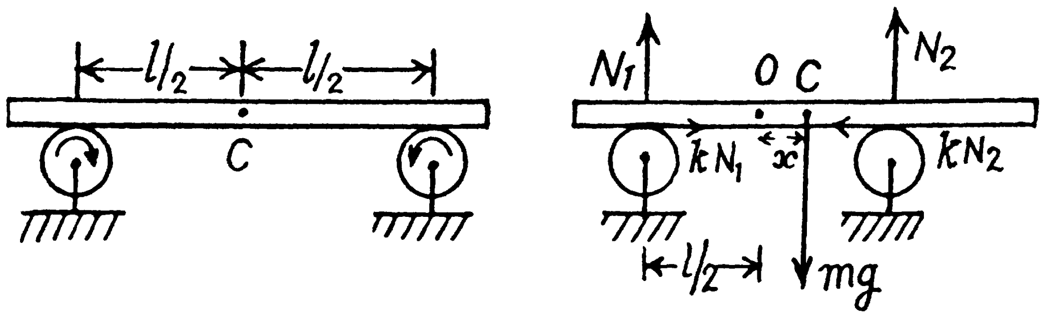
So mass times acceleration must be : $m \ddot{x}$

Hence $$m \ddot{x} + \rho g S (1 + \cos \theta) x = 0$$

This is S.H.M. with a time period

$$T = 2\pi \sqrt{\frac{m}{\rho g S (1 + \cos \theta)}}.$$

4.28 In the equilibrium position the C.M. of the rod lies nid way between the two rotating wheels. Let us displace the rod horizontally by some small distance and then release it. Let us depict the forces acting on the rod when its C.M. is at distance x from its equilibrium position (Fig.). Since there is no net vertical force acting on the rod, Newton's second law gives :



$$N_1 + N_2 = mg \tag{1}$$

For the translational motion of the rod from the Eqn. : $F_x = m\, w_{cx}$

$$k N_1 - k N_2 = m \ddot{x} \tag{2}$$

As the rod experiences no net torque about an axis perpendicular to the plane of the Fig. through the C.M. of the rod.

$$N_1 \left(\frac{l + x}{2} \right) = N_2 \left(\frac{l - x}{2} \right) \tag{3}$$

Solving Eqns. (1), (2) and (3) simultaneously we get

$$\ddot{x} = - k \frac{2g}{l} x$$

Hence the sought time period

$$T = 2\pi \sqrt{\frac{l}{2kg}} = \pi \sqrt{\frac{2l}{kg}} = 1{\cdot}5 \text{ s}$$

4.29 (a) The only force acting on the ball is the gravitational force $\vec{F}$, of magnitude $\gamma \frac{4}{3} \pi \rho m r$, where γ is the gravitational constant ρ, the density of the Earth and r is the distance of the body from the centre of the Earth.

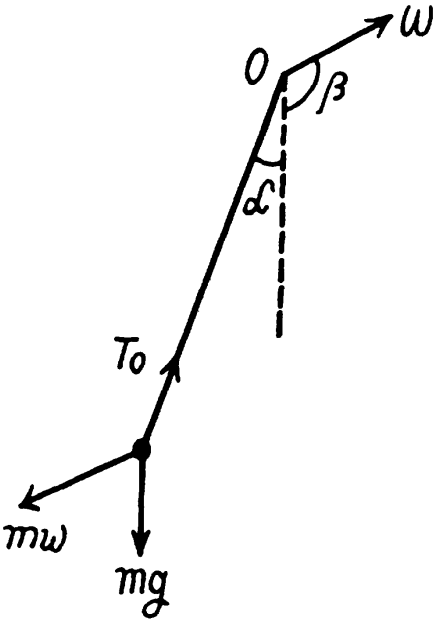
But, $g = \gamma \frac{4\pi}{3} \rho R$, so the expression for $\vec{F}$ can be written as,

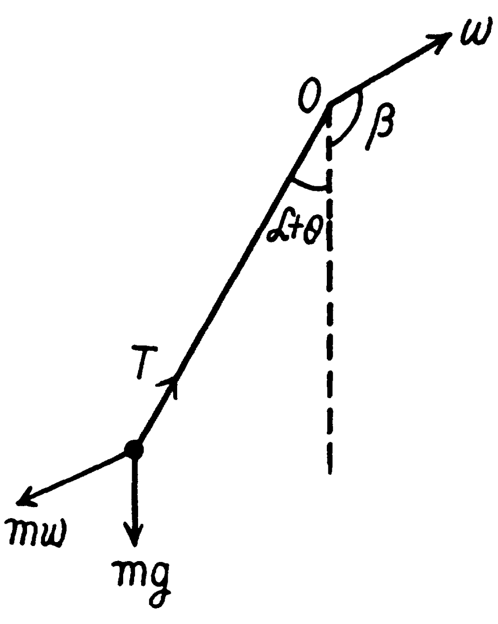
$\vec{F} = -mg\frac{\vec{r}}{R}$, here R is the radius of the Earth and the equation of motion in projection form has the form, or, $m\ddot{x} + \frac{mg}{R}x = 0$

(b) The equation, obtained above has the form of an equation of S.H.M. having the time period,
$$T = 2\pi\sqrt{\frac{R}{g}},$$
Hence the body will reach the other end of the shaft in the time,
$$t = \frac{T}{2} = \pi\sqrt{\frac{R}{g}} = 42 \text{ min}.$$

(c) From the conditions of S.H.M., the speed of the body at the centre of the Earth will be maximum, having the magnitude,
$$v = R\omega = R\sqrt{g/R} = \sqrt{gR} = 7{\cdot}9 \text{ km/s}.$$

4.30 In the frame of point of suspension the mathematical pendulum of mass m (say) will oscillate. In this frame, the body m will experience the inertial force $m(-\vec{w})$ in addition to the real forces during its oscillations. Therefore in equilibrium position m is deviated by some angle say α. In equilibrium position
$$T_0 \cos\alpha = mg + mw\cos(\pi - \beta) \quad \text{and} \quad T_0 \sin\alpha = mw\sin(\pi - \beta)$$
So, from these two Eqns
$$\left.\begin{aligned} \tan\alpha &= \frac{g - w\cos\beta}{w\sin\beta} \\ \text{and } \cos\alpha &= \frac{\sqrt{m^2 w^2 \sin^2\beta + (mg - mw\cos\beta)^2}}{mg - mw\cos\beta} \end{aligned}\right| \qquad (1)$$





Let us displace the bob m from its equilibrium position by some small angle and then release it. Now locate the ball at an angular position $(\alpha + \theta)$ from vertical as shown in the figure.

From the Eqn. : $$N_{0z} = I\,\beta_z$$

$$-m\,g\,l\sin(\alpha+\theta) - m\,w\cos(\pi-\beta)\,l\sin(\alpha+\theta) + m\,w\sin(\pi-\beta)\,l\cos(\alpha+\theta) = m\,l^2\,\ddot{\theta}$$

or, $$-g(\sin\alpha\cos\theta+\cos\alpha\sin\theta) - w\cos(\pi-\beta)(\sin\alpha\cos\theta+\cos\alpha\sin\theta) + w\sin\beta\,(\cos\alpha\cos\theta-\sin\alpha\sin\theta) = l\,\ddot{\theta}$$

But for small θ, $\sin\theta \approx \theta$ $\cos\theta \approx 1$

So, $$-g(\sin\alpha+\cos\alpha\,\theta) - w\cos(\pi-\beta)(\sin\alpha+\cos\alpha\,\theta) + w\sin\beta\,(\cos\alpha-\sin\alpha\,\theta) = l\,\ddot{\theta}$$

or, $$(\tan\alpha+\theta)(w\cos\beta-g) + w\sin\beta\,(1-\tan\alpha\,\theta) = \frac{l}{\cos\alpha}\ddot{\theta} \qquad (2)$$

Solving Eqns (1) and (2) simultaneously we get

$$-(g^2-2\,w\,g\cos\beta+w^2)\,\theta = l\sqrt{g^2+w^2-2\,w\,g\cos\beta}\;\ddot{\theta}$$

Thus $$\ddot{\theta} = -\frac{|\vec{g}-\vec{w}|}{l}\theta$$

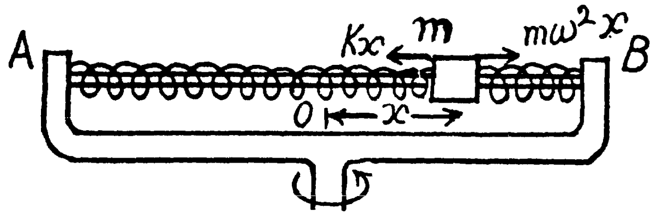
Hence the sought time period $T = \dfrac{2\pi}{\omega_0} = 2\pi\sqrt{\dfrac{l}{|\vec{g}-\vec{w}|}}$

4.31 Obviously the sleeve performs small oscillations in the frame of rotating rod. In the rod's frame let us depict the forces acting on the sleeve along the length of the rod while the sleeve is at a small distance x towards right from its equilibrium position. The free body diagram of block does not contain Coriolis force, because it is perpendicualr to the length of the rod.

From $F_x = m\,w_x$ for the sleeve in the frame of rod

$$-\kappa x + m\,\omega^2 x = m\ddot{x}$$

or, $$\ddot{x} = -\left(\frac{\kappa}{m}-\omega^2\right)x \qquad (1)$$



Thus the sought time period

$$T = \frac{2\pi}{\sqrt{\dfrac{\kappa}{m}-\omega^2}} = 0{\cdot}7\text{ s}$$

It is obvious from Eqn (1) that the sleeve will not perform small oscillations if $\omega \geq \sqrt{\dfrac{\kappa}{m}}$ 10 rod/s.

4.32 When the bar is about to start sliding along the plank, it experiences the maximum restoring force which is being provided by the limiting friction,

Thus

$$kN = m\,\omega_0^2\,a \qquad \text{or,}\quad k\,mg = m\,\omega_0^2\,a$$

or, $$k = \frac{\omega_0^2 a}{g} = \frac{a}{g}\left(\frac{2\pi}{T}\right)^2 = 4\,s.$$

4.33 The natural angular frequency of a mathematical pendulum equals $\omega_0 = \sqrt{g/l}$

(a) We have the solution of S.H.M. equation in angular form :

$$\theta = \theta_m \cos(\omega_0 t + \alpha)$$

If at the initial moment i.e. at $t = 0$, $\theta = \theta_m$ than $\alpha = 0$.

Thus the above equation takes the form

$$\theta = \theta_m \cos \omega_0 t$$

$$= \theta_m \cos \sqrt{\frac{g}{l}}\, t = 3° \cos \sqrt{\frac{9{\cdot}8}{0{\cdot}8}}\, t$$

Thus $$\theta = 3° \cos 3{\cdot}5\ t$$

(b) The S.H.M. equation in angular form :

$$\theta = \theta_m \sin(\omega_0 t + \alpha)$$

If at the initial moment $t = 0$, $\theta = 0$, then $\alpha = 0$ .Then the above equation takes the form

$$\theta = \theta_m \sin \omega_0 t$$

Let v_0 be the velocity of the lower end of pendulam at $\theta = 0$, then from conserved of mechanical energy of oscillaton

$$E_{mean} = E_{extreme} \quad \text{or} \quad T_{mean} = U_{extrem}$$

or, $$\frac{1}{2} m v_0^2 = m g l (1 - \cos \theta_m)$$

Thus

$$\theta_m = \cos^{-1}\left(1 - \frac{v_0^2}{2gl}\right) = \cos^{-1}\left[1 - \frac{(0 \cdot 22)^2}{2 \times 9 \cdot 8 \times 0 \cdot 8}\right] = 4 \cdot 5°$$

Thus the sought equation becomes

$$\theta = \theta_m \sin \omega_0 t = 4{\cdot}5° \sin 3{\cdot}5\, t$$

(c) Let θ_0 and v_0 be the angular deviation and linear velocity at $t = 0$.
As the mechanical energy of oscillation of the mathematical pendulum is conservation

$$\frac{1}{2} m v_0^2 + m g l (1 - \cos \theta_0) = m g l (1 - \cos \theta_m)$$

or, $$\frac{v_0^2}{2} = g l (\cos \theta_0 - \cos \theta_m)$$

Thus $$\theta_m = \cos^{-1}\left\{\cos \theta_0 - \frac{v_0^2}{2gl}\right\} = \cos^{-1}\left\{\cos 3° - \frac{(0{\cdot}22)^2}{2 \times 9{\cdot}8 \times 0{\cdot}8}\right\} = 5{\cdot}4°$$

Then from $\theta = 5.4° \sin(3.5\,t + \alpha)$, we see that $\sin\alpha = \frac{3}{5.4}$ and $\cos\alpha < 0$ because the velovity is directed towards the centre. Thus $\alpha = \frac{\pi}{2} + 1.0$ radians and we get the answer.

4.34 While the body A is at its upper extreme position, the spring is obviously elongated by the amount

$$\left(a - \frac{m_1 g}{\kappa}\right).$$

If we indicate y-axis in vertically downward direction, Newton's second law of motion in projection form i.e. $F_y = m\,w_y$ for body A gives :

$$m_1 g + \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_1 \omega^2 a \quad \text{or, } \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_1(\omega^2 a - g) \tag{1}$$

(Because at any extreme position the magnitude of acceleration of an oscillating body equals $\omega^2 a$ and is restoring in nature.)

If N be the normal force exerted by the floor on the body B, while the body A is at its upper extreme position, from Newton's second law for body B

$$N + \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_2 g$$

or, $$N = m_2 g - \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_2 g - m_1(\omega^2 a - g) \text{ (using Eqn. 1)}$$

Hence $N = (m_1 + m_2)\,g - m_1 \omega^2 a$

When the body A is at its lower extreme position, the spring is compresed by the distance $\left(a + \frac{m_1 g}{\kappa}\right)$.

From Newton's second law in projeciton form i.e. $F_y = m\,w_y$ for body A at this state:

$$m_1 g - \kappa\left(a + \frac{m_1 g}{\kappa}\right) = m_1(-\omega^2 a) \quad \text{or, } \kappa\left(a + \frac{m_1 g}{\kappa}\right) = m_1(g + \omega^2 a) \tag{3}$$

In this case if N' be the normal force exerted by the floor on the body B, From Newton's second law

for body B we get: $N' = \kappa\left(a + \frac{m_1 g}{\kappa}\right) + m_2 g = m_1(g + \omega^2 a) + m_2 g$ (using Eqn. 3)

Hence $$N' = (m_1 + m_2)\,g + m_1 \omega^2 a$$

From Newton's third law the magnitude of sought forces are N' and N, respectively.

4.35 (a) For the block from Newton's second law in projection form $F_y = m\,w_y$

$$N - mg = m\ddot{y} \tag{1}$$

But from $$y = a(1 - \cos\omega t)$$

We get $$\ddot{y} = \omega^2 a \cos \omega t \qquad (2$$

From Eqns (1) and (2)

$$N = m g \left(1 + \frac{\omega^2 a}{g} \cos \omega t \right) \qquad (3)$$

From Newtons's third law the force by which the body m exerts on the block is directed vertically downward and equls $N = m g \left(1 + \frac{\omega^2 a}{g} \cos \omega t \right)$

(b) When the body m starts, falling behind the plank or loosing contact, $N = 0$, (because the normal reaction is the contact force). Thus from Eqn. (3)

$$m g \left(1 + \frac{\omega^2 a}{g} \cos \omega t \right) = 0 \quad \text{for some } t.$$

Hence $$a_{min} = g/\omega^2 = 8 \text{ cm}.$$

(c) We observe that the motion takes place about the mean position $y = a$. At the initial instant $y = 0$. As shown in (b) the normal reaction vanishes at a height (g/ω^2) above the position of equilibrium and the body flies off as a free body. The speed of the body at a distance (g/ω^2) from the equilibrium position is $\omega \sqrt{a^2 - (g/\omega^2)^2}$, so that the condition of the problem gives

$$\frac{[\omega \sqrt{a^2 - (g/\omega^2)^2}\,]^2}{2g} + \frac{g}{\omega^2} + a = h$$

Hence solving the resulting quadratic equation and taking the positive roof,

$$a = -\frac{g}{\omega^2} + \sqrt{\frac{2hg}{\omega^2}} = 20 \text{ cm}.$$

4.36 (a) Let $y(t)$ = displacement of the body from the end of the unstreched position of the spring (not the equilibrium position). Then

$$m\ddot{y} = -\kappa y + m g$$

This equation has the solution of the form

$$y = A + B \cos(\omega t + \alpha)$$

if $$-m \omega^2 B \cos(\omega t + \alpha) = -\kappa [A + B \cos(\omega t + \alpha)] + m g$$

Then $$\omega^2 = \frac{\kappa}{m} \quad \text{and} \quad A = \frac{m g}{\kappa}$$

we have $$y = 0 \quad \text{and} \quad \dot{y} = 0 \quad \text{at} \quad t = 0. \text{ So}$$

$$-\omega B \sin \alpha = 0$$

$$A + B \cos \alpha = 0$$

Since $B > 0$ and $A > 0$ we must have $\alpha = \pi$

$$B = A = \frac{m g}{\kappa}$$

and $$y = \frac{mg}{\kappa}(1 - \cos\omega t)$$

(b) Tension in the spring is

$$T = \kappa y = mg(1 - \cos\omega t)$$

so $$T_{max} = 2mg, \; T_{min} = 0$$

4.37 In accordance with the problem

$$\vec{F} = -\alpha m \vec{r}$$

So, $$m(\ddot{x}\vec{i} + \ddot{y}\vec{j}) = -\alpha m(x\vec{i} + y\vec{j})$$

Thus $$\ddot{x} = -\alpha x \text{ and } \ddot{y} = -\alpha y$$

Hence the solution of the differential equation

$$\ddot{x} = -\alpha x \quad \text{becomes } x = a\cos(\omega_0 t + \delta), \text{ where } \omega_0^2 = \alpha \tag{1}$$

So, $$\dot{x} = -a\omega_0 \sin(\omega_0 t + \alpha) \tag{2}$$

From the initial conditions of the problem, $v_x = 0$ and $x = r_0$ at $t = 0$

So from Eqn. (2) $\alpha = 0$, and Eqn takes the form

$$x = r_0 \cos\omega_0 t \quad \text{so, } \cos\omega_0 t = x/r_0 \tag{3}$$

One of the solution of the other differential Eqn $\ddot{y} = -\alpha y$, becomes

$$y = a'\sin(\omega_0 t + \delta'), \text{ where } \omega_0^2 = \alpha \tag{4}$$

From the initial condition, $y = 0$ at $t = 0$, so $\delta' = 0$ and Eqn (4) becomes :

$y = a'\sin\omega_0 t$(5)

Differentiating w.r.t. time we get

$$\dot{y} = a'\omega_0 \cos\omega_0 t \tag{6}$$

But from the initial condition of the problem, $\dot{y} = v_0$ at $t = 0$,

So, from Eqn (6) $\quad v_0 = a'\omega_0 \quad$ or, $\quad a' = v_0/\omega_0$

Using it in Eqn (5), we get

$$y = \frac{v_0}{\omega_0}\sin\omega_0 t \quad \text{or} \quad \sin\omega_0 t = \frac{\omega_0 y}{v_0} \tag{7}$$

Squaring and adding Eqns (3) and (7) we get :

$$\sin^2\omega_0 t + \cos^2\omega_0 t = \frac{\omega_0^2 y^2}{v_0^2} + \frac{x^2}{r_0^2}$$

or, $$\left(\frac{x}{r_0}\right)^2 + \alpha\left(\frac{y}{v_0}\right)^2 = 1 \left(\text{as } \alpha, = \omega_0^2\right)$$

4.38 (a) As the elevator car is a translating non-inertial frame, therefore the body m will experience an inertial force mw directed downward in addition to the real forces in the elevator's frame. From the Newton's second law in projection form

$F_y = m w_y$ for the body in the frame of elevator car:

$$-\kappa\left(\frac{mg}{\kappa} + y\right) + mg + mw = m\ddot{y} \tag{A}$$

(Because the initial elongation in the spring is $m\,g/\kappa$)

so, $$m\ddot{y} = -\kappa y + m w = -\kappa\left(y - \frac{m w}{\kappa}\right)$$

or, $$\frac{d^2}{dt^2}\left(y - \frac{m w}{\kappa}\right) = -\frac{\kappa}{m}\left(y - \frac{m w}{\kappa}\right) \tag{1}$$

Eqn. (1) shows that the motion of the body m is S.H.M. and its solution becomes

$$y - \frac{m w}{\kappa} = a \sin\left(\sqrt{\frac{\kappa}{m}}\, t + \alpha\right) \tag{2}$$

Differentiating Eqn (2) w.r.t. time

$$\dot{y} = a\sqrt{\frac{\kappa}{m}} \cos\left(\sqrt{\frac{\kappa}{m}}\, t + \alpha\right) \tag{3}$$

Using the initial condition $y(0) = 0$ in Eqn (2), we get :

$$a \sin \alpha = -\frac{m w}{\kappa}$$

and using the other initial condition $\dot{y}(0) = 0$ in Eqn (3)

we get $$a\sqrt{\frac{\kappa}{m}} \cos \alpha = 0$$

Thus $$\alpha = -\alpha/2 \text{ and } a = \frac{m w}{\kappa}$$

Hence using these values in Eqn (2), we get

$$y = \frac{m w}{\kappa}\left(1 - \cos\sqrt{\frac{\kappa}{m}}\, t\right)$$

(b) Proceed up to Eqn.(1). The solution of this differential Eqn be of the form :

$$y - \frac{m w}{\kappa} = a \sin\left(\sqrt{\frac{\kappa}{m}}\, t + \delta\right)$$

or, $$y - \frac{\alpha t}{\kappa/m} = a \sin\left(\sqrt{\frac{\kappa}{m}}\, t + \delta\right)$$

or, $$y - \frac{\alpha t}{\omega_0^2} = a \sin(\omega_0 t + \delta) \left(\text{wher } \omega_0 = \sqrt{\frac{\kappa}{m}}\right) \tag{4}$$

From the initial condition that at$t = 0$, $y(0) = 0$, so $0 = a \sin \delta$ or $\delta = 0$

Thus Eqn.(4) takes the from : $y - \dfrac{\alpha t}{\omega_0^2} = a \sin \omega_0 t$ (5)

Differentiating Eqn. (5) we get : $\dot{y} - \dfrac{\alpha}{\omega_0^2} = a\,\omega_0 \cos \omega t$ (6)

But from the other initial condition $\dot{y}(0) = 0$ at $t = 0$.

So, from Eqn.(6) $\quad -\frac{\alpha}{\omega_0^2} = a\,\omega_0$ or $a = -\alpha/\omega_0^3$

Putting the value of a in Eqn. (5), we get the sought $y(t)$. i.e.

$$y - \frac{\alpha t}{\omega_0^2} = -\frac{\alpha}{\omega_0^3}\sin\omega_0 t \quad \text{or} \quad y = \frac{\alpha}{\omega_0^3}(\omega_0 t - \sin\omega_0 t)$$

4.39 There is an important difference between a rubber cord or steel coire and a spring. A spring can be pulled or compressed and in both cases, obey's Hooke's law. But a rubber cord becomes loose when one tries to compress it and does not then obey Hooke's law. Thus if we suspend a body by a rubber cord it stretches by a distance mg/κ in reaching the equilibrium configuration. If we further strech it by a distance Δh it will execute harmonic oscillations when released if $\Delta h \le mg/\kappa$ because only in this case will the cord remain taut and obey Hooke's law.

Thus $$\Delta h_{max} = mg/\kappa$$

The energy of oscillation in this case is

$$\frac{1}{2}\kappa(\Delta h_{max})^2 = \frac{1}{2}\frac{m^2 g^2}{\kappa}$$

4.40 As the pan is of negligible mass, there is no loss of kinetic energy even though the collision is inelastic. The mechanical energy of the body m in the field generated by the joint action of both the gravity force and the elastic force is conserved i.e. $\Delta E = 0$. During the motion of the body m from the initial to the final (position of maximum compression of the spring) position $\Delta T = 0$, and therefore $\Delta U = \Delta U_{gr} + \Delta U_{sp} = 0$

or $$-mg(h+x) + \frac{1}{2}\kappa x^2 = 0$$

On solving the quadratic equation :

$$x = \frac{mg}{\kappa} \pm \sqrt{\frac{m^2g^2}{\kappa^2} + \frac{2mgh}{\kappa}}$$

As minus sign is not acceptable

$$x = \frac{mg}{\kappa} + \sqrt{\frac{m^2g^2}{\kappa^2} + \frac{2mgh}{\kappa}}$$

If the body m were at rest on the spring, the corresponding position of m will be its equilibrium position and at this position the resultant force on the body m will be zero. Therefore the equilibrium compression Δx (say) due to the body m will be given by

$$\kappa\,\Delta x = mg \quad \text{or} \quad \Delta x = mg/\kappa$$

Therefore seperation between the equilibrium position and one of the extreme position i.e. the sought amplitude

$$a = x - \Delta x = \sqrt{\frac{m^2g^2}{\kappa^2} + \frac{2mgh}{\kappa}}$$

The mechanical energy of oscillation which is conserved equals $E = U_{extreme}$,because at the extreme position kinetic energy becomes zero.

Although the weight of body m is a conservative force , it is not restoring in this problem, hence $U_{extreme}$ is only concerned with the spring force. Therefore

$$E = U_{extreme} = \frac{1}{2}\kappa a^2 = m g h + \frac{m^2 g^2}{2\kappa}$$

4.41 Unlike the previous (4.40) problem the kinetic energy of body m decreases due to the perfectly inelastic collision with the pan.Obviously the body m comes to strike the pan with velocity $v_0 = \sqrt{2gh}$ .If v be the common velocity of the " body $m +$ pan " system due to the collision then from the conservation of linear momentum

$$m v_0 = (M+m) v$$

or

$$v = \frac{m v_0}{(M+m)} = \frac{m\sqrt{2gh}}{(M+m)} \qquad 1)$$

At the moment the body m strikes the pan, the spring is compressed due to the weight of the pan by the amount Mg/κ . If l be the further compression of the spring due to the velocity acquired by the "pan – body m " system, then from the conservation of mechanical energy of the said system in the field generatad by the joint action of both the gravity and spring forces

$$\frac{1}{2}(M+m)v^2 + (M+m) g l = \frac{1}{2}\kappa\left(\frac{Mg}{\kappa}+l\right)^2 - \frac{1}{2}\kappa\left(\frac{Mg}{\kappa}\right)^2$$

or, $$\frac{1}{2}(M+m)\frac{m^2 2gh}{(M+m)} + (M+m) g l = \frac{1}{2}\kappa\left(\frac{Mg}{\kappa}\right)^2 + \frac{1}{2}\kappa l^2 + M g l - \frac{1}{2}\kappa\left(\frac{Mg}{\kappa}\right)^2 \qquad \text{(Using 1)}$$

or, $$\frac{1}{2}\kappa l^2 - m g l - \frac{m^2 g h}{(m+M)} = 0$$

Thus $$l = \frac{mg \pm \sqrt{m^2 g^2 + \frac{2\kappa g h m^2}{M+m}}}{\kappa}$$

As minus sign is not acceptable

$$l = \frac{mg}{\kappa} + \frac{1}{\kappa}\sqrt{m^2 g^2 + \frac{2\kappa m^2 g h}{(M+m)}}$$

If the oscillating "pan + body m" system were at rest it correspond to their equilbrium position i.e. the spring were compressed by $\frac{(M+m)g}{\kappa}$ therefore the amplitude of oscillation

$$a = l - \frac{mg}{\kappa} = \frac{mg}{\kappa}\sqrt{1 + \frac{2h\kappa}{mg}}$$

The mechanical energy of oscillation which is only conserved with the restoring forces becomes $E = U_{extreme} = \frac{1}{2}\kappa a^2$ (Because spring force is the only restoring force not the weight of the body)

Alternately $$E = T_{mean} = \frac{1}{2}(M+m)\,a^2\omega^2$$

thus $$E = \frac{1}{2}(M+m)\,a^2\left(\frac{\kappa}{M+m}\right) = \frac{1}{2}\kappa a^2$$

4.42 We have $\vec{F} = a(\dot{y}\,\vec{i} - \dot{x}\,\vec{j})$

or, $$m(\ddot{x}\,\vec{i} + \ddot{y}\,\vec{j}) = a(\dot{y}\,\vec{i} - \dot{x}\,\vec{j})$$

So, $$m\ddot{x} = a\dot{y} \text{ and } m\ddot{y} = -a\dot{x} \tag{1}$$

From the initial condition,at $t = 0$, $\dot{x} = 0$ and $y = 0$

So, integrating Eqn,$m\ddot{x} = a\dot{y}$

we get $$m\dot{x} = ay \text{ or } \dot{x} = \frac{a}{m}y \tag{2}$$

Using Eqn (2) in the Eqn $m\ddot{y} = -a\dot{x}$, we get

$$m\ddot{y} = -\frac{a^2}{m}y \text{ or } \ddot{y} = -\left(\frac{a}{m}\right)^2 y \tag{3}$$

one of the solution of differential Eqn (3) is

$$y = A\sin(\omega_0 t + \alpha), \text{ where } \omega_0 = a/m.$$

As at $t = 0$, $y = 0$, so the solution takes the form $y = A\sin\omega_0 t$

On differentiating w.r.t. time $\dot{y} = A\,\omega_0\cos\omega_0 t$

From the initial condition of the problem, at $t = 0$, $\dot{y} = v_0$

So, $$v_0 = A\,\omega_0 \text{ or } A = v_0/\omega_0$$

Thus $$y = (v_0/\omega_0)\sin\omega_0 t \tag{4}$$

Thus from (2) $\dot{x} = v_0\sin\omega_0 t$ so integrating

$$x = B - \frac{v_0}{\omega_0}\cos\omega_0 t \tag{5}$$

On using $$x = 0 \text{ at } t = 0,\ B = \frac{v_0}{\omega_0}$$

Hence finally $$x = \frac{v_0}{\omega_0}(1 - \cos\omega_0 t) \tag{6}$$

Hence from Eqns (4) and (6) we get

$$[x - (v_0/\omega_0)]^2 + y^2 = (v_0/\omega_0)^2$$

which is the equation of a circle of radius (v_0/ω_0) with the centre at the point $x_0 = v_0/\omega_0$, $y_0 = 0$

4.43 If water has frozen, the system consisting of the light rod and the frozen water in the hollow sphere constitute a compound (physical) pendulum to a very good approximation because we can take the whole system to be rigid. For such systems the time period is given by

$T_1 = 2\pi\sqrt{\frac{l}{g}}\sqrt{1+\frac{k^2}{l^2}}$ where $k^2 = \frac{2}{5}R^2$ is the radius of gyration of the sphere.

The situation is different when water is unfrozen. When dissipative forces (viscosity) are neglected, we are dealing with ideal fluids. Such fluids instantaneously respond to (unbalanced) internal stresses. Suppose the sphere with liquid water actually executes small rigid oscillations. Then the portion of the fluid above the centre of the sphere will have a greater acceleration than the portion below the centre because the linear acceleration of any element is in this case, equal to angular acceleration of the element multiplied by the distance of the element from the centre of suspension (Recall that we are considering small oscillations). Then, as is obvious in a frame moving with the centre of mass, there will appear an unbalanced couple (not negated by any pseudoforces) which will cause the fluid to move rotationally so as to destroy differences in acceleration. Thus for this case of ideal fluids the pendulum must move in such a way that the elements of the fluid all undergo the same acceleration. This implies that we have a simple (mathematical) pendulum with the time period :

$$T_0 = 2\pi\sqrt{\frac{l}{g}}$$

Thus
$$T_1 = T_0\sqrt{1+\frac{2}{5}\left(\frac{R}{l}\right)^2}$$

(One expects that a liquid with very small viscosity will have a time period close T_0 while one with high viscosity will have a time period closer to T_1.)

4.44 Let us locate the rod at the position when it makes an angle θ from the vertical. In this problem both, the gravity and spring forces are restoring conservative forces, thus from the conservation of mechanical energy of oscillation of the oscillating system :

$$\frac{1}{2}\frac{m l^2}{3}(\dot{\theta})^2 + m g\frac{l}{2}(1-\cos\theta) + \frac{1}{2}\kappa(l\theta)^2 = \text{constant}$$

Differentiating w.r.t. time, we get :

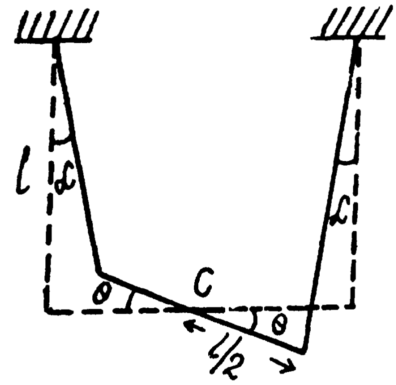
$$\frac{1}{2}\frac{m l^2}{3}2\dot{\theta}\ddot{\theta} + \frac{m g l}{2}\sin\theta\,\dot{\theta} + \frac{1}{2}\kappa l^2 2\theta\dot{\theta} = 0$$

Thus for very small θ

$$\ddot{\theta} = -\frac{3g}{2l}\left(1+\frac{\kappa l}{m g}\right)\theta$$

Hence,
$$\omega_0 = \sqrt{\frac{3g}{2l}\left(1+\frac{\kappa l}{mg}\right)}.$$

4.45 (a) Let us locate the system when the threads are deviated through an angle $\alpha' < \alpha$, during the oscillations of the system (Fig.). From the conservation of mechanical energy of the system :



$$\frac{1}{2}\frac{mL^2}{12}\dot{\theta}^2 + mgl(1-\cos\alpha') = \text{constant} \qquad (1)$$

Where L is the length of the rod, θ is the angular deviation of the rod from its equilibrium position i.e. $\theta = 0$.

Differentiating Eqn. (1) w.r.t. time

$$\frac{1}{2}\frac{mL^2}{12}2\dot{\theta}\ddot{\theta} + mgl\sin\alpha'\dot{\alpha}' = 0$$

So, $$\frac{L^2}{12}\dot{\theta}\ddot{\theta} + gl\alpha'\dot{\alpha}' = 0 \text{ (for small } \alpha', \sin\alpha' \approx \alpha' \text{)} \qquad (2)$$

But from the Fig.

$$\frac{L}{2}\theta = l\alpha' \text{ or } \alpha' = \frac{L}{2l}\theta$$

So, $$\dot{\alpha}' = \frac{L}{2l}\dot{\theta}$$

Putting these values of α' and $\frac{d\alpha'}{dt}$ in Eqn. (2) we get

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{l}\theta$$

Thus the sought time period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{3g}}$$

(b) The sought oscillation energy

$$E = U_{extreme} = mgl(1-\cos\alpha) = mgl2\sin^2\frac{\alpha}{2}$$

$$\approx mgl2\frac{\alpha^2}{4} = \frac{mgl\alpha^2}{2} \text{ (because for small angle } \sin\theta \approx \theta \text{)}$$

4.46 The K.E. of the disc is $\frac{1}{2}I\dot{\varphi}^2 = \frac{1}{2}\left(\frac{mR^2}{2}\right)\dot{\varphi}^2 = \frac{1}{4}mR^2\dot{\varphi}^2$

The torsional potential energy is $\frac{1}{2}k\varphi^2$. Thus the total energy is :

$$\frac{1}{4}mR^2\dot{\varphi}^2 + \frac{1}{2}k\varphi^2 = \frac{1}{4}mR^2\dot{\varphi}_0^2 + \frac{1}{2}k\varphi_0^2$$

By definition of the amplitude φ_m, $\dot{\varphi} = 0$ when $\varphi = \varphi_m$. Thus total energy is

$$\frac{1}{2}k\varphi_m^2 = \frac{1}{4}mR^2\dot{\varphi}_0^2 + \frac{1}{2}k\varphi_0^2$$

or
$$\varphi_m = \varphi_0\sqrt{1+\frac{mR^2}{2k}\frac{\dot{\varphi}_0^2}{\varphi_0^2}}$$

4.47 Moment of inertia of the rod equals $\frac{ml^2}{3}$ about its one end and perpendicular to its length

Thus rotational kinetic energy of the rod $= \frac{1}{2}\left(\frac{ml^2}{3}\right)\dot{\theta}^2 = \frac{ml^2}{6}\dot{\theta}^2$

when the rod is displaced by an angle θ its C.G. goes up by a distance $\frac{l}{2}(1-\cos\theta) = \frac{l\theta^2}{4}$ for small θ.

Thus the P.E. becomes : $mg\frac{l\theta^2}{4}$

As the mechanical energy of oscillation of the rod is conserved.

$$\frac{1}{2}\left(\frac{ml^2}{3}\right)\dot{\theta}^2 + \frac{1}{2}\left(\frac{mgl}{2}\right)\theta^2 = \text{Constant}$$

on differentiating w.r.t. time and for the simplifies we get : $\ddot{\theta} = -\frac{3g}{2l}\theta$ for small θ.

we see that the angular frequency ω is

$$= \sqrt{3g/2l}$$

we write the general solution of the angular oscillation as :

$$\theta = A\cos\omega t + B\sin\omega t$$

But $\quad \theta = \theta_0 \quad \text{at} \quad t = 0, \quad \text{so } A = \theta_0$

and $\quad \dot{\theta} = \dot{\theta}_0 \quad \text{at} \quad t = 0, \text{ so}$

$$B = \dot{\theta}_0/\omega$$

Thus
$$\theta = \theta_0\cos\omega t + \frac{\dot{\theta}_0}{\omega}\sin\omega t$$

Thus the K.E. of the rod

$$T = \frac{ml^2}{6}\dot{\theta}^2 = [-\omega\theta_0\sin\omega t + \dot{\theta}_0\cos\omega t]^2$$

$$= \frac{ml^2}{6}[\dot{\theta}_0^2\cos^2\omega t + \omega^2\theta_0^2\sin^2\omega t - 2\omega\theta_0\dot{\theta}_0\sin\omega t\cos\omega t]$$

On averaging over one time period the last term vanishes and $<\sin^2\omega t> = <\cos^2\omega t> = 1/2$. Thus

$$<T> = \frac{1}{12}ml^2\dot{\theta}_0^2 + \frac{1}{8}mgl^2\theta_0^2 \quad (\text{where } \omega^2 = 3g/2l)$$

4.48 Let l = distance between the C.G. (C) of the pendulum and its point of suspension O. Originally the pendulum is in inverted position and its C.G. is above O. When it falls to the normal (stable) position of equilibrium its C.G. has fallen by a distance $2l$. In the equilibrium position the total energy is equal to K.E. $= \frac{1}{2} I \omega^2$ and we have from energy conservation :

$$\frac{1}{2} I \omega^2 = m g 2 l \quad \text{or} \quad I = \frac{4 m g l}{\omega^2}$$

Angular frequency of oscillation for a physical pendulum is given by $\omega_0^2 = m g l / I$

Thus $$T = 2\pi \sqrt{\frac{I}{m g l}} = 2\pi \sqrt{\frac{4 m g l/\omega^2}{m g l}} = \frac{4\pi}{3}$$

4.49 Let, moment of inertia of the pendulum, about the axis, concerned is I, then writing $N_z = I \beta_z$, for the pendulum,

$$-m g x \sin a\, \theta = I \dot{\theta} \quad \text{or,} \quad \dot{\theta} = -\frac{m g x}{I} \theta \qquad \text{(For small } \theta \text{)}$$

which is the required equation for S.H.M. So, the frequency of oscillation,

$$\omega_1 = \sqrt{\frac{M g x}{I}} \quad \text{or,} \quad x = \frac{I}{M g} \sqrt{\omega_1^2} \tag{1}$$

Now, when the mass m is attached to the pendulum, at a distance l below the oscillating axis,

$$-M g x \sin \theta' - m g l \sin \theta' = (I + m l^2) \frac{d^2 \theta'}{d t^2}$$

or, $$-\frac{g (M x + m l)}{(I + m l^2)} \theta' = \frac{d^2 \theta}{d t^2}, \text{ (For small } \theta \text{)}$$

which is again the equation of S.H.M., So, the new frequency,

$$\omega_2 = \sqrt{\frac{g (M x + m l)}{(I + m l^2)}} \tag{2}$$

Solving Eqns. (1) and (2),

$$\omega_2 = \sqrt{\frac{g ((I/g) \omega_1^2 + m l)}{(I + m l^2)}}$$

or, $$\omega_2^2 = \frac{I \omega_1^2 + m g l}{I + m l^2}$$

or, $$I (\omega_2^2 - \omega_1^2) = m g l - m \omega_2^2 l^2$$

and hence, $$I = m l^2 (\omega_2^2 - g/l) \big/ (\omega_1^2 - \omega_2^2) = 0{\cdot}8 \text{ g} \cdot \text{m}^2$$

4.50 When the two pendulums are joined rigidly and set to oscillate, each exert torques on the other, these torques are equal and opposite. We write the law of motion for the two pendulums as

$$I_1 \ddot{\theta} = -\omega_1^2 I_1 \theta + G$$

$$I_2 \ddot{\theta} = -\omega_2^2 I_2 \theta - G$$

where $\pm G$ is the torque of mutual interactions. We have written the restoring forces on each pendulum in the absence of the other as $-\omega_1^2 I_1 \theta$ and $-\omega_2^2 I_2 \theta$ respectively. Then

$$\ddot{\theta} = -\frac{I_1 \omega_1^2 + I_2 \omega_2^2}{I_1 + I_2} \theta = -\omega^2 \theta$$

Hence $$\omega = \sqrt{\frac{I_1 \omega_1^2 + I_2 \omega_2^2}{I_1 + I_2}}$$

4.51 Let us locate the rod when it is at small angular position θ relative to its equilibrium position. If a be the sought distance, then from the conservation of mechanical energy of oscillation

$$m g a (1 - \cos \theta) + \frac{1}{2} I_{OO'} (\dot{\theta})^2 = \text{constant}$$

Differentiating w.r.t. time we get :

$$m g a \sin \theta \, \dot{\theta} + \frac{1}{2} I_{OO'} 2 \dot{\theta} \ddot{\theta} = 0$$

But $I_{OO'} = \frac{m l^2}{12} + m a^2$ and for small θ, $\sin \theta = \theta$, we get

$$\ddot{\theta} = -\left(\frac{g a}{\frac{l^2}{12} + a^2}\right) \theta$$

Hence the time period of one full osscillation becomes

$$T = 2\pi \sqrt{\frac{\frac{l^2}{12} + a^2}{a g}} \quad \text{or} \quad T^2 = \frac{4\pi^2}{g} \left(\frac{l^2}{12 a} + a\right)$$

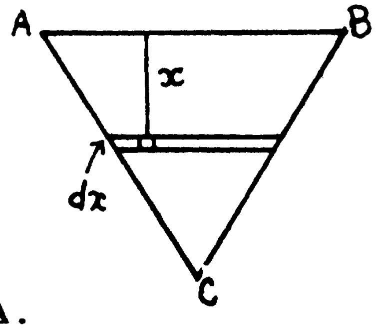
For T_{min}, obviously $\frac{d}{da}\left(\frac{l^2}{12 a} + a\right) = 0$

So, $$-\frac{l^2}{12 a^2} + 1 = 0 \quad \text{or} \quad a = \frac{l}{2\sqrt{3}}$$

Hence $$T_{min} = 2\pi \sqrt{\frac{l}{g\sqrt{3}}}$$

4.52 Consider the moment of inertia of the triangular plate about AB.

$$I = \iint x^2\, dm = \iint x^2 \rho\, dx\, dy$$



$$= \int_0^h x^2 \rho\, dx \frac{h-x}{h} \cdot \frac{2h}{\sqrt{3}} = \int_0^h x^2 \frac{2\rho}{\sqrt{3}} (h-x)\, dx$$

$$= \frac{2\rho}{\sqrt{3}} \left(\frac{h^4}{3} - \frac{h^4}{4} \right) = \frac{\rho h^4}{6\sqrt{3}} = \frac{m h^2}{6}$$

On using the area of the triangle $\Delta ABC = \frac{h^2}{\sqrt{3}}$ and $m = \rho \Delta$.

Thus K.E. $= \frac{1}{2} \frac{m h^2}{6} \dot{\theta}^2$

P.E. $= m g \frac{h}{3} (1 - \cos\theta) = \frac{1}{2} m g h \frac{\theta^2}{3}$

Here θ is the angle that the instantaneous plane of the plate makes with the equilibrium position which is vertical. (The plate rotates as a rigid body)

Thus $$E = \frac{1}{2} \frac{m h^2}{6} \dot{\theta}^2 + \frac{1}{2} \frac{m g h}{3} \theta^2$$

Hence $$\omega^2 = \frac{2g}{h} = \frac{m g h}{3} \Big/ \frac{m h^2}{6}$$

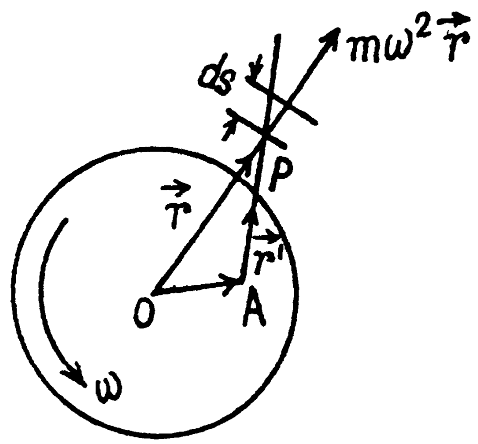
So $$T = 2\pi \sqrt{\frac{h}{2g}} = \pi \sqrt{\frac{2h}{g}} \,.$$ and $l_{reduced} = h/2$.

4.53 Let us go to the rotating frame, in which the disc is stationary. In this frame the rod is subjected to coriolis and centrifugal forces, $\mathbf{F}_{cor}$ and $\mathbf{F}_{cf}$, where

$$\mathbf{F}_{cor} = \int 2\, dm\, (\mathbf{v}' \times \vec{\omega}_0) \text{ and } \mathbf{F}_{cf} = \int dm\, \omega_0^2 \mathbf{r},$$

where $\mathbf{r}$ is the position of an elemental mass of the rod (Fig.) with respect to point O (disc's centre) and

$$\mathbf{v}' = \frac{d\mathbf{r}'}{dt}$$



As $$\mathbf{r} = \mathbf{OP} = \mathbf{OA} + \mathbf{AP}$$

So, $$\frac{d\mathbf{r}}{dt} = \frac{d(\mathbf{AP})}{dt} = \mathbf{v}'$$ (as OA is constant)

As the rod is vibrating transversely, so $\mathbf{v}'$ is directed perpendicular to the length of the rod. Hence $2\, dm\, (\mathbf{v}' \times \vec{\omega})$ for each elemental mass of the rod is directed along PA. Therefore the net torque of coriolis about A becomes zero. The not torque of centrifulgal force about point A :

Now, $$\vec{\tau}_{cf(A)} = \int \mathbf{AP} \times dm\, \omega_0^2 \mathbf{r} = \int \mathbf{AP} \times \left(\frac{m}{l}\right) ds\, \omega_0^2 (\mathbf{OA} + \mathbf{AP})$$

$$= \int \mathbf{AP} \times \left(\frac{m}{l} ds\right) \omega_0^2 \, \mathbf{OA} = \int \frac{m}{l} ds \, \omega_0^2 \, s \, a \sin\theta \, (-\mathbf{k})$$

$$= \frac{m}{l} \omega_0^2 \, a \sin\theta \, (-\mathbf{k}) \int_0^l s \, ds = m \, \omega_0^2 \, a \frac{l}{2} \sin\theta \, (-\mathbf{k})$$

So, $$\tau_{cf(Z)} = \vec{\tau}_{cf(A)} \cdot \mathbf{k} = -m \, \omega_0^2 \, a \frac{l}{2} \sin\theta$$

According to the equation of rotational dynamics : $\tau_{A(Z)} = I_A \, \alpha_Z$

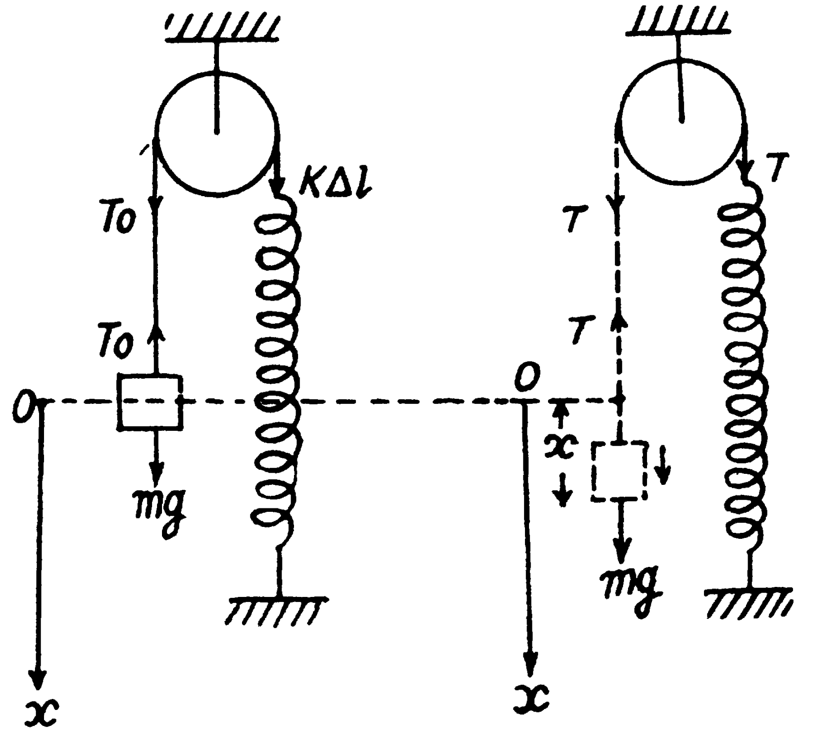
or, $$-m \, \omega_0^2 \, a \frac{l}{2} \sin\theta = \frac{m l^2}{3} \ddot{\theta}$$

or, $$\ddot{\theta} = -\frac{3}{2} \frac{\omega_0^2 \, a}{l} \sin\theta$$

Thus, for small θ, $$\ddot{\theta} = -\frac{3}{2} \frac{\omega^2 a}{2l} \theta$$

This implies that the frequency ω_0 of oscillation is $\omega_0 = \sqrt{\dfrac{3 \omega^2 a}{2l}}$

4.54 The physical system consists with a pulley and the block. Choosing an intertial frame, let us direct the x-axis as shown in the figure.



Initially the system is in equilibrium position. Now from the condition of translation equilibrium for the block

$$T_0 = m g \tag{1}$$

Similarly for the rotational equilibrium of the pulley

$$\kappa \, \Delta l R = T_0 R$$

or. $$T_0 = \kappa \, \Delta \, l \tag{2}$$

from Eqns. (1) and (2) $$\Delta l = \frac{m g}{\kappa} \tag{3}$$

Now let us disturb the equilibrium of the system no matter in which way to analyse its motion. At an arbitrary position shown in the figure, from Newton's second law of motion for the block

$$F_x = m w_x$$

$$m g - T = m w = m \ddot{x} \tag{4}$$

Similarly for the pulley

$$N_z = I \beta_z$$

$$T R - \kappa (\Delta l + x) R = I \ddot{\theta} \tag{5}$$

But $$w = \beta R \quad \text{or,} \quad \ddot{x} = R \ddot{\theta} \tag{6}$$

from (5) and (6) $$T R - \kappa (\Delta l + x) R = \frac{I}{R} \ddot{x} \tag{7}$$

Solving (4) and (7) using the initial condition of the problem

$$- \kappa R x = \left(m R + \frac{I}{R} \right) \ddot{x}$$

or, $$\ddot{x} = - \left(\frac{\kappa}{m + \frac{I}{R^2}} \right) x$$

Hence the sought time period, $T = \dfrac{2\pi}{\omega_0} = 2\pi \sqrt{\dfrac{m + I/R^2}{\kappa}}$

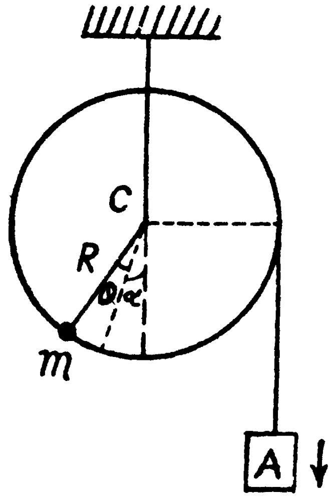
Note : we may solve this problem by using the conservation of mechanical energy also

4.55 At the equilbrium position, $N_{oz} = 0$ (Net torque about 0)

So, $$m_A g R - m g R \sin \alpha = 0 \quad \text{or} \quad m_A = m \sin \alpha \tag{1}$$

From the equation of rotational dynamics of a solid body about the stationary axis (say z-axis) of rotation i.e. from $N_z = I \beta_z$

when the pulley is rotated by the small angular displacement θ in clockwise sense relative to the equilibrium position (Fig.), we get :



$$m_A g R - m g R \sin (\alpha + \theta)$$

$$= \left[\frac{M R^2}{2} + m R^2 + m_A R^2 \right] \ddot{\theta}$$

Using Eqn. (1)

$$m g \sin \alpha - m g (\sin \alpha \cos \theta + \cos \alpha \sin \theta)$$

$$= \left\{ \frac{M R + 2 m (1 + \sin \alpha) R}{2} \right\} \ddot{\theta}$$

But for small θ, we may write $\cos\theta \approx 1$ and $\sin\theta \approx \theta$

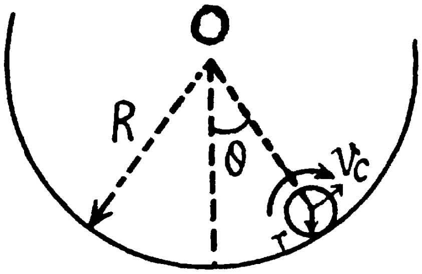
Thus we have

$$m\,g\sin\alpha - m\,g\,(\sin\alpha + \cos\alpha\,\theta) = \frac{\{MR + 2\,m\,(1+\sin\alpha)R\}}{2}\ddot{\theta}$$

Hence, $$\ddot{\theta} = -\frac{2\,m\,g\cos\alpha}{[MR + 2\,m\,(1+\sin\alpha)R]}\theta$$

Hence the sought angular frequency $\omega_0 = \sqrt{\dfrac{2\,m\,g\cos\alpha}{MR + 2\,m\,R\,(1+\sin\alpha)}}$

4.56 Let us locate solid cylinder when it is displaced from its stable equilibrium position by the small angle θ during its oscillations (Fig.). If v_c be the instantaneous speed of the C.M. (C) of the solid cylinder which is in pure rolling, then its angular velocity about its own centre C is



$$\omega = v_c/r \tag{1}$$

Since C moves in a circle of radius $(R-r)$, the speed of C at the same moment can be written as

$$v_c = \dot{\theta}\,(R-r) \tag{2}$$

Thus from Eqns (1) and (2)

$$\omega = \dot{\theta}\,\frac{(R-r)}{r} \tag{3}$$

As the mechanical energy of oscillation of the solid cylinder is conserved, i.e. $E = T + U =$ constant

So, $\dfrac{1}{2}m\,v_c^2 + \dfrac{1}{2}I_c\,\omega^2 + m\,g\,(R-r)\,(1-\cos\theta) =$ constant

(Where m is the mass of solid cylinder and I_c is the moment of inertia of the solid cylinder about an axis passing through its C.M. (C) and perpendicular to the plane of Fig. of solid cylinder)

or, $\dfrac{1}{2}m\,\omega^2 r^2 + \dfrac{1}{2}\dfrac{m\,r^2}{2}\omega^2 + m\,g\,(R-r)\,(1-\cos\theta) =$ constant (using Eqn (1) and $I_c = m\,r^2/2$)

$$\frac{3}{4}r^2(\dot{\theta})^2\frac{(R-r)^2}{r^2} + g\,(R-r)\,(1-\cos\theta) = \text{constant, (using Eqn. 3)}$$

Differentiating w.r.t. time

$$\frac{3}{4}(R-r)\,2\,\dot{\theta}\,\ddot{\theta} + g\sin\theta\,\dot{\theta} = 0$$

So, $$\ddot{\theta} = -\frac{2g}{3\,(R-r)}\theta, \text{ (because for small } \theta,\ \sin\theta \approx \theta\text{)}$$

Thus $$\omega_0 = \sqrt{\frac{2g}{3\,(R-r)}}$$

Hence the sought time period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{3(R-r)}{2g}}$$

4.57 Let κ_1 and κ_2 be the spring constant of left and right sides springs. As the rolling of the solid cylinder is pure its lowest point becomes the instanteneous centre of rotation. If θ be the small angular displacement of its upper most point relative to its equilibrium position, the deformation of each spring becomes $(2R\theta)$. Since the mechanical energy of oscillation of the solid cylinder is conserved, $E = T + U =$ constant

i.e. $$\frac{1}{2}I_p(\dot{\theta})^2 + \frac{1}{2}\kappa_1(2R\theta)^2 + \frac{1}{2}\kappa_2(2R\theta)^2 = \text{constant}$$

Differentianting w.r.t. time

$$\frac{1}{2}I_p 2\dot{\theta}\ddot{\theta} + \frac{1}{2}(\kappa_1+\kappa_2)4R^2 2\theta\dot{\theta} = 0$$

or, $$\left(\frac{mR^2}{2} + mR^2\right)\ddot{\theta} + 4R^2\kappa\theta = 0$$

$$\left(\text{Because } I_P = I_C + mR^2 = \frac{mR^2}{2} + mR^2\right)$$

Hence $$\ddot{\theta} = -\frac{8}{3}\frac{\kappa}{m}\theta$$

Thus $\omega_0 = \dfrac{8\kappa}{3m}$ and sought time period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{3m}{8\kappa}} = \pi\sqrt{\frac{3m}{2\kappa}}$$

4.58 In the C.M. frame (which is rigidly attached with the centre of mass of the two cubes) the cubes oscillates. We know that the kinetic energy of two body system equals $\frac{1}{2}\mu v_{rel}^2$, where μ is the reduced mass and v_{rel} is the modulus of velocity of any one body particle relative to other. From the conservation of mechanical energy of oscillation :

$$\frac{1}{2}\kappa x^2 + \frac{1}{2}\mu\left\{\frac{d}{dt}(l_0 + x)\right\}^2 = \text{constant}$$

Here l_0 is the natural length of the spring.

Differenting the above equation w.r.t time, we get :

$$\frac{1}{2}\kappa 2x\dot{x} + \frac{1}{2}\mu 2\dot{x}\ddot{x} = 0 \left[\text{becomes } \frac{d(l_0+x)}{dt} = \dot{x}\right]$$

Thus $\ddot{x} = -\dfrac{\kappa}{\mu}x \quad \left(\text{where } \mu = \dfrac{m_1 m_2}{m_1 + m_2}\right)$

Hence the natural frequency of oscillation : $\omega_0 = \sqrt{\dfrac{\kappa}{\mu}}$ where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

4.59 Suppose the balls 1 & 2 are displaced by x_1, x_2 from their initial position. Then the energy is : $E = \frac{1}{2} m_1 \dot{x}_1^2 + m_2 \dot{x}^2 + \frac{1}{2} k (x_1 - x_2)^2 = \frac{1}{2} m_1 v_1^2$

Also total momentum is : $m_1 \dot{x}_1 + m_2 \dot{x}_2 = m_1 v_1$

Define $$X = \frac{m_1 x_1 + m_1 x_2}{m_1 + m_2},\ x = x_1 - x_2$$

Then $$x_1 = X + \frac{m_2}{m_1 + m_2} x,\ x_2 = X - \frac{m_1}{m_1 + m_2} x$$

$$E = \frac{1}{2}(m_1 + m_2)\dot{X}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{x}^2 + \frac{1}{2} k x^2$$

Hence $$\dot{X} = \frac{m_1 v_1}{m_1 + m_2}$$

So $$\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{x}^2 + \frac{1}{2} k x^2 = \frac{1}{2} m_1 v_1^2 - \frac{1}{2}\frac{m_1^2 v_1^2}{m_1 + m_2} = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v_1^2$$

(a) From the above equation

We see $\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{3 \times 24}{2}} = 6\ s^{-1}$, when $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{2}{3}$ kg.

(b) The energy of oscillation is

$$\frac{1}{2}\frac{m_1 m_2}{m_2 + m_2} v_1^2 = \frac{1}{2}\frac{2}{3} \times (0.12)^2 = 48 \times 10^{-4} = 4.8 \text{ mJ}$$

We have $x = a \sin(\omega t + \alpha)$

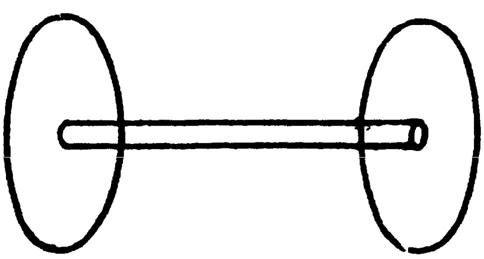
Initially $x = 0$ at $t = 0$ so $\alpha = 0$

Then $x = a \sin \omega t$. Also $\dot{x} = v_1$ at $t = 0$.

So $\omega a = v_1$ and hence $a = \frac{v_1}{\omega} = \frac{12}{6} = 2$ cm.

4.60 Suppose the disc 1 rotates by angle θ_1 and the disc 2 by angle θ_2 in the opposite sense. Then total torsion of the rod $= \theta_1 + \theta_2$

and torsional P.E. $= \frac{1}{2}\kappa(\theta_1 + \theta_2)^2$



The K.E. of the system (neglecting the moment of inertia of the rod) is

$$\frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2$$

So total energy of the rod

$$E = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2}\kappa(\theta_1 + \theta_2)^2$$

We can put the total angular momentum of the rod equal to zero since the frequency associated with the rigid rotation of the whole system must be zero (and is known).

Thus $\qquad I_1\,\theta_1 = I_2\,\dot{\theta}_2 \quad \text{or} \quad \dfrac{\dot{\theta}_1}{1/I_1} = \dfrac{\dot{\theta}_2}{1/I_2} = \dfrac{\dot{\theta}_1 + \dot{\theta}_2}{1/I_1 + 1/I_2}$

So $\qquad \dot{\theta}_1 = \dfrac{I_2}{I_1 + I_2}\,(\dot{\theta}_1 + \dot{\theta}_2) \quad \text{and} \quad \dot{\theta}_2 = \dfrac{I_1}{I_1 + I_2}\,(\dot{\theta}_1 + \dot{\theta}_2)$

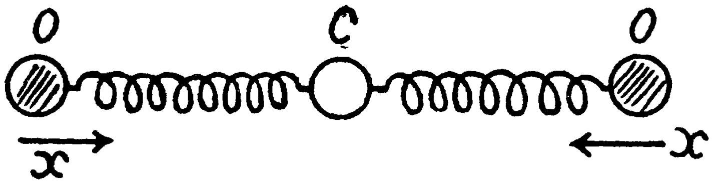
and $\qquad E = \dfrac{1}{2}\,\dfrac{I_1 I_2}{I_1 + I_2}\,(\dot{\theta}_1 + \dot{\theta}_2)^2 + \dfrac{1}{2}\kappa\,(\theta_1 + \theta_2)^2$

The angular oscillation, frequency corresponding to this is

$$\omega^2 = \kappa \Big/ \frac{I_1 I_2}{I_1 + I_2} = \kappa/I' \quad \text{and} \quad T = 2\pi\sqrt{\frac{I'}{\kappa}}\,, \text{ where } I' = \frac{I_1 I_2}{I_1 + I_2}$$

4.61 In the first mode the carbon atom remains fixed and the oxygen atoms move in equal & opposite steps. Then total energy is

(1)

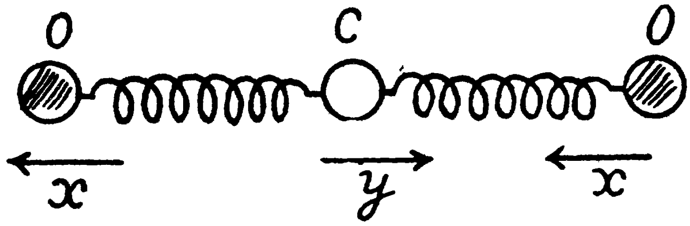


$$\frac{1}{2}2\,m_0\,\dot{x}^2 + \frac{1}{2}2\,\kappa\,x^2$$

where x is the displacement of one of the 0 atom (say left one). Thus

$$\omega_1^2 = \kappa/m_0\,.$$

(2)



In this mode the oxygen atoms move in equal steps in the same direction but the carbon atom moves in such a way as to keep the centre of mass fixed.

Thus $\qquad 2\,m_0\,x + m_c\,y = 0 \quad \text{or, } y = -\dfrac{2\,m_0}{m_c}x$

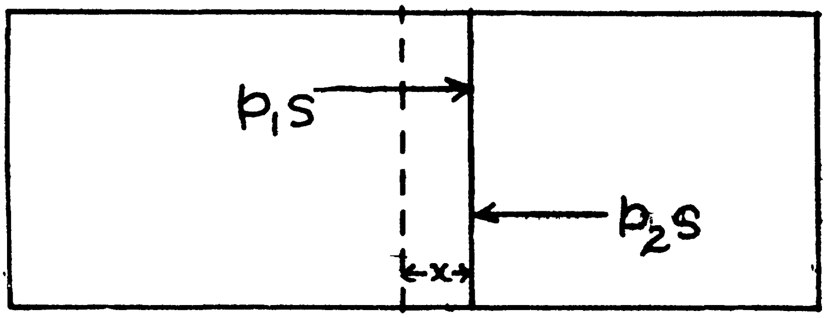
$$\text{KE.} = \frac{1}{2}2\,m_0\,\dot{x}^2 + \frac{1}{2}m_c\left(\frac{2\,m_0}{m_c}\dot{x}\right)^2 = \frac{1}{2}2\,m_0\,\dot{x}^2 + \frac{1}{2}2\,m_0\frac{2\,m_0}{m_c}\dot{x}^2 \qquad = \frac{1}{2}2\,m_0\left(1 + \frac{2\,m_0}{m_c}\right)\dot{x}^2$$

$$\text{P.E.} = \frac{1}{2}k\left(1 + \frac{2\,m_0}{m_c}\right)^2 x^2 \qquad + \frac{1}{2}\kappa\left(1 + \frac{2\,m_0}{m_c}\right)^2 x^2 = \frac{1}{2}2\,\kappa\left(1 + \frac{2\,m_0}{m_c}\right)^2 x^2$$

Thus $\qquad \omega_2^2 = \dfrac{\kappa}{m_0}\left(1 + \dfrac{2\,m_0}{m_c}\right) \quad \text{and } \omega_2 = \omega_1\sqrt{1 + \dfrac{2\,m_0}{m_c}}$

Hence, $\qquad \omega_2 = \omega_1\sqrt{1 + \dfrac{32}{12}} = \omega_1\sqrt{\dfrac{11}{3}} \approx 1{\cdot}91\,\omega_1$

4.62 Let, us displace the piston through small distance x, towards right, then from $F_x = m\,w_x$



or, $$(p_1 - p_1)\,S = -m\ddot{x} \qquad (1)$$

But, the process is adiabatic, so from $P\,V^{\gamma} =$ const.

$$p_2 = \frac{p_0\,V_0^{\gamma}}{(V_0 - S\,x)^{\gamma}} \quad \text{and} \quad p_1 = \frac{p_0\,V_0^{\gamma}}{(V_0 + S\,x)^{\gamma}},$$

as the new volumes of the left and the right parts are now $(V_0 + S\,x)$ and $(V_0 - S\,x)$ respectively.

So, the Eqn (1) becomes.

$$\frac{p_0\,V_0^{\gamma}\,S}{m}\left\{\frac{1}{(V_0 - S\,x)^{\gamma}} - \frac{1}{(V_0 + S\,x)^{\gamma}}\right\} = -\ddot{x}$$

or, $$\frac{p_0\,V_0^{\gamma}\,S}{m}\left\{\frac{(V_0 + S\,x)^{\gamma} - (V_0 - S\,x)^{\gamma}}{(V_0^2 - S^2x^2)^{\gamma}}\right\} = -\ddot{x}$$

or, $$\frac{p_0\,V_0^{\gamma}\,S}{m}\left\{\frac{\left(1 + \frac{\gamma\,S\,x}{V_0}\right) - \left(1 - \frac{\gamma\,S\,x}{V_0}\right)}{V_0^{\gamma}\left(1 - \frac{\gamma\,S^2x^2}{V_0^2}\right)}\right\} = -\ddot{x}$$

Neglecting the term $\frac{\gamma\,S^2x^2}{V_0^2}$ in the denominator, as it is very small, we get,

$$\ddot{x} = -\frac{2\,p_0\,S^2\,\gamma\,x}{m\,V_0},$$

which is the equation for S.H.M. and hence the oscillating frequency.

$$\omega_0 = S\sqrt{\frac{2\,p_0\,\gamma}{m\,V_0}}$$

4.63 In the absence of the charge, the oscillation period of the ball

$$T = 2\pi\sqrt{l/g}$$

when we impart the charge q to the ball, it will be influenced by the induced charges on the conducting plane. From the electric image method the electric force on the ball by the plane equals $\frac{q^2}{4\pi\varepsilon_0(2h)^2}$ and is directed downward. Thus in this case the effective acceleration of the ball

$$g' = g + \frac{q^2}{16\pi\varepsilon_0 m h^2}$$

and the corresponding time period

$$T' = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{g + \dfrac{q^2}{16\pi\varepsilon_0 m h^2}}}$$

From the conditon of the problem

$$T = \eta T'$$

So, $$T^2 = \eta^2 T'^2 \quad \text{or} \quad \frac{1}{g} = \eta^2\left(\frac{1}{g + \dfrac{q^2}{16\pi\varepsilon_0 m h^2}}\right)$$

Thus on solving

$$q = 4h\sqrt{\pi\varepsilon_0 m g(\eta^2 - 1)} = 2\,\mu C$$

4.64 In a magnetic field of induction B the couple on the magnet is $-MB\sin\theta = -MB\theta$ equating this to $I\ddot{\theta}$ we get

$$I\ddot{\theta} + MB\theta = 0$$

or $$\omega^2 = \frac{MB}{I} \quad \text{or} \quad T = 2\pi\sqrt{\frac{I}{MB}}.$$

Given $$T_2 = T_1/\eta$$

o. $$\sqrt{\frac{1}{B_2}} = \sqrt{\frac{1}{B_1}}\cdot\frac{1}{\eta} \quad \text{or} \quad \frac{1}{B_2} = \frac{1}{B_1}\cdot\frac{1}{\eta^2}$$

or $$B_2 = \eta^2 B_1$$

The induction of the field increased η^2 times.

4.65 We have in the circuit at a certain instant of time (t), from Faraday's law of electromagnetic induction :

$$L\frac{di}{dt} = Bl\frac{dx}{dt} \quad \text{or} \quad L\,di = Bl\,dx$$

As at $t = 0,\ x = 0,$ so $Li = Blx$ or $i = \frac{Bl}{L}x$ (1)

For the rod from the second law of motion $F_x = m\,w_x$

$$-ilB = m\ddot{x}$$

Using Eqn. (1), we get : $$\ddot{x} = -\left(\frac{l^2B^2}{mL}\right)x = -\omega_0^2 x \qquad (2)$$

where $$\omega_0 = lB/\sqrt{mL}$$

The solution of the above differential equation is of the form

$$x = a \sin(\omega_0 t + \alpha)$$

From the initial condition, at $t = 0$, $x = 0$, so $\alpha = 0$

Hence, $$x = a \sin \omega_0 t \qquad (3)$$

Differentiating w.r.t. time, $\dot{x} = a\,\omega_0 \cos \omega_0 t$

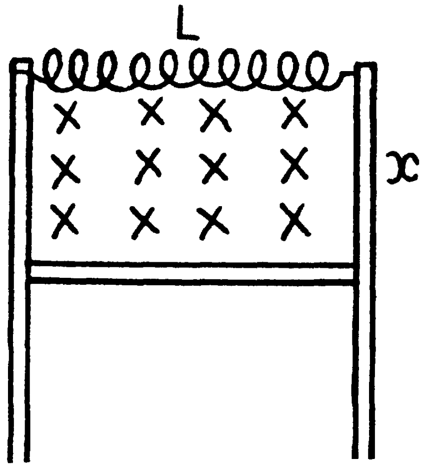
But from the initial condition of the problem at $t = 0$, $\dot{x} = v_0$

Thus $$v_0 = a\,\omega_0 \quad \text{or} \quad a = v_0/\omega_0 \qquad (4)$$

Putting the value of a from Eqn. (4) into Eqn. (3), we obtained

$$x = \frac{v_0}{\omega_0} \sin \omega_0 t \left(\text{where } \omega_0 = \frac{lB}{\sqrt{mL}} \right)$$

4.66 As the connector moves, an emf is set up in the circuit and a current flows, since the emf is $\xi = -B\,l\,\dot{x}$, we must have : $-B\,l\,\dot{x} + L\dfrac{dI}{dt} = 0$



so, $I = B\,l\,x/L$

provided x is measured from the initial position.

We then have

$$m\ddot{x} = -\frac{B\,l\,x}{L}.B.l + mg$$

for by Lenz's law the induced current will oppose downward sliding. Finally

$$\ddot{x} + \frac{(B\,l)^2}{mL}x = g$$

on putting $$\omega_0 = \frac{B\,l}{\sqrt{mL}}$$

$$\ddot{x} + \omega_0^2 x = g$$

A solution of this equation is $x = \dfrac{g}{\omega_0^2} + A\cos(\omega_0 t + \alpha)$

But $x = 0$ and $\dot{x} = 0$ at $t = 0$. This gives

$$x = \frac{g}{\omega_0^2}(1 - \cos \omega_0 t).$$

4.67 We are given $x = a_0 e^{-\beta t} \sin \omega t$

(a) The velocity of the point at $t = 0$ is obtained from

$$v_0 = (\dot{x})_{t=0} = \omega\, a_0$$

The term "oscillation amplitude at the moment $t = 0$" is meaningless. Probably the implication is the amplitude for $t << \dfrac{1}{\beta}$. Then $x = a_0 \sin \omega t$ and amplitude is a_0.

(b) $\dot{x} = (-\beta\, a_0 \sin \omega t + \omega\, a_0 \cos \omega t)\, e^{-\beta t} = 0$

when the displacement is an extremum. Then

$$\tan \omega t = \frac{\omega}{\beta}$$

or

$$\omega t = \tan^{-1}\frac{\omega}{\beta} + n\pi,\ n = 0, 1, 2, \ldots$$

4.68 Given $\quad \varphi = \varphi_0 e^{-\beta t} \cos \omega t$

we have $\dot{\varphi} = -\beta \varphi - \omega \varphi_0 e^{-\beta t} \sin \omega t$

$$\ddot{\varphi} = -\beta \dot{\varphi} + \beta \omega \varphi_0 e^{-\beta t} \sin \omega t - \omega^2 \varphi_0 e^{-\beta t} \cos \omega t$$

$$= \beta^2 \varphi + 2\beta \omega \varphi_0 e^{-\beta t} \sin \omega t - \omega^2 \varphi$$

so

(a) $(\dot{\varphi})_0 = -\beta \varphi_0,\ (\ddot{\varphi})_0 = (\beta^2 - \omega^2)\varphi_0$

(b) $\dot{\varphi} = -\varphi_0 e^{-\beta t}(\beta \cos \omega t + \omega \sin \omega t)$ becomes maximum (or minimum) when

$$\ddot{\varphi} = \varphi_0 (\beta^2 - \omega^2) e^{-\beta t} \cos \omega t + 2\beta \omega \varphi_0 e^{-\beta t} \sin \omega t = 0$$

or

$$\tan \omega t = \frac{\omega^2 - \beta^2}{2\beta\omega}$$

and

$$t_n = \frac{1}{\omega}\left[\tan^{-1}\frac{\omega^2 - \beta^2}{2\beta\omega} + n\pi\right],\ n = 0, 1, 2, \ldots$$

4.69 We write $x = a_0 e^{-\beta t} \cos(\omega t + \alpha)$.

(a) $x(0) = 0 \Rightarrow \alpha = \pm\frac{\pi}{2} \Rightarrow x = \mp a_0 e^{-\beta t} \sin \omega t$

$$\dot{x}(0) = (\dot{x})_{t=0} = \mp \omega a_0$$

Since a_0 is $+ve$, we must choose the upper sign if $\dot{x}(0) < 0$ and the lower sign if $\dot{x}(0) > 0$. Thus

$$a_0 = \frac{|\dot{x}(0)|}{\omega} \quad \text{and} \quad \alpha = \begin{cases} +\frac{\pi}{2} & \text{if } \dot{x}(0) < 0 \\ -\frac{\pi}{2} & \text{if } \dot{x}(0) > 0 \end{cases}$$

(b) we write $x = \mathrm{Re}\, A\, e^{-\beta t + i\omega t},\ A = a_0 e^{i\alpha}$

Then $\quad \dot{x} = v_x = \mathrm{Re}\,(-\beta + i\omega) A e^{-\beta t + i\omega t}$

From $v_x(0) = 0$ we get $Re\,(-\beta + i\omega)A = 0$

This implies $A = \pm i(\beta + i\omega)B$ where B is real and positive. Also

$$x_0 = \mathrm{Re}\, A = \mp \omega B$$

Thus $\quad B = \frac{|x_0|}{\omega}$ with $+$ sign in A if $x_0 < 0$

$$- \text{ sign in } A \text{ if } x_0 > 0$$

So $$A = \pm i \frac{\beta + i\omega}{\omega} |x_0| = \left(\mp 1 + \pm \frac{i\beta}{\omega} \right) |x_0|$$

Finally $$a_0 = \sqrt{1 + \left(\frac{\beta}{\omega} \right)^2} \, |x_0|$$

$$\tan \alpha = \frac{-\beta}{\omega}, \; \alpha = \tan^{-1} \left(\frac{-\beta}{\omega} \right)$$

α is in the 4^{th} quadrant $\left(-\frac{\pi}{2} < \alpha < 0 \right)$ if $x_0 > 0$ and α is in the 2^{nd} quadrant $\left(\frac{\pi}{2} < \alpha < \pi \right)$ if $x_0 < 0$.

4.70 $x = a_0 e^{-\beta t} \cos(\omega t + \alpha)$

Then $$(\dot{x})_{t=0} = -\beta a_0 \cos \alpha - \omega a_0 \sin \alpha = 0$$

or $$\tan \alpha = -\frac{\beta}{\omega}.$$

Also $$(x)_{t=0} = a_0 \cos \alpha = \frac{a_0}{\eta}$$

$$\sec^2 \alpha = \eta^2, \; \tan \alpha = -\sqrt{\eta^2 - 1}$$

Thus $$\beta = \omega \sqrt{\eta^2 - 1}$$

(We have taken the amplitude at $t = 0$ to be a_0).

4.71 We write $x = a_0 e^{-\beta t} \cos(\omega t + \alpha)$

$$= Re\, A e^{-\beta t + i\omega t}, \; A = a_0 e^{i\alpha}$$

$$\dot{x} = Re\, A(-\beta + i\omega) e^{-\beta t + i\omega t}$$

Velocity amplitude as a function of time is defined in the following manner. Put $t = t_0 + \tau$, then

$$x = Re\, A e^{-\beta(t_0 + \tau)} e^{i\omega(t_0 + \tau)}$$

$$\approx Re\, A e^{-\beta t_0} e^{i\omega t_0 + i\omega\tau} \approx Re A\, e^{-\beta t_0} e^{i\omega t}$$

for $\tau << \frac{1}{\beta}$. This means that the displacement amplitude around the time t_0 is $a_0 e^{-\beta t_0}$ and we can say that the displacement amplitude at time t is $a_0 e^{-\beta t}$. Similarly for the velocity amplitude.

Clearly

(a) Velocity amplitude at time $t = a_0 \sqrt{\beta^2 + \omega^2}\, e^{-\beta t}$

Since $$A(-\beta + i\omega) = a_0 e^{i\alpha}(-\beta + i\omega)$$

$$= a_0 \sqrt{\beta^2 + \omega^2}\, e^{i\gamma}$$

where γ is anotner constant.

(b) $x(0) = 0 \Rightarrow Re\, A = 0$ or $A = \pm\, i\, a_0$

where a_0 is real and positive.

Also $$v_x(0) = \dot{x}_0 = Re \pm i\, a_0(-\beta + i\omega)$$
$$= \mp\, \omega\, a_0$$

Thus $a_0 = \dfrac{|\dot{x}_0|}{\omega}$ and we take $-(+)$ sign if x_0 is negative (positive). Finally the velocity amplitude is obtained as

$$\frac{|\dot{x}_0|}{\omega}\sqrt{\beta^2 + \omega^2}\, e^{-\beta t}.$$

4.72 The first oscillation decays faster in time. But if one takes the natural time scale, the period T for each oscillation, the second oscillation attenuates faster during that period.

4.73 By definition of the logarithemic decrement $\left(\lambda = \beta \dfrac{2\pi}{\omega}\right)$ we get for the original decrement λ_0

$$\lambda_0 = \beta \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} \quad \text{and finally } \lambda = \frac{2\pi n \beta}{\sqrt{\omega_0^2 - n^2 \beta^2}}$$

Now $$\frac{\beta}{\sqrt{\omega_0^2 - \beta^2}} = \frac{\lambda_0}{2\pi} \quad \text{or} \quad \frac{\beta}{\omega_0} = \frac{\lambda_0/2\pi}{\sqrt{1 + \left(\dfrac{\lambda_0}{2\pi}\right)^2}}$$

so $$\frac{\lambda/2\pi}{\sqrt{1 + \left(\dfrac{\lambda}{2\pi}\right)^2}} = \frac{n\dfrac{\lambda_0}{2\pi}}{\sqrt{1 + \left(\dfrac{\lambda_0}{2\pi}\right)^2}}$$

Hence $$\frac{\lambda}{2\pi} = \frac{n\lambda_0/2\pi}{\sqrt{1 - (n^2 - 1)\left(\dfrac{\lambda_0}{2\pi}\right)^2}}$$

For critical damping $$\omega_0 = n_c \beta$$

$$\frac{1}{n_c} = \frac{\beta}{\omega_0} = \frac{\lambda_0/2\pi}{\sqrt{1 + \left(\dfrac{\lambda_0}{2\pi}\right)^2}} \quad \text{or} \quad n_c = \sqrt{1 + \left(\frac{2\pi}{\lambda_0}\right)^2}$$

4.74 The Eqn of the dead weight is

$$m\ddot{x} + 2\beta m\dot{x} + m\omega_0^2 x = mg$$

so
$$\Delta x = \frac{g}{\omega_0^2} \quad \text{or} \quad \omega_0^2 = \frac{g}{\Delta x}.$$

Now
$$\lambda = \frac{2\pi\beta}{\omega} = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}} \quad \text{or} \quad \frac{\omega_0}{\sqrt{\omega_0^2 - \beta^2}} = \sqrt{1 + \left(\frac{\lambda}{2\pi}\right)^2}$$

Thus
$$T = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} = \frac{2\pi}{\omega_0}\sqrt{1 + \left(\frac{\lambda}{2\pi}\right)^2}$$

$$= 2\pi\sqrt{\frac{\Delta x}{g}}\sqrt{1 + \left(\frac{\lambda}{2\pi}\right)^2} = \sqrt{\frac{\Delta x}{g}(4\pi^2 + \lambda^2)} = 0.70 \text{ sec.}$$

4.75 The displacement amplitude decrease η times every n oscillations. Thus
$$\frac{1}{\eta} = e^{-\beta \cdot \frac{2\pi}{\omega} \cdot n}$$

or
$$\frac{2\pi n\beta}{\omega} = \ln\eta \quad \text{or} \quad \frac{\beta}{\omega} = \frac{\ln\eta}{2\pi n}.$$

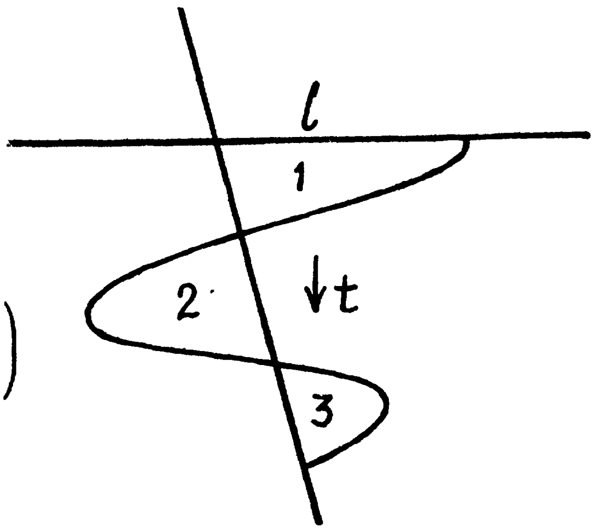
So
$$Q = \frac{\omega}{2\beta} = \frac{\pi n}{\ln\eta} \approx 499.$$

4.76 From $x = a_0 e^{-\beta t}\cos(\omega t + \alpha)$, we get using

$$(x)_{t=0} = l = a_0\cos\alpha$$

$$0 = (\dot{x})_{t=0} = -\beta a_0\cos\alpha - \omega a_0\sin\alpha$$

Then $\tan\alpha = -\dfrac{\beta}{\omega}$ or $\cos\alpha = \dfrac{\omega}{\sqrt{\omega^2 + \beta^2}}$



and $x = \dfrac{l\sqrt{\omega^2 + \beta^2}}{\omega} e^{-\beta t}\cos\left(\omega t - \tan^{-1}\dfrac{\beta}{\omega}\right)$

$x = 0$ at $t = \dfrac{1}{\omega}\left(n\pi + \dfrac{\pi}{2} + \tan^{-1}\dfrac{\beta}{\omega}\right)$

Total distance travelled in the first lap $= l$

To get the maximum displacement in the second lap we note that

$$\dot{x} = \left[-\beta\cos\left(\omega t - \tan^{-1}\frac{\beta}{\omega}\right) - \omega\sin\left(\omega t - \tan^{-1}\frac{\beta}{\omega}\right)\right]$$

$$\times \frac{l\sqrt{\omega^2 + \beta^2}}{\omega} e^{-\beta t} = 0$$

when
$$\omega t = \pi, 2\pi, 3\pi, \dots \text{ etc}.$$

Thus $\qquad \ddot{x}_{max} = -a_0 e^{-\pi\beta/\omega} \cos\alpha = -l e^{-\pi\beta/\omega}$ for $t = \pi/\omega$

so, distance traversed in the 2^{nd} lap $= 2 l e^{-\pi\beta/\omega}$

Continuing total distance traversed $= l + 2 l e^{-\pi\beta/\omega} + 2 l e^{-2\pi\beta/\omega} + \ldots\ldots$

$$= l + \frac{2 l e^{-\pi\beta/\omega}}{1 - e^{-\beta\pi/\omega}} = l + \frac{2 l}{e^{\beta\pi/\omega} - 1}$$

$$= l \frac{e^{\beta\pi/\omega} + 1}{e^{\beta\pi/\omega} - 1} = l \frac{1 + e^{\lambda/2}}{e^{\lambda/2} - 1}$$

where $\lambda = \dfrac{2\pi\beta}{\omega}$ is the logarithemic decrement. Substitution gives 2 metres.

4.77 For an undamped oscillator the mechanical energy $E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2$ is conserved. For a damped oscillator.

$$x = a_0 e^{-\beta t} \cos(\omega t + \alpha), \quad \omega = \sqrt{\omega_0^2 - \beta^2}$$

and $$E(t) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2$$

$$= \frac{1}{2} m a_0^2 e^{-2\beta t} \left[\beta^2 \cos^2(\omega t + \alpha) + 2\beta\omega \cos(\omega t + \alpha) \times \sin(\omega t + \alpha) + \omega^2 \sin^2(\omega t + \alpha)\right]$$

$$+ \frac{1}{2} m a_0^2 \omega_0^2 e^{-2\beta t} \cos^2(\omega t + \alpha)$$

$$= \frac{1}{2} m a_0^2 \omega_0^2 e^{-2\beta t} + \frac{1}{2} m a_0^2 \beta^2 e^{-2\beta t} \cos(2\omega t + 2\alpha) + \frac{1}{2} m a_0^2 \beta\omega e^{-2\beta t} \sin(2\omega t + 2\alpha)$$

If $\beta << \omega$, then the average of the last two terms over many oscillations about the time t will vanish and

$$\langle E(t) \rangle = \frac{1}{2} m a_0^2 \omega_0^2 e^{-2\beta t}$$

and this is the relevant mechanical energy.

In time τ this decreases by a factor $\dfrac{1}{\eta}$ so

$$e^{-2\beta\tau} = \frac{1}{\eta} \quad \text{or} \quad \tau = \frac{\ln\eta}{2\beta}.$$

$$\beta = \frac{\ln\eta}{2\tau}$$

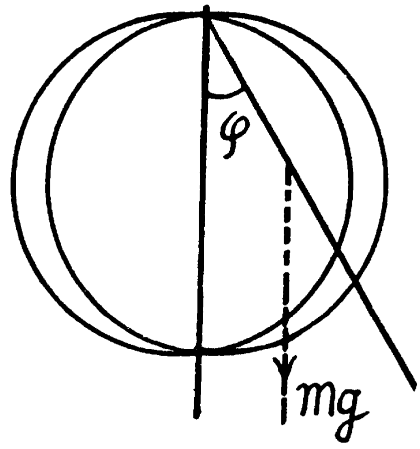
and $$\lambda = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}} = \frac{2\pi}{\sqrt{\left(\dfrac{\omega_0}{\beta}\right)^2 - 1}} = \frac{2\pi}{\sqrt{\dfrac{4 g \tau^2}{l \ln^2\eta} - 1}} \qquad \text{since } \omega_0^2 = \frac{g}{l}.$$

and $$Q = \frac{\pi}{\lambda} = \frac{1}{2}\sqrt{\frac{4 g \tau^2}{l \ln^2\eta} - 1} = 130.$$

4.78 The restoring couple is

$$\Gamma = -mgR\sin\varphi \approx -mgR\varphi$$

The moment of inertia is



$$I = \frac{3mR^2}{2}$$

Thus for undamped oscillations

$$\frac{3mR^2}{2}\ddot{\varphi} + mgR\varphi = 0$$

so, $\omega_0^2 = \frac{2g}{3R}$

Also
$$\lambda = \frac{2\pi\beta}{\omega} = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}}$$

Hence
$$\frac{\beta}{\sqrt{\omega_0^2 - \beta^2}} = \frac{\lambda}{2\pi} \quad \text{or} \quad \frac{\omega_0}{\sqrt{\omega_0 - \beta^2}} = \sqrt{1 + \left(\frac{\lambda}{2\pi}\right)^2}$$

Hence finally the period T of small oscillation comes to

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0} \times \frac{\omega_0}{\sqrt{\omega_0^2 - \beta^2}}\, 2\pi\sqrt{\frac{3R}{2g}\left(1 + \left(\frac{\lambda}{2\pi}\right)^2\right)}$$

$$= \sqrt{\frac{3R}{2g}(4\pi^2 + \lambda^2)} = 0.90 \text{ sec.}$$

4.79 Let us calculate the moment G_1 of all the resistive forces on the disc. When the disc rotates an element ($r\,dr\,d\theta$) with coordinates (r, θ) has a velocity $r\dot{\varphi}$, where φ is the instantaneous angle of rotation from the eqüilibrium position and r is measured from the centre. Then

$$G_1 = \int_0^{2\pi} d\theta \int_0^R dr \cdot r \cdot (F_1 \times r)$$

$$= \int_0^R \eta\, r\dot{\varphi}\, r^2\, d\gamma \times 2\pi = \frac{\eta\pi R^4}{2}\dot{\varphi}$$

Also moment of inertia $= \frac{mR^2}{2}$

Thus
$$\frac{mR^2}{2}\ddot{\varphi} + \frac{\pi\eta R^4}{2}\dot{\varphi} + \alpha\varphi = 0$$

or
$$\ddot{\varphi} + 2\frac{\pi\eta R^2}{2m}\dot{\varphi} + \frac{2\alpha}{mR^2}\varphi = 0$$

Hence
$$\omega_0^2 = \frac{2\alpha}{mR^2} \quad \text{and} \quad \beta = \frac{\pi\eta R^2}{2m}$$

and angular frequency $$\omega = \sqrt{\left(\frac{2\alpha}{mR^2}\right) - \left(\frac{\pi\eta R^2}{2m}\right)^2}$$

Note :- normally by frequency we mean $\frac{\omega}{2\pi}$.

4.80 From the law of viscosity, force per unit area $= \eta \frac{dv}{dx}$

so when the disc executes torsional oscillations the resistive couple on it is

$$= \int_0^R \eta \cdot 2\pi r . \frac{r\dot{\varphi}}{h} \cdot r \cdot dr \times 2 = \frac{\eta\pi R^4}{h}\dot{\varphi}$$

(factor 2 for the two sides of the disc; see the figure in the book)

where φ is torsion. The equation of motion is

$$I\ddot{\varphi} + \frac{\eta\pi R^4}{h}\dot{\varphi} + c\varphi = 0$$

Comparing with $$\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2\varphi = 0 \text{ we get}$$

$$\beta = \eta\pi R^4/2hI$$

Now the logarithmic decrement λ is given by $\lambda = \beta T$, $T =$ time period

Thus $$\eta = 2\lambda hI/\pi R^4 T$$

4.81 If $\varphi =$ angle of deviation of the frame from its normal position, then an e.m.f.

$$\varepsilon = Ba^2\dot{\varphi}$$

is induced in the frame in the displaced position and a current $\frac{\varepsilon}{R} = \frac{Ba^2\dot{\varphi}}{R}$ flows in it. A couple

$$\frac{Ba^2\dot{\varphi}}{R} \cdot B \cdot a \cdot a = \frac{B^2a^4}{R}\dot{\varphi}$$

then acts on the frame in addition to any elastic restoring couple $c\varphi$. We write the equation of the frame as

$$I\ddot{\varphi} + \frac{B^2a^4}{R}\dot{\varphi} + c\varphi = 0$$

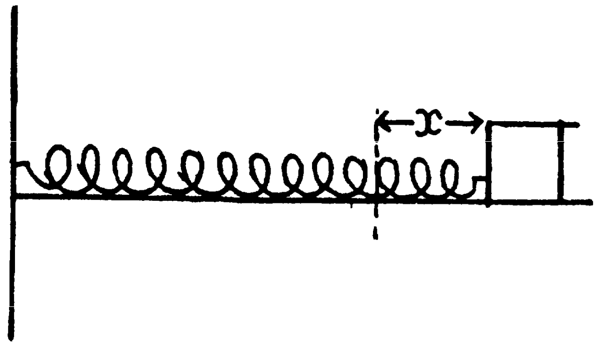
Thus $\beta = \frac{B^2a^4}{2IR}$ where β is defined in the book.

Amplitude of oscillation die out according to $e^{-\beta t}$ so time required for the oscillations to decrease to $\frac{1}{e}$ of its value is

$$\frac{1}{\beta} = \frac{2IR}{B^2a^4}$$

4.82 We shall denote the stiffness constant by κ. Suppose the spring is stretched by x_0. The bar in then subject to two horizontal forces (1) restoring force $-\kappa x$ and (2) friction kmg opposing motion. If

$$x_0 > \frac{k mg}{\kappa} = \Delta$$



the bar will come back.
(If $x_0 \leq \Delta$, the bar will stay put.)
The equation of the bar when it is moving to the left is

$$m\ddot{x} = -\kappa x + k mg$$

This equation has the solution

$$x = \Delta + (x_0 - \Delta)\cos\sqrt{\frac{k}{m}}\, t$$

where we have used $x = x_0, \dot{x} = 0$ at $t = 0$. This solution is only valid till the bar comes to rest. This happens at

$$t_1 = \pi \Big/ \sqrt{\frac{k}{m}}$$

and at that time $x = x_1 = 2\Delta - x_0$. if $x_0 > 2\Delta$ the tendency of the rod will now be to move to the right .(if $\Delta < x_0 < 2\Delta$ the rod will stay put now) Now the equation for rightward motion becomes

$$m\ddot{x} = -\kappa x - k mg$$

(the friction force has reversed).
We notice that the rod will move to the right only if

$$\kappa (x_0 - 2\Delta) > k mg \quad \text{i.e. } x_0 > 3\Delta$$

In this case the solution is

$$x = -\Delta + (x_0 - 3\Delta)\cos\sqrt{\frac{k}{m}}\, t$$

Since $\quad x = 2\Delta - x_0$ and $\dot{x} = 0$ at $t = t_1 = \pi \Big/ \sqrt{\frac{k}{m}}$.

The rod will next come to rest at

$$t = t_2 = 2\pi \Big/ \sqrt{\frac{k}{m}}$$

and at that instant $x = x_2 = x_0 - 4\Delta$. However the rod will stay put unless $x_0 > 5\Delta$. Thus

(a) time period of one full oscillation $= 2\pi \Big/ \sqrt{\frac{k}{m}}$.

(b) There is no oscillation if $0 < x_0 < \Delta$
One half oscillation if $\quad \Delta < x_0 < 3\Delta$

2 half oscillation if $3\,\Delta < x_0 < 5\,\Delta$ etc.

We can say that the number of full oscillations is one half of the integer n

where $$n = \left[\frac{x_0 - \Delta}{2\,\Delta}\right]$$

where $[\,x\,]$ = smallest non-negative integer greater than x.

4.83 The equation of motion of the ball is

$$m\,(\ddot{x} + \omega_0^2 x) = F_0 \cos \omega t$$

This equation has the solution

$$x = A \cos(\omega_0 t + \alpha) + B \cos \omega t$$

where A and α are arbitrary and B is obtained by substitution in the above equation

$$B = \frac{F_0/m}{\omega_0^2 - \omega^2}$$

The conditions $x = 0$, $\dot{x} = 0$ at $t = 0$ give

$$A \cos \alpha + \frac{F_0/m}{\omega_0^2 - \omega^2} = 0 \quad \text{and} \quad -\omega_0 A \sin \alpha = 0$$

This gives $\alpha = 0$, $$A = -\frac{F_0/m}{\omega_0^2 - \omega^2} = \frac{F_0/m}{\omega^2 - \omega_0^2}$$

Finally, $$x = \frac{F_0/m}{\omega^2 - \omega_0^2}\,(\cos \omega_0 t - \cos \omega t)$$

4.84 We have to look for solutions of the equation

$$m\ddot{x} + kx = F,\ 0 < t_1 < \tau,$$

$$m\ddot{x} + kx = 0,\ t > \tau$$

subject to $x(0) = \dot{x}(0) = 0$ where F is constant.

The solution of this equation will be sought in the form

$$x = \frac{F}{k} + A \cos(\omega_0 t + \alpha),\ 0 \le t \le \tau$$

$$x = B \cos(\omega_0 (t - \tau) + \beta),\ t > \tau$$

A and α will be determined from the boundary condition at $t = 0$.

$$0 = \frac{F}{k} + A \cos \alpha$$

$$0 = -\omega_0 A \sin \alpha$$

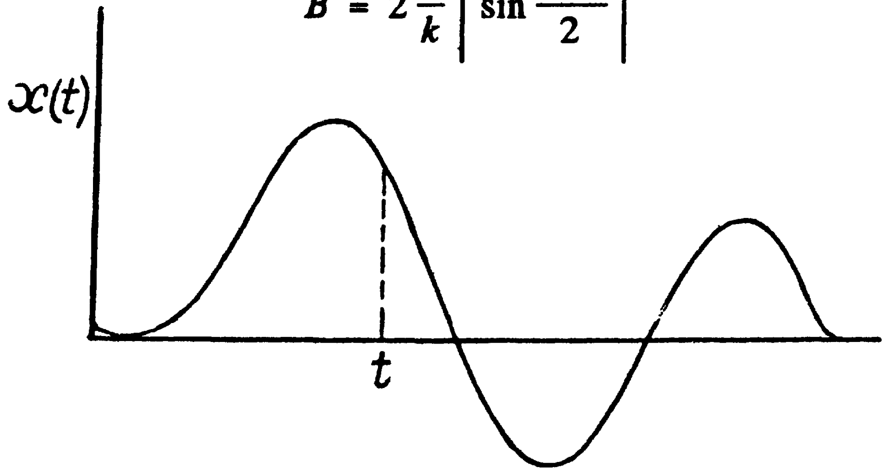
Thus $\alpha = 0$ and $A = -\dfrac{F}{k}$ and $x = \dfrac{F}{k}(1 - \cos \omega_0 t)\ \ 0 \le t < \tau$.

B and β will be determined by the continuity of x and $\dot{x}$ at $t = \tau$. Thus

$$\frac{F}{k}(1 - \cos \omega_0 \tau) = B \cos \beta \quad \text{and} \quad \cancel{\omega_0}\,\frac{F}{k} \sin \omega_0 \tau = -\cancel{\omega_0}\, B \sin \beta$$

Thus
$$B^2 = \left(\frac{F}{k}\right)^2 (2 - 2\cos\omega_0 \tau)$$

or
$$B = 2\frac{F}{k}\left|\sin\frac{\omega_0 \tau}{2}\right|$$



4.85 Foı ıhe spring $mg = \kappa \Delta l$
where κ is its stifness coefficient. Thus

$$\omega_0^2 = \frac{\kappa}{m} = \frac{g}{\Delta l},$$

The equation of motion of the ball is

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

Here
$$\lambda = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}} \quad \text{or} \quad \frac{\beta}{\omega} = \frac{\lambda/2\pi}{\sqrt{1 + (\lambda/2\pi)^2}}$$

To find the solution of the above equation we look for the solution of the auxiliary equation

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = \frac{F_0}{m}e^{i\omega t}$$

Clearly we can take $\operatorname{Re} z = x$. Now we look for a particular integral for z of the form

$$z = A e^{i\omega t}$$

Thus, substitution gives A and we get

$$z = \frac{(F_0/m)e^{i\omega t}}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$

so taking the real part

$$x = \frac{(F_0/m)\left[(\omega_0^2 - \omega^2)\cos\omega t + 2\beta\omega\sin\omega t\right]}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

$$= \frac{F_0}{m}\frac{\cos(\omega t - \varphi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}, \quad \varphi = \tan^{-1}\frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

The amplitude of this oscillation is maximum when the denominator is minimum.
Tthis happens when

$\omega^4 - 2\,\omega_0^2\,\omega^2 + 4\,\beta^2\,\omega^2 + \omega_0^4 = (\,\omega^2 - \omega_0^2 + 2\,\beta^2\,) + 4\,\beta^2\,\omega_0^2 - 4\,\beta^4$ is minimum. i.e for $\omega^2 = \omega_0^2 - 2\,\beta^2$

Thus
$$\omega_{res}^2 = \omega_0^2\left(1 - \frac{2\,\beta^2}{\omega_0^2}\right)$$

$$= \frac{g}{\Delta l}\left[1 - \frac{2\left(\dfrac{\lambda}{2\,\pi}\right)^2}{1 + \left(\dfrac{\lambda}{2\,\pi}\right)^2}\right] = \frac{g}{\Delta l}\,\frac{1 - \left(\dfrac{\lambda}{2\,\pi}\right)^2}{1 + \left(\dfrac{\lambda}{2\,\pi}\right)^2}$$

and
$$a_{res} = \frac{F_0/m}{\sqrt{4\,\beta^2\,\omega_0^2 - 4\,\beta^4}} = \frac{F_0/m}{2\,\beta\sqrt{\omega_0^2 - \beta^2}} = \frac{F_0/m}{2\,\beta^2}\cdot\frac{\lambda}{2\,\pi}$$

$$= \frac{F_0}{2\,m\,\omega_0^2}\cdot\frac{1 + \left(\dfrac{\lambda}{2\,\pi}\right)^2}{\lambda/2\,\pi} = \frac{F_0\,\Delta\,l\,\lambda}{4\,\pi\,m\,g}\left(1 + \frac{4\,\pi^2}{\lambda^2}\right)$$

4.86 Since $a = \dfrac{F_0/m}{\sqrt{(\,\omega^2 - \omega_0^2 + 2\,\beta^2\,)^2 + 4\,\beta^2\,(\,\omega_0^2 - \beta^2\,)}}$

we must have
$$\omega_1^2 - \omega_0^2 + 2\,\beta^2 = -\,(\,\omega_2^2 - \omega_0^2 + 2\,\beta^2\,)$$

or
$$\omega_0^2 - 2\,\beta^2 = \frac{\omega_1^2 + \omega_2^2}{2} = \omega_{res}^2$$

4.87
$$x = \frac{F_0}{m}\,\frac{(\,\omega_0^2 - \omega^2\,)\cos\omega\,t + 2\,\beta\,\omega\sin\omega\,t}{\sqrt{(\,\omega^2 - \omega_0^2\,)^2 + 4\,\beta^2\,\omega^2}}$$

Then
$$\dot{x} = \frac{F_0\,\omega}{m}\,\frac{2\,\beta\,\omega\cos\omega\,t + (\,\omega^2 - \omega_0^2\,)\sin\omega\,t}{(\,\omega_0^2 - \omega^2\,)^2 + 4\,\beta^2\,\omega^2}$$

Thus the velocity amplitude is
$$V_0 = \frac{F_0\,\omega}{m\sqrt{(\,\omega_0^2 - \omega^2\,)^2 + 4\,\beta^2\,\omega^2}}$$

$$= \frac{F_0}{m\sqrt{\left(\dfrac{\omega_0^2}{\omega} - \omega\right)^2 + 4\,\beta^2}}$$

This is maximum when
$$\omega^2 = \omega_0^2 = \omega_{res}^2$$

and then
$$V_{0\,res} = \frac{F_0}{2\,m\,\beta}\,.$$

Now at half maximum $$\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 = 12\,\beta^2$$

or $$\omega^2 \pm 2\sqrt{3}\,\beta\,\omega - \omega_0^2 = 0$$

$$\omega = \mp\ \beta\sqrt{3} + \sqrt{\omega_0^2 + 3\,\beta^2}$$

where we have rejected a solution with $-ve$ sign before there dical. Writing

$$\omega_1 = \sqrt{\omega_0^2 + 3\,\beta^2} + \beta\sqrt{3}\ ,\quad \omega_2 = \sqrt{\omega_0^2 + 3\,\beta^2} - \beta\sqrt{3}$$

we get (a) $\omega_{res} = \omega_0 = \sqrt{\omega_1\,\omega_2}$ (Velocity resonance frequency)

(b) $\beta = \dfrac{|\,\omega_1 - \omega_2\,|}{2\sqrt{3}}$ and damped oscillation frequency

$$\sqrt{\omega_0^2 - \beta^2} = \sqrt{\omega_1\,\omega_2 - \frac{(\,\omega_1 - \omega_2\,)^2}{12}}$$

4.88 In general for displacement amplitude

$$a = \frac{F_0}{m}\,\frac{1}{\sqrt{(\,\omega_0^2 - \omega^2\,)^2 + 4\,\beta^2\,\omega^2}}$$

$$= \frac{F_0}{m}\,\frac{1}{\sqrt{(\,\omega^2 - \omega_0^2 + 2\,\beta^2\,)^2 + 4\,\beta^2\,(\,\omega_0^2 - \beta^2\,)}}$$

Thus $$\eta = \frac{a_{res}}{a_{low}} = \frac{\omega_0^2}{\sqrt{4\,\beta^2\,(\,\omega_0^2 - \beta^2\,)}} = \frac{\omega_0^2}{2\,\beta\sqrt{\omega_0^2 - \beta^2}}$$

But $$\frac{\beta}{\omega_0} = \frac{\lambda/2\,\pi}{\sqrt{1 + (\,\lambda/2\,\pi\,)^2}}\ ,\quad \frac{\lambda}{2\,\pi} = \frac{\beta}{\sqrt{\omega_0^2 - \beta^2}}$$

Hence $$\eta = \frac{\omega_0^2}{2\,\beta^2}\cdot\frac{\lambda}{2\,\pi} = \frac{1}{2}\,\frac{1 + \left(\dfrac{\lambda}{2\,\pi}\right)^2}{\dfrac{\lambda}{2\,\pi}} = 2.90$$

4.89 The work done in one cycle is

$$A = \int F\,dx = \int_0^T F\,\mathrm{v}\,dt = \int_0^T F_0\cos\omega\,t\,(\,-\,\omega\,a\sin(\,\omega\,t - \varphi\,)\,)\,dt$$

$$= \int_0^T F_0\,\omega\,a\,(\,-\cos\omega\,t\sin\omega\,t\cos\varphi + \cos^2\omega\,t\sin\varphi\,)\,dt$$

$$= \frac{1}{2}\,F_0\,\omega\,a\,\frac{T}{2}\sin\varphi = \pi\,a\,F_0\sin\varphi$$

4.90 In the formula $x = a\cos(\omega t - \varphi)$

we have
$$a = \frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$$
$$\tan\varphi = \frac{2\beta\omega}{\omega_0^2-\omega^2}$$

Thus
$$\beta = \frac{(\omega_0^2-\omega^2)\tan\varphi}{2\omega}.$$

Hence
$$\omega_0 = \sqrt{K/m} = 20\ \mathrm{s}^{-1}.$$

and (a) the quality factor
$$Q = \frac{\pi}{\beta T} = \frac{\sqrt{\omega_0^2-\beta^2}}{2\beta} = \frac{1}{2}\sqrt{\frac{4\omega^2\omega_0^2}{(\omega_0^2-\omega^2)^2\tan^2\varphi}-1} = 2.17$$

(b) work done is $A = \pi a F_0 \sin\varphi$
$$= \pi m a^2\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}\sin\varphi = \pi m a^2\times 2\beta\omega$$
$$= \pi m a^2(\omega_0^2-\omega^2)\tan\varphi = 6\ \mathrm{mJ}.$$

4.91 Here as usual $\tan\varphi = \dfrac{2\beta\omega}{\omega_0^2-\omega^2}$ where φ is the phase lag of the displacement
$$x = a\cos(\omega t - \varphi),\ a = \frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$$

(a) Mean power developed by the force over one oscillation period
$$= \frac{\pi F_0 a\sin\varphi}{T} = \frac{1}{2}F_0 a\omega\sin\varphi$$
$$= \frac{F_0^2}{m}\frac{\beta\omega^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2} = \frac{F_0^2\beta}{m}\frac{1}{\left(\dfrac{\omega_0^2}{\omega}-\omega\right)^2+4\beta^2}$$

(b) Mean power $\langle P\rangle$ is maximum when $\omega = \omega_0$ (for the denominator is then minimum
Also
$$\langle P\rangle_{\max} = \frac{F_0^2}{4m\beta}$$

4.92 Given $\beta = \omega_0/\eta$. Then from the previous problem
$$\langle P\rangle = \frac{F_0^2\omega_0}{\eta m}\cdot\frac{1}{\left(\dfrac{\omega_0^2}{\omega}-\omega\right)^2+4\dfrac{\omega_0^2}{\eta^2}}$$

At displacement resonance $\omega = \sqrt{\omega_0^2 - 2\beta^2}$

$$<P>_{res} = \frac{F_0^2\,\omega_0}{\eta\, m} \frac{1}{\dfrac{4\beta^4}{\omega_0^2 - 2\beta^2} + \dfrac{4\omega_0^2}{\eta^2}} = \frac{F_0^2\,\omega_0}{\eta\, m} \frac{1}{\dfrac{4\omega_0^4/\eta^4}{\omega_0^2\left(1 - \dfrac{2}{\eta^2}\right)} + 4\dfrac{\omega_0^2}{\eta^2}}$$

$$= \frac{F_0^2}{4\eta\, m\,\omega_0} \frac{\eta^2}{\dfrac{1}{\eta^2 - 2} + 1} = \frac{F_0^2\,\eta}{4\, m\,\omega_0} \frac{\eta^2 - 2}{\eta^2 - 1}$$

while
$$<P>_{\max} = \frac{F_0^2\,\eta}{4\, m\,\omega_0}.$$

Thus
$$\frac{<P>_{\max} - <P>_{res}}{<P>_{\max}} = \frac{100}{\eta^2 - 1}\ \%$$

4.93 The equation of the disc is $\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2\varphi = \dfrac{N_m \cos\omega t}{I}$

Then as before
$$\varphi = \varphi_m \cos(\omega t - \alpha)$$

where
$$\varphi_m = \frac{N_m}{I[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]^{1/2}},\quad \tan\alpha = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

(a) Work performed by frictional forces

$$= -\int N_r\, d\varphi \quad \text{where } N_r = -2I\beta\dot{\varphi} = -\int_0^T 2\beta I\dot{\varphi}^2\, dt = -2\pi\beta\omega I\varphi_m^2$$

$$= -\pi I\varphi_m^2[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]^{1/2}\sin\alpha = -\pi N_m\varphi_m\sin\alpha$$

(b) The quality factor

$$Q = \frac{\pi}{\lambda} = \frac{\pi}{\beta T} = \frac{\sqrt{\omega_0^2 - \beta^2}}{2\beta} = \frac{\omega\sqrt{\omega_0^2 - \beta^2}}{(\omega_0^2 - \omega^2)\tan\alpha} = \frac{1}{2\tan\alpha}\left\{\frac{4\omega^2\omega_0^2}{(\omega_0^2 - \omega^2)^2} - \frac{4\beta^2\omega^2}{(\omega_0^2 - \omega^2)^2}\right\}^{1/2}$$

$$= \frac{1}{2\tan\alpha}\left\{\frac{4\omega^2\omega_0^2 I^2\varphi_m^2}{N_m^2\cos^2\alpha} - \tan^2\alpha\right\}^{1/2} \quad \text{since} \quad \omega_0^2 = \omega^2 + \frac{N_m}{I\varphi_m}\cos\alpha$$

$$= \frac{1}{2\sin\alpha}\left\{\frac{4\omega^2\omega_0^2 I^2\varphi_m^2}{N_m^2} - \sin^2\alpha\right\}^{1/2}$$

$$= \frac{1}{2\sin\alpha}\left\{\frac{4\omega^2 I^2\varphi_m^2}{N_m^2}\left(\omega^2 + \frac{N_m\cos\alpha}{I\varphi_m}\right) + 1 - \cos^2\alpha\right\}^{1/2}$$

$$= \frac{1}{2\sin\alpha}\left\{\frac{4I^2\varphi_m^2}{N_m^2}\omega^4 + \frac{4I\varphi_m}{N_m}\omega^2\cos\alpha + \cos^2\alpha - 1\right\}^{1/2} = \frac{1}{2\sin\alpha}\left\{\left(\frac{2I\varphi_m\omega^2}{N_m} + \cos\alpha\right)^2 - 1\right\}^{1/2}$$

4.2 ELECTRIC OSCILLATIONS

4.94 If the electron (charge of each electron = $-e$) are shifted by a sinau distance x, a net $+ve$ charge density (per unit area) is induced on the surface. This will result in an electric field $E = n e x/\varepsilon_0$ in the direction of x and a restoring force on an electron of

$$-\frac{n e^2 x}{\varepsilon_0},$$

Thus $$m \ddot{x} = -\frac{n e^2 x}{\varepsilon_0}$$

or $$\ddot{x} + \frac{n e^2}{m \varepsilon_0} x = 0$$

This gives $$\omega_p = \sqrt{\frac{n e^2}{m \varepsilon_0}} = 1.645 \times 10^{16} \text{ s}^{-1}.$$

as the plasma frequency for the problem.

4.95 Since there are no sources of emf in the circuit, Ohm's 1 law reads

$$\frac{q}{c} = -L \frac{dI}{dt}$$

where q = change on the capacitor, $I = \frac{dq}{dt}$ = current through the coil. Then

$$\frac{d^2 q}{d t^2} + \omega_0^2 q = 0, \ \omega_0^2 = \frac{1}{L C}.$$

The solution fo this equation is

$$q = q_m \cos(\omega_0 t + \alpha)$$

From the problem $V_m = \frac{q_m}{C}$. Then

$$I = -\omega_0 C V_m \sin(\omega_0 t + \alpha)$$

and $$V = V_m \cos(\omega_0 t - \alpha)$$

$$V^2 + \frac{I^2}{\omega_0^2 C^2} = V_m^2$$

or $$V^2 + \frac{L I^2}{C} = V_m^2.$$

By energy conservation $$\frac{1}{2} L I^2 + \frac{q^2}{2 C} = \text{constant}$$

When the P.D. across the capacitor takes its maximum value V_m, the current I must be zero.

Thus "constant" $= \frac{1}{2} C V_m^2$

Hence $$\frac{L I^2}{C} + V^2 = V_m^2 \text{ once again.}$$

4.96 After the switch was closed, the circuit satisfies

$$-L\frac{dI}{dt} = \frac{q}{C}$$

or
$$\frac{d^2q}{dt^2} + \omega_0^2 q = 0 \Rightarrow q = C\,V_m \cos\omega_0 t$$

where we have used the fact that when the switch is closed we must have

$$V = \frac{q}{C} = V_m,\ I = \frac{dq}{dt} = 0 \text{ at } t = 0.$$

Thus (a)
$$I = \frac{dq}{dt} = -C\,V_m\,\omega_0 \sin\omega_0 t$$

$$= -V_m\sqrt{\frac{C}{L}}\sin\omega_0 t$$

(b) The electrical energy of the capacitor is $\frac{q^2}{2C}\ \alpha \cos^2\omega_0 t$ and of the inductor is

$$\frac{1}{2}L I^2\ \alpha \sin^2\omega_0 t.$$

The two are equal when

$$\omega_0 t = \frac{\pi}{4}$$

At that instant the emf of the self-inductance is

$$-L\frac{di}{dt} = V_m \cos\omega_0 t = V_m/\sqrt{2}$$

4.97 In the oscillating circuit, let

$$q = q_m \cos\omega t$$

be the change on the condenser where

$\omega^2 = \frac{1}{LC}$ and C is the instantaneous capacity of the condenser (S = area of plates)

$$C = \frac{\varepsilon_0 S}{y}$$

y = distance between the plates. Since the oscillation frequency increases η fold, the quantity

$$\omega^2 = \frac{y}{\varepsilon_0 S L}$$

changes η^2 fold and so does y i.e. changes from y_0 initially to $\eta^2 y_0$ finally. Now the P.D. across the condenser is

$$V = \frac{q_m}{C}\cos\omega t = \frac{y\,q_m}{\varepsilon_0 S}\cos\omega t$$

and hence the electric field between the plates is

$$E = \frac{q_m}{\varepsilon_0 S}\cos\omega t$$

Thus, the charge on the plate being $q_m \cos \omega t$, the force on the plate is

$$F = \frac{q_m^2}{\varepsilon_0 S} \cos^2 \omega t$$

Since this force is always positive and the plate is pulled slowly we can use the average force

$$\overline{F} = \frac{q_m^2}{2 \varepsilon_0 S}$$

and work done is $$A = \overline{F} (\eta^2 y_0 - y_0) = (\eta^2 - 1) \frac{q_m^2 y_0}{2 \varepsilon_0 S}$$

But $\dfrac{q_m^2 y_0}{2 \varepsilon_0 S} = \dfrac{q_m^2}{2 C_0} = W$ the initial stored energy. Thus.

$$A = (\eta^2 - 1) W .$$

4.98 The equations of the $L - C$ circuit are

$$L \frac{d}{dt} (I_1 + I_) = \frac{C_1 V - \int I_1 \, dt}{C_1} = \frac{C_2 V - \int I_2 \, dt}{C_2}$$

Differentiating again $$L (I_1 + I_2) = - \frac{1}{C_1} I_1 = - \frac{1}{C_2} I_2$$

Then $$I_1 = \frac{C_1}{C_1 - C_2} I , \; I_2 = \frac{C_2}{C_1 + C_2} I ,$$

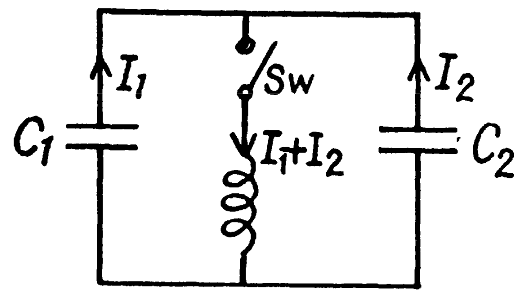
$$I = I_1 + I_2$$

so $\quad L (C_1 + C_2) I + I = 0$

or $\quad I = I_0 \sin (\omega_0 t + \alpha)$

where $\quad \omega_0^2 = \dfrac{1}{L (C_1 + C_2)}$ (Part a)

(Hence $T = \dfrac{2 \pi}{\omega_0} = 0.7$ ms)



At $\quad t = 0 , I = 0$ so $\alpha = 0$

$$I = I_0 \sin \omega_0 t$$

The peak value of the current is I_0 and it is related to the voltage V by the first equation

$$L I = V - \int I \, dt / (C_1 + C_2)$$

or $$+ L \, \omega_0 I_0 \cos \omega_0 t = V - \frac{1}{C_1 + C_2} \int_0^t I_0 \sin \omega_0 t \, dt$$

(The P.D. across the inductance is V at $t = 0$)

$$= V + \frac{1}{C_1 + C_2} \cdot \frac{I_0}{\omega_0} (\cos \omega_0 t - 1)$$

Hence $$I_0 = (C_1 + C_2) \omega_0 V = V \sqrt{\frac{C_1 + C_2}{L}} = 8.05 \text{ A.}$$

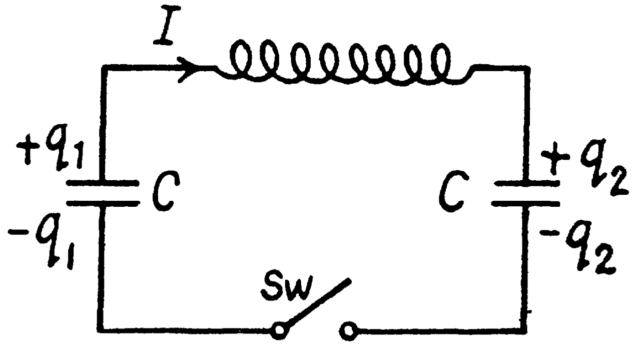
4.99 Initially $q_1 = C V_0$ and $q_2 = 0$. After the switch is closed change flows and we get

$$q_1 + q_2 = C V_0$$

$$\frac{q_1}{C} + L\frac{dI}{dt} - \frac{q_2}{C} = 0 \qquad (1)$$

Also $I = \dot{q}_1 = -\dot{q}_2$. Thus

$$L\ddot{I} + \frac{2I}{C} = 0$$



Hence $\ddot{I} + \omega_0^2 I = 0 \quad \omega_0^2 = \dfrac{2}{LC}$,

The soloution of this equation subject to

$I = 0$ at $t = 0$

is $I = I_0 \sin \omega_0 t$.

Integrating
$$q_1 = A - \frac{I_0}{\omega_0}\cos\omega_0 t$$

$$q_2 = B + \frac{I_0}{\omega_0}\cos\omega_0 t$$

Finally substituting in (1)

$$\frac{A-B}{C} - \frac{2I_0}{\omega_0 C}\cos\omega_0 t + L I_0 \omega_0 \cos\omega_0 t = 0$$

Thus
$$A = B = \frac{C V_0}{2} \text{ and}$$

$$\frac{C V_0}{2} + \frac{I_0}{\omega_0} = 0$$

so
$$q_1 = \frac{C V_0}{2}(1 + \cos\omega_0 t)$$

$$q_2 = \frac{C V_0}{2}(1 - \cos\omega_0 t)$$

4.100 The flux in the coil is

$$\Phi(t) = \begin{cases} \Phi & t < 0 \\ 0 & t > 0 \end{cases}$$

The equation of the current is
$$-L\frac{dI}{dt} = \frac{\int_0^t I\,dt}{C} \qquad (1)$$

This mean that
$$LC\frac{d^2 I}{dt^2} + I = 0$$

or with $\omega_0^2 = \dfrac{1}{LC}$
$$I = I_0 \sin(\omega_0 t + \alpha)$$

Putting in (1) $\quad -L I_0 \omega_0 \cos(\omega_0 t + \alpha) = -\dfrac{I_0}{\omega_0 C}\left[\cos(\omega_0 t + \alpha) - \cos\alpha\right]$

This implies $\cos\alpha = 0 \quad \therefore \quad I = \pm I_0 \cos\omega_0 t$. From Faraday's law

$$\varepsilon = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$

or integrating from $t = -\varepsilon$ to $-\varepsilon$ where $\varepsilon \to 0$

$$\Phi = L I_0 \text{ with + sign in } I$$

so,
$$I = \frac{\Phi}{L}\cos\omega_0 t\,.$$

4.101 Given $V = V_m e^{-\beta t}\cos\omega t$

(a) The phrase 'peak values' is not clear. The answer is obtained on taking $|\cos\omega t| = 1$

i.e
$$t = \frac{\pi n}{\omega}.$$

(b) For extrema $\dfrac{dV}{dt} = 0$

$$-\beta\cos\omega t - \omega\sin\omega t = 0$$

or
$$\tan\omega t = -\beta/\omega$$

i.e.
$$\omega t = n\pi + \tan^{-1}\left(\frac{-\beta}{\omega}\right).$$

4.102 The equation of the circuit is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

where Q = charge on the capacitor,

This has the solution $\quad Q = Q_m e^{-\beta t}\sin(\omega t + \alpha)$

where
$$\beta = \frac{R}{2L},\ \omega = \sqrt{\omega_0^2 - \beta^2},\ \omega_0^2 = \frac{1}{LC}.$$

Now
$$I = \frac{dQ}{dt} = 0 \quad \text{at } t = 0$$

so,
$$Q_m e^{-\beta t}(-\beta\sin(\omega t + \alpha) + \omega\cos(\omega t + \alpha)) = 0 \quad \text{at } t = 0$$

Thus
$$\omega\cos\alpha = \beta\sin\alpha \quad \text{or} \quad \alpha = \tan^{-1}\frac{\omega}{\beta}$$

Now
$$V_m = \frac{Q_m}{C} \quad \text{and} \quad V_0 = \text{P.D. at } t = 0 = \frac{Q_m}{C}\sin\alpha$$

$$\therefore \quad \frac{V_0}{V_m} = \sin\alpha = \frac{\omega}{\sqrt{\omega^2 + \beta^2}} = \frac{\omega}{\omega_0} = \sqrt{1 - \beta^2/\omega_0^2} = \sqrt{1 - \frac{R^2C}{4L^2}}$$

4.103 We write

$$-\frac{dQ}{dt} = I = I_m\, e^{-\beta t} \sin \omega t$$

$$= gm\; I_m\, e^{-\beta t + i \omega t} \qquad (gm \text{ means imaginary part})$$

Then

$$Q = gm\; I_m \frac{e^{-\beta t + i\omega t}}{-\beta + i\omega}$$

$$Q = gm\; I_m \frac{e^{-\beta t + i\omega t}}{\beta - i\omega}$$

$$= gm\; I_m \frac{(\beta + i\omega)\, e^{-\beta t + i\omega t}}{\beta^2 + \omega^2}$$

$$= I_m\, e^{-\beta t} \frac{\beta \sin \omega t + \omega \cos \omega t}{\beta^2 + \omega^2}$$

$$= I_m\, e^{-\beta t} \frac{\sin(\omega t + \delta)}{\sqrt{\beta^2 + \omega^2}}, \quad \tan \delta = \frac{\omega}{\beta}.$$

(An arbitrary constant of integration has been put equal to zero.)

Thus

$$V = \frac{Q}{C} = I_m \sqrt{\frac{L}{C}}\, e^{-\beta t} \sin(\omega t + \delta)$$

$$V(0) = I_m \sqrt{\frac{L}{C}} \sin \delta = I_m \sqrt{\frac{L}{C}} \frac{\omega}{\sqrt{\omega^2 + \beta^2}}$$

$$= I_m \sqrt{\frac{L}{C(1 + \beta^2/\omega^2)}}.$$

4.104 $I = I_m\, e^{-\beta t} \sin \omega t$

$$\beta = \frac{R}{2L}, \quad \omega_0 = \sqrt{\frac{1}{LC}}, \quad \omega = \sqrt{\omega_0^2 - \beta^2}$$

$I = -\dot{q}$, $q =$ charge on the capacitor

Then
$$q = I_m\, e^{-\beta t} \frac{\sin(\omega t + \delta)}{\sqrt{\omega^2 + \beta^2}}, \quad \tan \delta = \frac{\omega}{\beta}.$$

Thus
$$W_M = \frac{1}{2} L\, I_m^2\, e^{-2\beta t} \sin^2 \omega t$$

$$W_E = \frac{I_m^2}{2C} \frac{e^{-2\beta t} \sin^2(\omega t + \delta)}{\omega^2 + \beta^2} = \frac{L I_m^2}{2} e^{-2\beta t} \sin^2(\omega t + \delta)$$

Current is maximum when $$\frac{d}{dt}e^{-\beta t}\sin\omega t = 0$$

Thus $$-\beta\sin\omega t + \omega\cos\omega t = 0$$

or $$\tan\omega t = \frac{\omega}{\beta} = \tan\delta$$

i.e. $$\omega t = n\pi + \delta$$

and hence $$\frac{W_M}{W_E} = \frac{\sin^2(\omega t)}{\sin^2(\omega t + \delta)} = \frac{\sin^2\delta}{\sin^2 2\delta} = \frac{1}{4\cos^2\delta}$$

$$= \frac{1}{4\beta^2/\omega_0^2} = \frac{\omega_0^2}{4\beta^2} = \frac{1}{LC}\times\frac{L^2}{R^2} = \frac{L}{CR^2} = 5.$$

(W_M is the magnetic energy of the inductance coil and W_E is the electric energy of t capacitor.)

4.105 Clearly

$$L = L_1 + L_2,\ R = R_1 + R_2$$

4.106 $Q = \frac{\pi}{\beta T}$ or $\beta = \frac{\pi}{QT}$

Now $$\beta t = \ln\eta \quad \text{so} \quad t = \frac{\ln\eta}{\pi}QT$$

$$= \frac{Q\ln\eta}{\pi\nu} = 0.5\ \text{ms}$$

4.107 Current decreases e fold in time

$$t = \frac{1}{\beta} = \frac{2L}{R}\ \text{sec} = \frac{2L}{RT}\ \text{oscillations}$$

$$= \frac{2L}{R}\frac{\omega}{2\pi}$$

$$= \frac{L}{\pi R}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{1}{2\pi}\sqrt{\frac{4L}{R^2C} - 1} = 15.9\ \text{oscillations}$$

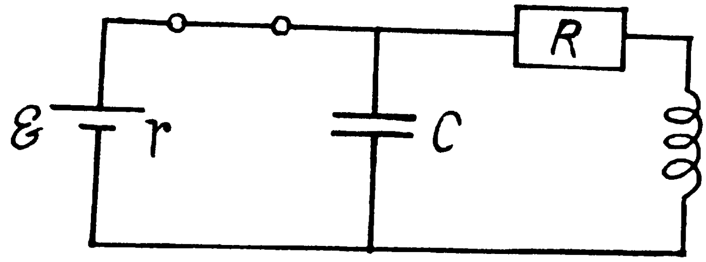
4.108 $Q = \frac{\pi}{\beta T} = \frac{\omega}{2\beta}$

$$\therefore\ \omega = 2\beta Q,\ \beta = \frac{\omega}{2Q}.$$

Now $$\omega_0 = \omega\sqrt{1 + \frac{1}{4Q^2}} \quad \text{or} \quad \omega = \frac{\omega_0}{\sqrt{1 + \frac{1}{4Q^2}}}$$

so $$\left|\frac{\omega_0 - \omega}{\omega_0}\right| \times 100\ \% = \frac{1}{8Q^2}\times 100\ \% = 0.5\%$$

4.109



At $t = 0$ current through the coil $= \dfrac{\varepsilon}{R + r}$

P.D. across the condenser $= \dfrac{\varepsilon}{R + r}$

(a) At $t = 0$, energy stored $= W_0$

$$= \frac{1}{2} L \left(\frac{\varepsilon}{R + r} \right)^2 + \frac{1}{2} C \left(\frac{\varepsilon R}{R + r} \right)^2 = \frac{1}{2} \varepsilon^2 \frac{(L + C R^2)}{(R + r)^2} = 2.0 \text{ mJ.}$$

(b) The current and the change stored decrease as $e^{-tR/2L}$ so energy decreases as $e^{-tR/L}$

$$\therefore \quad W = W_0 \, e^{-tR/L} = 0.10 \text{ mJ.}$$

4.110 $Q = \dfrac{\pi}{\beta T} = \dfrac{\pi \nu}{\beta} = \dfrac{\omega}{2 \beta} = \dfrac{\sqrt{\omega_0^2 - \beta^2}}{2 \beta}$

or $$\frac{\omega_0}{\beta} = \sqrt{1 + 4 Q^2} \quad \text{or} \quad \beta = \frac{\omega_0}{\sqrt{1 + Q^2}}$$

Now $$W = W_0 \, e^{-2 \beta t}$$

Thus energy decreases η times in $\dfrac{\ln \eta}{2 \beta}$ sec.

$$= \ln \eta \, \frac{\sqrt{1 + 4 Q^2}}{2 \omega_0} \approx \frac{Q \ln \eta}{2 \pi \nu_0} \text{ sec.} = 1.033 \text{ ms.}$$

4.111 In a leaky condenser

$$\frac{dq}{dt} = I - I' \quad \text{where} \quad I' = \frac{V}{R} = \text{leak current}$$

Now $$V = \frac{q}{C} = - L \frac{dI}{dt} = - L \frac{d}{dt} \left(\frac{dq}{dt} + \frac{V}{R} \right)$$

$$= - L \frac{d^2 q}{d t^2} - \frac{L}{R C} \frac{d q}{d t}$$

or $$\ddot{q} + \frac{1}{R C} \frac{d Q}{d t} + \frac{1}{L C} q = 0$$

Then $$q = q_m e^{-\beta t} \sin (\omega t + \alpha)$$

(a) $\beta = \frac{1}{2RC}$, $\omega_0^2 = \frac{1}{LC}$, $\omega = \sqrt{\omega_0^2 - \beta^2}$

$$= \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$$

(b) $$Q = \frac{\omega}{2\beta} = RC\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$$

$$= \frac{1}{2}\sqrt{\frac{4CR^2}{L} - 1}$$

4.112 Given $V = V_m e^{-\beta t} \sin \omega t$, $\omega \approx \omega_0$ $\beta T << 1$

$$\text{Power loss} = \frac{\text{Energy loss per cycle}}{T}$$

$$\approx \frac{1}{2} C V_m^2 \times 2\beta$$

(energy decreases as $W_0 e^{-2\beta t}$ so loss per cycle is $W_0 \times 2\beta T$)

Thus $$<P> = \frac{1}{2} C V_m^2 \times \frac{R}{L}$$

or $$R = \frac{2<P>}{V_m^2} \frac{L}{C}$$

Hence $$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \sqrt{\frac{C}{L}} \frac{V_m^2}{2<P>} = 100$$ on putting the vales.

4.113 Energy is lost across the resistance and the mean power lass is

$$<P> = R<I^2> = \frac{1}{2} R I_m^2 = 0.2 \text{ mW.}$$

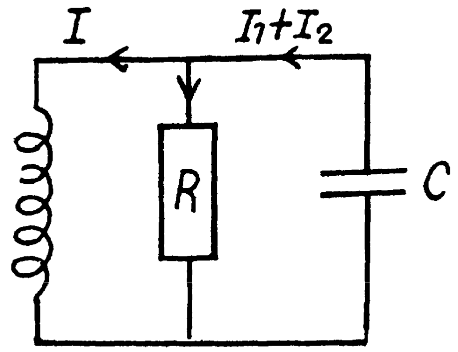
This power should be fed to the circuit to maintain undamped oscillations.

4.114 $<P> = \frac{RCV_m^2}{2L}$ as in (4.112). We get $<P> = 5$ mW.

4.115 Given $q = q_1 + q_2$

$I_1 = -\dot{q}_1$, $I_2 = -\dot{q}_2$

$L\dot{I}_1 = RI_2 = \frac{q}{C}$.



Thus $CL\ddot{q}_1 + (q_1 + q_2) = 0$

$RC\dot{q}_2 + q_1 + q_2 = 0$

Putting $q_1 = Ae^{i\omega t}$ $q_2 = Be^{+i\omega t}$

$$(1 - \omega^2 LC)A + B = 0$$

$$A + (1 + i\omega RC)B = 0$$

A solution exists only if

$$(1 - \omega^2 LC)(1 + i\omega RC) = 1$$

or

$$i\omega RC - \omega^2 LC - i\omega^3 LRC^2 = 0$$

or

$$LRC^2\omega^2 - i\omega LC - RC = 0$$

$$\omega^2 - i\omega\frac{1}{RC} - \frac{1}{LC} = 0$$

$$\omega = \frac{i}{2RC} \pm \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}} = i\beta \pm \omega_0$$

Thus $\quad q_1 = (A_1 \cos\omega_0 t + A_2 \sin\omega_0 t)\, e^{-\beta t}$ etc.

ω_0 is the oscillation frequency. Oscillations are possible only if $\omega_0^2 > 0$

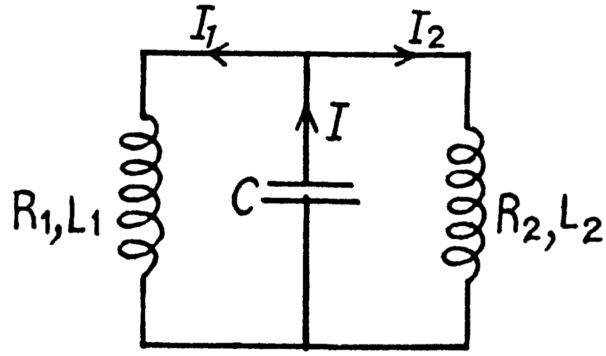
i.e. $$\frac{1}{4R^2} < \frac{C}{L}.$$

4.116 We have

$$L_1 \dot{I}_1 + R_1 I_1 = L_2 \dot{I}_2 + R_2 I_2$$

$$= -\frac{\int I\,dt}{C}$$

$$I = I_1 + I_2$$



Then differentiating we have the equations

$$L_1 C \ddot{I}_1 + R_1 C \dot{I}_1 + (I_1 + I_2) = 0$$

$$L_2 C \ddot{I}_2 + R_2 C \dot{I}_2 + (I_1 + I_2) = 0$$

Look for a solution

$$I_1 = A_1 e^{\alpha t},\; I_2 = A_2 e^{\alpha t}$$

Then $$(1 + \alpha^2 L_1 C + \alpha R_1 C)A_1 + A_2 = 0$$

$$A_1 + (1 + \alpha^2 L_2 C + \alpha R_2 C)A_2 = 0$$

This set of simultaneous equations has a nontrivial solution only if

$$(1 + \alpha^2 L_1 C + \alpha R_1 C)(1 + \alpha^2 L_2 C + \alpha R_2 C) = 1$$

or $$\alpha^3 + \alpha^2\frac{L_1 R_2 + L_2 R_1}{L_1 L_2} + \alpha\frac{L_1 + L_2 + R_1 R_2 C}{L_1 L_2 C} + \frac{R_1 + R_2}{L_1 L_2 C} = 0$$

This cubic equation has one real root which we ignore and two complex conjugate roots. We require the condition that this pair of complex conjugate roots is identical with the roots of the equation

$$\alpha^2 LC + \alpha RC + 1 = 0$$

The general solution of this problem is not easy.We look for special cases. If $R_1 = R_2 = 0$, then $R = 0$ and $L = \dfrac{L_1 L_2}{L_1 + L_2}$. If $L_1 = L_2 = 0$, then

$L = 0$ and $R = R_1 R_2 / (R_1 + R_2)$ .These are the quoted solution but they are misleading. We shall give the solution for small R_1, R_2 . Then we put $\alpha = -\beta + i\omega$ when β is small

We get $(1 - \omega^2 L_1 C - 2 i \beta \omega L_1 C - \beta \cancel{R}_1 C + i \omega R_1 C)$

$$(1 - \omega^2 L_2 C - 2 i \beta \omega L_2 C - \beta \cancel{R}_2 C + i \omega R_2 C) = 1$$

(we neglect β^2 & βR_1, βR_2). Then

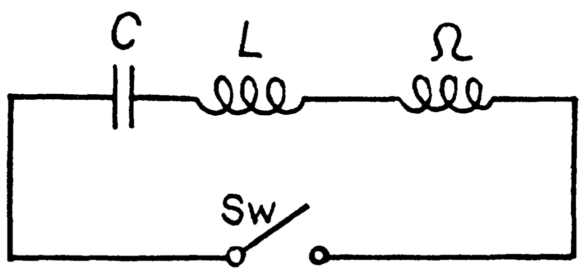
$$(1 - \omega^2 L_1 C)(1 - \omega^2 L_2 C) = 1 \Rightarrow \omega^2 = \frac{L_1 + L_2}{L_1 L_2 C}$$

This is identical with $\omega^2 = \dfrac{1}{LC}$ if $L = \dfrac{L_1 L_2}{L_1 + L_2}$.

also $\quad (2\beta L_1 - R_1)(1 - \omega^2 L_2 C) + (2\beta L_2 - R_2)(1 - \omega^2 L_2 C) = 0$

This gives $\beta = \dfrac{R}{2L} = \dfrac{R_1 L_2^2 + R_2 L_1^2}{2 L_1 L_2 (L_1 + L_2)} \Rightarrow R = \dfrac{R_1 L_2^2 + R_2 L_1^2}{(L_1 + L_2)^2}$.

4.117 $o = \dfrac{q}{C} + L\dfrac{dI}{dt} + RI$, $I = +\dfrac{dq}{dt}$



For the critical case $R = 2\sqrt{\dfrac{L}{C}}$

Thus $LC\,\ddot{q} + 2\sqrt{LC}\,\dot{q} + q = 0$

Look for a solution with $q \propto e^{\alpha t}$

$$\alpha = -\frac{1}{\sqrt{LC}}.$$

An independent solution is $t e^{\alpha t}$. Thus

$$q = (A + Bt)\, e^{-t/\sqrt{LC}},$$

At $\quad t = 0$ $q = C V_0$ thus $A = C V_0$

Also at $\quad t = 0$ $\dot{q} = I = 0$

$$0 = B - A\frac{1}{\sqrt{IC}} \Rightarrow B = V_0\sqrt{\frac{C}{L}}$$

Thus finally
$$I = \frac{dq}{dt} = V_0 \sqrt{\frac{C}{L}}\; e^{-t/\sqrt{LC}}$$
$$- \frac{1}{\sqrt{LC}}\left(C V_0 + V_0 \sqrt{\frac{C}{L}}\, t\right) e^{-t/\sqrt{LC}}$$
$$= - \frac{V_0}{L} t\, e^{-t/\sqrt{LC}}$$

The current has been defined to increase the charge. Hence the minus sign.

The current is maximum when
$$\frac{dI}{dt} = -\frac{V_0}{L}\, e^{-t/\sqrt{LC}} \left(1 - \frac{t}{\sqrt{LC}}\right) = 0$$
This gives $t = \sqrt{LC}$ and the magnitude of the maximum current is
$$|I_{\max}| = \frac{V_0}{e}\sqrt{\frac{C}{L}}.$$

4.118 The equation of the circuit is (I is the current)
$$L \frac{dI}{dt} + RI = V_m \cos \omega t$$

From the theory of differential equations
$$I = I_P + I_C$$
where I_P is a particular integral and I_C is the complementary function (Solution of the differential equation with the RHS = 0). Now
$$I_C = I_{CO}\, e^{-tR/L}$$
and for I_P we write $I_P = I_m \cos(\omega t - \varphi)$

Substituting we get
$$I_m = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}, \quad \varphi = \tan^{-1}\frac{\omega L}{R}$$

Thus
$$I_m = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t - \varphi) + I_{CO}\, e^{-tR/L}$$

Now in an inductive circuit $I = 0$ at $t = 0$ because a current cannot change suddenly.

Thus
$$I_{CO} = -\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos \varphi$$

and so
$$I = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \left[\cos(\omega t - \varphi) - \cos \varphi\, e^{-tR/L}\right]$$

4.119 Here the equation is (Q is charge on the capacitor)

$$\frac{Q}{C} + R\frac{dQ}{dt} = V_m \cos \omega t$$

A solution subject to $Q = 0$ at $t = 0$ is of the form (as in the previous problem)

$$Q = Q_m\left[\cos(\omega t - \overline{\varphi}) - \cos\overline{\varphi}\, e^{-t/RC}\right]$$

Substituting back

$$\frac{Q_m}{C}\cos(\omega t - \overline{\varphi}) - \omega R Q_m \sin(\omega t - \overline{\varphi})$$

$$= V_m \cos \omega t$$

$$= V_m\left\{\cos\overline{\varphi}\cos(\omega t - \overline{\varphi}) - \sin\overline{\varphi}\sin(\omega t - \overline{\varphi})\right\}$$

so

$$Q_m = C V_m \cos\overline{\varphi}$$

$$\omega R Q_m = V_m \sin\overline{\varphi}$$

This leads to

$$Q_m = \frac{C V_m}{\sqrt{1 + (\omega R C)^2}}, \quad \tan\overline{\varphi} = \omega R C$$

Hence

$$I = \frac{dQ}{dt} = \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\left[-\sin(\omega t - \overline{\varphi}) + \frac{\cos^2\overline{\varphi}}{\sin\overline{\varphi}}\, e^{-t/RC}\right]$$

The solution given in the book satisfies $I = 0$ at $t = 0$. Then $Q = 0$ at $t = 0$ but this will not satisfy the equation at $t = 0$. Thus $I \neq 0$, (Equation will be satisfied with $I = 0$ only if $Q \neq 0$ at $t = 0$)

With our I, $$I(t = 0) = \frac{V_m}{R}$$

4.120 The current lags behind the voltage by the phase angle

$$\varphi = \tan^{-1}\frac{\omega L}{R}$$

Now $L = \mu_0 n^2 \pi a^2 l$, $\quad l$ = length of the solenoid

$R = \dfrac{\rho \cdot 2\pi a n \cdot l}{\pi b^2}$, $2b$ = diameter of the wire

But $$2bn = 1 \qquad \therefore \quad b = \frac{1}{2n}$$

Then $$\varphi = \tan^{-1}\frac{\mu_0 n^2 l \pi a^2 \cdot 2\pi\nu}{\rho \cdot 2\pi a n l} \times \pi\frac{1}{4n^2}$$

$$= \tan^{-1}\frac{\mu_0 \pi^2 a \nu}{4\rho n}.$$

4.121 Here $V = V_m \cos \omega t$

$I = I_m \cos(\omega t + \varphi)$

where $$I_m = \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}, \quad \tan\varphi = \frac{1}{\omega R C}$$

Now $$R^2 + \frac{1}{(\omega C)^2} = \left(\frac{V_m}{I_m}\right)^2$$

$$\frac{1}{\omega R C} = \sqrt{\left(\frac{V_m}{R I_m}\right)^2 - 1}$$

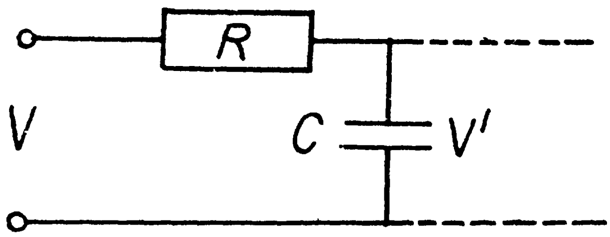
Thus the current is ahead of the voltage by

$$\varphi = \tan^{-1}\frac{1}{\omega R C} = \tan^{-1}\sqrt{\left(\frac{V_m}{R I_m}\right)^2 - 1} = 60°$$

4.122

Here $$V = IR + \frac{\int_0^t I\,dt}{C}$$

or $$R\dot{I} + \frac{1}{C}I = \dot{V} = -\omega V_0 \sin\omega t$$



Ignoring transients, a solution has the form

$I = I_0 \sin(\omega t - \alpha)$

$$\omega R I_0 \cos(\omega t - \alpha) + \frac{I_0}{C}\sin(\omega t - \alpha) = -\omega V_0 \sin\omega t$$

$$= -\omega V_0\left\{\sin(\omega t - \alpha)\cos\alpha + \cos(\omega t - \alpha)\sin\alpha\right\}$$

so $$R I_0 = -V_0 \sin\alpha$$

$$\frac{I_0}{\omega C} = -V_0\cos\alpha \qquad \alpha = \pi + \tan^{-1}(\omega R C)$$

$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}$$

$$I = I_0 \sin(\omega t - \tan^{-1}\omega R C - \pi) = -I_0\sin(\omega t - \tan^{-1}\omega R C)$$

Then $$Q = \int_0^t I\,dt = Q_0 + \frac{I_o}{\omega}\cos(\omega t - \tan^{-1}\omega R C)$$

It satisfies $$V_0(1 + \cos\omega t) = R\frac{dQ}{dt} + \frac{Q}{C}$$

if $$V_0(1+\cos\omega t) = -RI_0\sin(\omega t - \tan^{-1}\omega RC) + \frac{Q_0}{C} + \frac{I_0}{\omega C}\cos(\omega t - \tan^{-1}\omega RC)$$

Thus $$Q_0 = CV_0$$

and $$\left.\begin{aligned} \frac{I_0}{\omega C} &= V_0 \Big/ \sqrt{1+(\omega RC)^2} \\ RI_0 &= \frac{V_0\,\omega RC}{\sqrt{1+(\omega RC)^2}} \end{aligned}\right\} \text{checks}$$

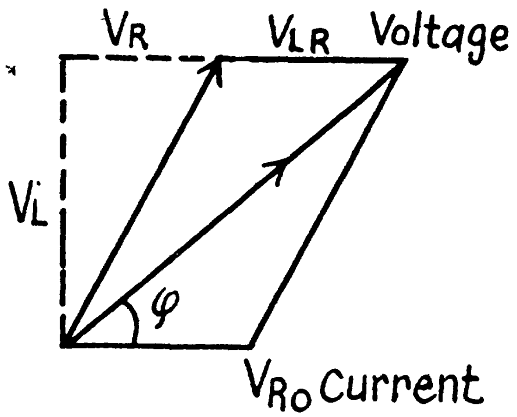
Hence $$V' = \frac{Q}{C} = V_0 + \frac{V_0}{\sqrt{1+(\omega RC)^2}}\cos(\omega t - \alpha)$$

(b) $$\frac{V_0}{\eta} = \frac{V_0}{\sqrt{1+(\omega RC)^2}}$$

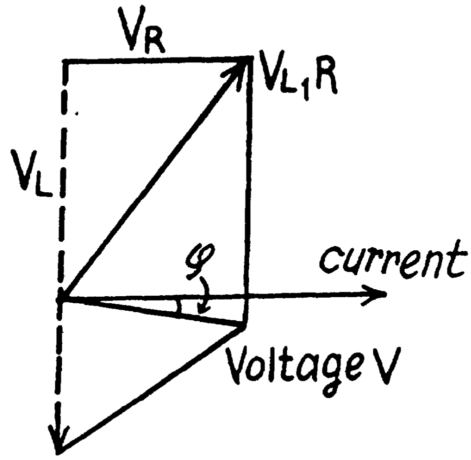
or $$\eta^2 - 1 = \omega^2 (RC)^2$$

or $$RC = \sqrt{\eta^2 - 1}\Big/\omega = 22 \text{ ms.}$$

4.123



(a)



(b)

(b) $$\tan\varphi = \frac{\omega L - \dfrac{1}{\omega C}}{R} = -ve$$

as $$\omega^2 < \frac{1}{LC}$$

4.124 (a) $$I_m = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} = 4{\cdot}48 \text{ A}$$

(b) $$\tan\varphi = \frac{\omega L - \dfrac{1}{\omega C}}{R}, \quad \varphi = -60°$$

Current lags behind the voltage V by φ

(c) $V_C = \frac{I_m}{\omega C} = 0 \cdot 65 \text{ kV}$

$$V_{L/R} = I_m \sqrt{R^2 + \omega^2 L^2} = 0 \cdot 5 \text{ kV}$$

4.125 (a) $V_C = \frac{1}{\omega C} \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$

$$= \frac{V_m}{\sqrt{(\omega R C)^2 + (\omega^2 L C - 1)^2}} = \frac{V_m}{\sqrt{\left(\frac{\omega^2}{\omega_0^2} - 1\right)^2 + 4 \beta^2 \omega^2 / \omega_0^4}}$$

$$= \frac{V_m}{\sqrt{\left(\frac{\omega^2}{\omega_0^2} - 1 + \frac{2 \beta^2}{\omega_0^2}\right)^2 + \frac{4 \beta^2}{\omega_0^2} - \frac{4 \beta^4}{\omega_0^4}}}$$

This is maximum when $\omega^2 = \omega_0^2 - 2 \beta^2 = \frac{1}{L C} - \frac{R^2}{2 L^2}$

(b) $V_L = I_m \omega L = V_m \frac{\omega L}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$

$$= \frac{V_m L}{\sqrt{\frac{R^2}{\omega^2} + \left(L - \frac{1}{\omega^2 C}\right)^2}} = \frac{V_m L}{\sqrt{L^2 - \frac{1}{\omega^2}\left(\frac{2 L}{C} - R^2\right) + \frac{1}{\omega^4 C^2}}}$$

$$= \frac{V_m L}{\sqrt{\left(\frac{1}{\omega^2 C} - \left(L - \frac{C R^2}{2}\right)\right)^2 + L^2 - \left(L - \frac{1}{2} C R^2\right)^2}}$$

This is maximum when

$$\frac{1}{\omega^2 C} = L - \frac{1}{2} C R^2$$

or

$$\omega^2 = \frac{1}{L C - \frac{1}{2} C^2 R^2} = \frac{1}{\frac{1}{\omega_0^2} - \frac{2 \beta^2}{\omega_0^4}}$$

$$= \frac{\omega_0^4}{\omega_0^2 - 2 \beta^2} \quad \text{or} \quad \omega = \frac{\omega_0^2}{\sqrt{\omega_0^2 - 2 \beta^2}}.$$

4.126. $$V_L = I_m \sqrt{R^2 + \omega^2 L^2} = \frac{V_m \sqrt{R^2 + \omega^2 L^2}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

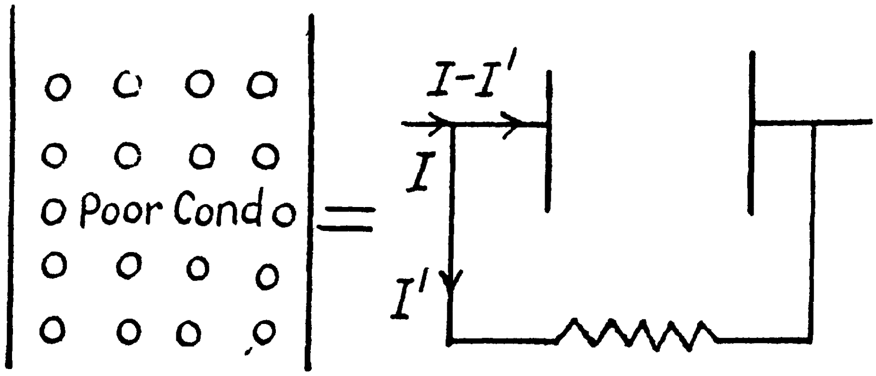
for a given ω, L, R, this is maximum when

$$\frac{1}{\omega C} = \omega L \quad \text{or} \quad C = \frac{1}{\omega^2 L} = 28.2\ \mu\text{F}.$$

For that C, $$V_L = \frac{V\sqrt{R^2 + \omega^2 L^2}}{R} = V\sqrt{1 + (\omega L/R)^2} = 0.540\ \text{kV}$$

At this C $$V_C = \frac{1}{\omega C}\frac{V_m}{R} = \frac{V_m \omega L}{R} = .509\ \text{kV}$$

4.127



We use the complex voltage $V = V_m e^{i\omega t}$. Then the voltage across the capacitor is

$$(I - I')\frac{1}{i\omega C}$$

and that across the resistance RI' and both equal V. Thus

$$I' = \frac{V_m}{R} e^{i\omega t}, \quad I - I' = i\omega C V_m e^{i\omega t}$$

Hence

$$I = \frac{V_m}{R}(1 + i\omega RC)\, e^{i\omega t}$$

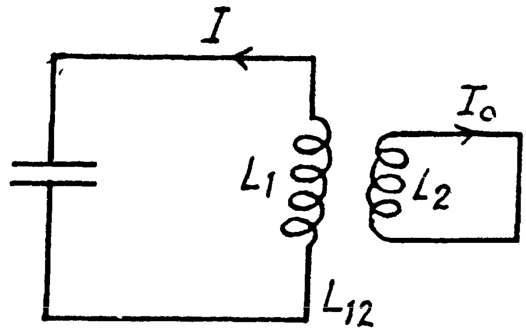
The actual voltage is obtained by taking the real part. Then

$$I = \frac{V_m}{R}\sqrt{1 + (\omega RC)^2}\cos(\omega t + \varphi)$$

Where $$\tan\varphi = \omega RC$$

Note → A condenser with poorly conducting material (dielectric of high resistance) be the plates is equvalent to an an ideal condenser with a high resistance joined in p between its plates.

4.128 $L_1 \dfrac{dI_1}{dt} + \dfrac{\int I_1\, dt}{C} = -L_{12}\dfrac{dI_2}{dt}$



$$L_2 \frac{dI_2}{dt} = -L_{12}\frac{dI_1}{dt}$$

from the second equation

$$L_2 I_2 = -L_{12} I_1$$

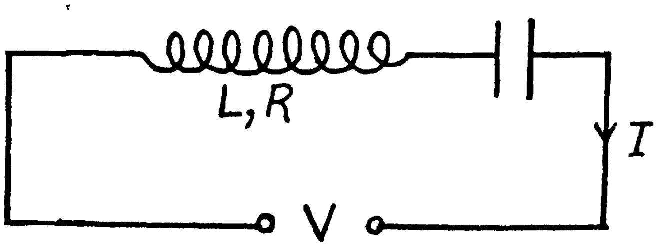
Then $$\left(L_1 - \frac{L_{12}^2}{L_2}\right)\ddot{I}_1 + \frac{I_1}{C} = 0$$

Thus the current oscillates with frequency

$$\omega = \frac{1}{\sqrt{C\left(L_1 - \dfrac{L_{12}^2}{L_2}\right)}}$$

4.129 Given $V = V_m \cos \omega t$

$$I = I_m \cos(\omega t - \varphi)$$



where

$$I_m = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}$$

Then, $$V_C = \frac{\int I\, dt}{C} = \frac{I_m \sin(\omega t - \varphi)}{\omega C}$$

$$= \frac{V_m}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}} \sin(\omega t - \varphi)$$

As resonance the voltage amplitude across the capacitor

$$= \frac{V_m}{RC\dfrac{1}{\sqrt{LC}}} = \sqrt{\frac{L}{CR^2}}\, V_m = n V_m$$

So $$\frac{L}{CR^2} = n^2$$

Now $$Q = \sqrt{\frac{L}{CR^2} - \frac{1}{4}} = \sqrt{n^2 - \frac{1}{4}}$$

4.130 For maximum current amplitude

$$I_m = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}$$

$$L = \frac{1}{\omega^2 C} \text{ and then } I_{m0} = \frac{V_m}{R}$$

Now
$$\frac{I_{m0}}{\eta} = \frac{V_m}{\sqrt{R^2 + \frac{(n-1)^2}{\omega^2 C^2}}}$$

So
$$\eta = \sqrt{1 + \frac{(n-1)^2}{(\omega R C)^2}}$$

$$\omega R C = \frac{n-1}{\sqrt{\eta^2 - 1}}$$

Now
$$Q = \sqrt{\left(\frac{L}{C R^2}\right)^2 - \frac{1}{4}} = \sqrt{\left(\frac{1}{\omega R C}\right)^2 - \frac{1}{4}} = \sqrt{\frac{\eta^2 - 1}{(n-1)^2} - \frac{1}{4}}$$

4.131 At resonance

$$\omega_0 L = (\omega_0 C)^{-1} \quad \text{or} \quad \omega_0 = \frac{1}{\sqrt{LC}},$$

and
$$(I_m)_{res} = \frac{V_m}{R}.$$

Now
$$\frac{V_m}{nR} = \frac{V_m}{\sqrt{R^2 + \left(\omega_1 L - \frac{1}{\omega_1 C}\right)^2}} = \frac{V_m}{\sqrt{R^2 + \left(\omega_2 L - \frac{1}{\omega_2 C}\right)^2}}$$

Then
$$\omega_1 L - \frac{1}{\omega_1 C} = \sqrt{n^2 - 1}\, R$$

$$\omega_2 L - \frac{1}{\omega_2 C} = + \sqrt{n^2 - 1}\, R \qquad \text{(assuming } \omega_2 > \omega_1\text{)}$$

or
$$\omega_1 - \frac{\omega_0^2}{\omega_1} = - \omega_2 + \frac{\omega_0^2}{\omega_2} = - \sqrt{n^2 - 1}\, \frac{R}{L}$$

or
$$\omega_1 + \omega_2 = \frac{\omega_0^2}{\omega_1 \omega_2}(\omega_1 + \omega_2) \Rightarrow \omega_0 = \sqrt{\omega_1 \omega_2}$$

and
$$\omega_2 - \omega_1 = \sqrt{n^2 - 1}\, \frac{R}{L}$$

$$\beta = \frac{R}{2L} = \frac{\omega_2 - \omega_1}{2\sqrt{n^2 - 1}}$$

and
$$Q = \sqrt{\frac{\omega_0^2}{4\beta^2} - \frac{1}{4}} = \sqrt{\frac{(n^2 - 1)\,\omega_1 \omega_2}{(\omega_2 - \omega_1)^2} - \frac{1}{4}}$$

4.132 $Q = \frac{\omega}{2\beta} \approx \frac{\omega_0}{2\beta}$ for low damping.

Now $\frac{I_m}{\sqrt{2}} = \frac{R I_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$, I_m= current amplitude at resouance

or $$\omega - \frac{\omega_0^2}{\omega} = \pm \frac{R}{L} = \pm 2\beta$$

Thus $$\omega \approx \omega_0 \pm \beta$$

So $$\Delta\omega = 2\beta \text{ and } Q = \frac{\omega_0}{\Delta\omega}.$$

4.133 At resonance $\omega = \omega_0$

$$I_m(\omega_0) = \frac{V_m}{R}$$

Then $$I_m(\eta\omega_0) = \frac{V_m}{\sqrt{R^2 + \left(\eta\omega_0 L - \frac{1}{\eta\omega_0 C}\right)^2}}$$

$$= \frac{V_m}{\sqrt{R^2 + \left(\eta - \frac{l}{\eta}\right)^2 \frac{L}{C}}} = \frac{V_m}{\sqrt{1 + \left(Q^2 + \frac{l}{4}\right)\left(\eta - \frac{1}{\eta}\right)^2 \frac{L}{C}}}$$

4.134 The a.c. current must be

$$I = I_0\sqrt{2}\sin\omega t$$

Then D.C. component of the rectified current is

$$\langle I' \rangle = \frac{1}{T}\int_0^{T/2} I_0\sqrt{2}\sin\omega t\, dt$$

$$= I_0\sqrt{2}\,\frac{1}{2\pi}\int_0^{\pi}\sin\theta\, d\theta$$

$$= \frac{I_0\sqrt{2}}{\pi}$$

Since the charge deposited must be the same

$$I_0 t_0 = \frac{I_0\sqrt{2}}{\pi}t \quad \text{or} \quad t = \frac{\pi t_0}{\sqrt{2}}$$

The answer is incorrect.

4.135 (a) $I(t) = I_1 \frac{t}{T} \quad 0 \le t < T$

$$I(t \pm T) = I(t)$$

Now mean current

$$\langle I \rangle = \frac{1}{T}\int_0^T I_1 \frac{t}{T}\, dt = I_1 \frac{T^2/2}{T^2} = I_1/2$$

Then $\qquad I_1 = 2I_0$ since $\langle I \rangle = I_0$.

Now mean square current $\langle I^2 \rangle$

$$= 4I_0^2 \frac{1}{T}\int_0^T \frac{t^2}{T^2}\, dt = \frac{4I_0^2}{3}$$

so effective current $= \dfrac{2I_0}{\sqrt{3}}$.

(b) In this case $I = I_1 |\sin \omega t|$

and $$I_0 = \frac{1}{T}\int_0^T I_1 |\sin \omega t|\, dt$$

$$= \frac{1}{2\pi} I_1 \int_0^{2\pi} |\sin\theta|\, d\theta = \frac{I_1}{\pi}\int_0^{\pi} \sin\theta\, d\theta = \frac{2I_1}{\pi}$$

So $$I_1 = \frac{\pi I_0}{2}$$

Then, mean square current $= \langle I^2 \rangle = \dfrac{\pi^2 I_0^2}{4T}\displaystyle\int_0^T \sin^2 \omega t\, dt$

$$= \frac{\pi^2 I_0^2}{4} \times \frac{1}{2\pi}\int_0^{2\pi} \sin^2\theta\, d\theta = \frac{\pi^2 I_0^2}{8}$$

so effective current $= \dfrac{\pi I_0}{\sqrt{8}}$.

4.136 $P_{d.c.} = \dfrac{V_0^2}{R}$

$$P_{a.c.} = \frac{V_0^2}{\sqrt{R^2 + \omega^2 L^2}} \cdot \frac{R}{\sqrt{R^2 + \omega^2 L^2}} = \frac{v_0^2/R}{1 + \left(\frac{\omega L}{R}\right)^2} = \frac{P_{d.c.}}{\eta}$$

Thus $$\frac{\omega L}{R} = \sqrt{\eta - 1}$$

or $$\omega = \frac{R}{L}\sqrt{\eta - 1}$$

$$\nu = \frac{R}{2\pi L}\sqrt{\eta - 1} = 2 \text{ kH of on putting the values.}$$

4.137 $Z = \sqrt{R^2 + X_L^2}$ or $R_0 = \sqrt{Z^2 - X_L^2}$

The $$\tan\theta = \frac{X_L}{\sqrt{Z^2 - X_L^2}}$$

So $$\cos\varphi = \frac{\sqrt{Z^2 - X_L^2}}{Z} = \sqrt{1 - \left(\frac{X_L}{Z}\right)^2}$$

$$\varphi = \cos^{-1}\sqrt{1 - \left(\frac{X_L}{Z}\right)^2} = 37°.$$

The current lags by φ behind the voltage.

also $$P = VI\cos\varphi = \frac{V^2}{Z^2}\sqrt{Z^2 - X_L^2} = .160 \text{ kW.}$$

4.138 $P = \dfrac{V^2(R + r)}{(R + r)^2 + \omega^2 L^2}$

This is maximum when $R + r = \omega L$ for

$$P = \frac{V^2}{R + r + \dfrac{(\omega L)^2}{R + r}} = \frac{V^2}{\left[\sqrt{R + r} - \dfrac{\omega L}{\sqrt{R + r}}\right]^2 + 2\omega L}$$

Thus $R = \omega L - r$ for maximum power and $P_{max} = \dfrac{V^2}{2\omega L}$.

Substituting the values, we get $R = 200\ \Omega$ and $P_{max} = .114$ kW.

4.139 $P = \dfrac{V^2 R}{R^2 + (X_L - X_C)^2}$

Varying the capacitor does not change R so if P increases n times

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \text{ must decreases } \sqrt{n} \text{ times}$$

Thus $$\cos\varphi = \frac{R}{Z} \text{ increases } \sqrt{n} \text{ times}$$

$\therefore$ % increase in $\cos\varphi = \left(\sqrt{n} - 1\right) \times 100\,\% = 30.4\%$.

4.140 $P = \dfrac{V^2 R}{R^2 + (X_L - X_C)^2}$

At resonance $X_L = X_C \Rightarrow \omega_0 = \dfrac{1}{\sqrt{LC}}$.

Power generated will decrease n times when

$$(X_L - X_C)^2 = \left(\omega L - \frac{1}{\omega C}\right)^2 = (n-1)R^2$$

or $$\omega - \frac{\omega_0^2}{\omega} = \pm\sqrt{n-1}\,\frac{R}{L} = \pm\sqrt{n-1}\,2\beta.$$

Thus $$\omega^2 \mp 2\sqrt{n-1}\,\beta\omega - \omega_0^2 = 0$$

$$\left(\omega \mp \sqrt{n-1}\,\beta\right)^2 = \omega_0^2 + (n-1)\beta^2$$

or $$\frac{\omega}{\omega_0} = \sqrt{1 + (n-1)\beta^2/\omega_0^2} \pm \sqrt{n-1}\,\beta/\omega_0$$

(taking only the positive sign in the first term to ensure positive value for $\dfrac{\omega}{\omega_0}$.)

Now $$Q = \frac{\omega}{2\beta} = \frac{1}{2}\sqrt{\left(\frac{\omega_0}{\beta}\right)^2 - 1}$$

$$\frac{\omega_0}{\beta} = \sqrt{1 + 4Q^2}$$

Thus $$\frac{\omega}{\omega_0} = \sqrt{1 + \frac{n-1}{(1+4Q^2)}} \pm \sqrt{n-1}\,/\sqrt{1+4Q^2}$$

For large Q

$$\left|\frac{\omega - \omega_0}{\omega_0}\right| = \frac{\sqrt{en-1}}{2Q} = \frac{\sqrt{en-1}}{2Q} \times 100\,\% = 0.5\%$$

4.141 We have

$$V_1 = \frac{VR}{\sqrt{(R+R_1)^2 + X_L^2}}, \quad V_2 = \frac{V\sqrt{R_1^2 + X_L^2}}{\sqrt{(R+R_1)^2 + X_L^2}}$$

so $$(R+R_1)^2 + X_L^2 = \left(\frac{VR}{V_1}\right)^2, \quad R_1^2 + X_L^2 = \left(\frac{V_2 R}{V_1}\right)^2$$

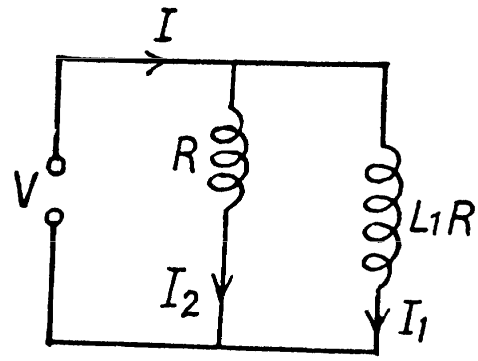
Hence $$R^2 + 2RR_1 = \frac{R^2}{V_1^2}(V^2 - V_2^2)$$

or $$R_1 = \frac{R}{2V_1^2}(V^2 - V_2^2 - V_1^2)$$

Heat generated in the coil $= \dfrac{V^2 R_1}{(R_1+R_2)^2+X_L^2} = \dfrac{V_1^2}{R^2}\times R_1 = \dfrac{V_1^2}{R^2}\times\dfrac{R^2}{2V_1^2}\,(V^2-V_1^2-V_2^2)$

$$= \frac{V^2-V_1^2-V_2^2}{2R} = 30 \text{ W}$$

4.142 Here $I_2 = \dfrac{V}{R}$, $V =$ effective voltage



$$I_1 = \frac{V}{\sqrt{R^2+X_L^2}}$$

and $I = \dfrac{V\sqrt{(R+R_1)^2+X_L^2}}{R\sqrt{R_1^2+X_L^2}} = \dfrac{V}{R_{eff}}$

R_{eff} is the impedance of the coil & the resistance in parallel.

Now $$\frac{I^2-I_2^2}{I_2^2} = \frac{R^2+2RR_1}{R_1^2+X_L^2} = \left(\frac{I_1}{I_2}\right)^2 + \frac{2RR_1}{R^2+X_L^2}$$

$$\frac{I^2-I_2^2-I_1^2}{I_2^2} = \frac{2RR_1}{R^2+X_L^2}$$

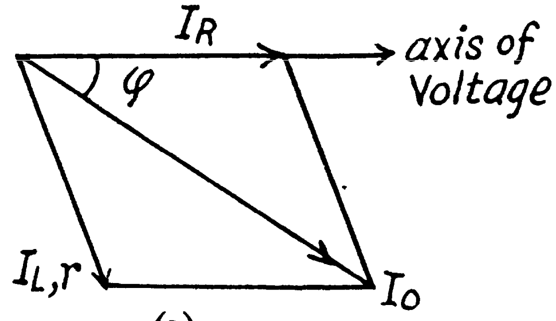
Now mean power consumed in the coil

$$= I_1^2 R_1 = \frac{V^2 R_1}{R_2+X_L^2} = I_2^2\,R\,.\,\frac{I^2-I_1^2-I_2^2}{2I_2^2} = \frac{1}{2}R\,(I^2-I_1^2-I_2^2) = 2.5 \text{ W.}$$

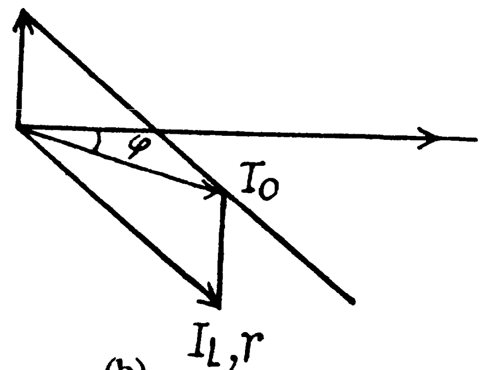
4.143 $\dfrac{1}{Z} = \dfrac{1}{R} + \dfrac{1}{\dfrac{1}{i\omega C}} = \dfrac{1}{R} + i\omega C = \dfrac{1+i\omega RC}{R}$

$$|Z| = \frac{R}{\sqrt{1+(\omega RC)^2}} = 40\ \Omega$$

4.144 (a) For the resistance, the voltage and the current are in phase. For the coil the voltage is ahead of the current by less than 90°. The current is obtained by addition because the elements are in parallel.



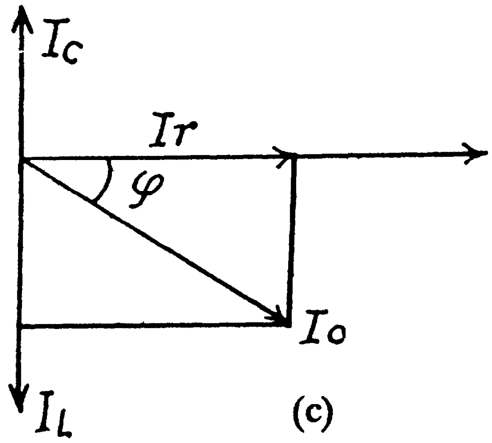
(a)



(b)

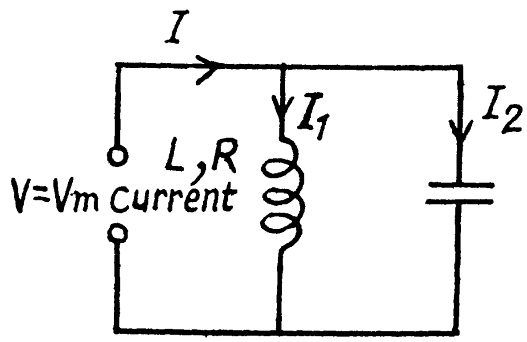
(b) I_C is ahead of the voltage by 90°.

(c) The coil has no resistance so I_L is 90° behind the voltage.



(c)

4.145 When the coil and the condenser are in parallel, the equation is



$$L\frac{dI_1}{dt}+RI_1=\frac{\int I_2\,dt}{C}=V_m\cos\omega t$$

$$I=I_1+I_2$$

Using complex voltages

$$I_1=\frac{V_m\,e^{i\omega t}}{R+i\omega L},\quad I_2=i\omega C\,V_m\,e^{i\omega t}$$

and

$$I=\left(\frac{1}{R+i\omega L}+i\omega C\right)V_m\,e^{i\omega t}=\left[\frac{R-i\omega L+i\omega C(R^2+\omega^2L^2)}{R^2+\omega^2L^2}\right]V_m\,e^{i\omega t}$$

Thus, taking real parts $\qquad I=\dfrac{V_m}{|Z(\omega)|}\cos(\omega t-\varphi)$

where $\qquad \dfrac{1}{|Z(\omega)|}=\dfrac{[R^2+\{\omega C(R^2+\omega^2L^2)-\omega L\}^2]}{(R^2+\omega^2L^2)^{1/2}}$

and $\qquad \tan\varphi=\dfrac{\omega L-\omega C(R^2+\omega^2L^2)}{R}$

(a) To get the frequency of resonance we must define what we mean by resouance. One definition requires the extremum (maximum or minimum) of current amplitude. The other definition requires rapid change of phase with φ passing through zero at resonance. For the series circuit.

$$I_m=\frac{V_m}{\left\{R^2+\left(\omega L-\dfrac{1}{\omega C}\right)^2\right\}^{1/2}}\quad\text{and}\quad\tan\varphi=\frac{\omega L-\dfrac{1}{\omega C}}{R}$$

both definitions give $\omega^2 = \dfrac{1}{LC}$ at resonance. In the present case the two definitions do not agree (except when $R = 0$). The definition that has been adopted in the answer given in the book is the vanishing of phase. This requires

$$C(R^2 + \omega^2 L^2) = L$$

or $$\omega^2 = \frac{1}{LC} - \frac{R^2}{L^2} = \omega_{res}^2, \quad \omega_{res} = 31.6 \times 10^3 \text{ rad/s.}$$

Note that for small R, φ rapidly changes from $\approx -\dfrac{\pi}{2}$ to $+\dfrac{\pi}{2}$ as ω passes through ω_{res} from $< \omega_{res}$ to $> \omega_{res}$.

(b) At resonance $$I_m = \frac{V_m R}{L/C} = V_m \frac{CR}{L}$$

so I = effective value of total current $= V \dfrac{CR}{L} = 3.1$ mA.

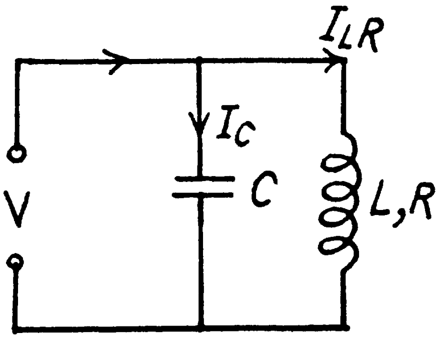
similarly $$I_L = \frac{V}{\sqrt{L/C}} = V\sqrt{\frac{C}{L}} = 0.98 \text{ A.}$$

$$I_C = \omega C V = V\sqrt{\frac{C}{L} - \frac{R^2C^2}{L^2}} = 0.98 \text{ A.}$$

Note :- The vanishing of phase (its passing through zero) is considered a more basic definition of resonance.

4.146 We use the method of complex voltage

$V = V_0 e^{i\omega t}$



Then $$I_C = \frac{V_0 e^{i\omega t}}{\dfrac{1}{i\omega C}} = i\omega C V_0 e^{i\omega t}$$

$$I_{L,R} = \frac{V_0 e^{i\omega t}}{R + i\omega L}$$

$$I = I_C + I_{L,R} = V_0 \frac{R - i\omega L + i\omega C(R^2 + \omega^2 L^2)}{R^2 + \omega^2 L^2} e^{i\omega t}$$

Then taking the real part

$$I = \frac{V_0\sqrt{R^2 + \{\omega C(R^2 + \omega^2 L^2) - \omega L\}^2}}{R^2 + \omega^2 L^2} \cos(\omega t - \varphi)$$

where $$\tan\varphi = \frac{\omega L - \omega C(R^2 + \omega^2 L^2)}{R}$$

4.147 From the previous problem

$$Z = \frac{R^2 + \omega^2 L^2}{\sqrt{R^2 + \left\{\omega C(R^2 + \omega^2 L^2) - \omega L\right\}^2}}$$

$$= \frac{R^2 + \omega^2 L^2}{\sqrt{(R^2 + \omega^2 L^2)(1 - 2\omega^2 LC) + \omega^2 C^2 (R^2 + \omega^2 L^2)^2}}$$

$$= \frac{\sqrt{R^2 + \omega^2 L^2}}{\sqrt{(1 - 2\omega^2 LC) + \omega^2 C^2 (R^2 + \omega^2 L^2)}} = \frac{\sqrt{R^2 + \omega^2 L^2}}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}}$$

4.148 (a) We have

$$\varepsilon = -\frac{d\Phi}{dt} = \omega \Phi_0 \sin \omega t = L\dot{I} + RI$$

Put $\quad I = I_m \sin(\omega t - \varphi)$. Then

$$\omega \Phi_0 \sin \omega t = \omega \Phi_0 \left\{\sin(\omega t - \varphi)\cos\varphi + \cos(\omega t - \varphi)\sin\varphi\right\}$$

$$= L I_m \omega \cos(\omega t - \varphi) + R I_m \sin(\omega t - \varphi)$$

so $\quad R I_m = \omega \Phi_0 \cos\varphi \quad$ and $\quad L I_m = \Phi_0 \sin\varphi$

or $\quad I_m = \dfrac{\omega \Phi_0}{\sqrt{R^2 + \omega^2 L^2}} \quad$ and $\quad \tan\varphi = \dfrac{\omega L}{R}$.

(b) Mean mechanical power required to maintain rotation = energy loss per unit time

$$= \frac{1}{T}\int_0^T R I^2 dt = \frac{1}{2} R I_m^2 = \frac{1}{2}\frac{\omega^2 \Phi_0^2 R}{R^2 + \omega^2 L^2}$$

4.149 We consider the force $\vec{F}_{12}$ that a circuit 1 exerts on another closed circuit 2 :-

$$\vec{F}_{12} = \oint I_2\, d\vec{l}_2 \times \vec{B}_{12}$$

Here $\vec{B}_{12}$ = magnetic field at the site of the current element $d\vec{l}_2$ due to the current I_1 flowing in 1.

$$= \frac{\mu_0}{4\pi}\int \frac{I_1\, d\vec{l}_1 \times \vec{r}_{12}}{r_{12}^3}$$

where $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$ = vector from current element $d\vec{l}_1$ to the current element $d\vec{l}_2$

Now

$$\vec{F}_{12} = \frac{\mu_0}{4\pi}\iint I_1 I_2 \frac{d\vec{l}_2 \times (d\vec{l}_1 \times \vec{r}_{12})}{r_{12}^3} = \frac{\mu_0}{4\pi}\iint I_1 I_2 \frac{d\vec{l}_1 (d\vec{l}_2 \cdot \vec{r}_{12}) - (d\vec{l}_1 \cdot d\vec{l}_2)\vec{r}_{12}}{r_{12}^3}$$

In the first term, we carry out the integration over $d\vec{l}_2$ first. Then

$$\iint \frac{d\vec{l_1}\,(d\vec{l_2}\cdot\vec{r_{12}})}{r_{12}^3} = \int d\vec{l_1}\oint \frac{d\vec{l_2}\cdot\vec{r_{12}}}{r_{12}^3} = -\int d\vec{l_1}\oint d\vec{l_2}\cdot\nabla_2\frac{1}{r_{12}} = 0$$

because $$\oint d\,l_2\cdot\nabla_2\frac{1}{r_{12}} = \int d\vec{S_2}\ \text{curl}\left(\nabla\frac{1}{r_{12}}\right) = 0$$

Thus $$F_{12} = -\frac{\mu_0}{4\pi}\iint I_1 I_2\ d\vec{l_1}\cdot d\vec{l_2}\,\frac{\vec{r_{12}}}{r_{12}^3}$$

The integral involved will depend on the vector $\vec{a}$ that defines the separation of the (suitably chosen)centre of the coils. Let C_1 and C_2 be the centres of the two coil suitably defined. Write

$$\vec{r_{12}} = \vec{r_2} - \vec{r_1} = \vec{\rho_2} - \vec{\rho_1} + \vec{a}$$

where $\vec{\rho_1}$ ($\vec{\rho_2}$) is the distance of $d\vec{l_1}(d\vec{l_2})$ from C_1 (C_2) and $\vec{a}$ stands for the vector $\overrightarrow{C_1C_2}$.

Then $$\frac{\vec{r_{12}}}{r_{12}^3} = -\vec{\nabla}_{\vec{a}}\frac{1}{r_{12}}$$

and $$\vec{F_{12}} = \vec{\nabla}_a\left[I_1 I_2\frac{\mu_0}{4\pi}\iint\frac{d\vec{l_1}\cdot d\vec{l_2}}{r_{12}}\right]$$

The bracket defines the mutual inductance L_{12}. Thus noting the definition of x

$$\langle F_x\rangle = \frac{\partial L_{12}}{\partial x}\langle I_1 I_2\rangle$$

where < > denotes time average. Now

$$I_1 = I_0\cos\omega t = \text{Real part of } I_0\,e^{i\omega t}$$

The current in the coil 2 satisfies $R I_2 + L_2\dfrac{dI_2}{dt} = -L_{12}\dfrac{dI_1}{dt}$

or $$I_2 = \frac{-i\omega L_{12}}{R+i\omega L_2}I_0\,e^{i\omega t}\ (\text{ in the complex case })$$

taking the real part

$$I_2 = -\frac{\omega L_{12}I_0}{R^2+\omega^2L_2^2}(\omega L_2\cos\omega t - R\sin\omega t) = -\frac{\omega L_{12}}{\sqrt{R^2+\omega^2L_\tau^2}}I_0\cos(\omega t+\varphi)$$

Where $\tan\varphi = \dfrac{R}{\omega L_2}$. Taking time average, we get

$$\langle F_x\rangle = \frac{\partial L_{12}}{\partial x}I_0\frac{\omega L_{12}I_0}{\sqrt{R^2+\omega^2L_2^2}}\cdot\frac{1}{2}\cos\varphi = \frac{\omega^2L_2L_{12}I_0^2}{2\left(R^2+\omega^2L_2^2\right)}\frac{\partial L_{12}}{\partial x}$$

The repulsive nature of the force is also consistent with Lenz's law, assuming, of comse, that L_{12} decreases with x.

4.3 ELASTIC WAVES. ACOUSTICS

4.150 Since the temperature varies linearly we can write the temperature as a function of x, which is, the distance from the point A towards B.

i.e.,
$$T = T_1 + \frac{T_2 - T_1}{l} x, \ [0 < x < l]$$

hence,
$$dT = \left(\frac{T_2 - T_1}{l} \right) dx \tag{1}$$

In order to travel an elemental distance of dx which is at a distance of x from A it will take a time

$$dt = \frac{dx}{\alpha \sqrt{T}} \tag{2}$$

From Eqns (1) and (2), expressing dx in terms of dT, we get

$$dt = \frac{l}{\alpha \sqrt{T}} \left(\frac{l \, dT}{T_2 - T_1} \right)$$

Which on integration gives

$$\int_0^t dt = \frac{l}{\alpha (T_2 - T_1)} \int_{T_1}^{T_2} \frac{dT}{\sqrt{T}}$$

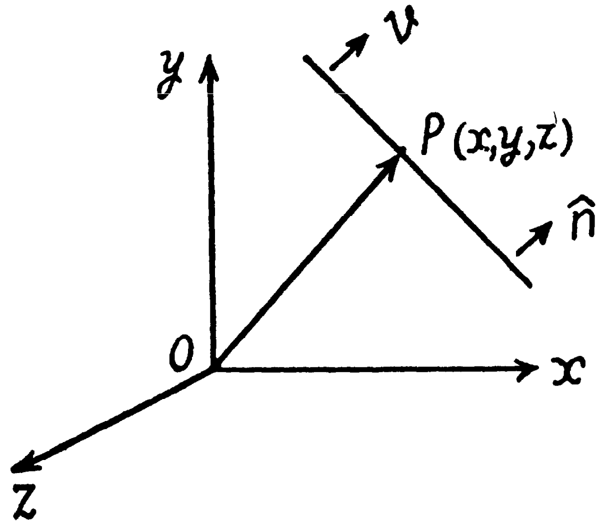
or,
$$t = \frac{2l}{(T_2 - T_1)} \left(\sqrt{T_2} - \sqrt{T_1} \right)$$

Hence the sought time $t = \dfrac{2l}{\alpha \left(\sqrt{T_1} + \sqrt{T_2} \right)}$

4.151 Equation of plane wave is given by

$\xi(r, t) = a \cos(\omega t - \vec{k} \cdot \vec{r})$, where $\vec{k} = \frac{\omega}{v} \hat{n}$ called the wave vector

and $\hat{n}$ is the unit vector normal to the wave surface in the direction of the propagation of wave.



or, $$\xi(x, y, z) = a\cos(\omega t - k_x x - k_y y - k_z z)$$
$$= a\cos(\omega t - kx\cos\alpha - ky\cos\beta - kz\cos\gamma)$$

Thus $$\xi(x_1, y_1, z_1, t) = a\cos(\omega t - kx_1\cos\alpha - ky_1\cos\beta - kz_1\cos\gamma)$$

and $$\xi(x_2, y_2, z_2, t) = a\cos(\omega t - kx_2\cos\alpha - ky_2\cos\beta - kz_2\cos\gamma)$$

Hence the sought wave phase difference
$$\varphi_2 - \varphi_1 = k\left[(x_1 - x_2)\cos\alpha + (y_1 - y_2)\cos\beta + (z_1 - z_2)\cos\gamma\right]$$

or $$\Delta\varphi = |\varphi_2 - \varphi_1| = k\left|\left[(x_1 - x_2)\cos\alpha + (y_1 - y_2)\cos\beta + (z_1 - z_2)\cos\gamma\right]\right|$$
$$= \frac{\omega}{v}\left|\left[(x_1 - x_2)\cos\alpha + (y_1 - y_2)\cos\beta + (z_1 - z_2)\cos\gamma\right]\right|$$

4.152 The phase of the oscillation can be written as
$$\Phi = \omega t - \vec{k}\cdot\vec{r}$$

When the wave moves along the x-axis
$$\Phi = \omega t - k_x x \quad \text{(On putting } k_y = k_z = 0\text{)}.$$

Since the velocity associated with this wave is v_1

We have $$k_x = \frac{\omega}{v_1}$$

Similarly $$k_y = \frac{\omega}{v_2} \text{ and } k_z = \frac{\omega}{v_3}$$

Thus $$\vec{k} = \frac{\omega}{v_1}\hat{e}_x + \frac{\omega}{v_2}\hat{e}_y + \frac{\omega}{v_3}\hat{e}_3.$$

4.153 The wave equation propagating in the direction of +ve x axis in medium K is give as
$$\xi = a\cos(\omega t - kx)$$

So, $\xi = a\cos k(vt - x)$, where $k = \dfrac{\omega}{v}$ and v is the wave velocity

In the refrence frame K', the wave velocity will be $(v - V)$ propagating in the direction of +ve x axis and x will be x'. Thus the sought wave equation.

$$\xi = a\cos k[(v - V)t - x']$$

or, $$\xi = a\cos\left[\left(\omega - \frac{\omega}{v}V\right)t - kx'\right] = a\cos\left[\omega t\left(1 - \frac{V}{v}\right) - kx'\right]$$

4.154 This follows on actually putting
$$\xi = f(t + \alpha x)$$

in the wave equation $$\frac{\partial^2\xi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\xi}{\partial t^2}$$

(We have written the one dimensional form of the wave equation.) Then
$$\frac{1}{v^2}f''(t + \alpha x) = \alpha^2 f''(t + \alpha x)$$

so the wave equation is satisfied if

$$\alpha = \pm \frac{1}{v'}$$

That is the physical meaning of the constant α .

4.155 The given wave equation

$$\xi = 60 \cos (1800\, t - 5{\cdot}3\, x)$$

is of the type

$$\xi = a \cos (\omega t - k x), \quad \text{where } a = 60 \times 10^{-6}\,\text{m}$$

$$\omega = 1800 \text{ per sec and } k = 5{\cdot}3 \text{ per metre}$$

As $$k = \frac{2\pi}{\lambda}, \quad \text{so } \lambda = \frac{2\pi}{k}$$

and also $$k = \frac{\omega}{v}, \quad \text{so } v = \frac{\omega}{k} = 340 \text{ m/s}$$

(a) Sought ratio $= \dfrac{a}{\lambda} = \dfrac{a k}{2\pi} = 5{\cdot}1 \times 10^{-5}$

(b) Since $\xi = a \cos (\omega t - k x)$

$$\frac{\partial \xi}{\partial t} = -a \omega \sin (\omega t - k x)$$

So velocity oscillation amplitude

$$\left(\frac{\partial \xi}{\partial t}\right)_m \quad \text{or} \quad v_m = a\omega = 0{\cdot}11 \text{ m/s} \tag{1}$$

and the sought ratio of velocity oscillation amplitude to the wave propagation velocity

$$= \frac{v_m}{v} = \frac{0{\cdot}11}{340} = 3{\cdot}2 \times 10^{-4}$$

(c) Relative deformation $= \dfrac{\partial \xi}{\partial x} = a k \sin (\omega t - k x)$

So, relative deformation amplitude

$$= \left(\frac{\partial \xi}{\partial x}\right)_m = a k = (60 \times 10^{-6} \times 5{\cdot}3)\,\text{m} = 3{\cdot}2 \times 10^{-4} \text{ m} \tag{2}$$

From Eqns (1) and (2)

$$\left(\frac{\partial \xi}{\partial x}\right)_m = a k = \frac{a \omega}{v} = \frac{1}{v}\left(\frac{\partial \xi}{\partial t}\right)_m$$

Thus $\left(\dfrac{\partial \xi}{\partial x}\right)_m = \dfrac{1}{v}\left(\dfrac{\partial \xi}{\partial t}\right)_m$, where $v = 340$ m/s is the wave velocity.

4.156 (a) The given equation is,

$$\xi = a \cos (\omega t - k x)$$

So at
$$t = o,$$
$$\xi = a \cos kx$$

Now,
$$\frac{d\xi}{dt} = -a\omega \sin(\omega t - kx)$$

and
$$\frac{d\xi}{dt} = a\omega \sin kx, \text{ at } t = 0.$$

Also,
$$\frac{d\xi}{dx} = +ak \sin(\omega t - kx)$$

and at
$$t = 0,$$
$$\frac{d\xi}{dx} = -ak \sin kx.$$

Hence all the graphs are similar having different amplitudes, as shown in the answer-sheet of the problem book.

(b) At the points, where $\xi = 0$, the velocity direction is positive, i.e., along + ve x – axis in the case of longitudinal and + ve y– axis in the case of transverse waves, where $\frac{d\xi}{dt}$ is positive and vice versa.

For sought plots see the answer-sheet of the problem book.

4.157 In the given wave equation the particle's displacement amplitude $= a e^{-\gamma x}$
Let two points x_1 and x_2, between which the displacement amplitude differ by $\eta = 1\,\%$

So,
$$a e^{-\gamma x_1} - a e^{-\gamma x_2} = \eta a e^{-\gamma x_1}$$

or
$$e^{-\gamma x_1}(1-\eta) = e^{-\gamma x_2}$$

or
$$\ln(1-\eta) - \gamma x_1 = -\gamma x_2$$

or,
$$x_2 - x_1 = -\frac{\ln(1-\eta)}{\gamma}$$

So path difference $= -\dfrac{\ln(1-\eta)}{\gamma}$

and phase difference $= \dfrac{2\pi}{\lambda} \times$ path difference

$$= -\frac{2\pi}{\lambda}\frac{\ln(1-\eta)}{\gamma} \approx \frac{2\pi\eta}{\lambda\cdot\gamma} = 0{\cdot}3 \text{ rad}$$

4.158 Let S be the source whose position vector relative to the reference point O is $\vec{r}$.
Since intensities are inversely proportional to the square of distances,

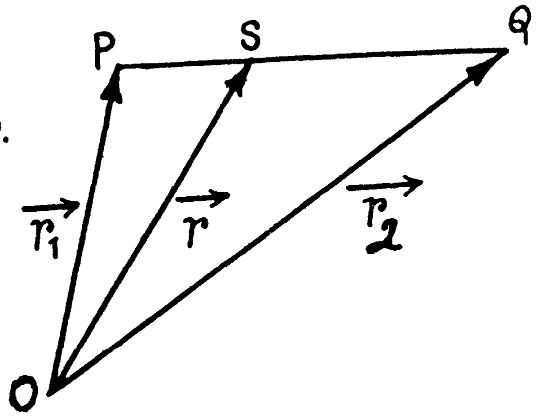
$$\frac{\text{Intensity at } P\,(I_1)}{\text{Intensity at } Q\,(I_2)} = \frac{d_2^2}{d_1^2}$$

where $d_1 = PS$ and $d_2 = QS$.

But intensity is proportional to the square of amplitude.

So, $\dfrac{a_1^2}{a_2^2} = \dfrac{d_2^2}{d_1^2}$ or $a_1 d_1 = a_2 d_2 = k$ (say)

Thus $\quad d_1 = \dfrac{k}{a_1}$ and $d_2 = \dfrac{k}{a_2}$



Let $\hat{n}$ be the unit vector along PQ directed from P to Q.

Then $$\overrightarrow{PS} = d_1 \hat{n} = \frac{k}{a_1}\hat{n}$$

and $$\overrightarrow{SQ} = d_2 \hat{n} = \frac{k}{a_2}\hat{n}$$

From the triangle law of vector addition.

$$\overrightarrow{OP} + \overrightarrow{PS} = \overrightarrow{OS} \quad \text{or} \quad \vec{r_1} + \frac{k}{a_1}\hat{n} = \vec{r}$$

or $$a_1 \vec{r_1} + k\hat{n} = a_1 \vec{r} \tag{1}$$

Similarly $$\vec{r} + \frac{k}{a_2}\hat{n} = \vec{r_2} \quad \text{or} \quad a_2 \vec{r_2} - k\hat{n} = a_2 \vec{r} \tag{2}$$

Adding (1) and (2),

$$a_1 \vec{r_1} + a_2 \vec{r_2} = (a_1 - a_2)\vec{r}$$

Hence $$\vec{r} = \frac{a_1 \vec{r_1} + a_2 \vec{r_2}}{a_1 + a_2}$$

4.159 (a) We know that the equation of a spherical wave in a homogeneous absorbing medium of wave damping coefficient γ is :

$$\xi = \frac{a'_0 e^{-\gamma r}}{r} \cos(\omega t - kr)$$

Thus particle's displacement amplitude equals

$$\frac{a'_0 e^{-\gamma r}}{r}$$

According to the conditions of the problem,

at $$r = r_0,\ a_0 = \frac{a'_0 e^{-\gamma r_0}}{r_0} \tag{1}$$

and when $$r = r, \quad \frac{a_0}{\eta} = \frac{a'^{-\gamma r}_0}{r} \tag{2}$$

Thus from Eqns (1) and (2)

$$e^{\gamma(r-r_0)} = \eta \frac{r_0}{r}$$

or, $$\gamma(r-r_0) = \ln(\eta r_0) - \ln r$$

or, $$\gamma = \frac{\ln\eta + \ln r_0 - \ln r}{r - r_0} = \frac{\ln 3 + \ln 5 - \ln 10}{5} = 0{\cdot}08\ \text{m}^{-1}$$

(b) As $\xi = \frac{a'_0 e^{-\gamma r}}{r} \cos(\omega t - k r)$

So, $$\frac{\partial \xi}{\partial t} = -\frac{a'_0 e^{-\gamma r}}{r} \omega \sin(\omega t - k r)$$

$$\left(\frac{\partial \xi}{\partial t}\right)_n = \frac{a'_0 e^{-\gamma r}}{r} \omega$$

But at point A, $\frac{a'_0 e^{-\gamma r}}{r} = \frac{a_0}{\eta}$

So, $\left(\frac{\partial \xi}{\partial t}\right)_m = \frac{a_0 \omega}{\eta} = \frac{a_0 2\pi}{\eta} = \frac{50 \times 10^{-6}}{3} \times 2 \times \frac{22}{7} \times 1{\cdot}45 \times 10^3 = 15\ \text{m/s}$

4.160 (a) Equation of the resultant wave,

$$\xi = \xi_1 + \xi_2 = 2a\cos k\left(\frac{y-x}{2}\right)\cos\left\{\omega t - \frac{k(x+y)}{2}\right\},$$

$$= a'\cos\left\{\omega t - \frac{k(x+y)}{2}\right\}, \quad \text{where} \quad a' = 2a\cos k'\left(\frac{y-x}{2}\right)$$

Now, the equation of wave pattern is,

$$x + y = k, \ (\text{a Const.})$$

For sought plots see the answer-sheet of the problem book.

For antinodes, i.e. maximum intensity

$$\cos\frac{k(y-x)}{2} = \pm 1 = \cos n\pi$$

or, $$\pm(x-y) = \frac{2n\pi}{k} = n\lambda$$

or, $$y = x \pm n\lambda, \ n = 0, 1, 2, \dots$$

Hence, the particles of the medium at the points, lying on the solid straight lines $(y = x \pm n\lambda)$, oscillate with maximum amplitude.

For nodes, i.e. minimum intensity,

$$\cos\frac{k(y-x)}{2} = 0$$

or, $$\pm\frac{k(y-x)}{2} = (2n+1)\frac{\pi}{2}$$

or, $$y = x \pm (2n+1)\lambda/2,$$

and hence the particles at the points, lying on dotted lines do not oscillate.

(b) When the waves are longitudinal,

For sought plots see the answer-sheet of the problem book.

$$k(y-x) = \cos^{-1}\frac{\xi_1}{a} - \cos^{-1}\frac{\xi_2}{a}$$

or, $$\frac{\xi_1}{a} = \cos\left\{k(y-x) + \cos^{-1}\frac{\xi_2}{a}\right\}$$

$$= \frac{\xi_2}{a}\cos k(y-x) - \sin k(y-x)\sin\left(\cos^{-1}\frac{\xi_2}{a}\right)$$

$$= \frac{\xi_2}{a}\cos k(y-x) - \sin k(y-x)\sqrt{1-\frac{\xi_2^2}{a^2}} \qquad (1)$$

from (1),

if $$\sin k(y-x) = 0 \sin(n\pi)$$

$$\xi_1 = \xi_2\,(-1)^n$$

thus, the particles of the medium at the points lying on the straight lines, $y = x \pm \frac{n\lambda}{2}$ will oscillate along those lines (even n), or at right angles to them (odd n).

Also from (1),

if $$\cos k(y-x) = 0 = \cos(2n+1)\frac{\pi}{2}$$

$$\frac{\xi_1^2}{a^2} = 1 - \xi_2^2/a^2, \text{ a circle.}$$

Thus the particles, at the points, where $y = x \pm (n \pm 1/4)\lambda$, will oscillate along circles. In general, all other particles will move along ellipses.

4.161 The displacement of oscillations is given by $\xi = a\cos(\omega t - kx)$
Without loss of generality, we confine ourselves to $x = 0$. Then the displacement maxima occur at $\omega t = n\pi$. Concentrate at $\omega t = 0$. Now the energy density is given by

$$w = \rho a^2 \omega^2 \sin^2 \omega t \qquad \text{at } x = 0$$

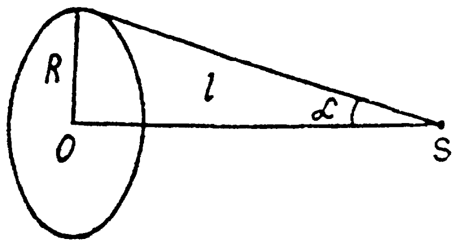
$T/6$ time later (where $T = \frac{2\pi}{\omega}$ is the time period) than $t = 0$.

$$w = \rho a^2 \omega^2 \sin^2\frac{\pi}{3} = \frac{3}{4}\rho a^2\omega^2 = w_0$$

Thus $$\langle w \rangle = \frac{1}{2}\rho a^2 \omega^2 = \frac{2w_0}{3}.$$

4.162 The power output of the source much be

$$4\pi l^2 I_0 = Q \text{ Watt.}$$



The required flux of accoustic power is then : $Q = \dfrac{\Omega}{4\pi}$

Where Ω is the solid angle subtended by the disc enclosed by the ring at S. This solid angle is

$$\Omega = 2\pi\,(1 - \cos\alpha)$$

So flux
$$\Phi = I_0\, I_0 \left(1 - \frac{l}{\sqrt{r^2 + R^2}}\right) 2\pi l^2$$

Substitution gives $\Phi = 2\Pi \times 30 \left(1 - \dfrac{l}{\sqrt{1 + \dfrac{1}{4}}}\right) \mu\text{W} = 1.99\ \mu\,\text{W}.$

Eqn. (1) is a well known result stich is derived as follows; Let SO be the polar axis. Then the required solid angle is the area of that part of the surface a sphere of much radius whose colatitude is $\leq \alpha$.

Thus
$$\Omega = \int_0^{\alpha} 2\pi\,\sin\theta\, d\theta = 2\pi\,(1 - \cos\alpha).$$

4.163 From the result of 4.162 power flowing out through anyone of the opening

$$= \frac{P}{2}\left(1 - \frac{h/2}{\sqrt{R^2 + (h/2)^2}}\right)$$
$$= \frac{P}{2}\left(1 - \frac{h}{\sqrt{4R^2 + h^2}}\right)$$

As total power output equals P, so the power reaching the lateral surface must be.

$$= P - 2\cdot\frac{P}{2}\left(1 - \frac{h}{\sqrt{4R^2 + h^2}}\right) = \frac{ph}{\sqrt{4R^2 + h^2}} = 0\cdot 07\text{W}$$

4.164 We are given

$$\xi = a\cos k x\,\omega t$$

so
$$\frac{\partial \xi}{\partial x} = -a k \sin kx \cos\omega t \quad\text{and}\quad \frac{\partial \xi}{\partial t} = -a\,\omega\cos kx \sin\omega t$$

Thus
$$(\xi)_{t=0} = a\cos kx\,,\ (\xi)_{t=T/2} = -a\cos kx$$
$$\left(\frac{\partial \xi}{\partial x}\right)_{t=0} = a k \sin kx\,,\ \left(\frac{\partial \xi}{\partial x}\right)_{t=T/2} = -a k \sin kx$$

(a) The graphs of (ξ) and $\left(\frac{\partial \xi}{\partial x}\right)$ are as shown in Fig. (35) of the book (p.332).

(b) We can calculate the density as follows :

Take a parallelopiped of cross section unity and length dx with its edges at x and $x + dx$.

After the oscillation the edge at x goes to $x + \xi(x)$ and the edge at $x + dx$ goes to $x + dx + \xi(x + dx)$

$= x + dx + \xi(x) + \frac{\partial \xi}{\partial x} dx$. Thus the volume of the element (originally dx) becomes

$$\left(1 + \frac{\partial \xi}{\partial x}\right) dx$$

and hence the density becomes $\rho = \frac{\rho_0}{1 + \frac{\partial \xi}{\partial x}}$.

On substituting we get for the density $\rho(x)$ the curves shown in Fig.(35). referred to above.

(c) The velocity $v(x)$ at time $t = T/4$ is

$$\left(\frac{\partial \xi}{\partial t}\right)_{t = T/4} = - a\,\omega \cos kx$$

On plotting we get the figure (36).

4.165 Given $\xi = a \cos kx \cos \omega t$

(a) The potential energy density (per unit volume) is the energy of longitudinal strain. This is

$$w_p = \left(\frac{1}{2}\text{stress} \times \text{strain}\right) = \frac{1}{2} E \left(\frac{\partial \xi}{\partial x}\right)^2, \quad \left(\frac{\partial \xi}{\partial x} \text{ is the longitudinal strain}\right)$$

$$w_p = \frac{1}{2} E\, a^2 k^2 \sin^2 kx \cos^2 \omega t$$

But $$\frac{\omega^2}{k^2} = \frac{E}{\rho} \quad \text{or} \quad E k^2 = \rho \omega^2$$

Thus $$w_p = \frac{1}{2} \rho\, a^2 \omega^2 \sin^2 kx \cos^2 \omega t$$

(b) The kinetic energy density is

$$= \frac{1}{2} \rho \left(\frac{\partial \xi}{\partial t}\right)^2 = \frac{1}{2} \rho\, a^2 \omega^2 \cos^2 kx \sin^2 \omega t\,.$$

On plotting we get Fig. 37 given in the book (p. 332). For example at $t = 0$

$$w = w_p + w_k = \frac{1}{2}\rho\, a^2\, \omega^2 \sin^2 k x$$

and the displacement nodes are at $x = \pm \frac{\pi}{2k}$ so we do get the figure.

4.166 Let us denote the displacement of the elements of the string by

$$\xi = a \sin k x \cos \omega t$$

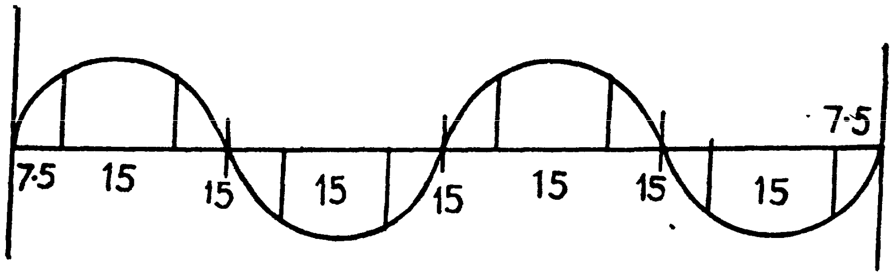
since the string is 120 cm long we must have $k \cdot 120 = n\pi$

If x_1 is the distance at which the displacement amplitude first equals 3·5 mm then

$$a \sin k x_1 = 3{\cdot}5 = a \sin(k x_1 + 15 k)$$

Then $$k x_1 + 15 k = \pi - k x_1 \quad \text{or} \quad k x_1 = \frac{\pi - 15 k}{2}$$

One can convince ourself that the string has the form shown below



It shows that $$k \times 120 = 4\pi, \quad \text{so} \quad k = \frac{\pi}{30}\,\text{cm}^{-1}$$

Thus we are dealing with the third overtone

Also $$k x_1 = \frac{\pi}{4} \quad \text{so} \quad a = 3{\cdot}5\sqrt{2}\ \text{mm} = 4{\cdot}949\ \text{mm}.$$

4.167 We have $n = \frac{1}{2l}\sqrt{\frac{T}{m}} = \frac{1}{2l}\sqrt{\frac{Tl}{M}}$ Where M = total mass of the wire. When the wire is stretched, total mass of the wire remains constant. For the first wire the new length $= l + \eta_1 l$ and for the record wire, the length is $l + \eta_2 l$. Also $T_1 = \alpha(\eta_1 l)$ where α is a constant and $T_2 = \alpha(\eta_2 l)$. Substituting in the above formula.

$$\nu_1 = \frac{1}{2(l + \eta_1 l)}\sqrt{\frac{(\alpha \eta_1 l)(l + \eta_1 l)}{M}}$$

$$\nu_2 = \frac{1}{2(l + \eta_2 l)}\sqrt{\frac{(\alpha \eta_2 l)(l + \eta_2 l)}{M}}$$

$$\therefore \quad \frac{\nu_2}{\nu_1} = \frac{1 + \eta_1}{1 + \eta_2}\sqrt{\frac{\eta_2}{\eta_1} \cdot \frac{1 + \eta_2}{1 + \eta_1}}$$

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{\eta_2(1 + \eta_1)}{\eta_2(1 + \eta_2)}} = \sqrt{\frac{0{\cdot}04(1 + 0{\cdot}02)}{0{\cdot}02(1 + 0{\cdot}04)}} = 1{\cdot}4$$

4.168 Let initial length and tension be l and T respectively.

So, $$\nu_1 = \frac{1}{2l}\sqrt{\frac{T}{\rho_1}}$$

In accordance with the problem, the new length

$$l' = l - \frac{l \times 35}{100} = 0 \cdot 65\, l$$

and new tension, $T' = T + \dfrac{T \times 70}{100} = 1 \cdot 7\, T$

Thus the new frequency

$$\nu_2 = \frac{1}{2l'}\sqrt{\frac{T'}{\rho_1}} = \frac{1}{2 \times 0 \cdot 65\, l}\sqrt{\frac{1 \cdot 7\, T}{\rho_1}}$$

Hence $$\frac{\nu_2}{\nu_1} = \frac{\sqrt{1 \cdot 7}}{0 \cdot 65} = \frac{1 \cdot 3}{0 \cdot 65} = 2$$

4.169 Obviously in this case the velocityof sound propagation

$$v = 2\nu\,(l_2 - l_1)$$

where l_2 and l_1 are consecutive lengths at which resonance occur

In our problem, $(l_2 - l_1) = l$

So $$v = 2\nu l = 2 \times 2000 \times 8 \cdot 5 \text{ cm/s} = 0 \cdot 34 \text{ km/s}.$$

4.170 (a) When the tube is closed at one end

$$\nu = \frac{v}{4l}(2n+1), \quad \text{where} \quad n = 0, 1, 2, \ldots$$

$$= \frac{340}{4 \times 0 \cdot 85}(2n+1) = 100\,(2n+1)$$

Thus for $n = 0, 1, 2, 3, 4, 5, 6, \ldots$, we get

$$n_1 = 100\,1\,H_z,\ n_2 = 300\,1\,H_z,\ n_3 = 500\,1\,H_z,\ n_4 = 700\,1\,H_z,$$

$$n_5 = 900\,1\,H_z,\ n_6 = 1100\,1\,H_z,\ n_7 = 1300\,1\,H_z$$

Since ν should be $< \nu_0 = 1250\,1\,H_z$, we need not go beyond n_6.

Thus 6 natural oscillations are possible.

(b) Organ pipe opened from both ends vibrates with all harmonics of the fundamental frequency. Now, the fundamental mode frequency is given as

$$\nu = v/\lambda$$

or, $$\nu = v/2l$$

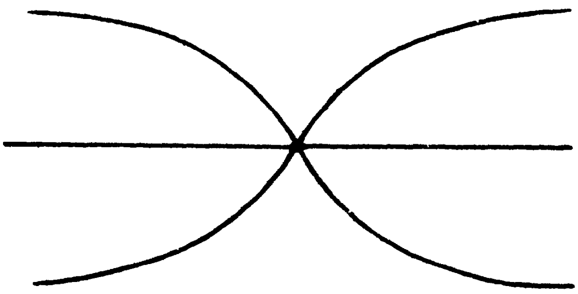
Here, also, end correction has been neglected. So, the frequencies of higher modes of vibrations are given by

$$\nu = n\,(v/2l) \tag{1}$$

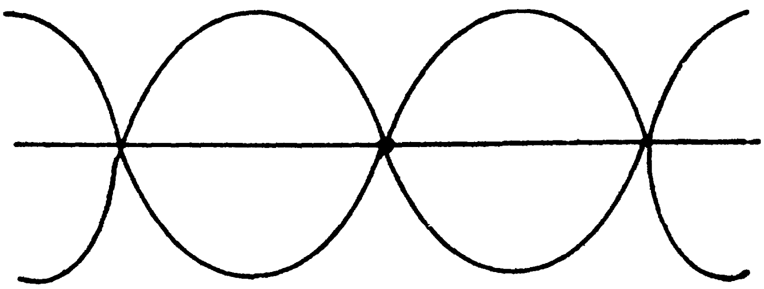
or, $\nu_1 = v/2l, \; \nu_2 = 2(v/2l), \; \nu_3 = 3(v/2l)$

It may be checked by putting the values of n in the equation (1) that below 1285 Hz, there are a total of six possible natural oscillation frequencies of air column in the open pipe.

4.171 Since the copper rod is clamped at mid point, it becomes a mode and the two free ends will be anitinodes. Thus the fundamental mode formed in the rod is as shown in the Fig. (a).



4.171 (a)



4.171 (b)

In this case, $$l = \frac{\lambda}{2}$$

So, $$\nu_0 = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{E}{\rho}}\sqrt{\frac{E}{e}}$$

where E = Young's modules and ρ is the density of the copper

Similarly the second mode or the first overtone in the rod is as shown above in Fig. (b).

Here $$l = \frac{3\lambda}{2}$$

Hence $$\nu_1 = \frac{3v}{2l} = \frac{3}{2l}\sqrt{\frac{E}{\rho}}$$

$$\nu = \frac{2n+1}{2l}\sqrt{\frac{E}{\rho}} \text{ where } n = 0, 1, 2 \ldots$$

Putting the given values of E and ρ in the general equation

$$\nu = 3{\cdot}8(2n+1)\,\text{k Hz}$$

Hence $\nu_0 = 3 \cdot 8$ k.Hz, $\nu_1 = (3{\cdot}8 \times 3)$ k Hz, $\nu_2 = (3{\cdot}8) \times 5 = 19$ k Hz,
$\nu_3 = (3{\cdot}8 \times 7) = 26{\cdot}6$ k Hz, $\nu_4 = (3{\cdot}8 \times 9) = 34{\cdot}2$ k Hz,
$\nu_5 = (3{\cdot}8 \times 11) = 41{\cdot}8$ k Hz, $\nu_6 = (3{\cdot}8) \times 13$ k Hz $= 49{\cdot}4$ k Hz and
$\nu_7 = (3{\cdot}8) \times 14$ k Hz > 50 k Hz.

Hence the sought number of frequencies between 20 to 50 k Hz equals 4.

4.172 Let two waves $\xi_1 = a\cos(\omega t - kx)$ and $\xi_2 = a\cos(\omega t + kx)$, superpose and as a result, we have a standing wave (the resultant wave) in the string of the form $\xi = 2a\cos kx \cos\omega t$.
According to the problem $2a = a_m$.

Hence the standing wave excited in the string is

$$\xi = a_m \cos kx \cos \omega t \tag{1}$$

or,
$$\frac{\partial \xi}{\partial t} = -\omega a_m \cos kx \sin \omega t \tag{2}$$

So the kinetic energy confined in the string element of length dx, is given by :

$$dT = \frac{1}{2}\left(\frac{m}{l}dx\right)\left(\frac{\partial \xi}{\partial t}\right)^2$$

or,
$$dT = \frac{1}{2}\left(\frac{m}{l}dx\right) a_m^2 \omega^2 \cos^2 kx \sin^2 \omega t$$

or,
$$dT = \frac{m a_m^2 \omega^2}{2l} \sin^2 \omega t \cos^2 \frac{2\pi}{\lambda} x \, dx$$

Hence the kinetic energy confined in the string corresponding to the fundamental tone

$$T = \int dT = \frac{m a_m^2 \omega^2}{2l} \sin^2 \omega t \int_0^{\lambda/2} \cos^2 \frac{2\pi}{\lambda} x \, dx$$

Because, for the fundamental tone, length of the string $l = \frac{\lambda}{2}$

Integrating we get, ; $T = \frac{1}{4} m a_m^2 \omega^2 \sin^2 \omega t$

Hence the sought maximum kinetic energy equals, $T_{max} = \frac{1}{4} m a_m^2 \omega^2$,

because for T_{max}, $\sin^2 \omega t = 1$

(ii) Mean kinetic energy averaged over one oscillation period

$$<T> = \frac{\int T\,dt}{\int dt} = \frac{1}{4} m a_m^2 \omega^2 \frac{\int_0^{2\pi/\omega} \sin^2 \omega t \, dt}{\int_0^{2\pi/\omega} dt}$$

or,
$$<T> = \frac{1}{8} m a_m^2 \omega^2 .$$

4.173 We have a standing wave given by the equation

$$\xi = a \sin kx \cos \omega t$$

So,
$$\frac{\partial \xi}{\partial t} = -a\omega \sin kx \sin \omega t \tag{1}$$

and
$$\frac{\partial \xi}{\partial t} = a k \cos kx \cos \omega t \tag{2}$$

The kinetic energy confined in an element of length dx of the rod

$$dT = \frac{1}{2}(\rho S\,dx)\left(\frac{\partial \xi}{\partial t}\right)^2 = \frac{1}{2}\rho S a^2 \omega^2 \sin^2 \omega t \,\sin^2 kx\,dx$$

So total kinetic energy confined into rod

$$T = \int dT = \frac{1}{2}\rho S a^2 \omega^2 \sin^2 \omega t \int_0^{\lambda/2} \sin^2 \frac{2\pi}{\lambda} x\,dx$$

or,
$$T = \frac{\pi S a^2 \omega^2 \rho \sin^2 \omega t}{4k} \tag{3}$$

The potential energy in the above rod element

$$dU = \int \partial U = -\int_0^{\xi} F_\xi \,d\xi, \text{ where } F_\xi = (\rho S\,dx)\frac{\partial^2 \xi}{\partial t^2}$$

or,
$$F_\xi = -(\rho S\,dx)\,\omega^2 \xi$$

so,
$$dU = \omega^2 \rho S\,dx \int_0^{\xi} \xi\,d\xi$$

or,
$$dU = \frac{\rho\,\omega^2 S \xi^2}{2}dx = \frac{\rho\,\omega^2 S a^2 \cos^2 \omega t\, \sin^2 kx\, dx}{2}$$

Thus the total potential energy stored in the rod $U = \int dU$

or,
$$U = \rho\,\omega^2 S a^2 \cos^2 \omega t \int_0^{\lambda/2} \sin^2 \frac{2\pi}{\lambda} x\,dx$$

So,
$$U = \frac{\pi \rho S a^2 \omega^2 \cos^2 \omega t}{4k}$$

To find the potential energy stored in the rod element we may adopt an easier way. We know that the potential energy density confined in a rod under elastic force equals :

$$U_D = \frac{1}{2}(\text{stress} \times \text{strain}) = \frac{1}{2}\sigma\,\varepsilon = \frac{1}{2}Y\varepsilon^2$$

$$= \frac{1}{2}\rho v^2 \varepsilon^2 = \frac{1}{2}\frac{\rho\,\omega^2}{k^2}\varepsilon^2$$

$$= \frac{1}{2}\frac{\rho\,\omega^2}{k^2}\left(\frac{\partial \xi}{\partial x}\right)^2 = \frac{1}{2}\rho a^2 \omega^2 \cos^2 \omega t \cos^2 kx$$

Hence the total potential energy stored in the rod

$$U = \int U_D\, dV = \int_0^{\lambda/2} \frac{1}{2}\rho a^2 \omega^2 \cos^2 \omega t \cos^2 kx\, S\, dx$$

$$= \frac{\pi \rho S a^2 \omega^2 \cos^2 \omega t}{4k} \qquad (4)$$

Hence the sought mechanical energy confined in the rod between the two adjacent nodes

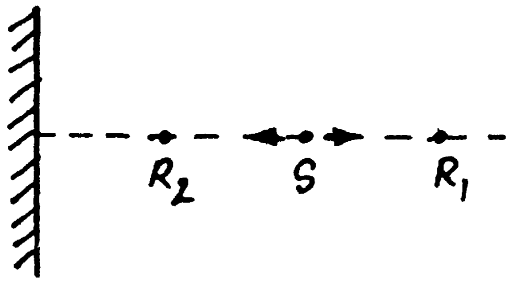
$$E = T + U = \frac{\pi \rho \omega^2 a^2 S}{4k}.$$

4.174 Receiver R_1 registers the beating, due to the sound waves reaching directly to it from source and the other due to the reflection from the wall.

Frequency of sound reaching directly from S to R_1

$$\nu_{S \to R_1} = \nu_0 \frac{v}{v-u} \quad \text{when } S \text{ moves towards } R_1$$

and $$\nu'_{S \to R_1} = \nu_0 \frac{v}{v+u} \quad \text{when } S \text{ moves towards the wall}$$



Now frequency reaching to R_1 after reflection from wall

$$\nu_{W \to R_1} = \nu_0 \frac{v}{v+u}, \text{ when } S \text{ moves towards } R_1$$

and $$\nu'_{W \to R_1} = \nu_0 \frac{v}{v-u}, \text{ when } S \text{ moves towards the wall}$$

Thus the sought beat frequency

$$\Delta\nu = \left(\nu_{S \to R_1} - \nu_{W \to R_1}\right) \text{ or } \left(\nu'_{W \to R_1} - \nu_{S \to R_1}\right)$$

$$= \nu_0 \frac{v}{v-u} - \nu_0 \frac{v}{v+u} = \frac{2\nu_0 v u}{v^2 - u^2} \approx \frac{2u\nu_0}{v} = 1 \text{ Hz}$$

4.175 Let the velocity of tuning fork is u. Thus frequency reaching to the observer due to the tuning fork that approaches the observer

$$\nu' = \nu_0 \frac{v}{v-u} \quad [v = \text{velocity of sound}]$$

Frequency reaching the observer due to the tunning fork that recedes from the observer

$$\nu'' = \nu_0 \frac{v}{v+u}$$

So, Beat frequency $\nu - \nu'' = \nu = \nu_0 v \left(\frac{1}{v-u} - \frac{1}{v+u}\right)$

or, $$\nu = \frac{2\nu_0 v u}{v^2 - u^2}$$

So, $$\nu u^2 + (2 v \nu_0) u - v^2 \nu = 0$$

Hence $$u = \frac{-2\,v\,\nu_0 \pm \sqrt{4\nu_0^2 v^2 + 4\nu^2 v^2}}{2\nu}$$

Hence the sought value of u, on simplifying and noting that $u > 0$

$$u = \frac{v\,\nu_0}{\nu}\left(\sqrt{1+\left(\frac{\nu}{\nu_0}\right)^2} - 1\right)$$

4.176 Obviously the maximum frequency will be heard when the source is moving with maximum velocity towards the receiver and minimum frequency will be heard when the source recedes with maximum velocity. As the source swing harmonically its maximum velocity equals $a\,\omega$. Hence

$$\nu_{max} = \nu_0 \frac{v}{v - a\,\omega} \text{ and } \nu_{min} = \nu_0 \frac{v}{v + a\,\omega}$$

So the frequency band width $\Delta\nu = \nu_{max} - \nu_{min} = \nu_0\,v\left(\frac{2\,a\,\omega}{v^2 - a^2\omega^2}\right)$

or, $$(\Delta\nu\,a^2)\,\omega^2 + (2\,\nu_0\,v\,a)\,\omega - \Delta\nu\,v^2 = 0$$

So, $$\omega = \frac{-2\,\nu_0\,v\,a \pm \sqrt{4\nu_0^2 v^2 a^2 + \Delta\nu^2 a^2 v^2}}{2\,\Delta\nu\,a^2}$$

On simplifying (and taking + sign as $\omega \to 0$ if $\Delta\nu \to 0$)

$$\omega = \frac{v\,\nu_0}{\Delta\nu\,a}\left(\sqrt{1+\left(\frac{\Delta\nu}{\nu_0}\right)^2} - 1\right)$$

4.177 It should be noted that the frequency emitted by the source at time t could not be received at the same moment by the receiver, becouse till that time the source will cover the distance $\frac{1}{2}w\,t^2$ and the sound wave will take the further time $\frac{1}{2}w\,t^2/v$ to reach the receiver. Therefore the frequency noted by the receiver at time t should be emitted by the source at the time $t_1 < t$. Therefore

$$t_1 + \left(\frac{1}{2}w\,t_1^2 / v\right) = t \qquad (1)$$

and the frequency noted by the receiver

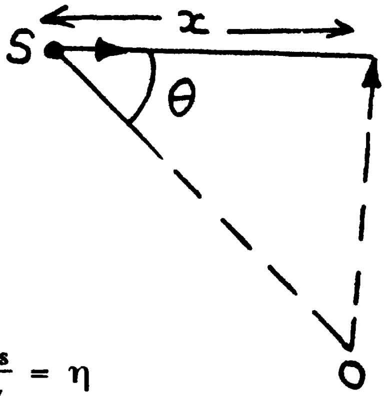
$$\nu = \nu_0 \frac{v}{v + w\,t_1} \qquad (2)$$

Solving Eqns (1) and (2), we get

$$\nu = \frac{\nu_0}{\sqrt{1 + \frac{2\,w\,t}{v}}} = 1\cdot 35 \text{ kHz.}$$

4.178 (a) When the observer receives the sound, the source is closest to him. It means, that frequency is emitted by the source sometimes before (Fig.) Figure shows that the source approaches the stationary observer with velocity $v_s \cos\theta$.

Hence the frequency noted by the observer



$$\nu = \nu_0\left(\frac{v}{v - v_s\cos\theta}\right)$$

$$= \nu_0\left(\frac{v}{v - \eta\, v\cos\theta}\right) = \frac{\nu_0}{1 - \eta\cos\theta} \quad (1)$$

But $\qquad \dfrac{x}{v_s} = \dfrac{\sqrt{l^2 + x^2}}{v}$, So, $\dfrac{x}{\sqrt{l^2 + x^2}} = \dfrac{v_s}{v} = \eta$

or, $$\cos\theta = \eta \quad (2)$$

Hence from Eqns. (1) and (2) the sought frequency

$$\nu = \frac{\nu_0}{1 - \eta^2} = 5 \text{ kHz}$$

(b) When the source is right in front of O, the sound emitted by it will not be Doppler shifted because $\theta = 90°$. This sound will be received at O at time $t = \dfrac{l}{v}$ after the source has passed it. The source will by then have moved ahead by a distance $v_s t = l\eta$. The distance between the source and the observer at this time will be $l\sqrt{1 + \eta^2} = 0.32$ km.

4.179 Frequency of sound when it reaches the wall

$$\nu' = \nu\frac{v + u}{v}$$

wall will reflect the sound with same frequency ν'. Thus frequency noticed by a stationary observer after reflection from wall

$\nu'' = \nu'\dfrac{v}{v - u}$, since wall behaves as a source of frequency ν'.

Thus, $$\nu'' = \nu\frac{v + u}{v}\cdot\frac{v}{v - u} = \nu\frac{v + u}{v - u}$$

or, $$\lambda'' = \lambda\frac{v - u}{v + u} \quad \text{or} \quad \frac{\lambda''}{\lambda} = \frac{v - u}{v + u}$$

So, $$1 - \frac{\lambda''}{\lambda} = 1 - \frac{v - u}{v + u} = \frac{2u}{v + u}$$

Hence the sought percentage change in wavelength

$$= \frac{\lambda - \lambda''}{\lambda} \times 100 = \frac{2u}{v + u} \times 100\,\% = 0.2\% \text{ decrease.}$$

4.180 Frequency of sound reaching the wall.

$$\nu = \nu_0\left(\frac{v-u}{v}\right) \tag{1}$$

Now for the observer the wall becomes a source of frequency ν receding from it with velocity u

Thus, the frequency reaching the observer

$$\nu' = \nu\left(\frac{v}{v+u}\right) = \nu_0\left(\frac{v-u}{v+u}\right) \qquad \text{[Using (1)]}$$

Hence the beat frequency registered by the receiver (observer)

$$\Delta\nu = \nu_0 - \nu' = \frac{2u\nu_0}{v+u} = 0.6 \text{ Hz.}$$

4.181 Intensity of a spherical sound wave emitted from a point source in a homogeneous absorbing medium of wave damping coefficient γ is given by

$$I = \frac{1}{2}\rho a^2 e^{-2\gamma r}\omega^2 v$$

So, Intensity of sound at a distance r_1 from the source

$$= \frac{I_1}{r_1^2} = \frac{1/2\,\rho a^2 e^{-2\gamma r_1}\omega^2 v}{r_1^2}$$

and intensity of sound at a distance r_2 from the source

$$= I_2/r_2^2 = \frac{1/2\,\rho a^2 e^{-2\gamma r_2}\omega^2 v}{r_2^2}$$

But according to the problem $\quad \frac{1}{\eta}\frac{I_1}{r_1^2} = \frac{I_2}{r_2^2}$

So, $\quad \frac{\eta r_1^2}{r_2^2} = e^{2\gamma(r_2-r_1)} \quad$ or $\quad \ln\frac{\eta r_2^2}{r_1^2} = 2\gamma(r_2-r_1)$

or, $$\gamma = \frac{\ln(\eta r_2^2/r_1^2)}{2(r_2-r_1)} = 6\times10^{-3}\text{ m}^{-1}$$

4.182 (a)Loudness level in bells = $\log\frac{I}{I_0}$. (I_0 is the theshold of audibility.)

So, loudness level in decibells, $L = 10\log\frac{I}{I_0}$

Thus loudness level at $x = x_1 = L_{x_1} = 10\log\frac{I_{x_1}}{I_0}$

Similarly $\quad L_{x_2} = 10\log\frac{I_{x_2}}{I_0}$

Thus $\quad L_{x_2} - L_{x_1} = 10\log\frac{I_{x_2}}{I_{x_1}}$

$$\text{or,}\quad L_{x_2} = L_{x_1} + 10\log\frac{1/2\,\rho\,a^2\,\omega^2\,v\,e^{-2\gamma x_2}}{1/2\,\rho\,a^2\,\omega^2\,v\,e^{-2\gamma x_1}} = L_{x_1} + 10\log\, e^{-2\gamma(x_2-x_1)}$$

$$L_{x_2} = L_{x_1} - 20\,\gamma\,(x_2 - x_1)\,\log e$$

$$\text{Hence}\qquad L' = L - 20\,\gamma\,x\log e \quad [\text{ since } (x_2 - x_1) = x\,]$$

$$= 20\,dB - 20 \times 0\cdot 23 \times 50 \times 0\cdot 4343\ dB$$

$$= 60\,dB - 10\,dB = 50\,dB$$

(b) The point at which the sound is not heard any more, the loudness level should be zero. Thus

$$0 = L - 20\,\gamma\,x\log e \quad \text{or} \quad x = \frac{L}{20\,\gamma\log e} = \frac{60}{20\times 0\cdot 23\times 0\cdot 4343} = 300\ \text{m}$$

4.183 (a) As there is no damping, so

$$L_{r_0} = 10\log\frac{I}{I_0} = 10\ \log\frac{1/2\,\rho\,a^2\,\omega^2\,v/r_0^2}{1/2\,\rho\,a^2\,\omega^2\,v} = -\ 20\ \log r_0$$

Similarly $L_r = -\ 20\log r$

$$\text{So,}\qquad L_r - L_{r_0} = 20\ \log(r_0/r)$$

$$\text{or, } L_r = L_{r_0} + 20\log\left(\frac{r_0}{r}\right) = 30 + 20\times\log\frac{20}{10} = 36\,dB$$

(b) Let r be the sought distance at which the sound is not heard.

$$\text{So,}\quad L_r = L_{r_0} + 20\log\frac{r_0}{r} = 0 \quad \text{or,}\quad L_{r_0} = 20\log\frac{r}{r_0} \quad \text{or}\quad 30 = 20\log\frac{r}{20}$$

$$\text{So,}\qquad \log_{10}\frac{r}{20} = 3/2 \quad \text{or}\quad 10^{(3/2)} = r/20$$

$$\text{Thus}\qquad r = 200\sqrt{10} = 0\cdot 63\ \text{Km.}$$

Thus for $r > 0\cdot 63$ km no sound will be heard.

4.184 We treat the fork as a point source. In the absence of damping the oscillation has the form

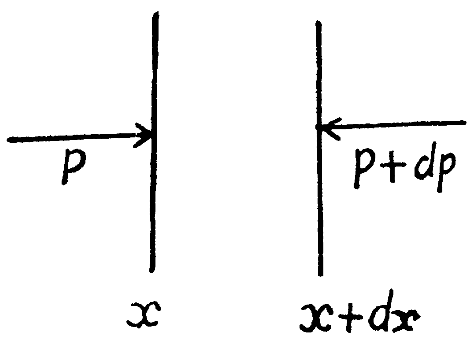
$$\frac{\text{Const.}}{r}\cos(\omega\,t - k\,r)$$

Because of the damping of the fork the amplitude of oscillation decreases exponentially with the retarded time (i.e. the time at which the wave started from the source.). Thus we write for the wave amplitude.

$$\xi = \frac{\text{Const.}}{r}\ e^{-\beta\left(t-\frac{r}{v}\right)}$$

This means that

$$\frac{e^{-\beta\left(t+\tau-\frac{r_A}{v}\right)}}{r_A} = \frac{e^{-\beta\left(t+\tau-\frac{r_B}{v}\right)}}{r_B}$$



Thus $$e^{-\beta\left(\tau+\frac{r_B-r_A}{v}\right)} = \frac{r_A}{r_B} \quad \text{or } \beta = \frac{\ln\frac{r_B}{r_A}}{\tau+\frac{r_B-r_A}{v}} = 0.12.\ s^{-1}$$

4.185 (a) Let us consider the motion of an element of the medium of thickness dx and unit area of cross-section. Let ξ = displacement of the particles of the medium at location x. Then by the equation of motion

$$\rho\, dx\, \ddot{\xi} = -dp$$

where dp is the pressure increment over the length dx

Recalling the wave equation

$$\ddot{\xi} = v^2 \frac{\partial^2 \xi}{\partial x^2}$$

we can write the foregoing equation as

$$\rho v^2 \frac{\partial^2 \xi}{\partial x^2} dx = -dp$$

Integrating this equation, we get

$$\Delta p = \text{surplus pressure} = -\rho v^2 \frac{\partial \xi}{\partial x} + \text{Const.}$$

In the absence of a deformation (a wave), the surplus pressure is $\Delta p = 0$. So 'Const' = 0 and

$$\Delta p = -\rho v^2 \frac{\partial \xi}{\partial x}.$$

(b) We have found earlier that

$$w = w_k + w_p = \text{total energy density}$$

$$w_k = \frac{1}{2}\rho\left(\frac{\partial \xi}{\partial t}\right)^2, \quad w_p = \frac{1}{2}E\left(\frac{\partial \xi}{\partial x}\right)^2 = \frac{1}{2}\rho v^2\left(\frac{\partial \xi}{\partial x}\right)^2$$

It is easy to see that the space-time average of both densities is the same and the space time average of total energy density is then

$$\langle w \rangle = \left\langle \rho v^2 \left(\frac{\partial \xi}{\partial x}\right)^2 \right\rangle$$

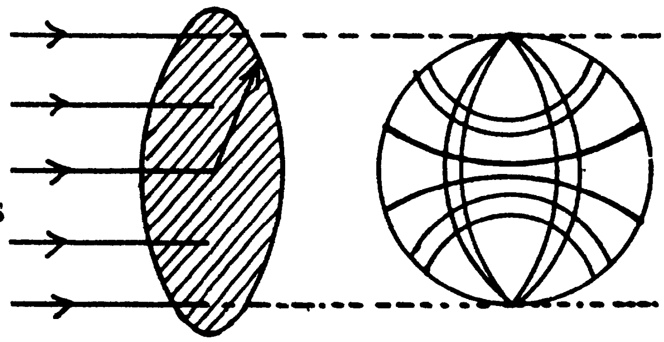
The intensity of the wave is

$$I = v \langle w \rangle = \left\langle \frac{(\Delta p)^2}{\rho v} \right\rangle$$

Using $$\left\langle (\Delta p)^2 \right\rangle = \frac{1}{2}(\Delta p)_m^2 \quad \text{we get} \quad I = \frac{(\Delta p)_m^2}{2\rho v}.$$

4.186 The intensity of the sound wave is

$$I = \frac{(\Delta p)_m^2}{2\rho v} = \frac{(\Delta p)_m^2}{2\rho\nu\lambda}$$



Using $v = \nu\lambda$, ρ is the density of air.

Thus the mean energy flow reaching the ball is

$$\pi R^2 I = \pi R^2 \frac{(\Delta p)_m^2}{2\rho\nu\lambda}$$

πR^2 being the effective area (area of cross section) of the ball.

Substitution gives 10.9 mW.

4.187 We have $\dfrac{P}{4\pi r^2}$ = intensity $= \dfrac{(\Delta p)_m^2}{2\rho v}$

or $$(\Delta p)_m = \sqrt{\frac{\rho v P}{2\pi r^2}}$$

$$= \sqrt{\frac{1{\cdot}293\ \text{kg/m}^3 \times 340\ \text{m/s} \times 0{\cdot}80\ \text{w}}{2\pi \times 1{\cdot}5 \times 1{\cdot}5\ \text{m}^2}} = \sqrt{\frac{1{\cdot}293 \times 340 \times {\cdot}8}{2\pi \times 1{\cdot}5 \times 1{\cdot}5}} \left(\frac{\text{kg kg m}^2\,\text{s}^{-3}\,\text{m s}^{-1}}{m^5}\right)^{\frac{1}{2}}$$

$$= 4{\cdot}9877 \left(\text{kg m}^{-1}\,\text{s}^{-2}\right) = 5\ \text{Pa}.$$

$$\frac{(\Delta p)_m}{p} = 5 \times 10^{-5}$$

(b) We have $$\Delta p = -\rho v^2 \frac{\partial \xi}{\partial x}$$

$$(\Delta p)_m = \rho v^2 k\,\xi_m = \rho v\, 2\pi\nu\,\xi_m$$

$$\xi_m = a = \frac{(\Delta p)_m}{2\pi\rho v\nu} = \frac{5}{2\pi \times 1{\cdot}293 \times 340 \times 600} = 3\ \mu\text{m}$$

$$\frac{\xi_m}{\lambda} = \frac{3 \times 10^{-6}}{340/600} = \frac{180\cancel{0}}{34\cancel{0}} \times 10^{-6} = 5 \times 10^{-6}$$

4.188 Express L in bels. (i.e. $L = 5$ bels).

Then the intensity at the relevant point (at a distance r from the source) is : $I_0{\cdot}10^L$

Had there been no damping the intensity would have been : $e^{2\gamma r}\, I_0{\cdot}10^L$

Now this must equal the quantity

$\dfrac{P}{4\pi r^2}$, where P = sonic power of the source.

Thus $$\frac{P}{4\pi r^2} = e^{2\gamma r}\, I_0{\cdot}10^L$$

or $$P = 4\pi r^2 e^{2\gamma r} I_0{\cdot}10^L = 1.39\ \text{W}.$$

4.4 ELECTROMAGNETIC WAVES. RADIATION

4.189 The velocity of light in a medium of relative permittivity ε is $\frac{c}{\sqrt{\varepsilon}}$. Thus the change in wavelength of light (from its value in vaccum to its value in the medium) is

$$\Delta\lambda = \frac{c/\sqrt{\varepsilon}}{v} - \frac{c}{v} = \frac{c}{v}\left(\frac{1}{\sqrt{\varepsilon}} - 1\right) = -50 \text{ m.}$$

4.190 From the data of the problem the relative permittivity of the medium varies as

$$\varepsilon(x) = \varepsilon_1 e^{-(x/l)\ln\frac{\varepsilon_1}{\varepsilon_2}}$$

Hence the local velocity of light

$$v(x) = \frac{c}{\sqrt{\varepsilon(x)}} = \frac{c}{\sqrt{\varepsilon_1}} e^{\frac{x}{2l}\ln\frac{\varepsilon_1}{\varepsilon_2}}$$

Thus the required time $t = \int_0^l \frac{dx}{v(x)} = \frac{\sqrt{\varepsilon_1}}{c}\int_0^l e^{-\frac{x}{2l}\ln\frac{\varepsilon_1}{\varepsilon_2}}$

$$dx = \frac{\sqrt{\varepsilon_1}}{c}\,\frac{-e^{-\frac{1}{2}\ln\frac{\varepsilon_1}{\varepsilon_2}} + 1}{\frac{1}{2l}\ln\frac{\varepsilon_1}{\varepsilon_2}} = \frac{2l}{c}\,\frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\ln\frac{\varepsilon_1}{\varepsilon_2}}$$

4.191 Conduction current density $= \sigma\vec{E}$

Displacement current density $= \frac{\partial\vec{D}}{\partial t} = \varepsilon\varepsilon_0\frac{\partial\vec{E}}{\partial t} = i\omega\varepsilon\varepsilon_0\vec{E}$

Ratio of magnitudes $= \frac{\sigma}{\omega\varepsilon\varepsilon_0} = \frac{j_c}{j_{dis}} = 2$, on puting the values.

4.192 $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu_0\frac{\partial\vec{H}}{\partial t}$

$$= \nabla\cos(\omega t - \vec{k}\cdot\vec{r})\times\vec{E}_m = \vec{k}\times\vec{E}_m\sin(\omega t - \vec{k}\cdot\vec{r})$$

At $\vec{r} = 0$

$$\frac{\partial\vec{H}}{\partial t} = -\frac{\vec{k}\times\vec{E}_m}{\mu_0}\sin\omega t$$

So integrating (ignoring a constant) and using $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$

$$\vec{H} = \frac{\vec{k}\times\vec{E}_m}{\mu_0}\cos ckt\times\frac{1}{ck} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,\frac{\vec{k}\times\vec{E}_m}{k}\cos ckt$$

4.193 As in the previous problem

$$\vec{H} = \frac{\vec{k} \times \vec{E}_m}{\mu_0 \omega} \cos(\omega t - \vec{k} \cdot \vec{r}) = \frac{E_m}{\mu_0 c} \hat{e}_z \cos(kx - \omega t)$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} E_m \hat{e}_z \cos(kx - \omega t)$$

Thus

(a) at $t = 0$ $\vec{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_m \hat{e}_z \cos kx$

(b)at $t = t_0$, $\vec{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_m \hat{e}_z \cos(kx - \omega t_0)$

4.194 $\xi_{ind} = \oint \vec{E} \cdot d\vec{l} = E_m l (\cos \omega t - \cos(\omega t - kl)$

$$= -2 E_m l \sin \frac{\omega l}{2c} \sin\left(\omega t - \frac{\omega l}{2c}\right)$$

Putting the values $E_m = 50$ m V/m, $l = \frac{1}{2}$ metre

$$\frac{\omega l}{c} = \frac{2\pi \nu l}{c} = \frac{\pi \times 10^8}{3 \times 10^8} = \frac{\pi}{3}$$

$$\xi_{ind} = 50 \text{ m} V\left(-\sin\frac{\pi}{6}\right) \sin\left(\omega t - \frac{\pi}{6}\right)$$

$$= -25 \sin\left(\omega t + \frac{\pi}{6} - \frac{\pi}{2}\right) = 25 \cos\left(\omega t - \frac{\pi}{3}\right) \text{m}V$$

4.195 $\vec{E} = \hat{j} E(t, x)$

$$\vec{B} = \hat{k} B(t_1 x)$$

and
$$\text{Curl } \vec{E} = \hat{k} \frac{\partial E}{\partial x} = -\frac{\partial \vec{B}}{\partial t} = -\hat{k} \frac{\partial B}{\partial t}$$

so
$$-\frac{\partial E}{\partial x} = \frac{\partial B}{\partial t}$$

Also
$$\text{Curl } \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

and
$$\text{Curl } \vec{B} = -\hat{j} \frac{\partial B}{\partial x} \quad \text{so} \quad \frac{\partial B}{\partial x} = -\frac{1}{c^2} \frac{\partial E}{\partial t}.$$

4.196 $\vec{E} = \vec{E}_m \cos(\omega t - \vec{k} \cdot \vec{r})$ then as before

$$\vec{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{\vec{k} \times \vec{E}_m}{k} \cos(\omega t - \vec{k} \cdot \vec{r})$$

so $$\vec{S} = \vec{E} \times \vec{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\ \vec{E}_m \times (\vec{k} \times \vec{E}_m) \frac{1}{k} \cos^2(\omega t - \vec{k} \cdot \vec{r})$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}}\ \vec{E}_m^2 \frac{\vec{k}}{k} \cos^2(\omega t - \vec{k} \cdot \vec{r})$$

$$\langle \vec{S} \rangle = \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}}\ \vec{E}_m^2 \frac{\vec{k}}{k}$$

4.197 $E = E_m \cos(2\pi \nu t - kx)$

(a) $$j_{dis} = \frac{\partial D}{\partial t} = -2\pi \varepsilon_0 \nu E_m \sin(\omega t - kx)$$

Thus $$(j_{dis})_{rms} = \left\langle j_{dis}^2 \right\rangle^{1/2}$$

$$= \sqrt{2}\ \pi \varepsilon_0 \nu E_m = 0.20\ \text{mA/m}^2.$$

(b) $\langle S_x \rangle = \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}}\ E_m^2$ as in (196). Thus $\langle S_x \rangle = 3.3\ \mu\,\text{W/m}^2$

4.198 For the Poynting vector we can derive as in (196)

$\langle S \rangle = \frac{1}{2} \sqrt{\frac{\varepsilon \varepsilon_0}{\mu_0}}\ E_m^2$ along the direction of propagation.

Hence in time t (which is much longer than the time period T of the wave), the energy reaching the ball is

$$\pi R^2 \times \frac{1}{2} \sqrt{\frac{\varepsilon \varepsilon_0}{\mu_0}}\ E_m^2 \times t = 5\ \text{kJ}.$$

4.199 Here $\vec{E} = \vec{E}_m \cos kx \cos \omega t$

From div $\vec{E} = 0$ we get $E_{mx} = 0$ so $\vec{E}_m$ is in the $y-z$ plane.

Also

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} = -\nabla \cos kx \times \vec{E}_m \cos \omega t$$

$$= \vec{k} \times \vec{E}_m \sin kx \cos \omega t$$

so $$\vec{B} = \frac{\vec{k} \times \vec{E}_m}{\omega} \sin kx \sin \omega t = \vec{B}_m \sin kx \sin \omega t$$

Where $\left| \vec{B}_m \right| = \frac{E_m}{c}$ and $\vec{B}_m \perp \vec{E}_m$ in the $y-z$ plane.

At $t = 0,\ \vec{B} = 0,\ E = E_m \cos kx$

At $t = T/4\ \ \vec{E} = 0,\ B = B_m \sin kx$

4.200 $\vec{E} = \vec{E}_m \cos kx\,\omega t$

$$\vec{H} = \frac{\vec{k}\times\vec{E}_m}{\mu_0\,\omega} \sin kx \sin\omega t \quad \text{(exactly as in 199)}$$

$$\vec{S} = \vec{E}\times\vec{H} = \frac{\vec{E}_m\times(\vec{k}\times\vec{E}_m)}{\mu_0\,\omega}\,\frac{1}{4}\sin 2kx\,\sin 2\omega t$$

Thus
$$S_x = \frac{1}{4}\,\varepsilon_0\, c\, E_m^2 \sin 2kx \sin 2\omega t \quad \left(\text{as } \frac{1}{\mu_0 c} = \varepsilon_0 c\right)$$

$$\langle S_x\rangle = 0$$

4.201 Inside the condenser the peak electrical energy $W_e = \frac{1}{2}\,C\,V_m^2$

$$= \frac{1}{2}V_m^2\,\frac{\varepsilon_0\,\pi R^2}{d}$$

(d = separation between the plates, πR^2 = area of each plate.).

$V = V_m \sin\omega t$, V_m is the maximum voltage

Changing electric field causes a displacement current

$$j_{dis} = \frac{\partial D}{\partial t} = \varepsilon_0 E_m\,\omega\cos\omega t$$

$$= \frac{\varepsilon_0\,\omega\, V_m}{d}\cos\omega t$$

This gives rise to a magnetic field $B(r)$ (at a radial distance r from the centre of the plate)

$$B(r)\cdot 2\pi r = \mu_0\,\pi r^2 j_{dis} = \mu_0\,\pi r^2\,\frac{\varepsilon_0\,\omega\,V_m}{d}\cos\omega t$$

$$B = \frac{1}{2}\,\varepsilon_0\,\mu_0\,\omega\,\frac{r}{d}\,V_m\cos\omega t$$

Energy associated with this field is

$$= \int d^3r\,\frac{B^2}{2\mu_0} = \frac{1}{8}\,\varepsilon_0^2\,\mu_0\,\frac{\omega^2}{d^2}\,2\pi\int_0^R r^2\,r\,dr\times d\times V_m^2\cos^2\omega t$$

$$= \frac{1}{16}\,\pi\,\varepsilon_0^2\,\mu_0\,\frac{\omega^2 R^4}{d}\,V_m^2\cos^2\omega t$$

Thus the maximum magnetic energy

$$W_m = \frac{\varepsilon_0^2\,\mu_0}{16}\,(\omega R)^2\,\frac{\pi R^2}{d}\,V_m^2$$

Hence
$$\frac{W_m}{W_e} = \frac{1}{8}\,\varepsilon_0\,\mu_0\,(\omega R)^2 = \frac{1}{8}\left(\frac{\omega R}{c}\right)^2 = 5\times 10^{-15}$$

The approximation are valid only if $\omega R \ll c$.

4.202 Here $I = I_m \cos \omega t$, then the peak magnetic energy is

$$W_m = \frac{1}{2} L I_m^2 = \frac{1}{2} \mu_0 n^2 I_m^2 \pi R^2 d$$

Changing magnetic field induces an electric field which by Faraday's law is given by

$$E \cdot 2\pi r = -\frac{d}{dt} \int \vec{B} \cdot d\vec{S} = \pi r^2 \mu_0 n I_m \omega \sin \omega t$$

$$E = \frac{1}{2} r \mu_0 n I_m \omega \sin \omega t$$

The associated peak electric energy is

$$W_e = \int \frac{1}{2} \varepsilon_0 E^2 d^3 r = \frac{1}{8} \varepsilon_0 \mu_0^2 n^2 I_m^2 \omega^2 \sin^2 \omega t \times \frac{\pi R^4}{2} d$$

Hence $$\frac{W_e}{W_m} = \frac{1}{8} \varepsilon_0 \mu_0 (\omega R)^2 = \frac{1}{8} \left(\frac{\omega R}{c} \right)^2$$

Again we expect the results to be valid if and only if

$$\left(\frac{\omega R}{c} \right) << 1$$

4.203 If the charge on the capacitor is Q, the rate of increase of the capacitor's energy

$$= \frac{d}{dt} \left(\frac{1}{2} \frac{Q^2}{C} \right) = \frac{Q \dot{Q}}{C} = \frac{d}{\varepsilon_0 \pi R^2} Q \dot{Q}$$

Now electric field between the plates (inside it) is, $E = \dfrac{Q}{\pi R^2 \varepsilon_0}$.

So displacement current $= \dfrac{\partial D}{\partial t} = \dfrac{\dot{Q}}{\pi R^2}$

This will lead to a magnetic field, (circuital) inside the plates. At a radial distance r

$$2 \pi r H_\theta (r) = \pi r^2 \frac{\dot{Q}}{\pi R^2} \quad \text{or} \quad H_\theta = \frac{\dot{Q} r}{2 \pi R^2}$$

Hence $H_\theta (R) = \dfrac{\dot{Q}}{2 \pi R}$ at the edge.

Thus inward Poynting vector $= S = \dfrac{\dot{Q}}{2 \pi R} \times \dfrac{Q}{\pi R^2 \varepsilon_0}$

Total flow $= 2 \pi R d \times S = \dfrac{Q \dot{Q} d}{\pi R^2 \varepsilon_0}$ Proved

4.204 Suppose the radius of the conductor is R_0. Then the conduction current density is

$$j_c = \frac{I}{\pi R_0^2} = \sigma E \quad \text{or} \quad E = \frac{I}{\pi R_0^2 \sigma} = \frac{\rho I}{\pi R_0^2}$$

where $\rho = \dfrac{1}{\sigma}$ is the resistivity.

Inside the conductor there is a magnetic field given by

$$H \cdot 2\pi R_0 = I \quad \text{or} \quad H = \frac{I}{2\pi R_0} \text{ at the edge}$$

$\therefore$ Energy flowing in per second in a section of length l is

$$EH \times 2\pi R_0 l = \frac{\rho I^2 l}{\pi R_0^2}$$

But the resistance $R = \dfrac{\rho l}{\pi R_0^2}$

Thus the energy flowing into the conductor $= I^2 R$.

4.205 Here $n e \mathrm{v} = I/\pi R^2$

where R = radius of cross section of the conductor and n = charge density (per unit volume)

Also $$\frac{1}{2} m \mathrm{v}^2 = eU \quad \text{or} \quad \mathrm{v} = \sqrt{\frac{2eU}{m}}\,.$$

Thus, the moving protons have a charge per unit length

$$= n e \pi R^2 = I \sqrt{\frac{m}{2eU}}\,.$$

This gives rise to an electric field at a distance r given by

$$E = \frac{I}{\varepsilon_0} \sqrt{\frac{m}{2eU}} \Big/ 2\pi r$$

The magnetic field is $H = \dfrac{I}{2\pi r}$ (for $r > R$)

Thus

$$S = \frac{I^2}{\varepsilon_0 4\pi^2 r^2} \sqrt{\frac{m}{2eU}} \text{ radially outward from the axis}$$

This is the Poynting vector.

4.206 Within the solenoid $B = \mu_0 n I$ and the rate of change of magnetic energy

$$= \dot{W}_m = \frac{d}{dt}\left(\frac{1}{2}\mu_0 n^2 I^2 \pi R^2 l\right) = \mu_0 n^2 \pi R^2 l I \dot{I}$$

where R = radius of cross section of the solenoid l = length.

Also $H = B/\mu_0 = nI$ along the axis within the solenoid.

By Faraday's law, the induced electric field is

$$E_\theta\, 2\pi r = \pi r^2 \dot{B} = \pi r^2 \mu_0 n \dot{I}$$

or $$E_\theta = \frac{1}{2}\mu_0 n \dot{I} r$$

so at the edge $E_\theta(R) = \frac{1}{2}\mu_0 n \dot{I} R$ (circuital)

Then $$S_r = E_\theta H_z \text{ (radially inward)}$$

and $$\dot{W}_m = \frac{1}{2}\mu_0 n^2 I \dot{I} R \times 2\pi R l = \mu_0 n^2 \pi R^2 l I \dot{I} \text{ as before.}$$

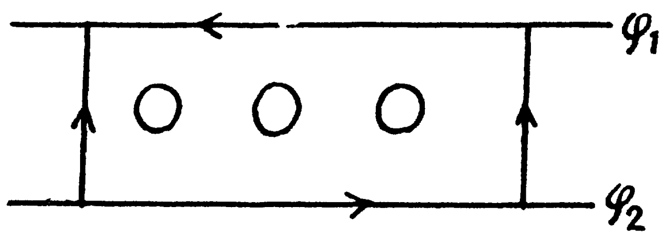
4.207 Given $\varphi_2 > \varphi_1$

The electric field is as shown by the dashed lines (----→-----).

The magnetic field is as shown (⊙) emerging out of the paper.

$\vec{S} = \vec{E} \times \vec{H}$ is parallel to the wires and towards right.

Hence source must be on the left.



4.208 The electric field (----→) and the magnetic field (H →) are as shown.

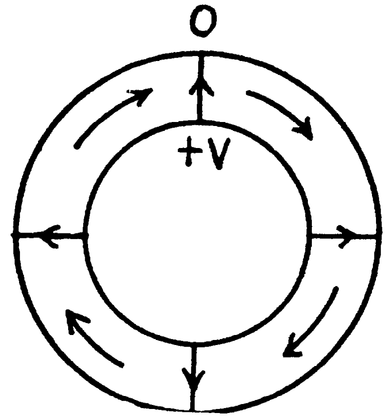
The electric field by Gauss's theorem is like

$$E_r = \frac{A}{r}$$

Integrating $$\varphi = A \ln\frac{r_2}{r}$$

so $$A = \frac{V}{\ln\frac{r_2}{r_1}} \quad (r_2 > r_1)$$

Then $$E = \frac{V}{r\ln\frac{r_2}{r_1}}$$



Magnetic field is $$H_\theta = \frac{I}{2\pi r}$$

The Poynting vector S is along the Z axis and non zero between the two wires $(r_1 < r < r_2)$. The total power flux is

$$= \int_{r_1}^{r_2} \frac{IV}{2\pi r^2 \ln\frac{r_2}{r_1}} \cdot 2\pi r\, dr = IV$$

4.209 As in the previous problem

$$E_r = \frac{V_0 \cos\omega t}{r\ln\frac{r_2}{r_1}} \quad \text{and} \quad H_\theta = \frac{I_0\cos(\omega t - \varphi)}{2\pi r}$$

Hence time averaged power flux (along the z axis) $= \frac{1}{2} V_0 I_0 \cos\varphi$

On using $$< \cos \omega t \cos(\omega t - \varphi) > = \frac{1}{2} \cos \varphi \ .$$

4.210 Let $\vec{n}$ be along the z axis. Then

$$S_{1n} = E_{1x} H_{1y} - E_{1y} H_{1x}$$

and $$S_{2n} = E_{2x} H_{2y} - E_{2y} H_{2x}$$

Using the boundary condition $E_{1t} = E_{2t}$, $H_{1t} = H_{2t}$ at the boundary ($t = x$ or y) we see that

$$S_{1n} = S_{2n} \ .$$

4.211 $P \cdot \alpha \ |\ddot{\vec{p}}|^2$ when

$$\vec{p} = \sum e_i \vec{r}_i = \sum \frac{e_i}{m_i} m_i \vec{r}_i = \frac{e}{m} \sum m_i \vec{r}_i$$

if $$\frac{e_i}{m_i} = \frac{e}{m} = \text{fixed}$$

But $$\frac{d^2}{dt^2} \sum m_i \vec{r}_i = 0 \quad \text{for a closed system}$$

Hence $$P = 0 \ .$$

4.212 $P = \frac{1}{4 \pi \varepsilon_0} \frac{2 (\ddot{\vec{p}})^2}{3 c^3}$

$$|\ddot{\vec{p}}|^2 = (e \omega^2 a)^2 \cos^2 \omega t$$

Thus $$< P > = \frac{1}{4 \pi \varepsilon_0} \frac{2}{3 c^3} (e \omega^2 a)^2 \times \frac{1}{2} = \frac{e^2 \omega^4 a^2}{12 \pi \varepsilon_0 c^3} = 5.1 \times 10^{-15} \text{ W.}$$

4.213 Here

$$\ddot{\vec{p}} = \frac{e}{m} \times \text{force} = \frac{e^2 q}{m R^2} \frac{1}{4 \pi \varepsilon_0} .$$

Thus $$P = \frac{1}{(4 \pi \varepsilon_0)^3} \left(\frac{e^2 q}{m R^2} \right)^2 \frac{2}{3 c^3} .$$

4.214 Most of the radiation occurs when the moving particle is closest to the stationary particle. In that region, we can write

$$R^2 = b^2 + \mathrm{v}^2 t^2$$

and apply the previous problem's formula

Thus $$\Delta W \approx \frac{1}{(4 \pi \varepsilon_0)^3} \frac{2}{3 c^3} \int_{-\infty}^{\infty} \left(\frac{q e^2}{m} \right)^2 \frac{dt}{(b^2 + \mathrm{v}^2 t^2)^2}$$

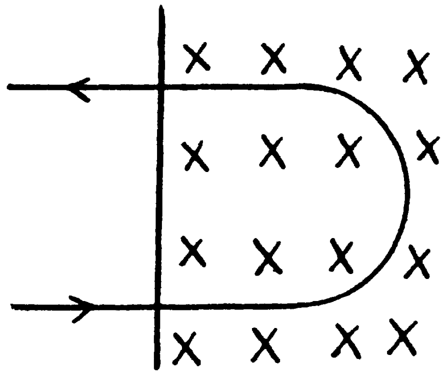
(the integral can be taken between $\pm \infty$ with little error.)

Now $$\int_{-\infty}^{\infty} \frac{dt}{(b^2+v^2t^2)^2} = \frac{1}{v}\int_{-\infty}^{\infty} \frac{dx}{(b^2+x^2)^2} = \frac{\pi}{2vb^3}.$$

Hence, $$\Delta W \approx \frac{1}{(4\pi\varepsilon_0)^3} \frac{\pi q^2 e^4}{3c^3 m^2 v b^3}.$$

4.215 For the semicircular path on the right

$$\frac{mv^2}{R} = Bev \quad \text{or} \quad v = \frac{BeR}{m}.$$



Thus K.E. $= T = \frac{1}{2}mv^2 = \frac{B^2e^2R^2}{2m}$.

Power radiated $= \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{ev^2}{R}\right)^2$

Hence energy radiated $= \Delta W$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{B^2e^3R}{m^2}\right)^2 \cdot \frac{\pi R}{BeR} m = \frac{B^3e^5R^2}{6\varepsilon_0 m^3 c^3}$$

So $$\frac{\Delta W}{T} = \frac{Be^3}{3\varepsilon_0 c^3 m^2} = 2.06 \times 10^{-18}.$$

(neglecting the change in v due to radiation, correct if $\Delta W/T \ll 1$).

4.216 $R = \frac{mv}{eB}$.

Then $$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{ev^2}{R}\right)^2 = \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{e^2Bv}{m}\right)^2$$
$$= \frac{1}{3\pi\varepsilon_0 c^3}\left(\frac{B^2e^4}{m^3}\right)T$$

This is the radiated power so

$$\frac{dT}{dt} = -\frac{B^2e^4}{3\pi\varepsilon_0 m^3 c^3}T$$

Integrating, $T = T_0\, e^{-t/\tau}$

$$\tau = \frac{3\pi\varepsilon_0 m^3 c^3}{B^2 e^4}$$

τ is $(1836)^3 \approx 10^{10}$ times less for an electron than for a proton so electrons radiate away their energy much faster in a magnetic field.

4.217 P is a fixed point at a distance l from the equilibrium position of the particle. Because $l \gg a$, to first order in $\frac{a}{l}$ the distance between P and the instantaneous position of the particle is still l. For the first case $y = 0$ so $t = T/4$

The corresponding retarded time is $t' = \frac{T}{4} - \frac{l}{c}$

Now $$\ddot{y}(t') = -\omega^2 a \cos \omega\left(\frac{I}{4} - \frac{l}{c}\right) = -\omega^2 a \sin \frac{\omega l}{c}$$

For the second case $y = a$ at $t = 0$ so at the retarded time $t' = -\frac{\omega l}{c}$

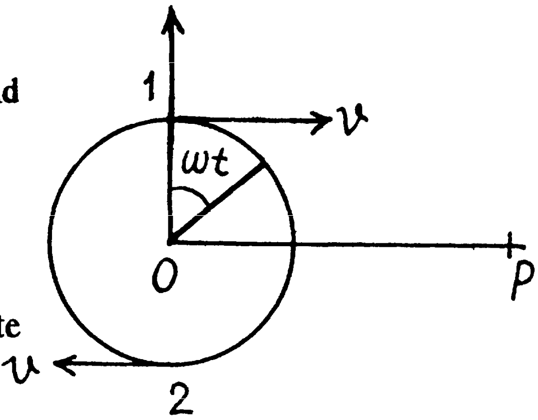
Thus $$\ddot{y}(t') = -\omega^2 a \cos \frac{\omega l}{c}$$

The radiation fluxes in the two cases are proportional to $(\ddot{y}(t'))^2$ so

$$\frac{S_1}{S_2} = \tan^2 \frac{\omega l}{c} = 3.06 \text{ on substitution.}$$

Note : The radiation received at P at time t depends on the acceleration of the charge at the retarded time.

4.218 Along the circle $x = R \sin \omega t$, $y = R \cos \omega t$



where $\omega = \frac{\text{v}}{R}$. If t is the parameter in $x(t), y(t)$ and t' is the observer time then

$$t' = t + \frac{l - x(t)}{c}$$

where we have neglected the effect of the y—cordinate which is of second order. The observed cordinate are

$$x'(t') = x(t),\ y'(t') = y(t)$$

Then $$\frac{dy'}{dt'} = \frac{dy}{dt'} = \frac{dt}{dt'}\frac{dy}{dt} = \frac{-\omega R \sin \omega t}{1 - \frac{\omega R}{c} \cos \omega t} = \frac{-\omega x}{1 - \frac{\omega y}{c}} = \frac{-\text{v}\, x/R}{1 - \frac{\text{v}\, y}{cR}}$$

and $$\frac{d^2 y'}{dt'^2} = \frac{dt}{dt'}\frac{d}{dt}\left(\frac{-\text{v}\, x/R}{1 - \frac{\text{v}\, y}{cR}}\right)$$

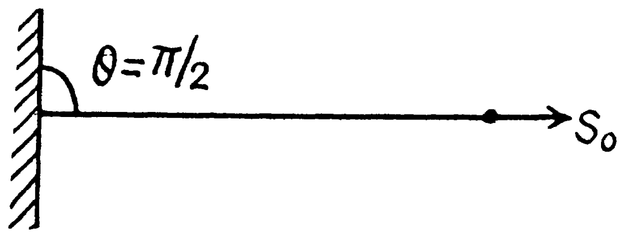
$$= \frac{1}{1 - \frac{\text{v}\, y}{cR}}\left\{\frac{-\frac{\text{v}^2}{R^2} y}{1 - \frac{\text{v}\, y}{cR}} + \frac{\frac{\text{v}\, x}{R}\left(\frac{\text{v}^2}{cR^2} x\right)}{\left(1 - \frac{\text{v}\, y}{cR}\right)^2}\right\} = \frac{\frac{\text{v}^2}{R}\left(\frac{\text{v}}{c} - \frac{y}{R}\right)}{\left(1 - \frac{\text{v}\, y}{cR}\right)^3}.$$

This is the observed acceleration.

(b) Energy flow density of EM radiation S is proportional to the square of the y– projection of the observed accèleration of the particle $\left(\text{i.e. } \dfrac{d^2 y'}{dt'^2}\right)$.

Thus
$$\frac{S_1}{S_2} = \left[\frac{\left(\frac{v}{c}-1\right)}{\left(1-\frac{v}{c}\right)^3} \Bigg/ \frac{\left(\frac{v}{c}+1\right)}{\left(1+\frac{v}{c}\right)^3}\right]^2 = \frac{\left(1+\frac{v}{c}\right)^4}{\left(1-\frac{v}{c}\right)^4}.$$

4.219 We know that $S_0(r) \propto \dfrac{1}{r^2}$



At other angles $S(r, \theta) \propto \sin^2\theta$

Thus $S(r, \theta) = S_0(r)\sin^2\theta = S_0 \sin^2\theta$

Average power radiated

$$= S_0 \times 4\pi r^2 \times \frac{2}{3} = \frac{8\pi}{3} S_0 r^2$$

$$\left(\text{Average of } \sin^2\theta \text{ over whole sphere is } \frac{2}{3}\right)$$

4.220 From the previous problem.

$$P_0 = \frac{8\pi S_0 r^2}{3}$$

or
$$S_0 = \frac{3P_0}{8\pi r^2}$$

Thus
$$\langle w \rangle = \frac{S_0}{c} = \frac{3P_0}{8\pi c r^2}$$

(Poynting flux vector is the energy contained is a box of unit cross section and length c).

4.221 The rotating dispole has moments

$$p_x = p\cos\omega t, \; p_y = p\sin\omega t$$

Thus
$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\omega^4 p^2 = \frac{p^2\omega^4}{6\pi\varepsilon_0 c^3}.$$

4.222 If the electric field of the wave is

$$\vec{E} = \vec{E}_0 \cos\omega t$$

then this induces a dipole moment whose second derivative is

$$\ddot{\vec{p}} = \frac{e^2\vec{E}_0}{m}\cos\omega t$$

Hence radiated mean power
$$\langle P \rangle = \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{e^2 E_0}{m}\right)^2 \times \frac{1}{2}$$

On the other hand the mean Poynting flux of the incident radiation is

$$<S_{inc}> = \sqrt{\frac{\varepsilon_0}{\mu_0}} \times \frac{1}{2} E_0^2$$

Thus $$\frac{P}{<S_{inc}>} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2}{3} (\varepsilon_0 \mu_0)^{3/2} \left(\frac{e^2}{m}\right)^2 \times \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$= \frac{\mu_0^2}{6\pi} \left(\frac{e^2}{m}\right)^2$$

4.223 For the elastically bound electron

$$m\ddot{\vec{r}} + m\,\omega_0^2\,\vec{r} = e\,\vec{E}_0 \cos\omega t$$

This equation has the particular integral

(i.e. neglecting the part which does not have the frequency of the impressed force)

$$\vec{r} = \frac{e\,\vec{E}_0}{m} \frac{\cos\omega t}{\omega_0^2 - \omega^2} \quad \text{so and} \quad \ddot{\vec{p}} = -\frac{e^2\,\vec{E}_0\,\omega^2}{(\omega_0^2 - \omega^2)\,m} \cos\omega t$$

Hence P = mean radiated power

$$= \frac{1}{4\pi\varepsilon_0} \frac{2}{3c^3} \left(\frac{e^2\omega^2}{m(\omega_0^2 - \omega^2)}\right)^2 \frac{1}{2} E_0^2$$

The mean incident poynting flux is

$$<S_{inc}> = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{1}{2} E_0^2$$

Thus $$\frac{P}{<S_{inc}>} = \frac{\mu_0^2}{6\pi} \left(\frac{e^2}{m}\right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2}.$$

4.224 Let r = radius of the ball

R = distance between the ball & the Sun $(r << R)$.

M = mass of the Sun

γ = gravitational constant

Then $$\frac{\gamma M}{R^2} \frac{4\pi}{3} r^3 \rho = \frac{P}{4\pi R^2} \pi r^2 \cdot \frac{1}{c}$$

(the factor $\frac{1}{c}$ converts the energy received on the right into momentum received. Then the right hand side is the momentum received per unit time and must equal the negative of the impressed force for equilibrium).

Thus $$r = \frac{3P}{16\pi\gamma M c \rho} = 0.606\ \mu\text{m}.$$

☆☆☆

PART FIVE

OPTICS

5.1 PHOTOMETRY AND GEOMETRICAL OPTICS

5.1 (a) The relative spectral response $V(\lambda)$ shown in Fig. (5.11) of the book is so defined that $A/V(\lambda)$ is the energy flux of light of wave length λ needed to produce a unit luminous flux at that wavelength. (A is the conversion factor defined in the book.)

At $\lambda = 0{\cdot}51\,\mu\,\text{m}$, we read from the figure

$$V(\lambda) = 0{\cdot}50 \text{ so}$$

energy flux corresponding to a luminous flux of 1 lumen $= \frac{1{\cdot}6}{0{\cdot}50} = 3{\cdot}2\,\text{mW}$

At $\lambda = 0{\cdot}64\,\mu\,\text{m}$, we read

$$V(\lambda) = 0{\cdot}17$$

and energy flux corresponding to a luminous flux of 1 lumen $= \frac{1{\cdot}6}{{\cdot}17} = 9{\cdot}4\,\text{mW}$

(b) Here $d\Phi_e(\lambda) = \frac{\Phi_e}{\lambda_2 - \lambda_1} d\lambda, \quad \lambda_1 \le \lambda \le \lambda_2$

since energy is distributed uniformly. Then

$$\Phi = \int_{\lambda_1}^{\lambda_2} V(\lambda)\, d\Phi_2(\lambda)/A = \frac{\Phi_e}{A(\lambda_2 - \lambda_1)} \int_{\lambda_1}^{\lambda_2} V(\lambda)\, d\lambda$$

since $V(\lambda)$ is assumed to vary linearly in the interval $\lambda_1 \le \lambda \le \lambda_2$, we have

$$\frac{1}{\lambda_1 - \lambda_2} \int_{\lambda_1}^{\lambda_2} V(\lambda)\, d\lambda = \frac{1}{2}(V(\lambda_1) + V(\lambda_2))$$

Thus $$\Phi = \frac{\Phi_e}{2A}(V(\lambda_1) + V(\lambda_2))$$

Using $$V(0{\cdot}58\,\mu\,\text{m}) = 0{\cdot}85$$

$$V(0{\cdot}63\,\mu\,\text{m}) = 0{\cdot}25$$

Thus $$\Phi = \frac{\Phi_e}{2 \times 1{\cdot}6} \times 1{\cdot}1 = 1{\cdot}55 \text{ lumen.}$$

5.2 We have $\Phi_e = \frac{\Phi A}{V(\lambda)}$

But $\Phi_e = \underbrace{\frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}}_{\substack{\downarrow \\ \text{mean energy} \\ \text{flux vector}}} E_m^2 \times \underbrace{4\pi r^2}_{\text{area}}$ or $E_m^2 = \frac{\Phi A}{2\pi r^2 V(\lambda)}\sqrt{\frac{\mu_0}{\varepsilon_0}}$

For $\lambda = 0{\cdot}59\ \mu\text{m}$ $V(\lambda) = 0{\cdot}74$ Thus

$$E_m = 1{\cdot}14\ \text{V/m}$$

Also $H_m = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E_m = 3{\cdot}02\ \text{m A/m}$

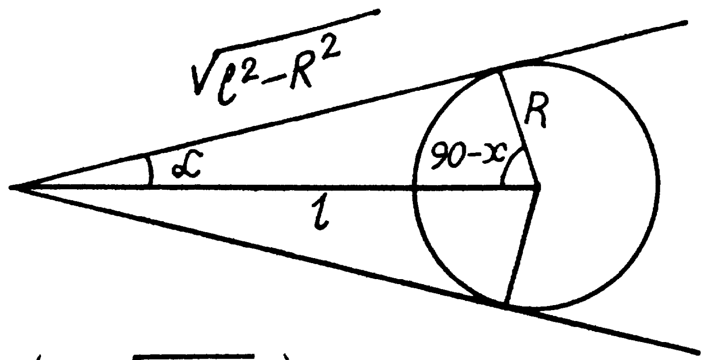
5.3 (a) Mean illuminance

$$= \frac{\text{Total luminous flux incident}}{\text{Total area illuminated}}.$$

Now, to calculate the total luminous flux incident on the sphere, we note that the illuminance at the point of normal incidence is E_0. Thus the incident flux is $E_o \cdot \pi R^2$. Thus

$$\text{Mean illuminance} = \frac{\pi R^2 \cdot E_0}{2\pi R^2}$$

or $$\langle E \rangle = \frac{1}{2} E_0 .$$



(b) The sphere subtends a solid angle

$$2\pi(1-\cos\alpha) = 2\pi\left(1 - \frac{\sqrt{l^2-R^2}}{l}\right)$$

at the point source and therefore receives a total flux of

$$2\pi I\left(1 - \frac{\sqrt{l^2-R^2}}{l}\right)$$

The area irradiated is : $2\pi R^2 \int_0^{90-\alpha} \sin\theta\, d\theta = 2\pi R^2(1-\sin\alpha) = 2\pi R^2\left(1-\frac{R}{l}\right)$

Thus $$\langle E \rangle = \frac{I}{R^2}\,\frac{1-\sqrt{1-(R/l)^2}}{1-\frac{R}{l}}$$

Substituting we get $\langle E \rangle = 50$ lux.

5.4 Luminance L is the light energy emitted per unit area of the emitting surface in a given direction per unit solid angle divided by $\cos\theta$. Luminosity M is simply energy emitted per unit area.

Thus
$$M = \int L \cdot \cos\theta \cdot d\Omega$$

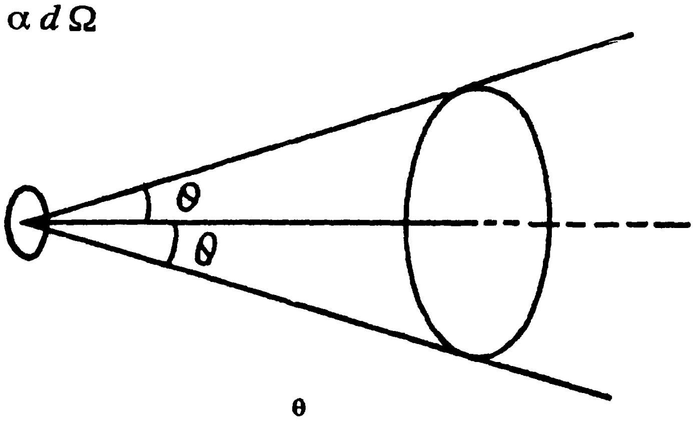
where the integration must be in the forward hemisphere of the emitting surface (assuming light is being emitted in only one direction say outward direction of the surface.) But

$$L = L_0 \cos\theta$$

Thus
$$M = \int L_0 \cos^2\theta \cdot d\Omega = 2\pi \int_0^{\pi/2} L_0 \cos^2\theta \sin\theta \, d\theta = \frac{2}{3}\pi L_0$$

5.5 (a) For a Lambert source $L = \text{Const}$
The flux emitted into the cone is

$$\Phi = L\,\Delta S \cos\alpha \, d\Omega$$



$$= L\,\Delta S \int_0^{\theta} 2\pi \cos\alpha \sin\alpha \, d\alpha$$

$$= L\,\Delta S\,\pi(1 - \cos^2\theta) = \pi L\,\Delta S \sin^2\theta$$

(b) The luminosity is obtained from the previous formula for $\theta = 90°$

$$M = \frac{\Phi(\theta = 90°)}{\Delta S} = \pi L$$

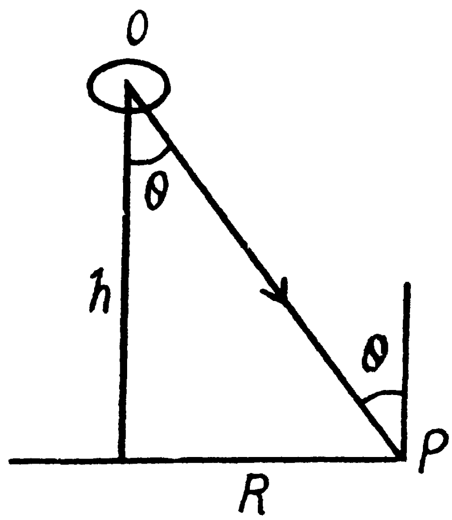
5.6 The equivalent luminous intensity in the direction OP is

$$L S \cos\theta$$

and the illuminance at P is

$$\frac{L S \cos\theta}{(R^2 + h^2)} \cos\theta = \frac{L S h^2}{(R^2 + h^2)^2}$$

$$= \frac{L S}{\left(\frac{R^2}{h} + h\right)^2} = \frac{L S}{\left[\left(\frac{R}{\sqrt{h}} - \sqrt{h}\right)^2 + 2R\right]^2}$$



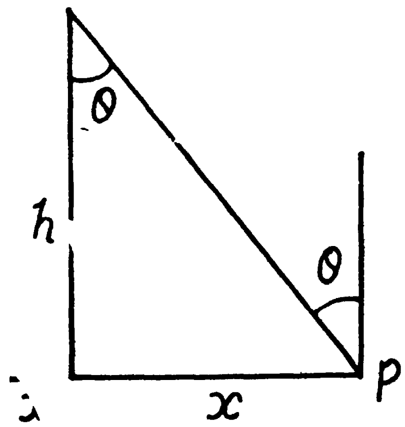
This is maximum when $R = h$

and the maximum illuminance is

$$\frac{LS}{4R^2} = \frac{1{\cdot}6 \times 10^2}{4} = 40 \text{ lux}$$

5.7 The illuminance at P is

$$E_p = \frac{I(\theta)}{(x^2 + h^2)} \cos\theta = \frac{I(\theta)\cos^3\theta}{h^2}$$



since this is constant at all x, we must have

$I(\theta)\cos^3\theta = \text{const} = I_0$

or $I(\theta) = I_0/\cos^3\theta$

The luminous flux reaching the table is

$\Phi = \pi R^2 \times \dfrac{I_0}{h^2} = 314$ lumen

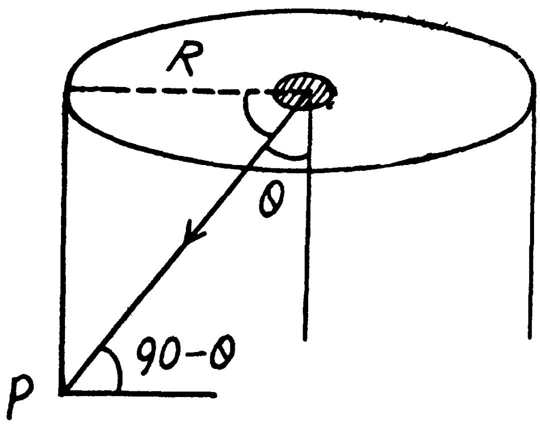
5.8 The illuminated area acts as a Lambert source of luminosity $M = \pi L$ where $MS = \rho ES$ = total reflected light

Thus, the luminance

$$L = \frac{\rho E}{\pi}$$

The equivalent luminous intensity in the direction making an angle θ from the vertical is

$$LS\cos\theta = \frac{\rho ES}{\pi}\cos\theta$$



and the illuminance at the point P is

$$\frac{\rho ES}{\pi}\cos\theta\sin\theta/R^2\operatorname{cosec}^2\theta = \frac{\rho ES}{\pi R^2}\cos\theta\sin^3\theta$$

This is maximum when

$$\frac{d}{d\theta}(\cos\theta\sin^3\theta) = -\sin^4\theta + 3\sin^2\theta\cos^2\theta = 0$$

or $\tan^2\theta = 3 \Rightarrow \tan\theta = \sqrt{3}$

Then the maximum illuminance is

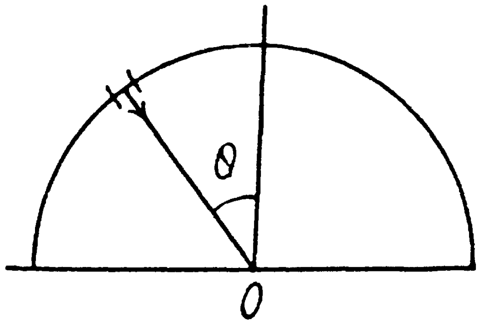
$$\frac{3\sqrt{3}}{16\pi}\frac{\rho ES}{R^2}$$

This illuminance is obtained at a distance $R\cot\theta = R/\sqrt{3}$ from the ceiling. Substitution gives the value

0·21 lux

5.9 From the definition of luminance, the energy emitted in the radial direction by an element dS of the surface of the dome is

$$d\Phi = L\,dS\,d\Omega$$



Here L = constant. The solid angle $d\Omega$ is given by

$$d\Omega = \frac{dA\cos\theta}{R^2}$$

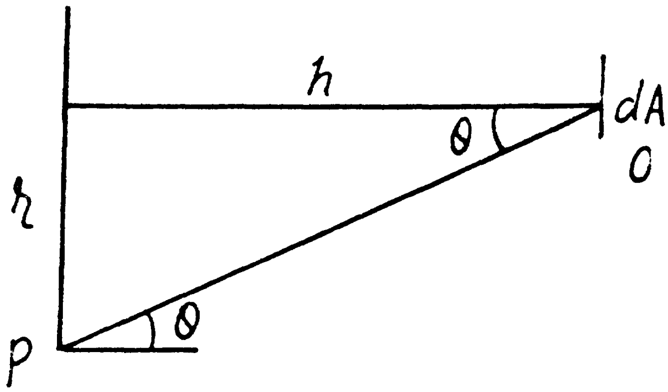
where dA is the area of an element on the plane illuminated by the radial light. Then

$$d\Phi = \frac{L\,dS\,dA}{R^2}\cos\theta$$

The illuminance at 0 is then

$$E = \int\frac{d\Phi}{dA} = \int_0^{\pi/2}\frac{L}{R^2}2\pi R^2\sin\theta\,d\theta\cos\theta = 2\pi L\int_0^1 x\,dx = \pi L$$

5.10 Consider an element of area dS at point P. It emits light of flux



$$\begin{aligned} d\Phi &= L\,dS\,d\Omega\cos\theta \\ &= L\,dS\frac{dA}{h^2\sec^2\theta}\cdot\cos^2\theta \\ &= \frac{L\,dS\,dA}{h^2}\cos^4\theta \end{aligned}$$

in the direction of the surface element dA at O. The total illuminance at O is then

$$E = \int\frac{L\,dS}{h^2}\cos^4\theta$$

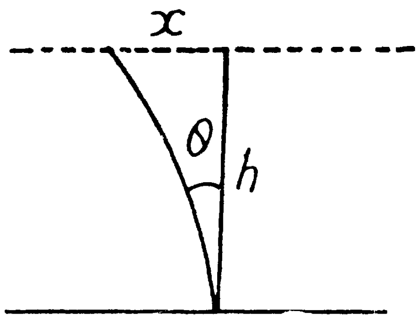
But $\quad dS = 2\pi r\,dr = 2\pi h\tan\theta\,d(h\tan\theta)$

$$= 2\pi h^2\sec^2\theta\tan\theta\,d\theta$$

Substitution gives $\quad E = 2\pi L\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \pi L$

5.11 Consider an angular element of area

$$2\pi x\,dx = 2\pi h^2\tan\theta\sec^2\theta\,d\theta$$



Light emitted from this ring is

$$d\Phi = L\,d\Omega\,(2\pi h^2\tan\theta\sec^2\theta\,d\theta)\cdot\cos\theta$$

Now $$d\Omega = \frac{dA\cos\theta}{h^2\sec^2\theta}$$

where dA = an element of area of the table just below the untre of the illuminant.

Then the illuminance at the element dA will be

$$E_0 = \int_{\theta=0}^{\theta=\alpha} 2\pi L\sin\theta\cos\theta\, d\theta$$

where $\sin\alpha = \dfrac{R}{\sqrt{h^2+R^2}}$. Finally using luminosity $M = \pi L$

$$E_0 = M\sin^2\alpha = M\frac{R^2}{h^2+R^2}$$

or $$M = E_0\left(1+\frac{h^2}{R^2}\right) = 700\ \text{lm/m}^2\ *\left(1\ \text{lx} = 1\frac{\text{lm}}{\text{m}^2}\ \text{dimensionally}\right).$$

5.12 See the figure below. The light emitted by an element of the illuminant towards the point O under consideration is

$$d\Phi = L\, dS\, d\Omega\cos(\alpha+\beta)$$

The element dS has the area

$$dS = 2\pi R^2\sin\alpha\, d\alpha$$

The distance

$$OA = \left[h^2+R^2-2hR\cos\alpha\right]^{1/2}$$

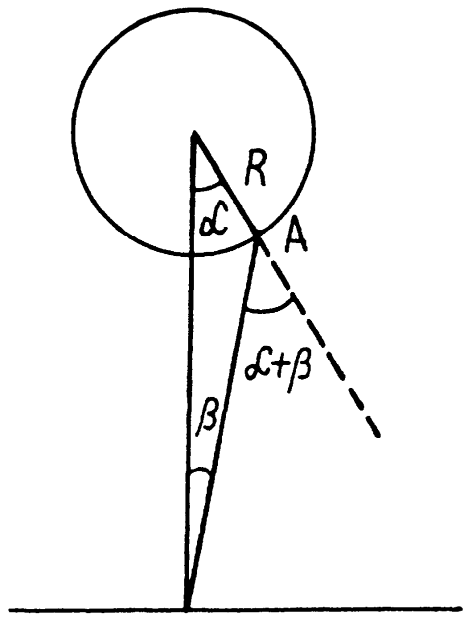
we also have

$$\frac{OA}{\sin\alpha} = \frac{h}{\sin(\alpha+\beta)} = \frac{R}{\sin\beta}$$

From the diagram

$$\cos(\alpha+\beta) = \frac{h\cos\alpha - R}{OA}$$

$$\cos\beta = \frac{h-R\cos\alpha}{OA}$$



If we imagine a small area $d\Sigma$ at O then

$$\frac{d\Sigma\cos\beta}{OA^2} = d\Omega$$

Hence, the illuminance at O is

$$\int\frac{d\Phi}{d\Sigma} = \int L\,2\pi R^2\sin\alpha\, d\alpha\,\frac{(h\cos\alpha - R)(h-R\cos\alpha)}{(OA)^4}$$

The limit of α is $\alpha = 0$ to that value for which $\alpha + \beta = 90°$, for then light is emitted tangentially. Thus

$$\alpha_{max} = \cos^{-1}\frac{R}{h}.$$

Thus
$$E = \int_0^{\cos^{-1}\frac{R}{h}} L \cdot 2\pi R^2 \sin\alpha \, d\alpha \, \frac{(h - R\cos\alpha)(h\cos\alpha - R)}{(h^2 + R^2 - 2hR\cos\alpha)^2}$$

we put
$$y = h^2 + R^2 - 2hR\cos\alpha$$

So,
$$dy = 2hR\sin\alpha \, d\alpha$$

$$E = \int_{(h-R)^2}^{h^2-R^2} L \cdot 2\pi R^2 \frac{dy}{2hR} \frac{\left(h - \frac{h^2 + R^2 - y}{2h}\right)\left(\frac{h^2 + R^2 - y}{2R} - R\right)}{y^2}$$

$$= \frac{L \cdot 2\pi R^2}{8h^2R^2} \int_{(h-R)^2}^{h^2-R^2} \frac{(h^2 - R^2 + y)(h^2 - R^2 - y)}{y^2} \, dy$$

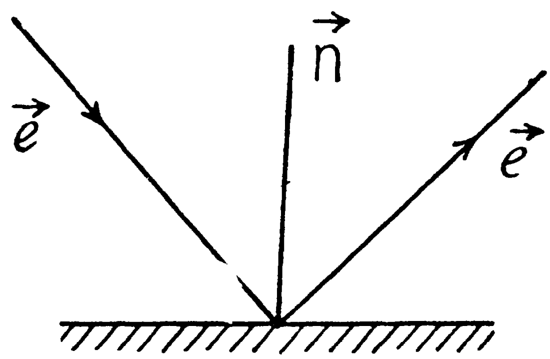
$$= \frac{\pi L}{4h^2} \int_{(h-R)^2}^{h^2-R^2} \left[\frac{(h^2 - R^2)^2}{y^2} - 1\right] dy = \frac{\pi L}{4h^2}\left[-\frac{(h^2 - R^2)^2}{y} - y\right]_{(h-R)^2}^{h^2-R^2}$$

$$= \frac{\pi L}{4h^2}\left[(h+R)^2 - (h^2 - R^2) - (h^2 - R^2) + (h-R)^2\right]$$

$$= \frac{\pi L}{4h^2}\left[2h^2 + 2R^2 - 2h^2 + 2R^2\right] = \frac{\pi L R^2}{h^2}$$

Substitution gives : $\quad E = 25 \cdot 1$ lux

5.13 We see from the diagram that because of the law of reflection, the component of the incident unit vector $\vec{e}$ along $\cdot \vec{n}$ changes sign on reflection while the component || to the mirror remains unchanged. Writing $\vec{e} = \vec{e}_{\|} + \vec{e}_{\perp}$



where $\vec{e}_{\perp} = \vec{n}(\vec{e} \cdot \vec{n})$

$\vec{e}_{\|} = \vec{e} - \vec{n}(\vec{e} \cdot \vec{n})$

we see that the reflected unit vector is

$$\vec{e}\,' = \vec{e}_{\|} - \vec{e}_{\perp} = \vec{e} - 2\vec{n}(\vec{e} \cdot \vec{n})$$

5.14 We choose the unit vectors perpendicular to the mirror as the x, y, z axes in space. Then after reflection from the mirror with normal along x axis

$$\vec{e}\,' = \vec{e} - 2\,\hat{i}\,(\hat{i}\cdot\vec{e}) = -\vec{e}_x\,\hat{i} + e_y\,\hat{j} + e_z\,\hat{k}$$

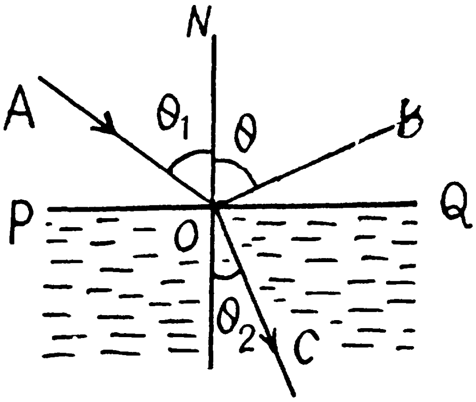
where $\hat{i}, \hat{j}, \hat{k}$ are the basic unit vectors. After a second reflection from the 2nd mirror say along y axis.

$$\vec{e}\,'' = \vec{e}\,' - 2\,\hat{j}\,(\hat{j}\cdot\vec{e}\,') = -e_x\,\hat{i} - e_y\,\hat{h} + e_z\,\hat{k}$$

Finally after the third reflection

$$\vec{e}\,''' = -e_x\,\hat{i} - e_y\,\hat{j} - e_z\,\hat{k} = -\vec{e}.$$

5.15 Let PQ be the surface of water and n be the R.I. of water. Let AO is the shaft of light with incident angle θ_1 and OB and OC are the reflected and refracted light rays at angles θ_1 and θ_2 respectively (Fig.). From the figure $\theta_2 = \frac{\pi}{2} - \theta_1$



From the law of refraction at the interface PQ

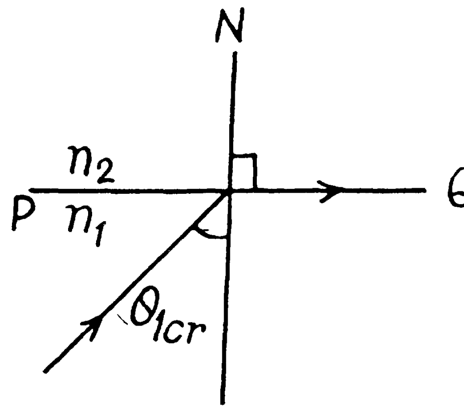
$$n = \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin\theta_1}{\sin\left(\frac{\pi}{2} - \theta_1\right)}$$

$$\text{or, } n = \frac{\sin\theta_1}{\cos\theta_1} = \tan\theta_1$$

Hence $\theta_1 = \tan^{-1} n$

5.16 Let two optical mediums of R.I. n_1 and n_2 respectively be such that $n_1 > n_2$. In the case when angle of incidence is $\theta_{1\,cr}$ (Fig.), from the law of refraction

$$n_1 \sin\theta_{1\,cr} = n_2 \qquad (1)$$



In the case , when the angle of incidence is θ_1, from the law of refraction at the interface of mediums 1 and 2 .

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

But in accordance with the problem $\theta_2 = (\pi/2 - \theta_1)$

$$\text{so,} \qquad n_1 \sin\theta_1 = n_2 \cos\theta_1 \qquad (2)$$

Dividing Eqn (1) by (2)

$$\frac{\sin\theta_{1\,cr}}{\sin\theta_1} = \frac{1}{\cos\theta_1}$$

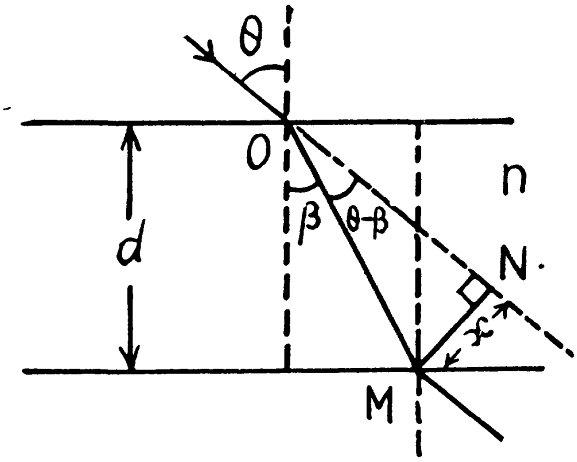
$$\text{or,} \qquad \eta = \frac{1}{\cos\theta_1}, \quad \text{so} \quad \cos\theta_1 = \frac{1}{\eta} \text{ and } \sin\theta_1 = \frac{\sqrt{\eta^2 - 1}}{\eta} \qquad (3)$$

$$\text{But} \qquad \frac{n_1}{n_2} = \frac{\cos\theta_1}{\sin\theta_1}$$

So, $$\frac{n_1}{n_2} = \frac{1}{\eta}\frac{\eta}{\sqrt{\eta^2-1}}$$ (Using 3)

Thus $$\frac{n_1}{n_2} = \frac{1}{\sqrt{\eta^2-1}}$$

5.17 From the Fig. the sought lateral shift



$$x = OM \sin(\theta-\beta)$$
$$= d\sec\beta\sin(\theta-\beta)$$
$$= d\sec\beta\,(\sin\theta\cos\beta - \cos\theta\sin\beta)$$
$$= d(\sin\theta - \cos\theta\tan\beta) \qquad (1)$$

But from the law of refraction

$$\sin\theta = n\sin\beta \quad \text{or,} \quad \sin\beta = \frac{\sin\theta}{n}$$

So, $$\cos\beta = \frac{\sqrt{n^2-\sin^2\theta}}{n} \quad \text{and} \quad \tan\beta = \frac{\sin\theta}{\sqrt{n^2-\sin^2\theta}}$$

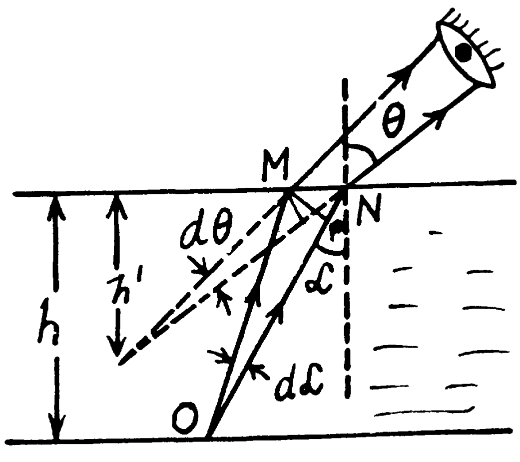
Thus $$x = d(\sin\theta - \cos\theta\tan\beta) = d\left(\sin\theta - \cos\theta\frac{\sin\theta}{\sqrt{n^2-\sin^2\theta}}\right)$$

$$= d\sin\theta\left[1-\sqrt{\frac{1-\sin^2\theta}{n^2-\sin^2\theta}}\right]$$

5.18 From the Fig.

$$\sin d\alpha = \frac{MP}{OM} = \frac{MN\cos\alpha}{h\sec(\alpha+d\alpha)}$$

As $d\alpha$ is very small, so



$$d\alpha \approx \frac{MN\cos\alpha}{h\sec\alpha} = \frac{MN\cos^2\alpha}{h} \qquad (1)$$

Similarly

$$d\theta = \frac{MN\cos^2\theta}{h'} \qquad (2)$$

From Eqns (1) and (2)

$$\frac{d\alpha}{d\theta} = \frac{h'\cos^2\alpha}{h\cos^2\theta} \quad \text{or,} \quad h' = \frac{h\cos^2\theta}{\cos^2\alpha}\frac{d\alpha}{d\theta} \qquad (3)$$

From the law of refraction

$$n\sin\alpha = \sin\theta \qquad (A)$$

$$\sin\alpha = \frac{\sin\theta}{n}, \quad \text{so}, \quad \cos\alpha = \sqrt{\frac{n^2 - \sin^2\theta}{n^2}} \tag{B}$$

Differentiating Eqn.(A)

$$n\cos\alpha\, d\alpha = \cos\theta\, d\theta \quad \text{or}, \quad \frac{d\alpha}{d\theta} = \frac{\cos\theta}{n\cos\alpha} \tag{4}$$

Using (4) in (3), we get

$$h' = \frac{h\cos^3\theta}{n\cos^3\alpha} \tag{5}$$

Hence $\quad h' = \dfrac{h\cos^3\theta}{n\left(\dfrac{n^2 - \sin^2\theta}{n^2}\right)^{3/2}} = \dfrac{n^2 h\cos^3\theta}{\left(n^2 - \sin^2\theta\right)^{3/2}}$ [Using Eqn.(B)]

5.19 The figure shows the passage of a monochromatic ray through the given prism, placed in air medium.

From the figure, we have

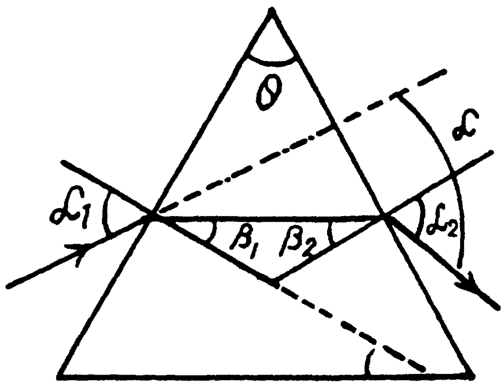
$$\theta = \beta_1 + \beta_2 \tag{A}$$

and $\quad \alpha = (\alpha_1 + \alpha_2) - (\beta_1 + \beta_2)$

$$\alpha = (\alpha_1 + \alpha_2) - \theta \tag{1}$$

From the Snell's law

$$\sin\alpha_1 = n\sin\beta_1$$



or $\quad \alpha_1 = n\,\beta_1$ (for small angles) $\quad$ (2)

and $\quad \sin\alpha_2 = n\sin\beta_2$

or, $\quad \alpha_2 = n\,\beta_2$ (for small angles) $\quad$ (3)

From Eqns (1), (2) and (3), we get

$$\alpha = n(\beta_1 + \beta_2) - \theta$$

So, $\quad \alpha = n(\theta) - \theta = (n-1)\theta$ [Using Eqn.A]

5.20 (a) In the general case, for the passage of a monochromatic ray through a prism as shown in the figure of the soln. of 5.19,

$$\alpha = (\alpha_1 + \alpha_2) - \theta \tag{1}$$

And from the Snell's law,

$$\left.\begin{array}{l} \sin\alpha_1 = n\sin\beta_1 \quad \text{or} \quad \alpha_1 = \sin^{-1}(n\sin\beta_1) \\ \text{Similarly } \alpha_2 = \sin^{-1}(n\sin\beta_2) = \sin^{-1}[n\sin(\theta - \beta_1)] \text{ (As } \theta = \beta_1 + \beta_2) \end{array}\right] \tag{2}$$

Using (2) in (1)

$$\alpha = \left[\sin^{-1}(n\sin\beta_1) + \sin^{-1}\left(n\sin(\theta - \beta_1)\right)\right] - \theta$$

For α to be minimum, $\dfrac{d\alpha}{d\beta_1} = 0$

or, $$\frac{n\cos\beta_1}{\sqrt{1-n^2\sin^2\beta_1}} - \frac{n\cos(\theta-\beta_1)}{\sqrt{1-n^2\sin^2(\theta-\beta_1)}} = 0$$

or, $$\frac{\cos^2\beta_1}{(1-n^2\sin^2\beta_1)} = \frac{\cos^2(\theta-\beta_1)}{1-n^2\sin^2(\theta-\beta_1)}$$

or, $$\cos^2\beta_1\left(1-n^2\sin^2(\theta-\beta_1)\right) = \cos^2(\theta-\beta_1)\,(1-n^2\sin^2\beta_1)$$

or, $$(1-\sin^2\beta_1)\left(1-n^2\sin^2(\theta-\beta_1)\right) = \left(1-\sin^2(\theta-\beta_1)\right)(1-n^2\sin^2\beta_1)$$

or, $$1-n^2\sin^2(\theta-\beta_1)-\sin^2\beta_1+\sin^2\beta_1\,n^2\sin^2(\theta-\beta_1)$$
$$= 1-n^2\sin^2\beta_1-\sin^2(\theta-\beta_1)+\sin^2\beta_1\,n^2\sin^2(\theta-\beta_1)$$

or, $$\sin^2(\theta-\beta_1)-n^2\sin^2(\theta-\beta_1) = \sin^2\beta_1(1-n^2)$$

or, $$\sin^2(\theta-\beta_1)(1-n^2) = \sin^2\beta_1(1-n^2)$$

or, $$\theta-\beta_1 = \beta_1 \quad \text{or} \quad \beta_1 = \theta/2$$

But $$\beta_1+\beta_2 = \theta, \quad \text{so,} \quad \beta_2 = \theta/2 = \beta_1$$

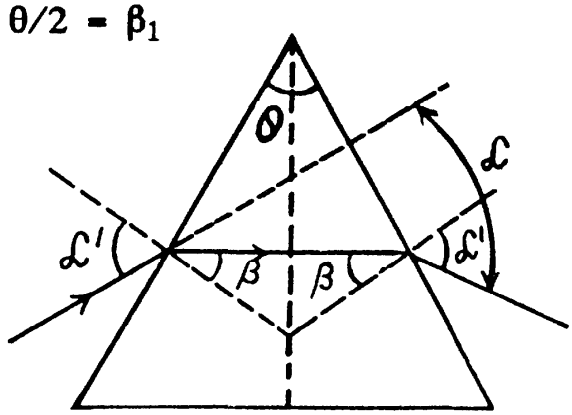
which is the case of symmetric passage of ray.
In the case of symmetric passage of ray

$\alpha_1 = \alpha_2 = \alpha'$ (say)

and $\beta_1 = \beta_2 = \beta = \theta/2$

Thus the total deviation

$\alpha = (\alpha_1+\alpha_2)-\theta$



$$\alpha = 2\alpha'-\theta \quad \text{or} \quad \alpha' = \frac{\alpha+\theta}{2} \tag{1}$$

But from the Snell's law $\sin\alpha = n\sin\beta$

So, $$\sin\frac{\alpha+\theta}{2} = n\sin\frac{\theta}{2}$$

5.21 In this case we have

$$\sin\frac{\alpha+\theta}{2} = n\sin\frac{\theta}{2} \text{ (see soln. of 5.20)}$$

In our problem $\alpha = \theta$

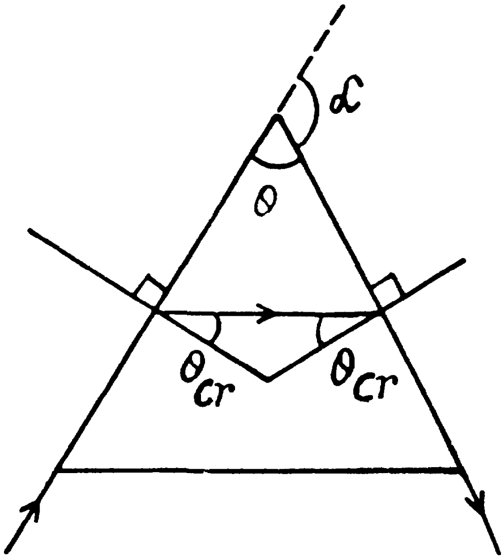
So, $\sin\theta = n\sin(\theta/2)$ or $2\sin(\theta/2)\cos(\theta/2) = n\sin(\theta/2)$

Hence $\cos(\theta/2) = \dfrac{n}{2}$ or $\theta = 2\cos^{-1}(n/2) = 83°$, where $n = 1{\cdot}5$

5.22 In the case of minimum deviation

$$\sin\frac{\alpha+\theta}{2} = n\sin\frac{\theta}{2}$$

So, $\alpha = 2\sin^{-1}\left\{n\sin\frac{\theta}{2}\right\} - \theta = 37°$, for $n = 1{\cdot}5$



Passage of ray for grazing incidence and grazing imergence is the condition for maximum deviation (Fig.). From Fig.

$$\alpha = \pi - \theta = \pi - 2\,\theta_{cr}$$

(where θ_{cr} is the critical angle)

So, $\alpha = \pi - 2\sin(1/n) = 58°$,
for $n = 1{\cdot}5$ = R.I. of glass.

5.23 The least deflection angle is given by the formula,
$\delta = 2\,\alpha - \theta$, where α is the angle of incidence at first surface and θ is the prism angle.

Also from Snell's law, $n_1\sin\alpha = n_2\sin(\theta/2)$, as the angle of refraction at first surface is equal to half the angle of prism for least deflection

so, $$\sin\alpha = \frac{n_2}{n_1}\sin(\theta/2) = \frac{1{\cdot}5}{1{\cdot}33}\sin 30° = {\cdot}5639$$

or, $$\alpha = \sin^{-1}({\cdot}5639) = 34{\cdot}3259°$$

Substituting in the above (1), we get, $\delta = 8{\cdot}65°$

5.24 From the Cauchy's formula, and also experimentally the R.I. of a medium depends upon the wavelength of the mochromatic ray i.e. $n = f(\lambda)$. In the case of least deviation of a monochromatic ray the passage a prism, we have:

$$n\sin\frac{\theta}{2} = \sin\frac{\alpha+\theta}{2} \qquad (1)$$

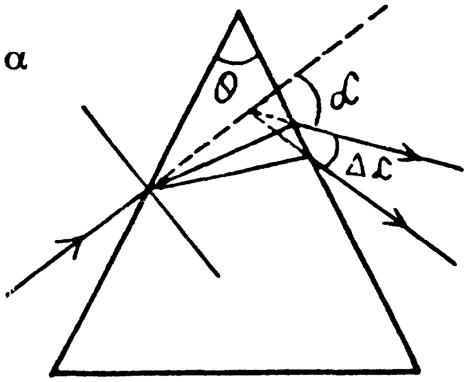
The above equation tales us that we have $n = n(\alpha)$, so we may write

$$\Delta n = \frac{dn}{d\alpha}\Delta\alpha \qquad (2)$$

From Eqn. (1)

$$dn\sin\frac{\theta}{2} = \frac{1}{2}\cos\frac{\alpha+\theta}{2}\,d\alpha$$

or, $$\frac{dn}{d\alpha} = \frac{\cos\frac{\alpha+\theta}{2}}{2\sin\frac{\theta}{2}} \qquad (3)$$



From Eqns (2) and (3)

$$\Delta n = \frac{\cos\frac{\alpha+\theta}{2}}{2\sin\frac{\theta}{2}}\Delta\alpha$$

or, $$\Delta n = \frac{\sqrt{1 - \sin^2\left(\frac{\alpha + \theta}{2}\right)}}{2 \sin \frac{\theta}{2}} \Delta \alpha = \frac{\sqrt{1 - n^2 \sin^2 \frac{\theta}{2}}}{2 \sin \frac{\theta}{2}} \Delta \alpha \text{ (Using Eqn. 1.)}$$

Thus $$\Delta \alpha = \frac{2 \sin \frac{\theta}{2}}{\sqrt{1 - n^2 \sin^2 \frac{\theta}{2}}} \Delta n = 0 \cdot 44$$

5.25 Fermat's principle : " The actual path of propagation of light (trajectory of a light ray) is the path which can be followed by light with in the lest time, in comparison with all other hypothetical paths between the same two points. "

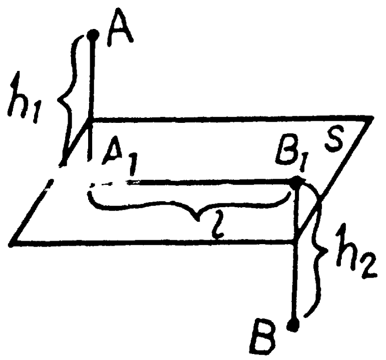
"Above statement is the original wordings of Fermat (A famous French scientist of 17th century)"

Deduction of the law of refraction from Fermat's principle :

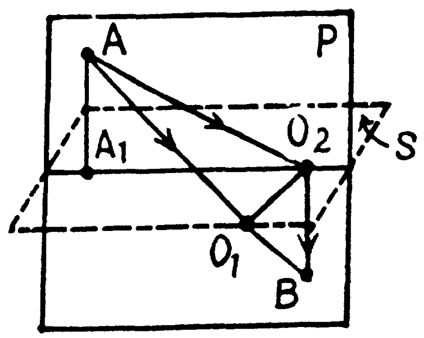
Let the plane S be the interface between medium 1 and medium 2 with the refractive indices $n_1 = c/v_1$ and $n_2 = c/v_2$ Fig. (a). Assume, as usual, that $n_1 < n_2$. Two points are given– one above the plane S (point A), the other under plane S (point B). The various distances are :

$A A_1 = h_1$, $B B_1 = h_2$, $A_1 B_1 = l$. We must find the path from A to B which can be covered by light faster than it can cover any other hypothetical path. Clearly, this path must consist of two straight lines, viz, AO in medium 1 and OB in medium 2; the point O in the plane S has to be found.

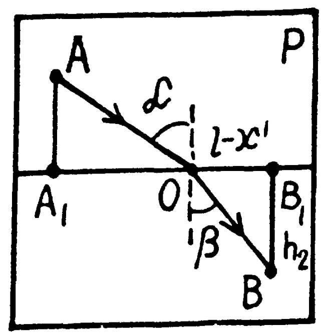
First of all, it follows from Fermat's priniciple that the point O must lie on the intersection of S and a plane P, which is perpendicular to S and passes through A and B.



(a)



(b)



(c)

Indeed, let us assume that this point does not lie in the plane P; let this be point O_1 in Fig. (b). Drop the perpendicular $O_1 O_2$ from O_1 onto P. Since $A O_2 < A O_1$ and $B O_2 < B O_1$, it is clear that the time required to traverse $A O_2 B$ is less than that needed to cover the path $A O_1 B$. Thus, using Fermat's principle, we see that the first law of refraction is observed : the incident and the refracted rays lie in the same plane as the perpendicular to the interface at the point

where the ray is refracted. This plane is the plane P in Fig. (b); it is called the plane of incidence.

Now let us consider light rays in the plane of incidence Fig. (c). Designate $A_1 O$ as x and $O B_1 = l - x$. The time it takes a ray to travel from A to O and then from O to B is

$$T = \frac{AO}{v_1} + \frac{OB}{v_2} = \frac{\sqrt{h_1^2 + x^2}}{v_1} + \frac{\sqrt{h_2^2 + (l - x)^2}}{v_2} \tag{1}$$

The time depends on the value of x. According to Fermat's principle, the value of x must minimize the time T. At this value of x the derivative dT/dx equals zero :

$$\frac{dT}{dx} = \frac{x}{v_1 \sqrt{h_1^2 + x^2}} - \frac{l - x}{v_2 \sqrt{h_2^2 + (l - x)^2}} = 0 . \tag{2}$$

Now,

$$\frac{x}{\sqrt{h_1^2 + x^2}} = \sin \alpha , \text{ and } \frac{l - x}{\sqrt{h_2^2 + (l - x)^2}} = \sin \beta ,$$

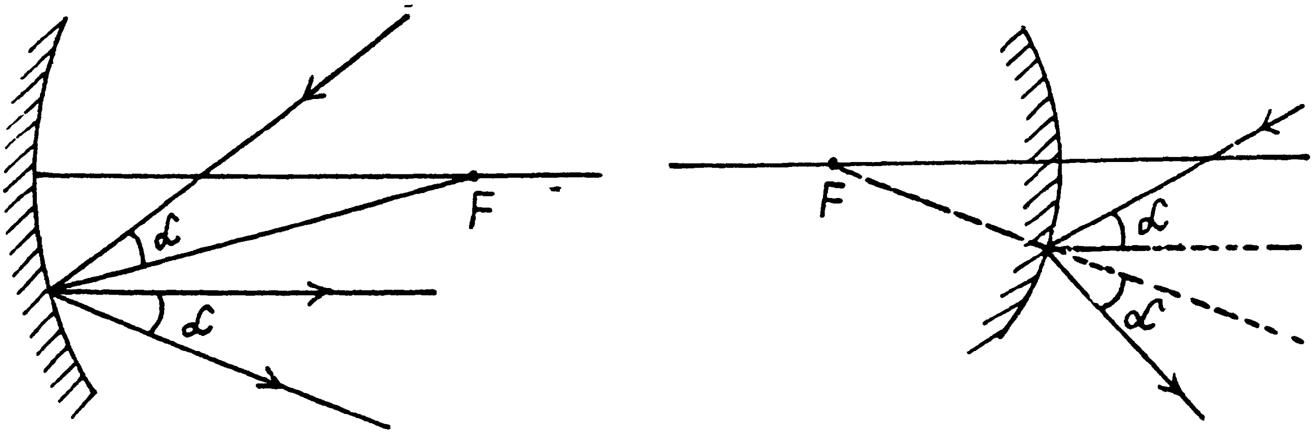
Consequently,

$$\frac{\sin \alpha}{v_1} - \frac{\sin \beta}{v_2} = 0 , \quad \text{or} \quad \frac{\sin \alpha}{\sin \beta} = \frac{v_1}{v_2}$$

So,
$$\frac{\sin \alpha}{\sin \beta} = \frac{c/n_1}{c/n_2} = \frac{n_2}{n_1}$$

Note : Fermat himself could not use Eqn. 2. as mathematical analysis was developed later by Newton and Leibniz. To deduce the law of the refraction of light, Fermat used his own maximum and minimum method of calculus, which, in fact, corresponded to the subsequently developed method of finding the minimum (maximum) of a function by differentiating it and equating the derivative to zero.

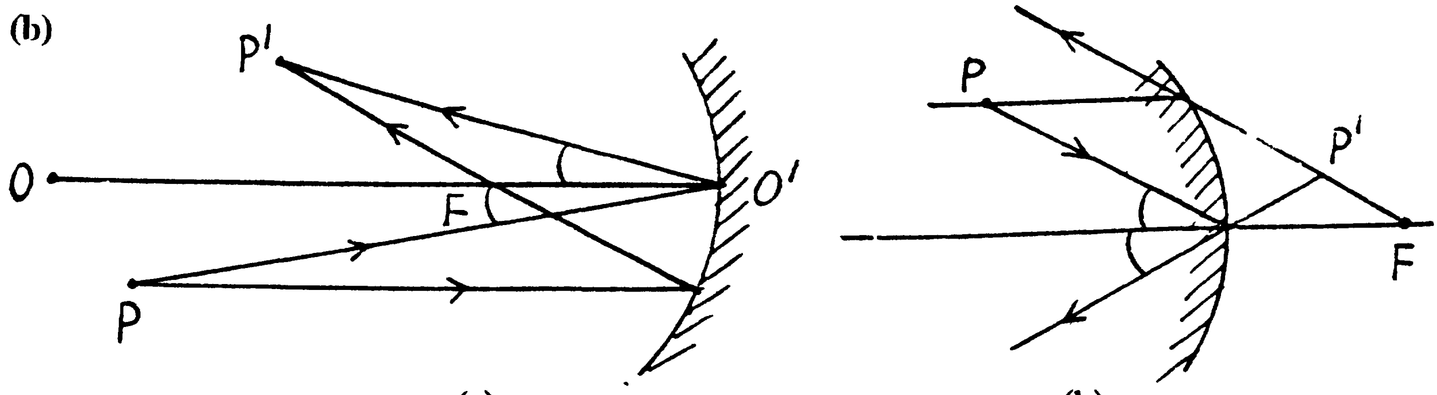
5.26 (a) Look for a point O' on the axis such that $O'P'$ and $O'P$ make equal angles with $O'O$. This determines the position of the mirror. Draw a ray from P parallel to the axis. This must on reflection pass through P'. The intersection of the reflected ray with principal axis determines the focus.



(a) (b)

(b) Suppose P is the object and P' is the image. Then the mirror is convex because the image is virtual, erect & diminished. Look for a point X (between P & P') on the axis such that PX and $P'X$ make equal angle with the axis.

(b)



(a) (b)

5.27 (a) From the mirror formula,

$$\frac{1}{s'} + \frac{1}{s} = \frac{1}{f} \quad \text{we get} \quad f = \frac{s' s}{s' + s} \tag{1}$$

In accordance with the problem $s - s' = l \quad \frac{s'}{s} = \beta$,

From these two relations, we get : $s = \frac{l}{1-\beta}, \quad s' = -\frac{l\beta}{1+\beta}$

Substituting it in the Eqn. (1),

$$f = \frac{\beta\left(\frac{l}{1-\beta}\right)^2}{l\left(\frac{1-\beta}{1-\beta}\right)} = \frac{l\beta}{(1-\beta^2)} = -10 \text{ cm}$$

(b) Again we have,
$$\frac{1}{s'} + \frac{1}{s} = \frac{1}{f} \quad \text{or,} \quad \frac{s}{s'} + 1 = \frac{s}{f}$$

or,
$$\frac{1}{\beta_1} = \frac{s}{f} - 1 = \frac{s-f}{f}$$

or,
$$\beta_1 = \frac{f}{s-f} \tag{2}$$

Now, it is clear from the above equation, that for smaller β, s must be large, so the object is displaced away from the mirror in second position.

i.e.
$$\beta_2 = \frac{f}{s+l-f} \tag{3}$$

Eliminating s from the Eqn. (2) and (3), we get,

$$f = \frac{l\beta_1\beta_2}{(\beta_2 - \beta_1)} = -2.5 \text{ cm}.$$

5.28 For a concave mirror as usual $\frac{1}{s'}+\frac{1}{s} = \frac{1}{f}$ so $s' = \frac{sf}{s-f}$

(In coordinate convention $s = -s$ is negative & $f = -|f|$ is also negative.)

If A is the area of the mirror (assumed small) and the object is on the principal axis, then the light incident on the mirror per second is $I_0 \frac{A}{s^2}$.

This follows from the definition of luminous intensity as light emitted per second per unit solid angle in a given direction and the fact that $\frac{A}{s^2}$ is the solid angle subtended by the mirror at the source. Of this a fraction ρ is reflected so if I is the luminous intensity of the image, then $I \frac{A}{s'^2} = \rho I_0 \frac{A}{s^2}$

Hence $$I = \rho I_0 \left(\frac{|f|}{|f|-s}\right)^2$$

(Because our convention makes f – ve for a concave mirror, we have to write $|f|$.)

Substitution gives $$I = 2.0 \times 10^3 \text{ cd.}$$

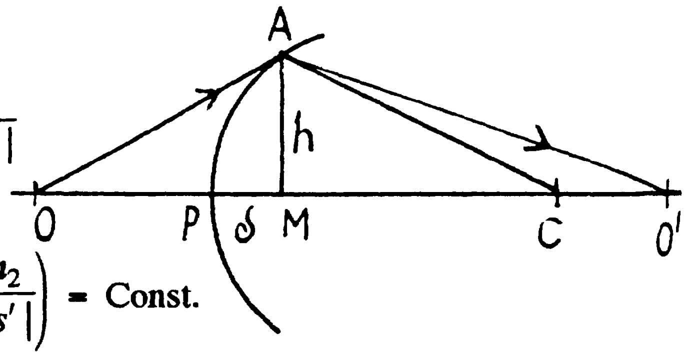
5.29 For O_1 to be the image, the optical paths of all rays OAO_1 must be equal upto terms of leading order in h. Thus

$$n_1 OA + n_2 AO_1 = \text{constant}$$

But, $OP = |s|$, $O_1P = |s'|$ and so

$$OA = \sqrt{h^2 + (|s|+\delta)^2} \cong |s| + \delta + \frac{h^2}{2|s|}$$

$$O_1A = \sqrt{h^2 + (|s'|-\delta)^2} \cong |s'| - \delta + \frac{h^2}{2|s'|}$$



(neglecting products $h^2\delta$). Then

$$n_1|s| + n_2|s'| + n_1\delta - n_2\delta + \frac{h^2}{2}\left(\frac{n_1}{|s|} + \frac{n_2}{|s'|}\right) = \text{Const.}$$

Now $(r-\delta)^2 + h^2 = r^2$

or $h^2 = 2r\delta$ or $\delta = \frac{h^2}{2r}$

Here $r = CP$.

Hence $n_1|s| + n_2|s'| + \frac{h^2}{2}\left\{\frac{n_1-n_2}{r} + \frac{n_1}{|s|} + \frac{n_2}{|s'|}\right\} = \text{Constant}$

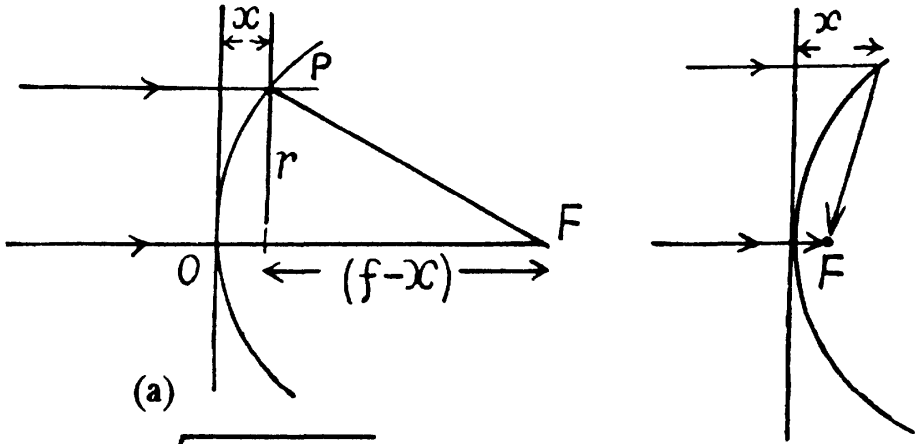
Since this must hold for all h, we have

$$\frac{n_2}{|s'|} + \frac{n_1}{|s|} = \frac{n_2-n_1}{r}$$

From our sign convention, $s' > 0$, $s < 0$ so we get

$$\frac{n_2}{s'} - \frac{n_1}{s} = \frac{n_2 - n_1}{r}.$$

5.30 All rays focusing at a point must have traversed the same optical path. Thus



$$x + n\sqrt{r^2 + (f-x)^2} = nf \quad \text{or} \quad (nf-x)^2 = n^2r^2 + n^2(f-x)^2$$

or, $$n^2r^2 = (nf-x)^2 - [n(f-x)]^2 = (nf - x + nf - nx)(nf - x - nf + nx)$$
$$= x(n-1)(2nf - (n+1)x)$$
$$= 2n(n-1)fx - (n+1)(n-1)x^2$$

Thus, $$(n+1)(n-1)x^2 - 2n(n-1)fx + n^2r^2 = 0$$

so, $$x = \frac{n(n-1)f \pm \sqrt{n^2(n-1)^2f^2 - n^2r^2(n+1)(n-1)}}{(n+1)(n-1)}$$
$$= \frac{nf}{n+1}\left[1 \pm \sqrt{1 - \frac{n+1}{n-1}\frac{r^2}{f^2}}\right]$$

Ray must move forward so $x < f$, for + sign $x > f$ for small r, so –sign.

(Also $x \to 0$ as $r \to 0$)

($x > f$ means ray turning back in the direction of incidence. (see Fig.)

Hence $$x = \frac{nf}{n+1}\left[1 - \sqrt{1 - \frac{n+1}{n-1}\frac{r^2}{f^2}}\right]$$

For the maximum value of r,

$$\sqrt{1 - \frac{n+1}{n-1}\frac{r^2}{f^2}} = 0 \tag{A}$$

because the expression under the radical sign must be non-negative, which gives the maximum value of r.

Hence from Eqn. (A), $r_{max} = f\sqrt{(n-1)/(n+1)}$

5.31 As the given lense has significant thickness, the thin lense, formula cannote be used.

For refraction at the front surface from the formula $\frac{n'}{s'} - \frac{n}{s} = \frac{n' - n}{R}$

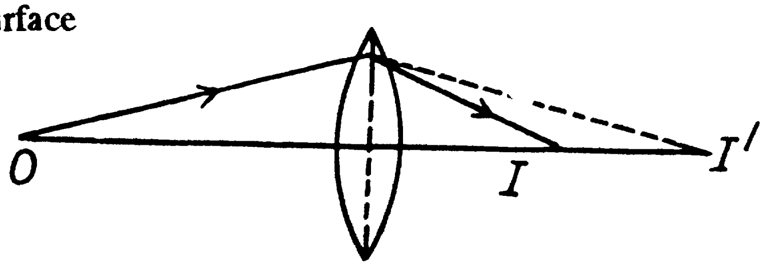
$$\frac{1 \cdot 5}{s'} - \frac{1}{-20} = \frac{1 \cdot 5 - 1}{5}$$

On simplifying we get , $s' = 30$ cm.

Thus the image I' produced by the front surface behaves as a virtual source for the rear surface at distance 25 cm from it, because the thickness of the lense is 5 cm. Again from the refraction formula at cerve surface

$$\frac{n'}{s'} - \frac{n}{s} = \frac{n' - n}{R}$$

$$\frac{1}{s'} - \frac{1 \cdot 5}{25} = \frac{1 - 1 \cdot 5}{-5}$$



On simplifying, $s' = +6 \cdot 25$ cm

Thus we get a real image I at a distance $6 \cdot 25$ cm beyond the rear surface (Fig.).

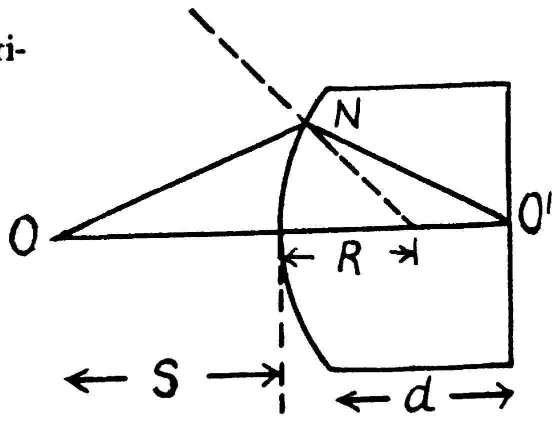
5.32 (a) The formation of the image of a source S, placed at a distance u from the pole of the convex surface of plano-convex lens of thickness d is shown in the figure.

On applying the formula for refraction through spherical surface, we get

$$\frac{n}{s'} - \frac{1}{s} = (n-1)/R \text{, (here } n_2 = n \text{ and } n_1 = 1)$$

or, $\frac{n}{d} - \frac{1}{s} = (n-1)/R$ or, $\frac{1}{s} = \frac{n}{d} - \frac{(n-1)}{R}$

or, $$\frac{s'}{s} = s' \left\{ \frac{n}{d} - \frac{(n-1)}{R} \right\}$$



But in this case optical path of the light, corresponding to the distance v in the medium is v/n, so the magnification produced will be,

$$\beta = \frac{s'}{ns} = \frac{s'}{n} \left\{ \frac{n}{d} - \frac{(n-1)}{R} \right\} = \frac{d}{n} \left\{ \frac{n}{d} - \frac{(n-1)}{R} \right\} = 1 - \frac{d(n-1)}{nR}$$

Substituting the values, we get magnification $\beta = -0{\cdot}20$.

(b) If the transverse area of the object is A (assumed small), the area of the image is $\beta^2 A$. We shall assume that $\frac{\pi D^2}{4} > A$. Then light falling on the lens is : $LA \frac{\pi D^2/4}{s^2}$

from the definition of luminance (See Eqn. (5.1c) of the book; here $\cos\theta \approx 1$ if $D^2 << s^2$ and $d\Omega = \frac{\pi D^2/4}{s^2}$). Then the illuminance of the image is

$$LA\,\frac{\pi D^2/4}{s^2} \bigg/ \beta^2 A = L\,n^2\,\pi D^2/4d^2$$

Substitution gives 42 1x.

5.33 (a) Optical power of a thin lens of R.I. n in a medium with R.I. n_0 is given by :

$$\Phi = (n - n_0)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{A}$$

From Eqn.(A), when the lens is placed in air :

$$\Phi_0 = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{1}$$

Similarly from Eqn.(A), when the lens is placed in liquid :

$$\Phi = (n - n_0)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{2}$$

Thus from Eqns (1) and (2)

$$\Phi = \frac{n - n_0}{n - 1}\,\Phi_0 = 2\,D$$

The second focal length, is given by

$f' = \frac{n'}{\Phi}$, where n' is the R.I. of the medium in which it is placed.

$$f' = \frac{n_0}{\Phi} = 85 \text{ cm}$$

(b) Optical power of a thin lens of R.I. n placed in a medium of R.I. n_0 is given by :

$$\Phi = (n - n_0)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{A}$$

For a biconvex lens placed in air medium from Eqn. (A)

$$\Phi_0 = (n - 1)\left(\frac{1}{R} - \frac{1}{-R}\right) = \frac{2(n-1)}{R} \tag{1}$$

where R is the radius of each curve surface of the lens

Optical power of a spherical refractive surface is given by :

$$\Phi = \frac{n' - n}{R} \tag{B}$$

For the rear surface of the lens which divides air and glass medium

$$\Phi_0 = \frac{n-1}{R} \text{ (Here } n \text{ is the R.I. (2) of glass)}$$

Similarly for the front surface which divides water and glass medium

$$\Phi_l = \frac{n-n}{-R} = \frac{n-n_0}{R} \tag{3}$$

Hence the optical power of the given optical system

$$\Phi = \Phi_a + \Phi_l = \frac{n-1}{R} + \frac{n-n_0}{R} = \frac{2n-n_0-1}{R} \tag{4}$$

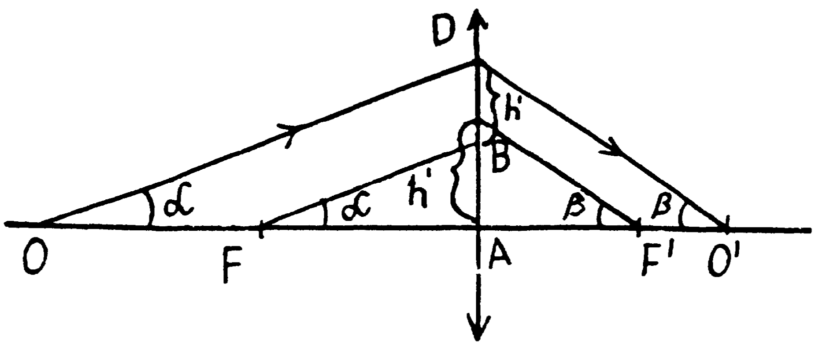
From Eqns (1) and (4)

$$\frac{\Phi}{\Phi_0} = \frac{2n-n_0-1}{2(n-1)} \quad \text{So} \quad \Phi = \frac{(2n-n_0-1)}{2(n-1)}\,\Phi_0$$

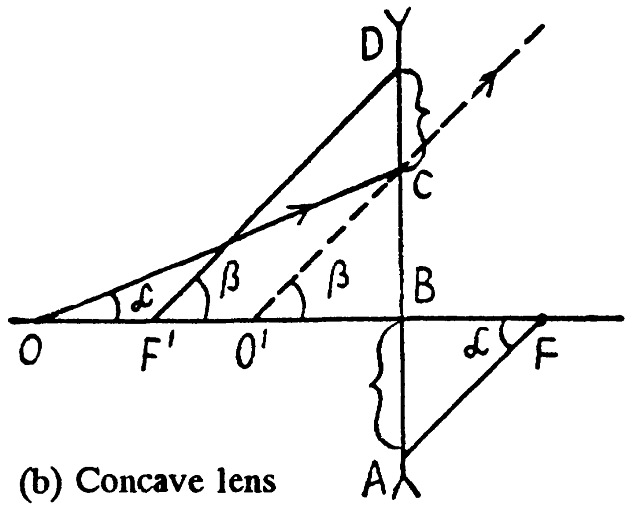
Focal length in air, $f = \dfrac{1}{\Phi} = 15$ cm

and focal length in water $= \dfrac{n_0}{\Phi} = 20$ cm for $n_0 = \dfrac{4}{3}$.

5.34 (a) Clearly the media on the sides are different. The front focus F is the position of the object (virtual or real) for which the image is formated at infinity. The rear focus F' is the position of the image (virtual or real) of the object at infinity. (a) Figures 5.7 (a) & (b). This geometrical construction ensines that the second of the equations (5.1g) is obeyed.

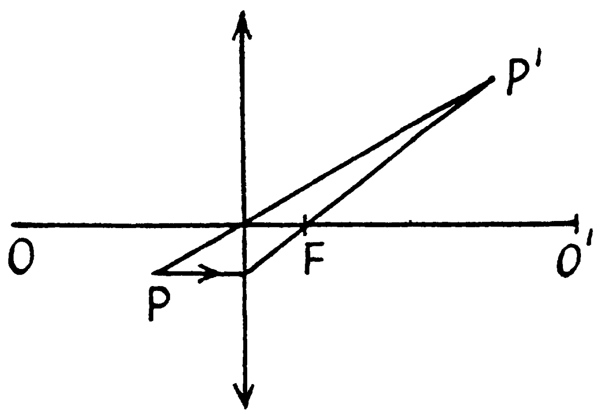


(a) Convex lens

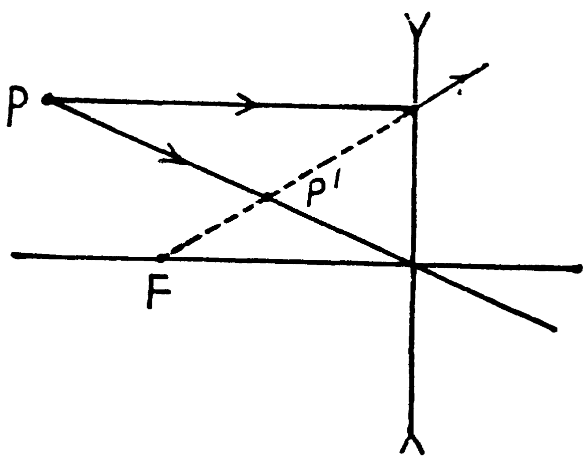


(b) Concave lens

(b) Figure 5.5 (a) & (b) with lens



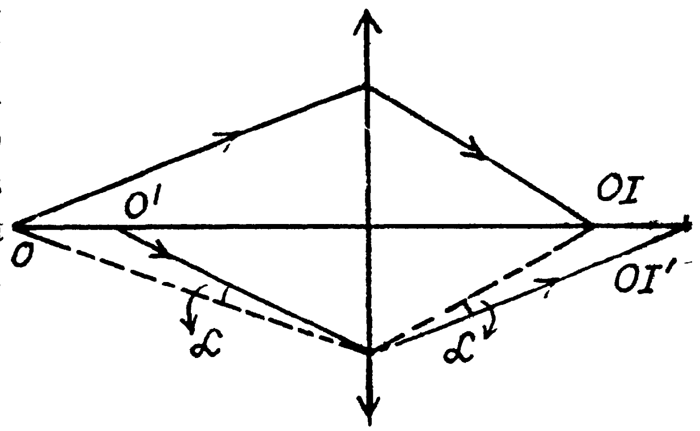
(a) Convex lens
(P is the object)



(b) Concave lens

(c) Figure (5.8) (a) & (b).

Clearly, the important case is that when the rays (1) & (2) are not symmetric about the principal axis, otherwise the figure can be completed by reflection in the principal axis. Knowing one path we know the path of all rays connecting the two points. For a different object. We proceed as shown below, we use the fact that a ray incident at a given height above the optic centre suffers a definite deviation.



The concave lens can be discussed similarly.

5.35 Since the image is formed on the screen, it is real, so for a conversing lens object is in the incident side.

Let s_1 and s_2 be the magnitudes of the object distance in the first and second case respectively. We have the lens formula

$$\frac{1}{s'} - \frac{1}{s} = \frac{1}{f} \tag{1}$$

In the first case from Eqn. (1)

$$\frac{1}{(+l)} - \frac{1}{(-s_1)} = \frac{1}{f} \quad \text{or,} \quad s_1 = \frac{f(l)}{l-f} = 26\cdot 31 \text{ cm}.$$

Similarly from Eqn.(1) in the second case

$$\frac{1}{(l-\Delta l)} - \frac{1}{(-s_2)} = \frac{1}{f} \quad \text{or,} \ s_2 = \frac{l f}{(l-\Delta l)-f} = 26\cdot 36 \text{ cm}.$$

Thus the sought distance $\Delta x = s_2 - s_1 = 0\cdot 5 \text{ mm} \approx \Delta l f^2/(l-f^2)$

5.36 The distance between the object and the image is l. Let x = distance between the object and the lens. Then, since the image is real, we have in our convention, $u = -x$, $v = l - x$

so
$$\frac{1}{x} + \frac{1}{l-x} = \frac{1}{f}$$

or
$$x(l-x) = lf \quad \text{or } x^2 - xl + lf = 0$$

Solving we get the roots

$$x = \frac{1}{2}\left[l \pm \sqrt{l^2 - 4lf}\right]$$

(We must have $l > 4f$ for real roots.)

(a) If the distance between the two positions of the lens is Δl, then clearly $\Delta l = x_2 - x_1$ = difference between roots $= \sqrt{l^2 - 4lf}$

so
$$f = \frac{l^2 - \Delta l^2}{4l} = 20 \text{ cm}.$$

(b) The two roots are conjugate in the sense that if one gives the object distance the other gives the corresponding image distance (in both cases). Thus the magnifications are

$$-\frac{l + \sqrt{l^2 - 4lf}}{l - \sqrt{l^2 - 4lf}} \text{ (enlarged) and } -\frac{l - \sqrt{l^2 - 4lf}}{l + \sqrt{l^2 - 4lf}} \text{ (diminished).}$$

The ratio of these magnification being η we have

$$\frac{l - \sqrt{l^2 - 4lf}}{l - \sqrt{l^2 - 4lf}} = \sqrt{\eta} \quad \text{or} \quad \frac{\sqrt{l^2 - 4lf}}{l} = \frac{\sqrt{\eta} - 1}{\sqrt{\eta} + 1}$$

or $$1 - \frac{4f}{l} = \left(\frac{\sqrt{\eta} - 1}{\sqrt{\eta} + 1}\right)^2 = 1 - 4\frac{\sqrt{\eta}}{(1 + \sqrt{\eta})^2}$$

Hence $$f = l\frac{\sqrt{\eta}}{(1 + \sqrt{\eta})^2} = 20 \text{ cm.}$$

5.37 We know from the previous problem that the two magnifications are reciprocals of each other ($\beta' \beta'' = 1$). If h is the size of the object then $h' = \beta' h$ and

$$h'' = \beta'' h$$

Hence $$h = \sqrt{h' h''}.$$

5.38 Refer to problem 5.32 (b). If A is the area of the object, then provided the angular diameter of the object at the lens is much smaller than other relevant angles like $\frac{D}{f}$ we calculate the light falling on the lens as $LA\frac{\pi D^2}{4s^2}$

where u^2 is the object distance squared. If β is the transverse magnification $\left(\beta = \frac{s'}{u}\right)$ then the area of the image is $\beta^2 A$. Hence the illuminance of the image (also taking account of the light lost in the lens)

$$E = (1 - \alpha)LA\frac{\pi D^2}{4s^2}\frac{1}{\beta^2 A} = \frac{(1-\alpha)\pi D^2 L}{4f^2}$$

since $s' = f$ for a distant object. Substitution gives $E = 15$ lx.

5.39 (a) If s = object distance, s' = average distance, L = luminance of the sounce, ΔS = area of the source assumed to be a plane surface held normal to the principal axis, then we find for the flux $\Delta\Phi$ incident on the lens

$$\Delta\Phi = \int L\,\Delta S \cos\theta\, d\Omega$$

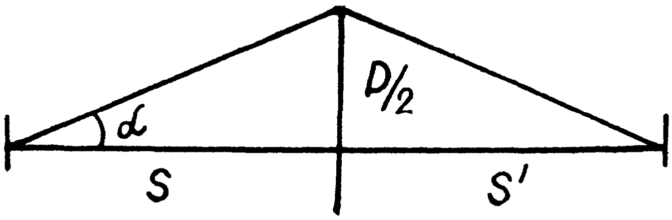
$$\approx L\,\Delta S\int_0^{\infty} \cos\theta\, 2\pi \sin\theta\, d\theta = L\,\Delta S\,\pi \sin^2\alpha \approx L\,\Delta S\frac{\pi D^2}{4s^2}$$

Here we are assuming $D << s$, and ignoring the variation of L since α is small

Then if L' is the luminance of the image, and $\Delta S' = \left(\frac{S'}{S}\right)^2 \Delta S$ is the area of the image then similarly

$$L' \Delta S' \frac{D^2}{4 s'^2} \pi = L' \Delta S \frac{D^2}{4 s^2} \pi = L \Delta S \frac{D^2}{4 s^2} \pi$$

or $\qquad L' = L$ irrespective of D.



(b) In this case the image on the white screen from a Lambert source. Then if its luminance is L_0 its luminosity will be the πL_0 and

$$\pi L_0 \frac{s'^2}{s^2} \Delta S = L \Delta S \frac{D^2}{4 s^2} \pi$$

or

$$L_0 \propto D^2$$

since s' depends on f, s but not on D.

5.40 Focal length of the converging lens, when it is submerged in water of R.I. n_0 (say) :

$$\frac{1}{f_1} = \left(\frac{n_1}{n_0} - 1\right)\left(\frac{1}{R} - \frac{1}{R}\right), \frac{2(n_1 - n_0)}{n_0 R} \tag{1}$$

Similarly, the focal length of diverging lens in water.

$$\frac{1}{f_2} = \left(\frac{n_2}{n_0} - 1\right)\left(\frac{1}{-R} - \frac{1}{R}\right) \frac{-2(n_2 - n_0)}{n_0 R} \tag{2}$$

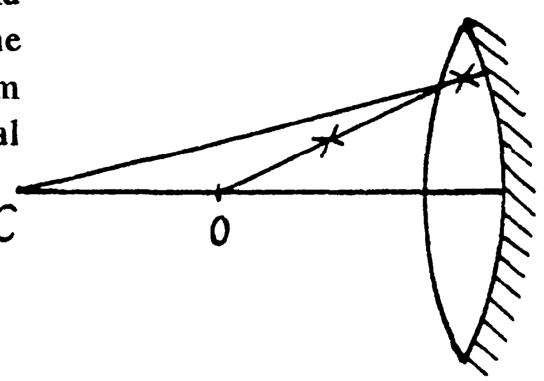
Now, when they are put together in the water, the focal length of the system,

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

$$= \frac{2(n_1 - n_2)}{n_0 R} - \frac{2(n_2 - n_0)}{n_0 R} = \frac{2(n_1 - n_2)}{n_0 R}$$

or, $$f = \frac{-n_0 R}{2(n_1 - n_2)} = 35 \text{ cm}$$

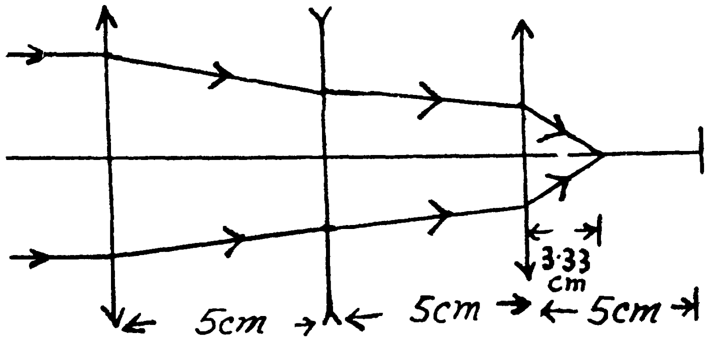
5.41 C is the centre of curvature of the silvered surface and O is the effective centre of the equivalent mirror in the sence that an object at O forms a coincident image. From the figure, using the formula for refraction at a spherical surface, we have



$$\frac{n}{-R} - \frac{1}{2f} = \frac{n-1}{R} \quad \text{or} \quad f = \frac{-R}{2(2n-1)}$$

(In our convention f is $-ve$).
Substitution gives $f = -10$ cm.

5.42 (a) Path of a ray, as it passes through the lens system is as shown below.



(a)

Focal length of all the three lenses,

$$f = \frac{1}{10}\text{ m} = 10\text{ cm, neglecting their signs.}$$

Applying lens formula for the first lens, considering a ray coming from infinity,

$$\frac{1}{s'} - \frac{1}{\infty} = \frac{1}{f} \quad \text{or,} \quad s' = f = 10\text{ cm},$$

and so the position of the image is 5 cm to the right of the second lens, when only the first one is present, but the ray again gets refracted while passing through the second, so,

$$\frac{1}{s'} - \frac{1}{5} = \frac{1}{f} = \frac{1}{-10}$$

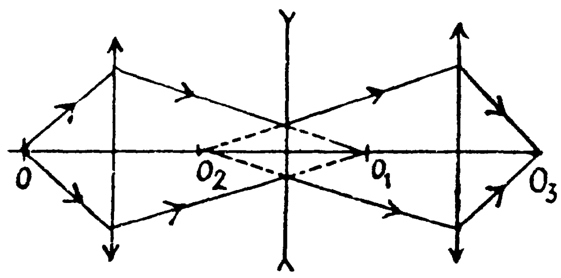
or, $s' = 10$ cm, which is now 5 cm left to the third lens so for this lens,

$$\frac{1}{s''} - \frac{1}{5} = \frac{1}{10} \quad \text{or} \quad \frac{1}{s''} = \frac{3}{10}$$

or, $$s'' = 10/3 = 3{\cdot}33\text{ cm. from the last lens.}$$

(b) This means that if the object is x cm to be left of the first lens on the axis OO' then the image is x on to the right of the 3rd (last) lens. Call the lenses 1,2,3 from the left and let O be the object, O_1 its image by the first lens, O_2 the image of O_1 by the 2nd lens and O_3, the image of O_2 by the third lens.

O_1 and O_2 must be symmetrically located with respect to the lens L_2 and since this lens is concave, O_1 must be at a distance $2\,|f_2|$ to be the right of L_2 and O_2 must be $2\,|f_2|$ to be the left of L_2. One can check that this satisfies lens equation for the third lens L_3



(b)

$$u = -(2\,|f_2| + 5) = -25\text{ cm.}$$

$$s' = x, \quad f_3 = 10\text{ cm.}$$

Hence $$\frac{1}{x} + \frac{1}{25} = \frac{1}{10} \quad \text{so } x = 16.67\text{ cm.}$$

5.43 (a) Angular magnification for Galilean telescope in normal adjustment is given as.

$$\Gamma = f_o/f_e$$

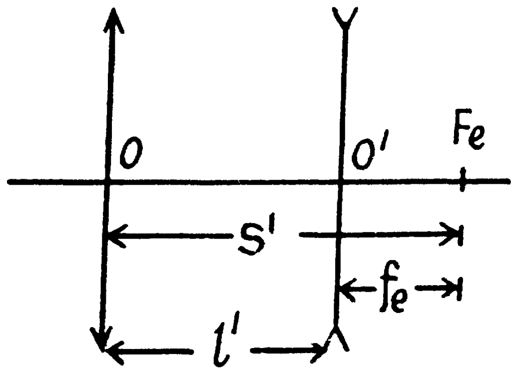
or, $$10 = f_o/f_e \quad \text{or} \quad f_o = 10 f_e \tag{1}$$

The length of the telescope in this case.

$$l = f_o - f_e = 45 \text{ cm. given,}$$

So, using (1), we get,

$$f_e = +5 \text{ and } f_o = +50 \text{ cm.}$$



(b) Using lens formula for the objective,

$$\frac{1}{s'_o} - \frac{1}{s_o} = \frac{1}{f_o} \quad \text{or,} \quad s'_o = \frac{s_o f_o}{s_o + f_o} = 50{\cdot}5 \text{ cm}$$

From the figure, it is clear that,

$s'_o = l' + f_e$, where l' is the new tube length.

or, $\quad l' = v_0 - f_e = 50{\cdot}5 - 5 = 45{\cdot}5$ cm.

So, the displacement of ocular is,

$$\Delta l = l' - l = 45{\cdot}5 - 45 = 0{\cdot}5 \text{ cm}$$

5.44 In the Keplerian telescope, in normal adjustment, the distance between the objective and eyepiece is $f_0 + f_e$. The image of the mounting produced by the eyepiece is formed at a distance v to the right where

$$\frac{1}{s'} - \frac{1}{s} = \frac{1}{f_e}$$

But
$$s = -(f_0 + f_e),$$

so
$$\frac{1}{s'} = \frac{1}{f_e} - \frac{1}{f_0 + f_e} = \frac{f_0}{f_e(f_0 + f_e)}$$

The linear magnification produced by the eyepiece of the mounting is, in magnitude,

$$|\beta| = \left|\frac{s'}{s}\right| = \frac{f_e}{f_0}$$

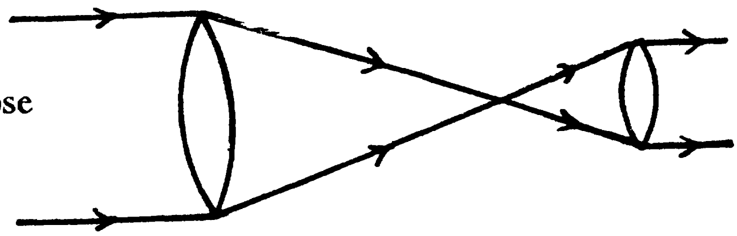
This equals $\dfrac{d}{D}$ according to the problem so

$$\Gamma = \frac{f_e}{f_e} = \frac{D}{d}.$$

5.45 It is clear from the figure that a parallel beam of light, originally of intensity I_0 has, on emerging from the telescope, an intensity.

$$I = I_0 \left(\frac{f_0}{f_e}\right)^2$$

because it is concentrated over a section whose diameter is f_e/f_0 of the diameter of the cross section of the incident beam.



Thus $$\eta = \left(\frac{f_0}{f_e}\right)^2$$

So $$\Gamma = \frac{f_0}{f_e} = \sqrt{\eta}$$

Now $$\Gamma = \frac{\tan \Psi'}{\tan \Psi} = \frac{\Psi'}{\Psi}$$

Hence $\Psi = \Psi'/\sqrt{\eta} = 0.6'$ on substitution.

5.46 When a glass lens is immersed in water its focal length increases approximately four times. We check this as follows as :

$$\frac{1}{f_a'} = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{f_w} = \left(\frac{n}{n_0} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{\frac{n}{n_0} - 1}{n - 1} \cdot \frac{1}{f_a} = \frac{n - n_0}{n_0 (n - 1)} \frac{1}{f_a}$$

Now back to the problem. Originally in air

$$\Gamma = \frac{f_0}{f_e} = 15 \text{ so } l = f_0 + f_e = f_e (\Gamma + 1)$$

In water, $$f_e' = \frac{n_0 (n - 1)}{n - n_0} f_e$$

and the focal length of the replaced objective is given by the condition

$$f_0' + f_e' = l = (\Gamma + 1) f_e$$

or $$f_0' = (\Gamma + 1) f_e - f_e'$$

Hence $$\Gamma' = \frac{f_0'}{f_e'} = (\Gamma + 1) \frac{n - n_0}{n_0 (n - 1)} - 1$$

Substitution gives ($n = 1.5$, $n_0 = 1.33$), $\Gamma' = 3.09$

5.47 If L is the luminance of the object, A is its area, s = distance of the object then light falling on the objective is

$$\frac{L \pi D^2}{4 s^2} A$$

The area of the image formed by the telescope (assuming that the image coincides with the object) is Γ^2 A and the area of the final image on the retina is

$$= \left(\frac{f}{s}\right)^2 \Gamma^2 A$$

Where f = focal length of the eye lens. Thus the illuminance of the image on the retin (when the object is observed through the telescope) is

$$\frac{L\pi D^2 A}{4u^2\left(\frac{f}{s}\right)^2 \Gamma^2 A} = \frac{L\pi D^2}{4f^2\Gamma^2}$$

When the object is viewed directly, the illuminance is, similarly, $\frac{L\pi d_0^2}{4f^2}$

We want $$\frac{L\pi D^2}{4f^2\Gamma^2} \geq \frac{L\pi d_0^2}{4f^2}$$

So, $\Gamma \leq \frac{D}{d_0} = 20$ on substitution of the values.

5.48 Obviously, $f_o = +1$ cm and $f_e = +5$ cm

Now, we know that, magnification of a microscope,

$$\Gamma = \left(\frac{s'_o}{f_o} - 1\right)\frac{D}{f_e}, \text{ for distinct vision}$$

or, $$50 = \left(\frac{s'_o}{1} - 1\right)\frac{25}{5} \quad \text{or, } v_o = 11 \text{ cm.}$$

Since distance between objective and ocular has increased by 2 cm, hence it will cause the increase of tube length by 2cm.

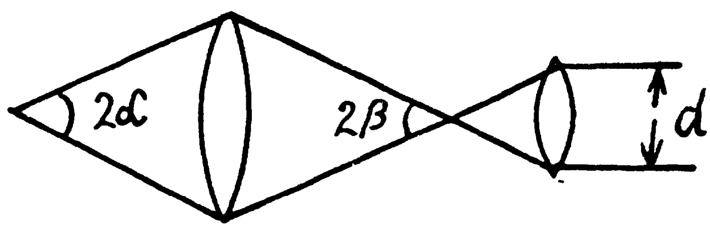
so, $$s'_o = s'_o + 2 = 13$$

and hence, : $$\Gamma' = \left(\frac{s'_o}{f_o} - 1\right)\frac{D}{f_e} = 60$$

5.49 It is implied in the problem that final image of the object is at infinity (otherwise light coming out of the eyepiece will not have a definite diameter).

(a) We see that $s'_0 2\beta = |s_0| 2\alpha$, then

$$\beta = \frac{|s_0|}{s'_0}\alpha$$



Then, from the figure

$$d = 2f_e\beta = 2f_e\alpha \Big/ \frac{s'_0}{|s_0|}$$

But when the final image is at infinity, the magnification Γ in a microscope is given by

$$\Gamma = \frac{s'_0}{|s_0|}\cdot\frac{l}{f_e}$$ (l = least distance of distinct vision) So $d = 2l\alpha/\Gamma$

So $d = d_0$ when $\Gamma = \Gamma_0 = \frac{2l\alpha}{d_0} = 15$ on putting the values.

(b) If Γ is the magnification produced by the microscope, then the area of the image produced on the retina (when we observe an object through a microscope) is : $\Gamma^2 \left(\frac{f}{s}\right)^2 A$

Where u = distance of the image produced by the microscope from the eye lens, f = focal length of the eye lens and A = area of the object. If Φ = luminous flux reaching the objective from the object and $d \le d_0$ so that the entire flux is admitted into the eye), then the illuminance of the final image on the retina

$$= \frac{\Phi}{\Gamma^2\,(f/s)^2 A}$$

But if $d \ge d_0^2$, then only a fraction $(d_0 \mid d)^2$ of light is admitted into the eye and the illuminance becomes

$$\frac{\Phi}{A\left(\frac{f}{s}\right)^2 \Gamma^2}\left(\frac{d_0}{d}\right)^2 = \frac{\Phi\, d_0^2}{A\left(\frac{f}{s}\right)^2 (2\,l\,\alpha)^2}$$

independent of Γ. The condition for this is then

$$d \ge d_0 \quad \text{or} \quad \Gamma \le \Gamma_0 = 15.$$

5.50 The primary and secondary focal length of a thick lens are given as,

$$f = -(n/\Phi)\{1 - (d/n')\,\Phi_2\}$$

and $$f' = +(n''/\Phi)\{1 - (d/n')\,\Phi_1\},$$

where Φ is the lens power n, n' and n'' are the refractive indices of first medium, lens material and the second medium beyond the lens. Φ_1 and Φ_2 are the powers of first and second spherical surface of the lens.

Here, $n = 1$, for lens, $n' = n$, for air

and $n'' = n_0$, for water.

So, $$\left.\begin{aligned} f &= -1/\Phi_1 \\ \text{and } f' &= +n_0/\Phi \end{aligned}\right\}, \text{ as } d \approx 0, \qquad (1)$$

Now, power of a thin lens,

$$\Phi = \Phi_1 + \Phi_2,$$

where, $$\Phi_1 = \frac{(n-1)}{R}$$

and $$\Phi_2 = \frac{(n_0 - n)}{-R}$$

So, $$\Phi = (2n - n_0 - 1)/R \qquad (2)$$

From equations (1) and (2), we get,

$$f = \frac{-R}{(2n - n - 1)} = -11.2 \text{ cm}$$

and $$f = \frac{n_0 R}{(2n - n_0 - 1)} = +14.9 \text{ cm.}$$

Since the distance between the primary principal point and primary nodal point is given as,

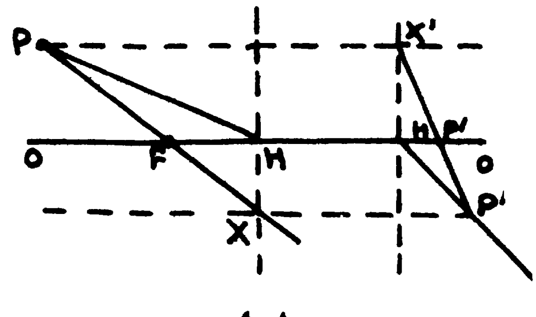
$$x = f\{(n'' - n)/n''\}$$

So, in this case, $$x = (n_0/\Phi)(n_0 - 1)/n_0 = (n_0 - 1)/\Phi$$

$$= \frac{n_0}{\Phi} - \frac{1}{\Phi} = f' + f = 3{\cdot}7 \text{ cm.}$$

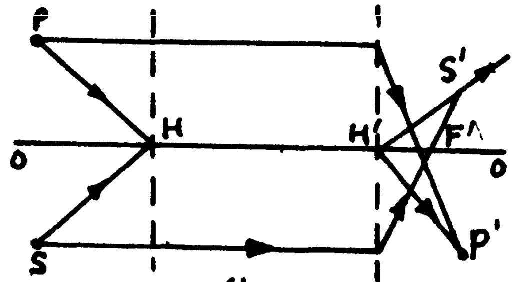
5.51 See the answersheet of problem book.

5.52 (a) Draw $P'X$ parallel to the axis OO' and let PF interest it at X. That determines the principal point H. As the medium on both sides of the system is the same, the principal point coincides with the nodal point. Draw a ray parallel to PH through P'. That determines H'. Draw a ray PX' parallel to the axis and join $P'X'$. That gives F'.



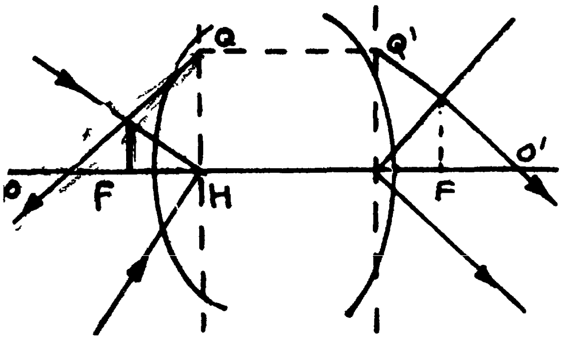
(a)

(b) We let H stand for the principal point (on the axis). Determine H' by drawing a ray $P'H'$ passing through P' and parallel to PH. One ray (conjugate to SH) can be obtained from this. To get the other ray one needs to know F or F'. This is easy because P and P' are known. Finally we get S'.



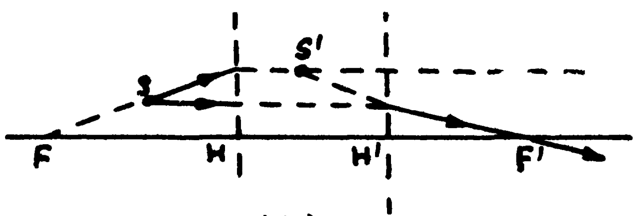
(b)

(c) From the incident ray we determine Q. A line parallel to OO' through Q determines Q and hence H'. H and H' are then also the nodal points. A ray parallel to the incident ray through H will emerge parallel to itself through H'. That determines F'. Similarly a ray parallel to the emergent ray through H determines F.

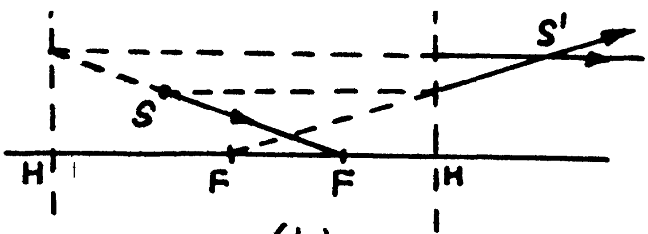


(c)

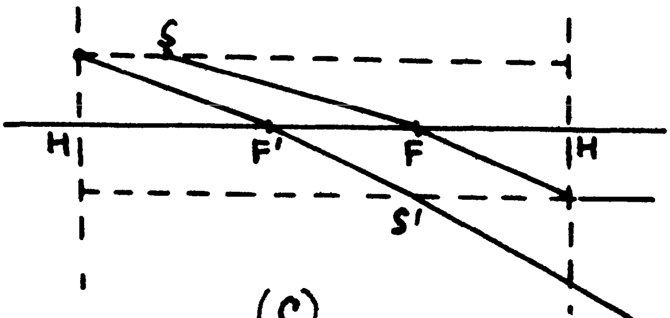
5.53 Here we do not assume that the media on the two sides of the system are the same.



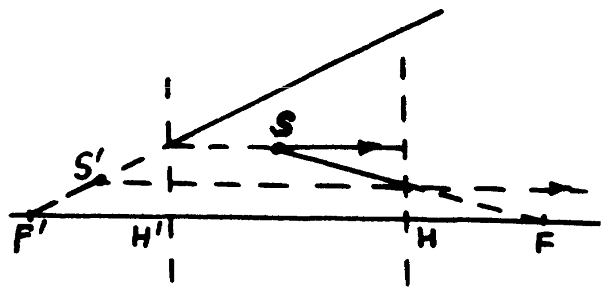
(a)



(b)



(c)



(d)

5.54 (a) Optical power of the system of combination of two lenses,

$$\Phi = \Phi_1 + \Phi_2 - d\,\Phi_1\,\Phi_2$$

on putting the values,

$$\Phi = 4\,\text{D}$$

or, $$f = \frac{1}{\Phi} = 25\text{ cm}$$

Now, the position of primary principal plane with respect to the vertex of converging lens,

$$X = \frac{d\,\Phi_2}{\Phi} = 10\text{ cm}$$

Similarly, the distance of secondary principal plane with respect to the vertex of diverging lens.

$$\text{'}X' = -\frac{d\,\Phi_1}{\Phi} = -10\text{ cm}\text{ , i.e. 10 cm left to it.}$$

(b) The distance between the rear principal focal point F' and the vertex of converging lens,

$$l = d + \left(\frac{1}{\Phi}\right)(-d\,\Phi_1) = \frac{\Phi\,d}{\Phi} + \left(\frac{-d\,\Phi_1}{\Phi}\right)$$

and $$f/l = \left(\frac{1}{\Phi}\right) \Big/ \frac{\Phi\,d}{\Phi} - \frac{d\,\Phi_1}{\Phi}, \quad \text{as} \quad f = \frac{1}{\Phi}$$

$$= 1/d\,\Phi - d\,\Phi_1$$

$$= 1/d\,(\Phi_1 + \Phi_2 - d\,\Phi_1\,\Phi_2) - d\,\Phi_1 = 1/d\,\Phi_2 - d^2\,\Phi_1\,\Phi_2$$

Now, if f/l is maximum for certain value of d then l/f will be minimum for the same value of d. And for minimum l/f,

$$d\,(l/f)/d\,d = \Phi_2 - 2\,d\,\Phi_1\,\Phi_2 = 0$$

or, $$d = \Phi_2/2\,\Phi_1\,\Phi_2$$

or, $$d = 1/2\,\Phi_1 = 5\text{ cm}$$

So, the required maximum ratio of $f/l = 4/3$.

5.55 The optical power of first convex surface is,

$$\Phi = \frac{P\,(n-1)}{R_1} = 5\text{ D}, \quad \text{as } R_1 = 10\text{ cm}$$

and the optical power of second concave surface is,

$$\Phi_2 = \frac{(1-n)}{R_2} = -10\,\text{D}$$

So, the optical power of the system,

$$\Phi = \Phi_1 + \Phi_2 - \frac{d}{n}\Phi_1 \Phi_2 = -4\text{ D}$$

Now, the distance of the primary principal plane from the vertex of convex surface is given as,

$$x = \left(\frac{1}{\Phi}\right)\left(\frac{d}{n}\right)\Phi_2 \text{ , here } n_1 = 1 \text{ and } n_2 = n.$$

$$= \frac{d\,\Phi_2}{\Phi\, n} = 5\text{ cm}$$

and the distance of secondary principal plane from the vertex of second concave surface,

$$x' = -\left(\frac{1}{\Phi}\right)\left(\frac{d}{n}\right)\Phi_1 = -\frac{d\,\Phi_1}{\Phi\, n} = 2{\cdot}5\text{ cm}$$

5.56 The optical power of the system of two thin lenses placed in air is given as,

$$\Phi = \Phi_1 + \Phi_2 - d\,\Phi_1 \Phi_2$$

or, $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$, where f is the equivalent focal length

So, $$\frac{1}{f} = \frac{f_2 + f_1 - d}{f_1 f_2}$$

or, $$f = \frac{f_1 f_2}{f_1 + f_2 - d} \qquad (1)$$

This equivalent focal length of the system of two lenses is measured from the primary principal plane.

As clear from the figure, the distance of the primary principal plane from the optical centre of the first is

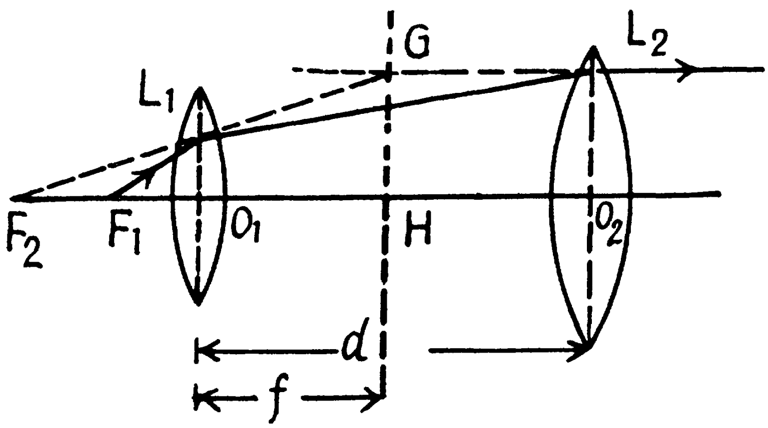
$O_1 H = x = +(n/\Phi)(d/n')\Phi_1$

$= \frac{d\,\Phi_1}{\Phi}$, as $n = n' = 1$, for air.

$= \frac{df}{f_1}$

$= \left(\frac{d}{f_1}\right)\left(\frac{f_1 f_2}{f_1 + f_2 - d}\right)$

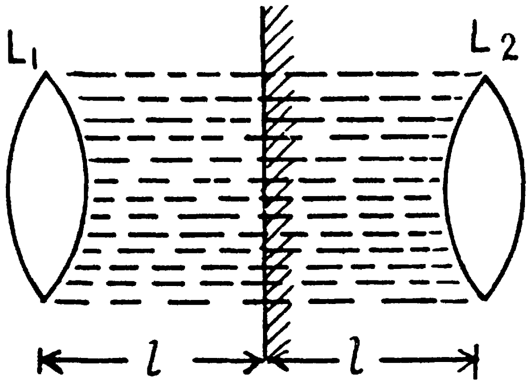
$= \frac{d f_2}{f_1 + f_2 - d}$



Now, if we place the equivalent lens at the primary principal plane of the lens system, it will provide the same transverse magnification as the system. So, the distance of equivalent lens from the vertex of the first lens is,

$$x = \frac{d f_2}{f_1 + f_2 - d}$$

5.57 The plane mirror forms the image of the lens, and water, filled in the space between the two, behind the mirror, as shown in the figure.



So, the whole optical system is equivalent to two similar lenses, seperated by a distance $2l$ and thus, the power of this system,

$\Phi = \Phi_1 + \Phi_2 - \frac{d\,\Phi_1\,\Phi_2}{n_0}$, where $\Phi_1 = \Phi_2 = \Phi'_1$

= optical power of individual lens and n_0 = R.I. of water.

Now, Φ' = optical power of first convex surface + optical power of second concave surface.

$= \frac{(n-1)}{R} + \frac{n_0 - n}{R}$, n is the refractive index of glass.

$$\frac{(2n - n_0 - 1)}{R} \qquad (1)$$

and so, the optical power of whole system,

$$\Phi = 2\,\Phi' - \frac{2\,d\,\Phi'^2}{n_0} = 3{\cdot}0\text{ D, substituting the values.}$$

5.58 (a) A telescope in normal adjustment is a zero power conbination of lenses. Thus we require

$$\Phi = O = \Phi_1 + \Phi_2 - \frac{d}{n}\,\Phi_1\,\Phi_2$$

But Φ_1 = Power of the convex surface $= \frac{n-1}{R_0 + \Delta R}$

Φ_2 = Power of the concave surface $= -\frac{n-1}{R_0}$

Thus, $$O = \frac{(n-1)\,\Delta R}{R_0\,(R_0 + \Delta R)} + \frac{d}{n}\,\frac{(n-1)^2}{R_0\,(R_0 + \Delta R)}$$

So $$d = \frac{n\,\Delta R}{n-1} = 4.5\text{ cm. on putting the values.}$$

(b) Here, $\Phi = -1 = \frac{.5}{.1} - \frac{.5}{.075} + \frac{d}{1.5} \times \frac{.5 \times .5}{.1 \times .075}$

$$= 5 - \frac{20}{3} + \frac{d \times 2}{3} \times \frac{5 \times 20}{3} = -\frac{5}{3} + \frac{200'd}{9}$$

$$= \frac{200\,d}{9} = \frac{2}{3} \text{ or } d = (3/100)\text{ m} = 3\text{ cm.}$$

5.59 (a) The power of the lens is (as in the previous problem)

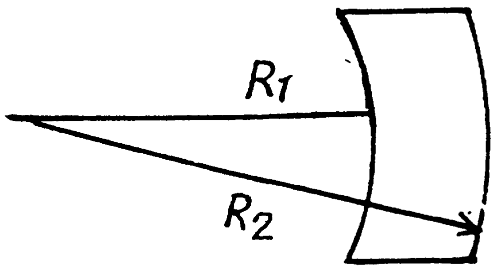
$$\Phi = \frac{n-1}{R} - \frac{n-1}{R} - \frac{d}{n}\left(\frac{n-1}{R}\right)\left(-\frac{n-1}{R}\right) = \frac{d(n-1)^2}{n R^2} > 0.$$

The principal planes are located on the side of the convex surface at a distance d from each other, with the front principal plane being removed from the convex surface of the lens by a distance $R/(n-1)$.

(b) Here $\Phi = -\frac{n-1}{R_1} + \frac{n+1}{R_2} + \frac{R_2 - R_1}{n}\frac{(n-1)^2}{R_1 R_2}$

$$= \frac{(n-1)(R_2 - R_1)}{R_2 R_1}\left[-1 + \frac{n-1}{n}\right]$$

$$= -\frac{n-1}{n}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) < 0$$



Both principal planes pass through the common centre of curvature of the surfaces of the lens.

5.60 Let the optical powers of the first and second surfaces of the ball of radius R_1 be Φ_1' and Φ_1'', then

$$\Phi_1' = (n-1)/R_1 \quad \text{and} \quad \Phi_1'' = (1-n)/(-R_1) = \frac{(n-1)}{R_1}$$

This ball may be treated as a thick spherical lens of thickness $2R_1$. So the optical power of this sphere is,

$$\Phi = \Phi_1' - \frac{2R_1 \Phi_1' \Phi_1''}{n} = 2(n-1)/nR_1 \tag{1}$$

Similarly, the optical power of second ball,

$$\Phi_2 = 2(n-1)/nR_2$$

If the distance between the centres of these balls be d. Then the optical power of whole system,

$$\Phi = \Phi_1 + \Phi_2 - d\,\Phi_1 \Phi_2$$

$$= \frac{2(n-1)}{nR_1} + \frac{2(n-1)}{nR_2} - \frac{4d(n-1)^2}{n^2 R_1 R_2}$$

$$= \frac{2(n-1)}{nR_1R_2}\left[(R_1 + R_2) - \frac{2d(n-1)}{n}\right].$$

Now, since this system serves as telescope, the optical power of the system must be equal to zero.

$$(R_1 + R_2) = \frac{2d(n-1)}{n}, \quad \text{as} \quad \frac{2(n-1)}{nR_1R_2} \neq 0.$$

or, $$d = \frac{n(R_1 + R_2)}{2(n-1)} = 9 \text{ cm}.$$

Since the diameter D of the objective is $2R_1$ and that of the eye-piece is $d = 2R_2$

So, the magnification,

$$\Gamma = D/d = \frac{2R_1}{2R_2} = R_1\ R_2 = 5.$$

5.61 Optical powers of the two surfaces of the lens are

$$\Phi_1 = (n-1)/R \text{ and } \Phi_2 = (1-n)/-R = \frac{n-1}{R}$$

So, the power of the lens of thickness d,

$$\Phi' = \Phi_1 + \Phi_2 - \frac{d\,\Phi_1 \Phi_2}{n} = \frac{n-1}{R} + \frac{n-1}{R} - \frac{d\,(n-1)^2/R^2}{n^2} = \frac{n^2-1}{nR}$$

and optical power of the combination of these two thick lenses,

$$\Phi = \Phi' + \Phi' = 2\,\Phi' = \frac{2\,(n^2-1)}{nR}$$

So, power of this system in air is, $\Phi_0 = \dfrac{\Phi}{n} = \dfrac{2\,(n^2-1)}{n^2 R} = 37\,\text{D}$.

5.62 We consider a ray QPR in a medium of gradually varying refractive index n. At P, the gradient of n is a vector with the given direction while is nearly the same at neighbouring points Q, R. The arc length QR is ds. We apply Snell's formula $n \sin\theta = \text{constant}$ where θ is to be measured from the direction $\vec{\nabla} n$. The refractive indices at Q,R whose mid point is P are $\quad \eta \pm \frac{1}{2}\,|\nabla\eta|\,d\theta\,\cos\theta$

so $\quad (\eta - \frac{1}{2}\,|\nabla n|\,d\theta\,\cos\theta)\,(\sin\theta + \frac{1}{2}\cos\theta\,d\theta)$

$= (\eta + \frac{1}{2}\,|\nabla n|\,d\theta\cos\theta)\,(\sin\theta - \frac{1}{2}\cos\theta\,d\theta)$ or $\quad n\cos\theta\,d\theta = |\nabla n|\,ds\,\cos\theta\,\sin\theta$

(we have used here $\sin(\theta \pm \frac{1}{2}d\theta) = \sin\theta \pm \frac{1}{2}\cos\theta\,d\theta$)

Now using the definition of the radius of curvature $\dfrac{1}{\rho} = \dfrac{d\theta}{ds}$

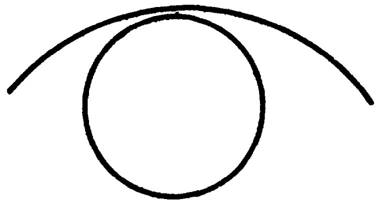
$$\frac{1}{\rho} = \frac{1}{\eta}\,|\nabla n|\,\sin\theta$$

The quantity $|\nabla n|\sin\theta$ can be called $\dfrac{\delta n}{\delta N}$ i.e. the derivative of n along the normal N to the ray. Then $\quad \dfrac{1}{\rho} = \dfrac{\delta}{\delta N}\ln n.$

5.63 From the above problem

$\dfrac{1}{\rho} = \dfrac{1}{n}\,\hat{p}\cdot\vec{\nabla} n \approx \hat{p}\cdot\nabla n \approx |\nabla n| = 3\times10^{-8}\,\text{m}^{-1}$

(since $\hat{p}\,||\,\vec{\nabla} n$ both being vertical). So $\rho = 3{\cdot}3\times10^{7}\,\text{m}$



For the ray of light to propagate all the way round the earth we must have

$\rho = R = 6400\,\text{km} = 6{\cdot}4\times10^{6}\,\text{m}$

Thus $\;|\nabla n| = 1{\cdot}6\times10^{-7}\,\text{m}^{-1}$

5.2 INTERFERENCE OF LIGHT

5.64 (a) In this case the net vibration is given by

$$x = a_1 \cos \omega t + a_2 \cos (\omega t + \delta)$$

where δ is the phase difference between the two vibrations which varies rapidly and randomly in the interval $(0, 2\pi)$. (This is what is meant by incoherence.)

Then $\quad x = (a_1 + a_2 \cos \delta) \cos \omega t + a_2 \sin \delta \sin \omega t$

The total energy will be taken to be proportional to the time average of the square of the displacement.

Thus $\quad E = \left\langle (a_1 + a_2 \cos \delta)^2 + a_2^2 \sin^2 \delta \right\rangle = a_1^2 + a_2^2$

as $\langle \cos \delta \rangle = 0$ and we have put $\langle \cos^2 \omega t \rangle = \langle \sin^2 \omega t \rangle = \frac{1}{2}$ and has been absorbed in the overall constant of proportionality.

In the same units the energies of the two oscillations are $\overline{a_1^2}$ and a_2^2 respectively so the proposition is proved.

(b) Here $\quad \vec{r} = a_1 \cos \omega t \, \hat{i} + a_2 \cos (\omega t + \delta) \hat{j}$

and the mean square displacement is $\alpha \, a_1^2 + a_2^2$

if δ is fixed but arbitrary. Then as in (a) we see that $E = E_1 + E_2$.

5.65 It is easier to do it analytically.

$$\xi_1 = a \cos \omega t, \; \xi_2 = 2 a \sin \omega t$$

$$\xi_3 = \frac{3}{2} a \left(\cos \frac{\pi}{3} \cos \omega t - \sin \frac{\pi}{3} \sin \omega t \right)$$

Resultant vibration is

$$\xi = \frac{7a}{4} \cos \omega t + a \left(2 - \frac{3\sqrt{3}}{4} \right) \sin \omega t$$

This has an amplitude $= \frac{a}{4} \sqrt{49 + (8 - 3\sqrt{3})^2} = 1{\cdot}89\, a$

5.66 We use the method of complex amplitudes. Then the amplitudes are

$$A_1 = a, \; A_2 = a e^{i\varphi}, \; \ldots \; A_N = a e^{i(N-1)\varphi}$$

and the resultant complex amplitude is

$$A = A_1 + A_2 + \ldots + A_N = a \left(1 + e^{i\varphi} + e^{2i\varphi} + \ldots + e^{i(N-1)\varphi} \right)$$

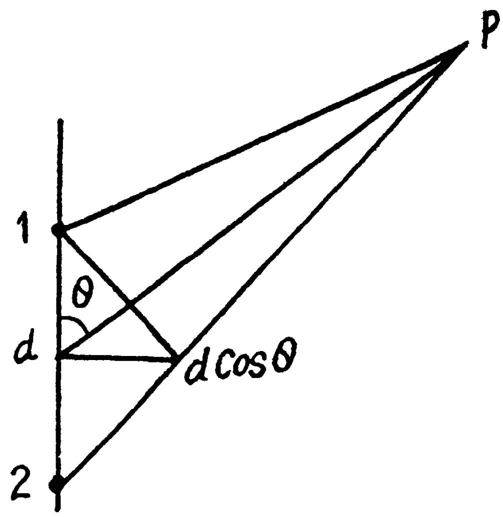
$$= a \frac{1 - e^{iN\varphi}}{1 - e^{i\varphi}}$$

The corresponding ordinary amplitude is

$$|A| = a\left|\frac{1-e^{iN\varphi}}{1-e^{i\varphi}}\right| = a\left[\frac{1-e^{iN\varphi}}{1-e^{i\varphi}} \times \frac{1-e^{-iN\varphi}}{1-e^{-i\varphi}}\right]^{1/2}$$

$$= a\left[\frac{2-2\cos N\varphi}{2-2\cos\varphi}\right]^{1/2} = a\,\frac{\sin\frac{N\varphi}{2}}{\sin\frac{\varphi}{2}}.$$

5.67 (a) With dipole moment $\perp^r$ to plane there is no variation with θ of individual radiation amplitude. Then the intensity variation is due to interference only.



In the direction given by angle θ the phase difference is

$$\frac{2\pi}{\lambda}(d\cos\theta) + \varphi = 2k\pi \quad \text{for maxima}$$

Thus $\quad d\cos\theta = \left(k - \frac{\varphi}{2\pi}\right)\lambda$

$$k = 0, \pm 1, \pm 2, \ldots$$

We have added φ to $\frac{2\pi}{\lambda} d\cos\theta$ because the extra path that the wave from 2 has to travel in going to P (as compared to 1) makes it lag more than it already is (due to φ).

(b) Maximum for $\theta = \pi$ gives $-d = \left(k - \frac{\varphi}{2\pi}\right)\lambda$

Minimum for $\theta = 0$ gives $d = \left(k' - \frac{\varphi}{2\pi} + \frac{1}{2}\right)\lambda$

Adding we get $\quad \left(k + k' - \frac{\varphi}{\pi} + \frac{1}{2}\right)\lambda = 0$

This can be true only if $\quad k' = -k,\ \varphi = \frac{\pi}{2}$

since $\ 0 < \varphi < \pi$

Then $\quad -d = \left(k - \frac{1}{4}\right)\lambda$

Here $\quad k = 0, -1, -2, -3, \ldots$

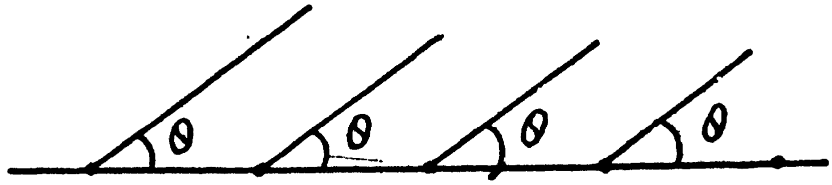
(Otherwise R.H.S. will become *+ve*).

Putting $k = -\bar{k}$, $\bar{k} = 0, +1, +2, +3, \ldots$

$$d = \left(\bar{k} + \frac{1}{4}\right)\lambda.$$

5.68 If $\Delta\varphi$ is the phase difference between neighbouring radiators then for a maximum in the direction θ we must have

$$\frac{2\pi}{\lambda} d \cos\theta + \Delta\varphi = 2\pi k$$



For scanning $\theta = \omega t + \beta$

Thus $$\frac{d}{\lambda}\cos(\omega t + \beta) + \frac{\Delta\varphi}{2\pi} = k$$

or $$\Delta\varphi = 2\pi\left[k - \frac{d}{\lambda}\cos(\omega t + \beta)\right]$$

To get the answer of the book, put $\beta = \alpha - \pi/2$.

5.69 From the general formula

$$\Delta x = \frac{l\lambda}{d}$$

we find that $$\frac{\Delta x}{\eta} = \frac{l\lambda}{d + 2\Delta h}$$

since d increases to $d + 2\Delta h$ when the source is moved away from the mirror plane by Δh.

Thus $$\eta d = d + 2\Delta h \quad \text{or} \quad d = 2\Delta h/(\eta - 1)$$

Finally $$\lambda = \frac{2\Delta h \Delta x}{(\eta - 1)l} = 0{\cdot}6\,\mu\text{m}.$$

5.70 We can think of the two coherent plane waves as emitted from two coherent point sources very far away. Then

$$\Delta x = \frac{l\lambda}{d} = \frac{\lambda}{d/l}$$

But $$\frac{d}{l} = \psi \text{ (if } \psi \ll 1\text{)}$$

so $$\Delta x = \frac{\lambda}{\psi}.$$

5.71 (a) Here $S'S'' = d = 2r\alpha$

Then $\Delta x = \dfrac{(b+r)\lambda}{2\alpha}$

Putting $b = 1{\cdot}3$ metre, $r = {\cdot}1$ metre

$\lambda = 0{\cdot}55\,\mu\text{m}$, $\alpha = 12' = \dfrac{1}{5 \times 57}$ radian

we get $\Delta x = 1{\cdot}1$ mm

Number of possible maxima $= \dfrac{2b\alpha}{\Delta x} + 1 \approx 8{\cdot}3 + 1 \approx 9$

($2b\alpha$ is the length of the spot on the screen which gets light after reflection from both mirror. We add 1 above to take account of the fact that in a distance Δx there are two maxima).

(b) When the silt moves by δl along the arc of radius r the incident ray on the mirror rotates by $\frac{\delta l}{r}$; this is the also the rotation of the reflected ray. There is then a shift of the fringe of magnitude.

$$b\frac{\delta l}{r} = 13 \text{ mm}.$$

(c) If the width of the slit is δ then we can imagine the slit to consist of two narrow slits with separation δ. The fringe pattern due to the wide slit is the superposition of the pattern due to these two narrow slits. The full pattern will not be sharp at all if the pattern due to the two narrow slits are $\frac{1}{2}\Delta x$ apart because then the maxima due to one will fill the minima due to the other. Thus we demand that

$$\frac{b\,\delta_{max}}{r} = \frac{1}{2}\Delta x = \frac{(b+r)\lambda}{4r\alpha}$$

or $$\delta_{max} = \left(1+\frac{r}{b}\right)\frac{\lambda}{4\alpha} = 42\,\mu\text{m}.$$

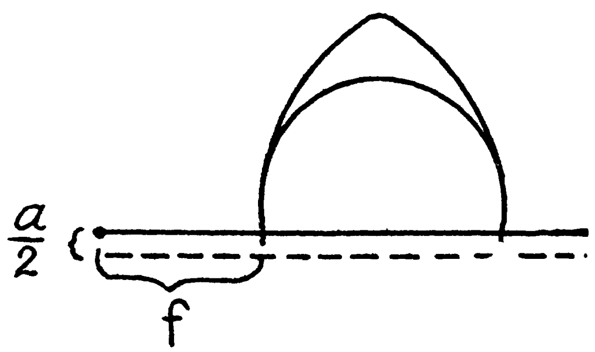
5.72 To get this case we must let $r \to \infty$ in the formula for Δx of the last example.

So $$\Delta x = \frac{(b+r)\lambda}{2\alpha r} \to \frac{\lambda}{2\alpha}.$$

(A plane wave is like light emitted from a point source at ∞).

Then $$\lambda = 2\alpha\Delta x = 0{\cdot}64\,\mu\text{m}.$$

5.73



(a) We show the upper half of the lens. The emergent light is at an angle $\frac{a}{2f}$ from the axis. Thus the divergence angle of the two incident light beams is

$$\psi = \frac{a}{f}$$

When they interfere the fringes produced have a width

$$\Delta x = \frac{\lambda}{\psi} = \frac{f\lambda}{a} = 0{\cdot}15 \text{ mm}.$$

The patch on the screen illuminated by both light has a width $b\psi$ and this contains

$$\frac{b\psi}{\Delta x} = \frac{b a^2}{f^2 \lambda} \text{ fringes} = 13 \text{ fringes}$$

(if we ignore 1 in comparis on to $\frac{b\psi}{\Delta x}$ (if 5.71 (a))

(b) We follow the logic of (5.71 c). From one edge of the slit to the other edge the distance is of magnitude δ $\left(\text{i.e. } \frac{a}{2} \text{ to } \frac{a}{2} + \delta\right)$.

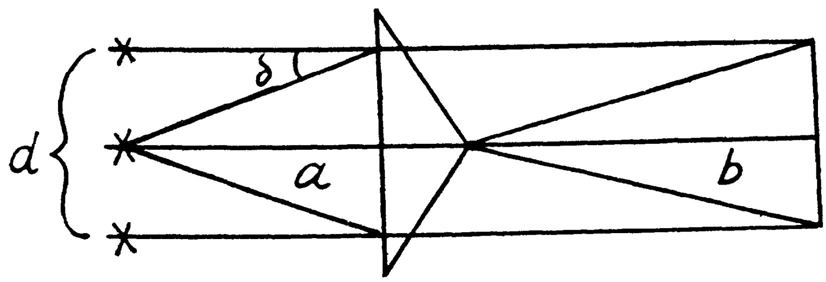
If we imagine the edge to shift by this distance, the angle $\psi/2$ will increase by $\frac{\Delta\psi}{2} = \frac{\delta}{2f}$

and the light will shift $\pm\, b\frac{\delta}{2f}$

The fringe pattern will therefore shift by $\frac{\delta \cdot b}{f}$

Equating this to $\frac{\Delta x}{2} = \frac{f\lambda}{2a}$ we get $\delta_{max} = \frac{f^2\lambda}{2ab} = 37{\cdot}5\ \mu\text{m}$.

5.74 $\Delta x = \frac{l\lambda}{d}$



$l = a + b$

$d = 2(n-1)\theta a$

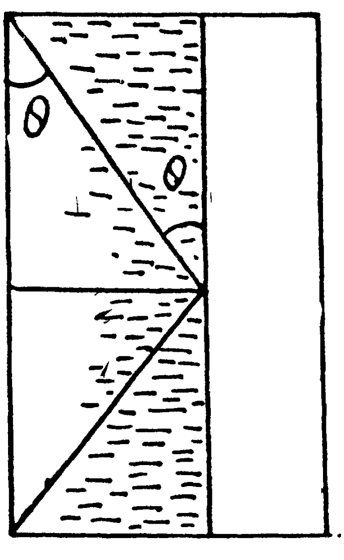
$\delta = (n-1)\theta$

$d = 2\delta \cdot a$

$n =$ R.I. of glass

Thus $$\lambda = \frac{2(n-1)\theta a \Delta x}{a+b} = 0{\cdot}64\ \mu\text{m}.$$

5.75 It will be assumed that the space between the biprism and the glass plate filled with benzene constitutes complementary prisms as shown.



Then the two prisms being oppositely placed, the net deviation produced by them is

$\delta = (n-1)\theta - (n'-1)\theta = (n-n')\theta$

Hence as in the previous problem

$d = 2a\delta = 2a\theta(n-n')$

So $$\Delta x = \frac{(a+b)\lambda}{2a\theta(n-n')}$$

For plane incident wave we let $a \to \infty$

so $$\Delta x = \frac{\lambda}{2\theta(n-n')} = 0{\cdot}2 \text{ mm}.$$

5.76 Extra phase difference introduced by the glass plate is

$$\frac{2\pi}{\lambda}(n-1)h$$

This will cause a shift equal to $(n-1)\frac{h}{\lambda}$ fringe widths

i.e. by $$(n-1)\frac{h}{\lambda} \times \frac{l\lambda}{d} = \frac{(n-1)hl}{d} = 2 \text{ mm}.$$

The fringes move down if the lower slit is covered by the plate to compensate for the extra phase shift introduced by the plate.

5.77 No. of fringes shifted $= (n'-n)\frac{l}{\lambda} = N$

so $$n' = n + \frac{N\lambda}{l} = 1{\cdot}000377.$$

5.78 (a) Suppose the vector $\vec{E}$, $\vec{E}'$, $\vec{E}''$ correspond to the incident, reflected and the transmitted wave. Due to the continuity of the tangential component of the electric field across the interface, it follows that

$$E_\tau + E'_\tau = E''_\tau \tag{1}$$

where the subscript τ means tangential.

The energy flux density is $\vec{E} \times \vec{H} = \vec{S}$.

Since $$H\sqrt{\mu\mu_0} = E\sqrt{\varepsilon\varepsilon_0}$$

$$H = E\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon/\mu} = n\sqrt{\frac{\varepsilon_0}{\mu_0}}E$$

Now $S \sim nE^2$ and since the light is incident normally

$$n_1 E_\tau^2 = n_1 E_\tau'^2 + n_2 E_\tau''^2 \tag{2}$$

or $$n_1(E_\tau^2 - E_\tau'^2) = n_2 E_\tau''^2$$

so $$n_1(E_\tau - E_\tau') = n_2 E_\tau'' \tag{3}$$

so $$E_\tau'' = \frac{2n_1}{n_1+n_2}E_\tau$$

Since E_τ'' and E_τ have the same sign, there is no phase change involved in this case.

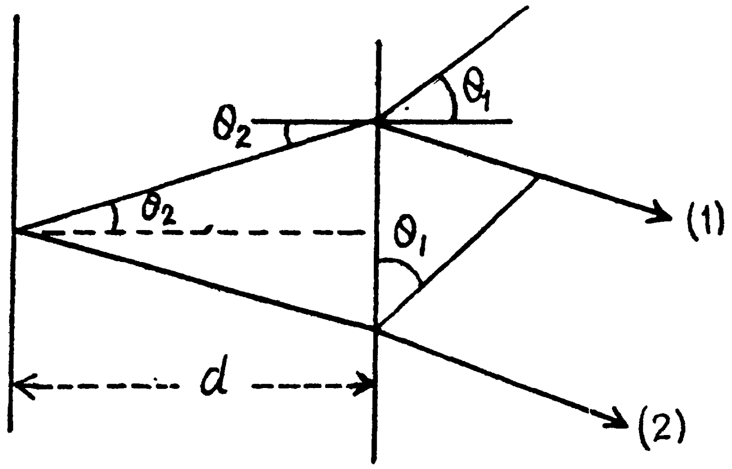
(b) From (1) & (3)

$$(n_2+n_1)E_\tau' + (n_2-n_1)E_\tau = 0$$

or $$E_\tau' = \frac{n_1-n_2}{n_1+n_2}E_\tau.$$

If $n_2 > n_1$, then E_τ' & E_τ have opposite signs. Thus the reflected wave has an abrupt change of phase by π if $n_2 > n_1$ i.e. on reflection from the interface between two media when light is incident from the rarer to denser medium.

5.79



Path difference between (1) & (2) is

$$2\,n\,d \sec\theta_2 - 2\,d\tan\theta_2\sin\theta_1$$

$$= 2d\frac{n - \dfrac{\sin^2\theta_1}{n}}{\sqrt{1-\dfrac{\sin^2\theta_1}{n^2}}} = 2d\sqrt{n^2-\sin^2\theta_1}$$

For bright fringes this must equal $\left(k+\frac{1}{2}\right)\lambda$ where $\frac{1}{2}$ comes from the phase change of π for (1).

Here $$k = 0, 1, 2, \ldots$$

Thus $$4\,d\sqrt{n^2-\sin^2\theta_1} = (2k+1)\lambda$$

or $$d = \frac{\lambda(1+2k)}{4\sqrt{n^2-\sin^2\theta_1}} = 0{\cdot}14\,(1+2k)\,\mu\,\text{m}\,..$$

5.80 Given

$$2\,d\sqrt{n^2-1/4} = \left(k+\frac{1}{2}\right)\times 0{\cdot}64\,\mu\,\text{m}\ \text{(bright fringe)}$$

$$2\,d\sqrt{n^2-1/4} = k'\times 0{\cdot}40\,\mu\,\text{m}\qquad\text{(dark fringe)}$$

where k, k' are integers.

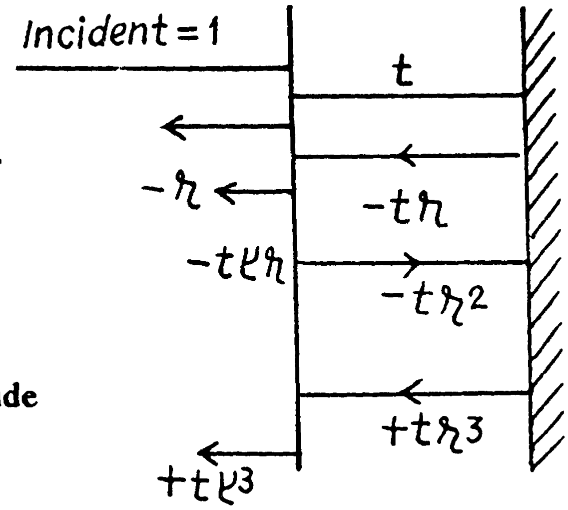
Thus $$64\left(k+\frac{1}{2}\right) = 40\,k' \ \text{ or }\ 4(2k+1) = 5\,k'$$

This means, for the smallest integer solutions

$$k = 2,\ k' = 4$$

Hence $$d = \frac{4\times 0{\cdot}40}{2\sqrt{n^2-1/4}} = 0{\cdot}65\,\mu\,\text{m}\,.$$

5.81 When the glass surface is coated with a material of R.I. $n' = \sqrt{n}$ (n = R.I. of glass) of appropriate thickness, reflection is zero because of interference between various multiply reflected waves. We show this below.



Let a wave of unit amplitude be normally incident from the left. The reflected amplitude is $-r$ where

$$r = \frac{\sqrt{n}-1}{\sqrt{n}+1}$$

Its phase is $-ve$ so we write the reflected wave as $-r$. The transmitted wave has amplitude t

$$t = \frac{2}{1+\sqrt{n}}$$

This wave is reflected at the second face and has amplitude $-tr$

$$\left(\text{because } \frac{n-\sqrt{n}}{n+\sqrt{n}} = \frac{\sqrt{n}-1}{\sqrt{n}+1}.\right)$$

The emergent wave has amplitude $-tt'r$.

We prove below that $-tt' = 1-r^2$. There is also a reflected part of emplitude $trr' = -tr^2$, where r' is the reflection coefficient for a ray incident from the coating towards air. After reflection from the second face a wave of amplitude

$$+tt'r^3 = +(1-r^2)r^3$$

emerges. Let δ be the phase of the wave after traversing the coating both ways. Then the complete reflected wave is

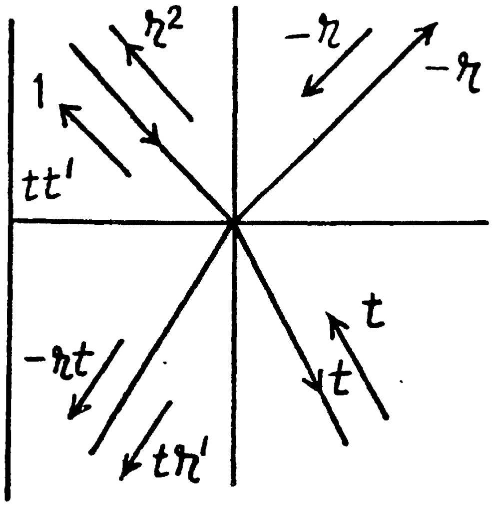
$$-r-(1-r^2)re^{i\delta}+(1-r^2)r^3e^{2i\delta}$$
$$-(1-r^2)r^5e^{3i\delta} \dots\dots$$
$$= -r-(1-r^2)re^{i\delta}\frac{1}{1+r^2e^{i\delta}}$$
$$= -r\left[1+r^2e^{i\delta}+(1-r^2)e^{i\delta}\right]\frac{1}{1+r^2e^{i\delta}}$$
$$= -r\frac{1+e^{i\delta}}{1+r^2e^{i\delta}}$$

This vanishes if $\delta = (2k+1)\pi$. But

$$\delta = \frac{2\pi}{\lambda}2\sqrt{n}\,d \text{ so}$$

$$d = \frac{\lambda}{4\sqrt{n}}(2k+1)$$

We now deduce $t\,t' = 1 - r^2$ and $r' = +r$. This follows from the principle of reversibility of light path as shown in the figure below.



$$t\,t' + r^2 = 1$$

$$-r\,t + r'\,t = 0$$

$$\therefore \quad t\,t' = 1 - r^2$$

$$r' = +r.$$

($-r$ is the reflection ratio for the wave entering a denser medium).

5.82 We have the condition for maxima

$$2d\sqrt{n^2 - \sin^2\theta_1} = \left(k + \frac{1}{2}\right)\lambda$$

This must hold for angle $\theta \pm \dfrac{\delta\theta}{2}$ with successive values of k. Thus

$$2d\sqrt{n^2 - \sin^2\left(\theta + \frac{\delta\theta}{2}\right)} = \left(k - \frac{1}{2}\right)\lambda$$

$$2d\sqrt{n^2 - \sin^2\left(\theta - \frac{\delta\theta}{2}\right)} = \left(k + \frac{1}{2}\right)\lambda$$

Thus
$$\lambda = 2d\left\{\sqrt{n^2 - \sin^2\theta + \delta\theta\sin\theta\cos\theta} - \sqrt{n^2 - \sin^2\theta - \delta\theta\sin\theta\cos\theta}\right\}$$

$$= 2d\,\frac{\delta\theta\sin\theta\cos\theta}{\sqrt{n^2 - \sin^2\theta}}$$

Thus
$$d = \frac{\sqrt{n^2 - \sin^2\theta}\,\lambda}{\sin 2\theta\,\delta\theta} = 15{\cdot}2\ \mu\text{m}$$

5.83 For small angles θ we write for dark fringes

$$2d\sqrt{n^2 - \sin^2\theta} = 2d\left(n - \frac{\sin^2\theta}{2n}\right) = (k + 0)\lambda$$

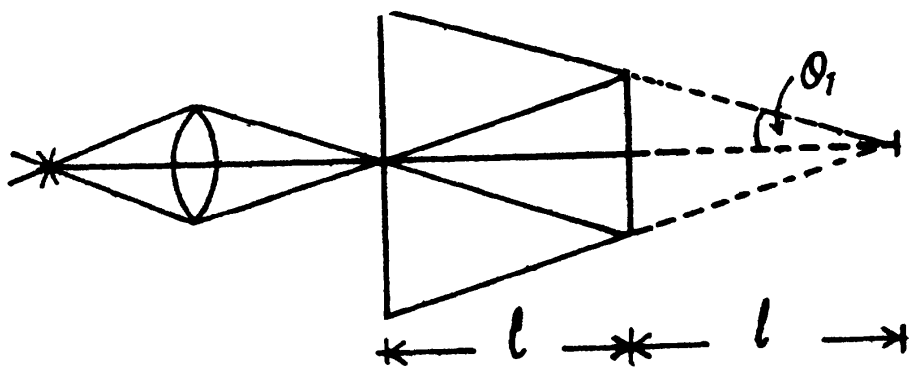
For the first dark fringe $\theta \simeq 0$ and

$$2dn = (k_0 + 0)\lambda$$

For the i^{th} dark fringe

$$2d\left(n - \frac{\sin^2\theta_i}{2n}\right) = (k_0 - i + 1)\lambda$$

or
$$\sin^2\theta_i = \frac{n\lambda}{d}(i - 1) = \frac{r_i^2}{4l^2}$$



Finally
$$\frac{n\lambda}{d}(i-k) = \frac{r_i^2 - r_k^2}{4l^2}$$

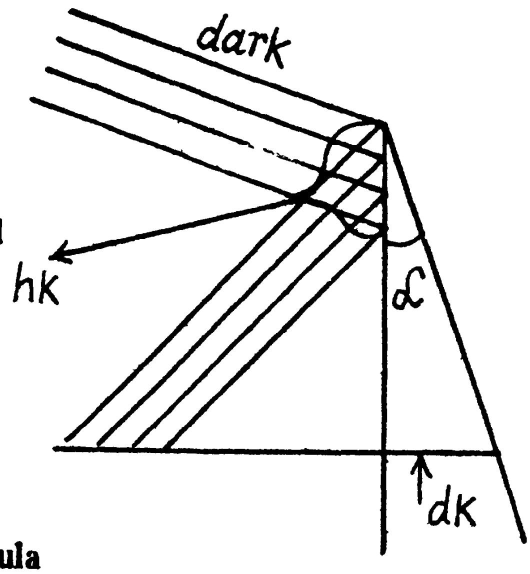
so
$$\lambda = \frac{d(r_i^2 - r_k^2)}{4l^2 n(i-k)}$$

5.84 We have the usual equation for maxima

$$2h_k \alpha \sqrt{n^2 - \sin^2\theta_1} = \left(k + \frac{1}{2}\right)\lambda$$

Here h_k = distance of the fringe from top

$h_k \alpha = d_k$ = thickness of the film



Thus on the screen placed at right angles to the reflected light

$$\Delta x = (h_k - h_{k-1})\cos\theta_1$$
$$= \frac{\lambda\cos\theta_1}{2\alpha\sqrt{n^2 - \sin^2\theta_1}}$$

5.85 (a) For normal incidence we have using the above formula

$$\Delta x = \frac{\lambda}{2n\alpha}$$

so
$$\alpha = \frac{\lambda}{2n\Delta x} = 3' \text{ on putting the values}$$

(b) In a distance l on the wedge there are $N = \dfrac{l}{\Delta x}$ fringes.

If the fringes disappear there, it must be due to the fact that the maxima due to the component of wavelength λ coincide with the mainima due to the component of wavelength $\lambda + \Delta\lambda$. Thus

$$N\lambda = \left(N - \frac{1}{2}\right)(\lambda + \Delta\lambda) \quad \text{or} \quad \Delta\lambda = \frac{\lambda}{2N}$$

so
$$\frac{\Delta\lambda}{\lambda} = \frac{1}{2N} = \frac{\Delta x}{2l} = \frac{0{\cdot}21}{30} = 0{\cdot}007.$$

The answer given in the book is off by a factor 2.

5.86 We have

$$r^2 = \frac{1}{2}k\lambda R$$

So for k differing by 1 ($\Delta k = 1$)

$$2r\Delta r = \frac{1}{2}\Delta k\lambda R = \frac{1}{2}\lambda R$$

or

$$\Delta r = \frac{\lambda R}{4r}.$$

5.87 The path traveresed in air film of the wave constituting the k^{th} ring is

$$\frac{r^2}{R} = \frac{1}{2}k\lambda$$

when the lens is moved a distance Δh the ring radius changes to r' and the path length becomes

$$\frac{r'^2}{R} + 2\Delta h = \frac{1}{2}k\lambda$$

Thus

$$r' = \sqrt{r^2 - 2R\Delta h} = 1{\cdot}5 \text{ mm}.$$

5.88 In this case the path difference is $\dfrac{r^2 - r_0^2}{R}$ for $r > r_0$ and zero for $r \le r_0$.

This must equals $(k - 1/2)\lambda$ (where $k = 6$ for the six[th] bright ring.)

Thus $r = \sqrt{r_0^2 + \left(k - \frac{1}{2}\right)\lambda R} = 3{\cdot}8$ mm

5.89 From the formula for Newton's rings we derive for dark rings

$$\frac{d_1^2}{4} = k_1 R\lambda, \quad \frac{d_2^2}{4} = k_2 R\lambda$$

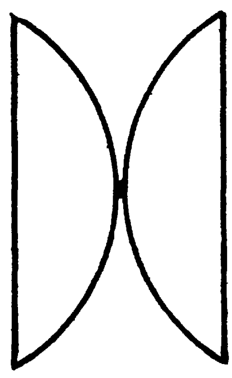
so

$$\frac{d_2^2 - d_1^2}{4(k_2 - k_1)R} = \lambda$$

Substituting the values, $\lambda = 0{\cdot}5\ \mu$ m.

5.90 Path difference between waves reflected by the two convex surfaces is

$$r^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$



Taking account of the phase change at the 2[nd] surface we write the condition of bright rings as

$$r^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{2k+1}{2}\lambda$$

$k = 4$ for the fifth bright ring.

Thus $$\frac{1}{R_1}+\frac{1}{R_2} = \frac{9}{2}\lambda\cdot\frac{4}{d^2} = \frac{18\lambda}{d^2}$$

Now $$\frac{1}{f_1} = (n-1)\frac{1}{R_1},\ \frac{1}{f_2} = (n-1)\frac{1}{R_2}$$

so $$\frac{1}{f} = \frac{1}{f_1}+\frac{1}{f_2} = (n-1)\frac{18\lambda}{d^2} = \Phi = 2{\cdot}40\,\mathrm{D}$$

Here n = R.I. of glass = 1·5.

5.91 Here $\Phi = (n-1)\left(\frac{2}{R_1}-\frac{2}{R_2}\right)$

so $$\frac{1}{R_1}-\frac{1}{R_2} = \frac{\Phi}{2(n-1)}.$$

As in the previous example, for the dark rings we have

$$r_k^2\left(\frac{1}{R_1}-\frac{1}{R_2}\right) = \frac{\Phi}{2(n-1)}r_k^2 = k\lambda$$

$k = 0$ is dark spot; excluding it, we take $k = 10$ hre.

Then $$r = \sqrt{\frac{20\lambda(n-1)}{\Phi}} = 3{\cdot}49\text{ mm}.$$

(b) Path difference in water film will be

$$n_0\bar{r}^2\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

where $\bar{r}$ = new radius of the ring. Thus

$$n_0\bar{r}^2 = r^2$$

or $$\bar{r} = r/\sqrt{n_0} = 3{\cdot}03\text{ mm}.$$

Where n_0 = R.I. of water = 1·33.

5.92 The condition for minima are

$$\frac{r^2}{R}n_2 = \left(k+\frac{1}{2}\right)\lambda,$$

(There occur phase changes at both surfaces on reflection, hence minima when path difference is half integer multiple of λ).

In this case $k = 4$ for the fifth dark ring

(Counting from $k = 0$ for the first dark ring).

Thus, we can write

$$r = \sqrt{(2K-1)\lambda R/2n_2}\ ,\ K = 5$$

Substituting we get $r = 1{\cdot}17$ mm.

5.93 Sharpness of the fringe pattern is the worst when the maxima and minima intermingle :-

$$n_1\lambda_1 = \left(n_1-\frac{1}{2}\right)\lambda_2$$

or putting $\lambda_1 = \lambda, \ \lambda_2 = \lambda + \Delta\lambda$

we get $$n_1 \Delta\lambda = \frac{\lambda}{2}$$

or $$n_1 = \frac{\lambda}{2\Delta\lambda} \approx \frac{\lambda_1}{2(\lambda_2 - \lambda_1)} = 140\,.$$

5.94 Interference pattern vanishes wnen the maxima due to one wavelength mingle with the minima due to the other. Thus

$$2\Delta h = k\lambda_2 = (k+1)\lambda_1$$

where Δh = displacement of the mirror between the sharpest patterns of rings

Thus $$k(\lambda_2 - \lambda_1) = \lambda_1$$

or $$k = \frac{\lambda_1}{\lambda_2 - \lambda_1}$$

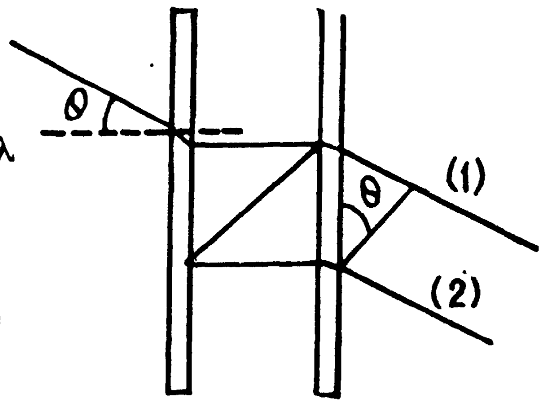
So $$\Delta h = \frac{\lambda_1 \lambda_2}{2(\lambda_2 - \lambda_1)} = \frac{\lambda^2}{2\Delta\lambda} \approx \cdot 29 \text{ mm}\,.$$

5.95 The path difference between (1) & (2) can be seen to be

$$\Delta = 2d\sec\theta - 2d\tan\theta\sin\theta$$
$$= 2d\cos\theta = k\lambda$$

for maxima. Here k = half-integer.

The order of interference decreases as θ increases i.e. as the radius of the rings increases.



(b) Differentiating

$$2d\sin\theta\,\delta\theta = \lambda$$

on putting $\delta k = -1$. Thus

$$\delta\theta = \frac{\lambda}{2d\sin\theta}$$

$\delta\theta$ decreases as θ increases.

5.96 (a) We have $k_{max} = \frac{2d}{\lambda}\cdot$ for $\theta = 0 .= 10^5$.

(b) We must have

$$2d\cos\theta = k\lambda = (k-1)(\lambda + \Delta\lambda)$$

Thus $\frac{1}{k} = \frac{\lambda}{2d}\cdot$ and $\Delta\lambda = \frac{\lambda}{k} = \frac{\lambda^2}{2d} = 5$ pm. on putting the values.

5.3 DIFFRACTION OF LIGHT

5.97 The radius of the periphery of the N^{th} Fresnel zone is

$$r_N = \sqrt{N b \lambda}$$

Then by conservation of energy

$$I_0 \pi (\sqrt{N b \lambda})^2 = \int_0^{\infty} 2 \pi r \, d r \, I(r)$$

Here r is the distance from the point P.

Thus
$$I_0 = \frac{2}{N b \lambda} \int_0^{\infty} r \, d r \, I(r).$$

5.98 By definition

$$r_k^2 = k \frac{a b \lambda}{a + b}$$

for the periphery of the kth zone. Then

$$a r_k^2 + b r_k^2 = k a b \lambda$$

So
$$b = \frac{a r_k^2}{k a \lambda - r_k^2} = \frac{a r^2}{k a \lambda - r^2} = 2 \text{ metre}.$$

on putting the values. (It is given that $r = r_k$) for $k = 3$).

5.99 Suppose maximum intensity is obtained when the aperture contains k zones. Then a minimum will be obtained for $k + 1$ zones. Another maximum will be obtained for $k + 2$ zones. Hence

$$r_1^2 = k \lambda \frac{a b}{a + b}$$

$$r_2^2 = (k + 2) \lambda \frac{a b}{a + b}$$

Thus
$$\lambda = \frac{a + b}{2 a b} (r_2^2 - r_1^2) = 0{\cdot}598 \, \mu \text{m}$$

On putting the values.

5.100 (a) When the aperture is equal to the first Fresnel Zone :-

The amplitude is A_1 and should be compared with the amplitude $\frac{A}{2}$ when the aperture is very wide. If I_0 is the intensity in the second case the intensity in the first case will be $4 I_0$.

When the aperture is equal to the internal half of the first zone :- Suppose A_{in} and A_{out} are the amplitudes due to the two halves of the first Fresnel zone. Clearly A_{in} and A_{out} differ in phase by $\frac{\pi}{2}$ because only half the Fresnel zone in involved. Also in magnitude $|A_{in}| = |A_{out}|$. Then

$$A_1^2 = 2\,|A_{in}|^2 \quad \text{so} \quad |A_{in}|^2 = \frac{A_1^2}{2}$$

Hence following the argument of the first case. $I_{in} = 2\,I_0$

(b) The aperture was made equal to the first Fresnel zone and then half of it was closed along a diameter. In this case the amplitude of vibration is $\frac{A_1}{2}$. Thus

$$I = I_0 .$$

5.101 (a) Suppose the disc does not obstruct light at all. Then

$$A_{disc} + A_{remainder} = \frac{1}{2} A_{disc}$$

(because the disc covers the first Fresnel zone only).

So $A_{remainder} = -\frac{1}{2} A_{disc}$

Hence the amplitude when half of the disc is removed along a diameter

$= \frac{1}{2} A_{disc} + A_{remainder} = \frac{1}{2} A_{disc} - \frac{1}{2} A_{disc} = 0$

Hence $I = 0$.

(b) In this case

$$A = \frac{1}{2} A_{external} + A_{remainder}$$

$$= \frac{1}{2} A_{external} - \frac{1}{2} A_{disc}$$

We write $$A_{disc} = A_{in} + i A_{out}$$

where $A_{in}\,(A_{out})$ stands for $A_{internal}\,(A_{external})$. The factor i takes account of the $\frac{\pi}{2}$ phase difference between two halves of the first Fresnel zone. Thus

$$A = -\frac{1}{2} A_{in} \quad \text{and} \quad I = \frac{1}{4} A_{in}^2$$

On the other hand $$I_0 = \frac{1}{4}(A_{in}^2 + A_{out}^2) = \frac{1}{2} A_{in}^2$$

so $$I = \frac{1}{2} I_0 .$$

5.102 When the screen is fully transparent, the amplitude of vibrations is $\frac{1}{2} A_1$ (with intensity $I_0 = \frac{1}{4} A_1^2$).

(a) (1) In this case $A = \frac{3}{4}\left(\frac{1}{2} A_1\right)$ so squaring $I = \frac{9}{16} I_0$

(2) In this case $\frac{1}{2}$ of the plane is blacked out so

$$A = \frac{1}{2}\left(\frac{1}{2}A_1\right) \quad \text{and} \quad I = \frac{1}{4}I_0$$

(3) In this case $A = \frac{1}{4}(A_1/2)$ and $I = \frac{1}{16}I_0$.

(4) *In this case* $A = \frac{1}{2}\left(\frac{1}{2}A_1\right)$ *again and* $I = \frac{1}{4}I_0$ *so* $I_4 = \frac{I}{2}$

In general we get $I(\varphi) = I_0\left(1 - \left(\frac{\varphi}{2\pi}\right)\right)^2$

where φ is the total angle blocked out by the screen.

(b) (5) Here $A = \frac{3}{4}\left(\frac{1}{2}A_1\right) + \frac{1}{4}A_1$

A_1 being the contribution of the first Fresnel zone.

Thus $$A = \frac{5}{8}A_1 \quad \text{and} \quad I = \frac{25}{16}I_0$$

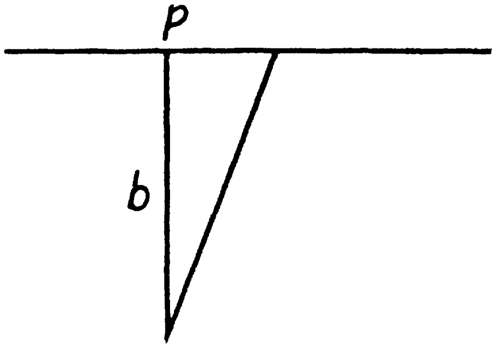
(6) $A = \frac{1}{2}\left(\frac{1}{2}A_1\right) + \frac{1}{2}A_1 = \frac{3}{4}A_1$ and $I = \frac{9}{4}I_0$

(7) $A = \frac{1}{4}\left(\frac{1}{2}A_1\right) + \frac{3}{4}A_1 = \frac{7}{8}A_1$ and $I = \frac{49}{16}I_0$

(8) $A = \frac{1}{2}\left(\frac{1}{2}A_1\right) + \frac{1}{2}A_1 = \frac{3}{4}A_1$ and $I = \frac{9}{4}I_0$ $(I_8 = I_6)$

In 5 to 8 the first term in the expression for the amplitude is the contribution of the plane part and the second term gives the expression for the Fresnel zone part. In general in (5) to (8) $I = I_0\left(1 + \left(\frac{\varphi}{2\pi}\right)\right)^2$ when φ is the angle covered by the screen.

5.103 We would require the contribution to the amplitude of a wave at a point from half a Fresnel zone. For this we proceed directly from the Fresnel Huyghens principle. The complex amplitude is written as



$$E = \int K(\varphi)\,\frac{a_0}{r}\,e^{-ikr}\,dS$$

Here $K(\varphi)$ is a factor which depends on the angle φ between a normal $\vec{n}$ to the area dS and the direction from dS to the point P and r is the distance from the element dS to P.

We see that for the first Fresnal zone

$$\left(\text{using } r \approx b+\frac{\rho^2}{2b}\left(\text{for } \sqrt{\rho^2+b^2}\right)\right)$$

$$E \approx \frac{a_0}{b}\int_0^{\sqrt{b\lambda}} e^{-ikb-ik\rho^2/2b}\, 2\pi\rho\, d\rho \quad (K(\varphi) \approx 1)$$

For the first Fresnel zone $r = b+\lambda/2$ so $r^2 \approx b^2+b\lambda$ and $\rho^2 = b\lambda$.

Thus
$$E \approx \frac{a_0}{b}e^{-ikb}\, 2\pi \int_0^{\frac{b\lambda}{2}} e^{-i\frac{kx}{b}}\, dx$$

$$= \frac{a_0}{b}2\pi e^{-ikb}\,\frac{e^{-ik\lambda/2}-1}{-ik/b}$$

$$= \frac{a_0}{k}2\pi i e^{-ikb}(-2) = -\frac{4\pi}{k} i a_0 e^{-ikb} \approx A_1$$

For the next half zone

$$E = \frac{a_0}{b}e^{-ikb}2\pi \int_{\frac{b\lambda}{2}}^{\frac{3b\lambda}{4}} e^{-ikx/b}\, dxx$$

$$= \frac{a_0}{k}2\pi i e^{-ikb}\left(e^{-i\frac{3k\lambda}{4}} - e^{-ik\lambda/2}\right)$$

$$= \frac{a_0}{k}2\pi i\, e^{-ikb}\,(+1+i) = -\frac{A_1(1+i)}{2}$$

If we calculate the contribution of the full 2nd Fresnel zone we will get $-A_1$. If we take account of the factors $K(\varphi)$ and $\frac{1}{r}$ which decrease monotonically we expect the contribution to change to $-A_2$. Thus we write for the contribution of the half zones in the 2nd Fresnel zone as

$$-\frac{A_2(1+i)}{2} \quad \text{and} \quad -\frac{A_2(1-i)}{2}$$

The part lying in the recess has an extra phase difference equal to $-\delta = -\frac{2\pi}{\lambda}(n-1)h$. Thus the full amplitude is (note that the correct form is e^{-ikr})

$$\left(A_1 - \frac{A_2}{2}(1+i)\right)e^{+i\delta} - \frac{A_2}{2}(1-i) + A_3 - A_4 + \dots$$

$$\approx \left(\frac{A_1}{2}(1-i)\right)e^{+i\delta} - \frac{A_2}{2}(1-i) + \frac{A_3}{2}$$

$$\simeq \left(\frac{A_1}{2}(1-i)\right)e^{+i\delta} + i\frac{A_1}{2} \text{ (as } A_2 \simeq A_3 \simeq A_1\text{) and } A_3 - A_4 + A_5 \ldots = \frac{A_3}{2}.$$

The corresponding intensity is

$$I = \frac{A_1^2}{4}\left[(1-i)e^{+i\delta} + \frac{i}{e}\right]\left[(1+i)e^{-i\delta} - i\right]$$

$$= I_0[3 - 2\cos\delta + 2\sin\delta] = I_0\left[3 + 2\sqrt{2}\sin\left(\delta - \frac{\pi}{4}\right)\right]$$

(a) For maximum intensity $\sin\left(\delta - \frac{\pi}{4}\right) = +1$

or
$$\delta - \frac{\pi}{4} = 2k\pi + \frac{\pi}{2}, \quad k = 0, 1, 2, \ldots$$

$$\delta = 2k\pi + \frac{3\pi}{4} = \frac{2\pi}{\lambda}(n-1)h$$

so
$$h = \frac{\lambda}{n-1}\left(k + \frac{3}{8}\right)$$

(b) For minimum intensity

$$\sin\left(\delta - \frac{\pi}{4}\right) = 1$$

$$\delta - \frac{\pi}{4} = 2k\pi + \frac{3\pi}{2} \quad \text{or} \quad \delta = 2k\pi + \frac{7\pi}{4}$$

so
$$h = \frac{\lambda}{n-1}\left(k + \frac{7\pi}{8}\right)$$

(c) For $I = I_0$, $\left.\begin{array}{l}\cos\delta = 0\\ \sin\delta = -1\end{array}\right\}$ or $\left\{\begin{array}{l}\sin\delta = 0\\ \cos\delta = +1\end{array}\right.$

Thus
$$\delta = 2k\pi \quad h = \frac{k\lambda}{n-1}$$

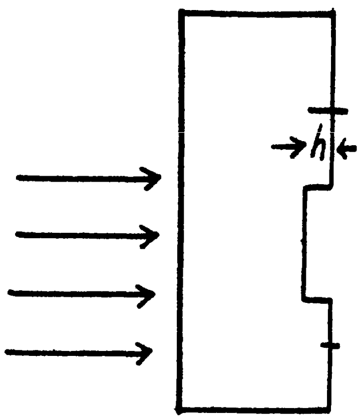
or
$$\delta = 2k\pi + \frac{3\pi}{2}, \quad h = \frac{\lambda}{n-1}\left(k + \frac{3\pi}{4}\right)$$

5.104 The contribution to the wave amplitude of the inner half-zone is

$$\frac{2\pi a_0 e^{-ikb}}{b}\int_0^{\sqrt{b\lambda/2}} e^{-ik\rho^2/2b}\rho\, d\rho$$

$$= \frac{2\pi a_0 e^{-ikb}}{b}\int_0^{b\lambda/4} e^{-ikx/b}dx$$

$$= \frac{2\pi a_0 e^{-ikb}}{b}\left(e^{-ik\lambda/4} - 1\right) \times \frac{1}{\frac{-ik}{b}}$$



$$= \frac{2\pi i a_0 e^{-ikb}}{k}(-i-1) = +\frac{A_1}{2}(1+i)$$

With phase factor this becomes $\frac{A_1}{2}(1+i)e^{i\delta}$ where $\delta = \frac{2\pi}{\lambda}(n-1)h$. The contribution of the remaining aperture is $\frac{A_1}{2}(1-i)$

(so that the sum of the two parts when $\delta = 0$ is A_1)

Thus the complete amplitude is

$$\frac{A_1}{2}(1+i)e^{i\delta}+\frac{A_1}{2}(1-i)$$

and the intensity is

$$\begin{aligned} I &= I_0[(1+i)e^{i\delta}+(1-i)][(1-i)e^{-i\delta}+(1+i)] \\ &= I_0[2+2+(1-i)^2e^{-i\delta}+(1+i)^2e^{i\delta}] \\ &= I_0[4-2ie^{-i\delta}+2ie^{i\delta}] = I_0(4-4\sin\delta) \end{aligned}$$

Here $I_0 = \frac{A_1^2}{4}$ is the intensity of the incident light which is the same as the intensity due to an aperture of infinite extent (and no recess). Now

I is maximum when $\sin\delta = -1$

or
$$\delta = 2k\pi+\frac{3\pi}{2}$$

so
$$h = \frac{\lambda}{n-1}\left(k+\frac{3}{4}\right) \quad \text{and (b)} \quad I_{max} = 8I_0.$$

5.105 We follow the argument of 5.103. we find that the contribution of the first Fresnel zone is

$$A_1 = -\frac{4\pi i}{k}a_0 e^{-ikb}$$

For the next half zone it is $-\frac{A_2}{2}(1+i)$

(The contribution of the remaining part of the 2[nd] Fresnel zone will be $-\frac{A_2}{2}(1-i)$)

If the disc has a thickness h, the extra phase difference suffered by the light wave in passing through the disc will be

$$\delta = \frac{2\pi}{\lambda}(n-1)h.$$

Thus the amplitude at P will be

$$\begin{aligned} E_P &= \left(A_1-\frac{A_2}{2}(1+i)\right)e^{-i\delta}-\frac{A_2}{2}(1-i)+A_3-A_4-A_5+\ldots \\ &\approx \left(\frac{A_1(1-i)}{2}\right)e^{-i\delta}+\frac{iA_1}{2} = \frac{A_1}{2}[(1-i)e^{-i\delta}+i] \end{aligned}$$

The corresponding intensity will be

$$I = I_0(3 - 2\cos\delta - 2\sin\delta) = I_0\left(3 - 2\sqrt{2}\sin\left(\delta + \frac{\pi}{4}\right)\right)$$

The intensity will be a maximum when

$$\sin\left(\delta + \frac{\pi}{4}\right) = -1$$

or

$$\delta + \frac{\pi}{4} = 2k\pi + \frac{3\pi}{2}$$

i.e.

$$\delta = \left(k + \frac{5}{8}\right).2\pi$$

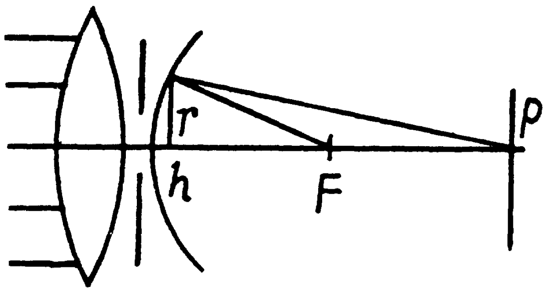
so

$$h = \frac{\lambda}{n-1}\left(k + \frac{5}{8}\right),\ k = 0, 1, 2, \ldots$$

Note :- It is not clear why $k = 2$ for h_{min}. The normal choice will be $k = 0$. If we take $k = 0$ we get $h_{min} = 0{\cdot}59\ \mu$ m.

5.106 Here the focal point acts as a virtual source of light. This means that we can take spherical waves converging towards F. Let us divide these waves into Fresnel zones just after they emerge from the stop. We write

$$r^2 = f^2 - (f-h)^2 = (b - m\lambda/2)^2 - (b-h)^2$$



Here r is the radius of the m^{th} fresnel zone and h is the distance to the left of the foot of the perpendicular. Thus

$$r^2 = 2fh = -bm\lambda + 2bh$$

So

$$h = bm\lambda/2(b-f)$$

and

$$r^2 = fbm\lambda/(b-f).$$

The intensity maxima are observed when an odd number of Fresnel zones are exposed by the stop. Thus

$$r_k = \sqrt{\frac{kbf\lambda}{b-f}} \quad \text{where} \quad k = 1, 3, 5, \ldots$$

5.107 For the radius of the periphery of the kth zone we have

$$r_k = \sqrt{k\lambda\frac{ab}{a+b}} = \sqrt{k\lambda b} \quad \text{if } a = \infty.$$

If the aperture diameter is reduced η times it will produce a similar deffraction pattern (reduced η times) if the radii of the Fresnel zones are also η times less. Thus

$$r'_k = r_k/\eta$$

This requires $b' = b/\eta^2$.

5.108 (a) If a point source is placecd before an opaque ball, the diffraction pattern consists of a bright spot inside a dark disc followed by fringes. The bright spot is on the line joining the point source and the centre of the ball. When the object is a finite source of transverse

diamension y, every point of the source has its corresponding image on the line joining that point and the centre of the ball. Thus the transverse dimension of the image is given by

$$y' = \frac{b}{a} y = 9 \text{ mm.}$$

(b) The minimum height of the irregularities covering the surface of the ball at random, at which the ball obstructs light is, according to the note at the end of the problem, comparable with the width of the Fresnel zone along which the edge of opaque screen passes. So

$$h_{min} \approx \Delta r$$

To find Δr we note that

$$r^2 = \frac{k \lambda a b}{a+b}$$

or

$$2 r \Delta r = D \Delta r = \frac{\lambda a b}{a+b} \Delta k$$

Where D = diameter of the disc (= diameter of the last Fresnel zone) and $\Delta k = 1$

Thus $h_{min} = \dfrac{\lambda a b}{D(a+b)} = 0{\cdot}099$ mm.

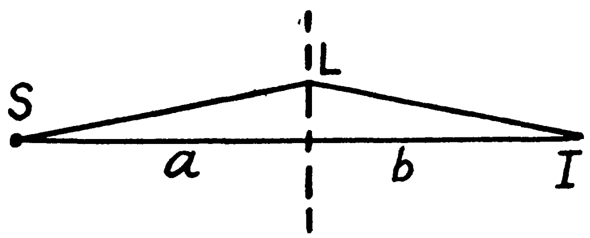
5.109 In a zone plate an undarkened circular disc is followed by a number of alternately undarkened and darkened rings. For the proper case, these correspond to 1st, 2nd, 3rd..... Fresnel zones.

Let r_1 = radius of the central undarkened circle. Then for this to be first Fresnel zone in the present case, we must have

$$SL + LI - SI = \lambda/2$$

Thus if r_1 is the radius of the periphery of the first zone

$$\sqrt{a^2 + r_1^2} + \sqrt{b^2 + r_1^2} - (a+b) = \frac{\lambda}{2}$$



or

$$\frac{r_1^2}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{\lambda}{2} \quad \text{or} \quad \frac{1}{a} + \frac{1}{b} = \frac{1}{r_1^2/\lambda}$$

It is clear that the plate is acting as a lens of focal length

$$f_1 = \frac{r_1^2}{\lambda} = \frac{a b}{a+b} = \cdot 6 \text{ metre.}$$

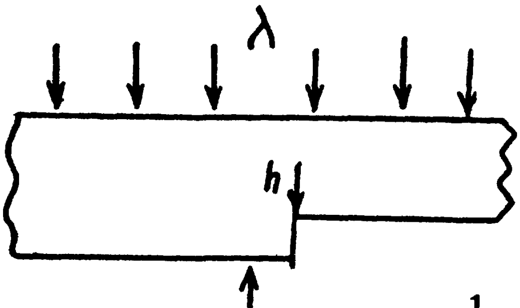
This is the principle focal length.

Other maxima are obtained when

$$SL + LI - SI = 3\frac{\lambda}{2}, 5\frac{\lambda}{2}, \ldots$$

These focal lengths are also $\quad \dfrac{r_1^2}{3\lambda}, \dfrac{r_1^2}{5\lambda}, \ldots$

5.110 Just below the edge the amplitude of the wave is given by



$$A = \frac{1}{2}(A_1 - A_2 + A_3 - A_4 + \dots)\,e^{-i\delta}$$

$$+\frac{1}{2}(A_1 - A_2 + A_3 - A_4 + \dots)$$

Here the quantity in the brackets is the contribution of various Fresnel zones; the factor $\frac{1}{2}$ is to take account of the division of the plate into two parts by the ledge; the phase factor δ is given by

$$\delta = \frac{2\pi}{\lambda} h (n-1)$$

and takes into account the extra length traversed by the waves on the left.

Using $$A_1 - A_2 + A_3 - A_4 + \dots \approx \frac{A_1}{2}$$

we get $$A = \frac{A_1}{4}(1 + e^{i\delta})$$

and the corresponding intensity is

$$I = I_0 \frac{1 + \cos\delta}{2}. \quad \text{where } I_0 \propto \left(\frac{A_1}{2}\right)^2$$

(a) This is minimum when

$$\cos\delta = -1$$

So $$\delta = (2k+1)\pi$$

and $$h = (2k+1)\frac{\lambda}{2(n-1)}, \quad k = 0, 1, 2, \dots$$

using $n = 1{\cdot}5$, $\lambda = 0{\cdot}60\ \mu m$

$$h = 0{\cdot}60\,(2k+1)\ \mu\text{m}.$$

(b) $I = I_0/2$ when $\cos\delta = 0$

or $$\delta = k\pi + \frac{\pi}{2} = (2k+1)\frac{\pi}{2}$$

Thus in this case $$h = 0{\cdot}30\,(2k+1)\ \mu\text{m}.$$

5.111 (a) From the Cornu's spiral, the intensity of the first maximum is given as

$$I_{max,1} = 1{\cdot}37\,I_0$$

and the intensity of the first minimum is given by

$$I_{min} = 0{\cdot}78\,I_0$$

so the required ratio is

$$\frac{I_{max}}{I_{min}} = 1{\cdot}76$$

(b) The value of the distance x is related to the parameter ν in Fresnel's integral by

$$\nu = x\sqrt{\frac{2}{b\lambda}}\,.$$

For the first two maxima the distances x_1, x_2 are related to the parameters ν_1, ν_2 by

$$x_1 = \sqrt{\frac{b\lambda}{2}}\,\nu_1\,,\; x_2 = \sqrt{\frac{b\lambda}{2}}\,\nu_2$$

Thus
$$(\nu_2 - \nu_1)\sqrt{\frac{b\lambda}{2}} = x_2 - x_1 = \Delta x$$

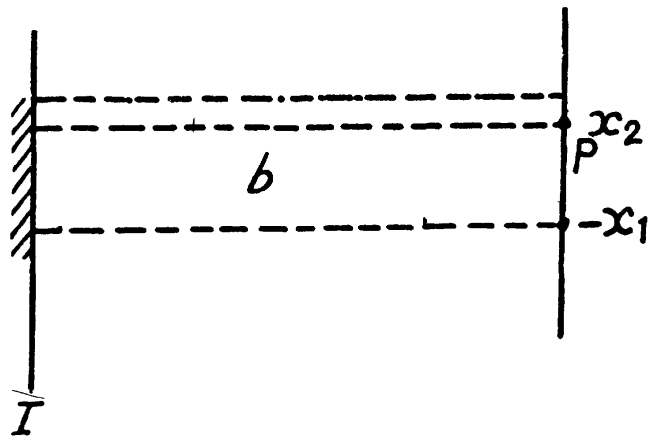
or
$$\lambda = \frac{2}{b}\left(\frac{\Delta x}{\nu_2 - \nu_1}\right)^2$$

From the Cornu's spiral the positions of the maxima are

$$\nu_1 = 1{\cdot}22\,,\; \nu_2 = 2{\cdot}34\,,\; \nu_3 = 3{\cdot}08 \text{ etc}$$

Thus
$$\lambda = \frac{2}{b}\left(\frac{\Delta x}{1{\cdot}12}\right)^2 = 0{\cdot}63\,\mu\text{m}\,.$$

5.112 We shall use the equation written down in 5.103, the Fresnel-Huyghens formula.



Suppose we want to find the intensity at P which is such that the coordinates of the edges (x-coordinates) with respect to P are x_2 and $-x_1$. Then, the amplitude at P is

$$E = \int K(\varphi)\,\frac{a_0}{r}\,e^{-ikr}\,dS$$

We write $dS = dx\,dy$, y is to integrated from $-\infty + 0 + \infty$. We write

$$r = b + \frac{x^2 + y^2}{2b} \tag{1}$$

(r is the distance of the element of surface on I from P. It is $\sqrt{b^2 + x^2 + y^2}$ and hence approximately (1)). We then get

$$E = A_0(b)\left[\int_{x_2}^{\infty} e^{-ikx^2/2b}\,dx + \int_{-\infty}^{-x_1} e^{-ikx^2/2b}\,dx\right]$$

$$= A'_0(b)\left[\int_{v_2}^{\infty} e^{-i\frac{\pi u^2}{2}}\,du + \int_{-\infty}^{-v_1} e^{-i\pi u^2/2}\,du\right]$$

where $$v_2 = \sqrt{\frac{2}{b\lambda}}\,x_2,\; v_1 = \sqrt{\frac{2}{b\lambda}}\,x_1$$

The intensity is the square of the amplitude. In our case, at the centre

$$v_1 = v_2 = \sqrt{\frac{2}{b\lambda}}\cdot\frac{a}{2} = \sqrt{\frac{a^2}{2b\lambda}} = 0{\cdot}64$$

(a = width of the strip = 0·7 mm, b = 100 cm, λ = 0·60 μm)

At, say, the lower edge $v_1 = 0,\ v_2 = 1{\cdot}28$

Thus

$$\frac{I_{\text{centre}}}{I_{\text{edge}}} = \left|\frac{\int_{0\cdot64}^{\infty} e^{-i\pi u^2/2}\,du + \int_{-\infty}^{-0\cdot64} e^{-i\pi u^2/2}\,du}{\int_{1\cdot28}^{\infty} e^{-i\pi u^2/2}\,du + \int_{-\infty}^{0} e^{-i\pi u^2/2}\,du}\right|^2 = 4\,\frac{\left(\frac{1}{2}-C(0{\cdot}64)\right)^2+\left(\frac{1}{2}-S(0{\cdot}64)\right)^2}{(1-C(1{\cdot}28))^2+(1-S(1{\cdot}28))^2}$$

where $$C(v) = \int_0^v \cos\frac{\pi u^2}{2}\,du$$

$$S(v) = \int_0^v \sin\frac{\pi u^2}{2}\,du$$

Rough evaluation of the integrals using cornu's spiral gives

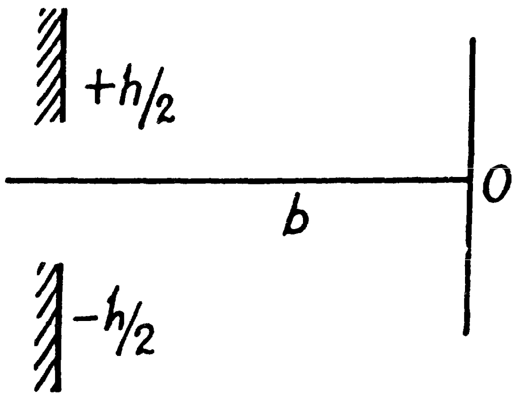
$$\frac{I_{\text{centre}}}{I_{\text{edge}}} \approx 2{\cdot}4$$

(We have used $\int_0^{\infty} \cos\frac{\pi u^2}{2}\,du = \int_0^{\infty} \sin\frac{\pi u^2}{2}\,du = \frac{1}{2}$

$C(0{\cdot}64) = 0{\cdot}62,\ S(0{\cdot}64) = 0{\cdot}15$

$C(1{\cdot}28) = 0{\cdot}65\ \ S(1{\cdot}28) = 0{\cdot}67$

5.113 If the aperture has width h then the parameters $(v, -v)$ associated with $\left(h/2, -\frac{h}{2}\right)$ are given by



$$v = \frac{h}{2}\sqrt{\frac{2}{b\lambda}} = h/\sqrt{2b\lambda}$$

The intensity of light at O on the screen is obtained as the square of the amplitude A of the wave at O which is

$$A \sim \text{const} \int_{-v}^{v} e^{-i\pi u^2/2}\, du$$

Thus $$I = 2I_0((C(v))^2 + (S(v))^2)$$

where $C(v)$ and $S(v)$ have been defined above and I_0 is the intensity at O due to an infinitety wide $(v = \infty)$ aperture for then

$$I = 2I_0\left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right) = 2I_0 \times \frac{1}{2} = I_0 .$$

By definition v corresponds to the first minimum of the intensity. This means

$$v = v_1 \approx \cdot 90$$

when we increase h to $h + \Delta h$, the corresponding $v_2 = \dfrac{h + \Delta h}{\sqrt{2b\lambda}}$ relates to the second minimum of intensity. From the cornu's spiral $v_2 \approx 2{\cdot}75$

Thus $$\Delta h = \sqrt{2b\lambda}\,(v_2 - v_1) = 0{\cdot}85\sqrt{2b\lambda}$$

or $$\lambda = \left(\frac{\Delta h}{0{\cdot}85}\right)^2 \frac{1}{2b} = \left(\frac{0{\cdot}70}{0{\cdot}85}\right)^2 \frac{1}{2 \times 0{\cdot}6}\,\mu\text{m} = 0{\cdot}565\,\mu\text{m}$$

5.114 Let a = width of the recess and

$$v = \frac{a}{2}\sqrt{\frac{2}{b\lambda}} = \frac{a}{\sqrt{2b\lambda}} = \frac{0{\cdot}6}{\sqrt{2 \times 0{\cdot}77 \times 0{\cdot}65}} \approx 0{\cdot}60$$

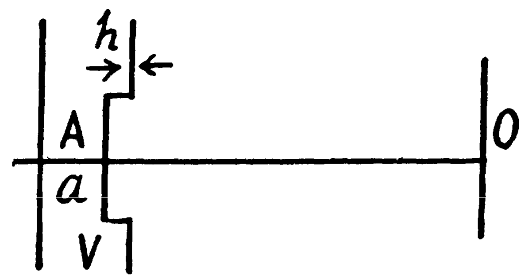
be the parameter along Cornu's spiral corresponding to the half-width of the recess.

The amplitude of the diffracted wave is given by

$$\sim \text{const}\left[e^{i\delta}\int_{-v}^{v} e^{-i\pi u^2/2}\, du + \int_{v}^{\infty} e^{-i\pi u^2/2}\, du + \int_{-\infty}^{v} e^{-i\pi u^2/2}\, du\right]$$

where $$\delta = \frac{2\pi}{\lambda}(n-1)h$$

is the extra phase due to the recess. (Actually an extra phase $e^{-i\delta}$ appears outside the recess. When we take it out and absorb it in the constant we get the expression written).



Thus the amplitude is

$$\sim \text{const}\left[(C(v)-iS(v))e^{i\delta}+\left(\frac{1}{2}-C(v)\right)-i\left(\frac{1}{2}-S(v)\right)\right]$$

From the Cornu's spiral, the coordinates corresponding to the parameter $v = 0{\cdot}60$ are

$$C(v) = 0{\cdot}57,\ S(v) = 0{\cdot}13$$

so the intensity at O is proportional to

$$\left|\left[(0{\cdot}57-0{\cdot}13\,i)e^{i\delta} - 0{\cdot}07 - i\,0{\cdot}37\right]\right|^2$$
$$= (0{\cdot}57^2+0{\cdot}13^2)+0{\cdot}07^2+0{\cdot}37^2$$
$$+ (0{\cdot}57-0{\cdot}13\,i)(-0{\cdot}07+0{\cdot}37\,i)e^{i\delta}$$
$$+ (0{\cdot}57+0{\cdot}13\,i)(-0{\cdot}07-i\,0{\cdot}37\,i)e^{-i\delta}$$

We write

$$0{\cdot}57 \mp 0{\cdot}13\,i = 0{\cdot}585\ e^{\mp i\alpha} \quad \alpha = 12{\cdot}8^\circ$$
$$-0{\cdot}07 \pm 0{\cdot}37\,i = 0{\cdot}377\ e^{\pm i\beta} \quad \beta = 100{\cdot}7^\circ$$

Thus the cross term is

$$2\times 0{\cdot}585\times 0{\cdot}377\ \cos(\delta+88^\circ)$$
$$\approx 2\times 0{\cdot}585\times 0{\cdot}377\ \cos\left(\delta+\frac{\pi}{2}\right)$$

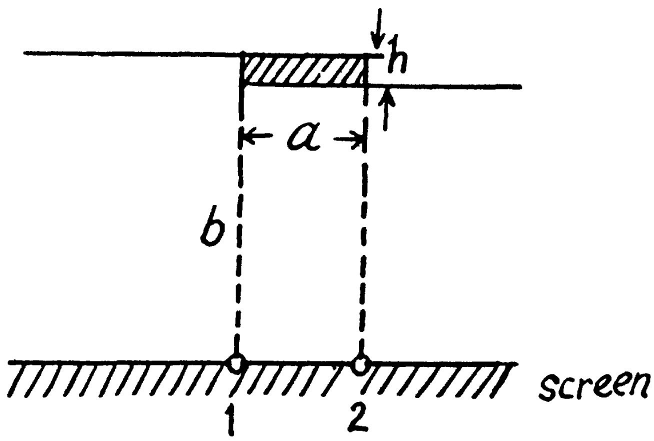
For maximum intensity

$$\delta+\frac{\pi}{2} = 2k'\pi,\quad k' = 1,2,3,4,\ldots$$
$$= 2(k+1)\pi,\ k = 0,1,2,3,\ldots$$

or
$$\delta = 2k\pi+\frac{3\pi}{2}$$

so
$$h = \frac{\lambda}{n-1}\left(k+\frac{3}{4}\right)$$

5.115



Using the method of problem 5.103 we can immediately write down the amplitudes at 1 and 2. We get :

At 1 amplitude $A_1 \sim \text{const}\left[\int_{-\infty}^{0} e^{-i\pi u^2/2}\,du + e^{-i\delta}\int_{v}^{\infty} e^{-i\pi u^2/2}\,du\right]$

At 2 amplitude $A_2 \sim$ const $\left[\int_{-\infty}^{-\nu} e^{-i\pi u^2/2} du + e^{-i\delta} \int_0^{\infty} e^{-i\pi u^2/2} du \right]$

where $$\nu = a\sqrt{\frac{2}{b\lambda}}$$

is the parameter of Cornu's spiral and constant factor is common to 1 and 2.

With the usual notation

$$C = C(\nu) = \int_0^{\nu} \cos\frac{\pi u^2}{2} du$$

$$S = S(\nu) = \int_0^{\nu} \sin\frac{\pi u^2}{2} du$$

and the result $$\int_0^{\infty} \cos\frac{\pi u^2}{2} du = \int_0^{\infty} \sin\frac{\pi u^2}{2} du = \frac{1}{2}$$

We find the ratio of intensities as

$$\frac{I_2}{I_1} = \left| \frac{\left(\frac{1}{2} - C\right) - i\left(\frac{1}{2} - S\right) + e^{-i\delta}\frac{(1-i)}{2}}{\frac{1-i}{2} + e^{-i\delta}\left\{\left(\frac{1}{2} - C\right) - i\left(\frac{1}{2} - S\right)\right\}} \right|^2$$

(The constants in A_1 and A_2 must be the same by symmetry)

In our case, $a = 0{\cdot}30$ mm, $\lambda = 0{\cdot}65\ \mu$m, $b = 1{\cdot}1$ m

$$\nu = 0{\cdot}30 \times \sqrt{\frac{2}{1{\cdot}1 \times 0{\cdot}65}} = 0{\cdot}50$$

$$C(0{\cdot}50) = 0{\cdot}48 \qquad S(0{\cdot}50) = 0{\cdot}06$$

$$\frac{I_2}{I_1} = \left| \frac{0{\cdot}02 - 0{\cdot}44\,i + e^{-i\delta}\frac{(1-i)}{2}}{\frac{1-i}{2}e^{i\delta} + 0{\cdot}02 - 0{\cdot}44\,i} \right|^2 = \left| \frac{1 + (0{\cdot}02 - 0{\cdot}44\,i)\sqrt{2}\,e^{i\delta + \frac{i\pi}{4}}}{1 + (0{\cdot}02 - 0{\cdot}44\,i)\sqrt{2}\,e^{-i\delta + \frac{i\pi}{4}}} \right|^2$$

But $0{\cdot}02 - 0{\cdot}44\,i = 0{\cdot}44\,e^{i\alpha}$, $\alpha = 1{\cdot}525$ rad ($\approx 87{\cdot}4°$)

So $$\frac{I_2}{I_1} = \left| \frac{1 + 0{\cdot}44 \times \sqrt{2} \times e^{i(\delta - 0{\cdot}740)}}{1 + 0{\cdot}44 \times \sqrt{2} \times e^{-i(\delta + 0{\cdot}740)}} \right|^2 = \frac{1 + 2(0{\cdot}44)^2 + 2\sqrt{2} \times 0{\cdot}44 \cos(\delta - 0{\cdot}740)}{1 + 2(0{\cdot}44)^2 + 2\sqrt{2} \times 0{\cdot}44 \cos(\delta + 0{\cdot}740)}$$

I_2 is maximum when $\delta - 0{\cdot}740 = 0$ (modulo 2π)

Thus in that case $$\frac{I_2}{I_1} = \frac{1{\cdot}387 + 1{\cdot}245}{1{\cdot}387 + 1{\cdot}245\cos(1{\cdot}48)} = \frac{2{\cdot}632}{1{\cdot}5} \approx 1{\cdot}75$$

5.116 We apply the formula of problem 5.103 and calculate

$$\int_{aperture} \frac{a_0}{r} e^{-ikr} dS = \int_{Semicircle} + \int_{Slit}$$

The contribution of the full 1st Fresnel zone has been evaluated in 5.103. The contribution of the semi-circle is one half of it and is

$$-\frac{2\pi}{k} i a_0 e^{-ikb} = -i a_0 \lambda e^{-ikb}$$

The contribution of the slit is

$$\frac{a_0}{b} \int_0^{0.90\sqrt{b\lambda}} e^{-ikb} e^{-ik\frac{x^2}{2b}} dx \int_{-\infty}^{\infty} e^{-iky^2/2b} dy$$

Now
$$\int_{-\infty}^{\infty} e^{-iky^2/2b} dy = \int_{-\infty}^{\infty} e^{-i\frac{\pi y^2}{b\lambda}} dy$$

$$\sqrt{\frac{b\lambda}{2}} \int_{-\infty}^{\infty} e^{-i\pi u^2/2} du = \sqrt{b\lambda}\ e^{-i\pi/4}$$

Thus the contribution of the slit is

$$\frac{a_0}{b}\sqrt{b\lambda}\ e^{-ikb-i\pi/4} \int_0^{0.9\times\sqrt{2}} e^{-i\pi u^2/2} du \sqrt{\frac{b\lambda}{2}}$$

$$= a_0 \lambda e^{-ikb-i\pi/4} \frac{1}{\sqrt{2}} \int_0^{1.27} e^{-i\pi u^2/2} du$$

Thus the intensity at the observation point P on the screen is

$$a_0^2\lambda^2 \left| -i + \frac{1-i}{2}(C(1.27) - iS(1.27)) \right|^2 = a_0^2\lambda^2 \left| -i + \frac{(1-i)(0.67 - 0.65\,i)}{2} \right|^2$$

(on using $C(1.27) = 0.67$ and $S(1.27) = 0.65$)

$$= a_0^2\lambda^2 \,|-i + 0.01 - 0.66\,i\,|^2$$

$$= a_0^2\lambda^2 \,|\,0.01 - 1.66\,i\,|^2$$

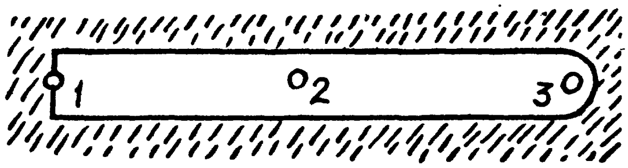
$$= 2.76\ a_0^2\lambda^2$$

Now $a_0^2\lambda^2$ is the intensity due to half of 1st Fresnel zone and is therefore equal to I_0. (It can also be obtained by doing the x-integral over $-\infty$ to $+\infty$).

Thus
$$I = 2.76 I_0 .$$

5.117 From the statement of the problem we know that the width of the slit = diameter of the first Fresnel zone = $2\sqrt{b\lambda}$ where b is the distance of the observation point from the slit.

We calculate the amplitudes by evaluating the integral of problem 5.103 We get



$$A_1 = \frac{a_0}{b} \int_{-\sqrt{b\lambda}}^{\sqrt{b\lambda}} e^{-ikb}\, e^{-ik\frac{x^2}{2b}}\, dx \int_0^{\infty} e^{-ik\frac{y^2}{2b}}\, dy$$

$$= \frac{a_0}{b}\, e^{-ikb}\, \frac{b\lambda}{2} \int_{-\sqrt{2}}^{\sqrt{2}} e^{-i\pi u^2/2}\, du \times \int_0^{\infty} e^{-i\pi u^2/2}\, du$$

$$= \frac{a_0 \lambda}{2}(1-i)\, e^{-ikb}\left(C\left(\sqrt{2}\right) - iS\left(\sqrt{2}\right)\right)$$

$$A_2 = \frac{a_0}{b} \int_{-\sqrt{b\lambda}}^{\sqrt{b\lambda}} e^{-ikb}\, e^{-ik\frac{x^2}{2b}}\, dx \int_{-\infty}^{\infty} e^{-iky^2/2b}\, dy$$

$$= 2A_1$$

$$A_3 = -i\, a_0 \lambda\, e^{-ikb} + \frac{a_0 \lambda (1-i)}{2}\left(C\left(\sqrt{2}\right) - iS\left(\sqrt{2}\right)\right) e^{-ikb}$$

where the contribution of the 1st half Fresnel zone (in A_3, first term) has been obtained from the last problem.

Thus $$I_1 = a_0^2 \lambda^2 \left| \frac{(1-i)(0{\cdot}53 - 0{\cdot}72\, i)}{2} \right|^2$$

$\left(\text{on using } C\left(\sqrt{2}\right) = 0{\cdot}53,\ S\left(\sqrt{2}\right) = 0{\cdot}72\right)$

$$= a_0^2 \lambda^2 \left| -0{\cdot}095 - 0{\cdot}625\, i \right|^2 = 0{\cdot}3996\, a_0^2 \lambda^2$$

$$I_2 = 4 I_1$$

$$I_3 = a_0^2 \lambda^2 \left| -0{\cdot}095 - 0{\cdot}625\, i - i \right|^2$$

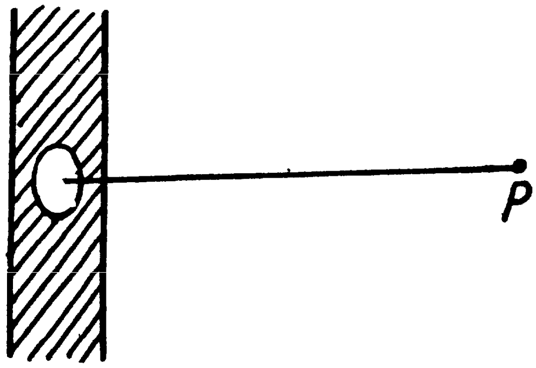
$$= a_0^2 \lambda^2 \left| -0{\cdot}095 - 1{\cdot}625\, i \right|^2$$

$$= 2{\cdot}6496\, a_0^2 \lambda^2$$

So $$I_3 = 6{\cdot}6\, I_1$$

Thus $$I_1 : I_2 : I_3 \approx 1 : 4 : 7$$

5.118 The radius of the first half Fresnel zone is $\sqrt{b\lambda/2}$ and the amplitude at P is obtained using problem 5.103.



$$A = \frac{a_0}{b}\left[\int_{-\infty}^{-\eta\sqrt{b\lambda/2}} + \int_{\eta\sqrt{b\lambda/2}}^{\infty}\right] e^{-ikb-\frac{kx^2}{2b}}\,dx$$

$$\int_{-\infty}^{\infty} e^{-iky^2/2b}\,dy + \frac{a_0}{b}e^{-ikb}\int_0^{\sqrt{b\lambda/2}} e^{-ik\rho^2/2b}\,2\pi\rho\,d\rho .$$

We use
$$\int_{-\infty}^{-\eta\sqrt{b\lambda/2}} e^{-ikx^2/2b}\,dx$$

$$= \int_{\eta\sqrt{b\lambda/2}}^{\infty} e^{-ikx^2/2b}\,dx = \int_{\eta\sqrt{b\lambda/2}}^{\infty} e^{-i\frac{\pi x^2}{b\lambda}}\,dx$$

$$= \int_{\eta}^{\infty} e^{-i\pi u^2/2}\sqrt{\frac{b\lambda}{2}}\,du = \sqrt{\frac{b\lambda}{2}}\left(\int_0^{\infty} - \int_0^{\eta}\right) e^{-i\pi u^2/2}\,du.$$

$$= \sqrt{\frac{b\lambda}{2}}\left(\left(\frac{1}{2}-C(\eta)\right) - i\left(\frac{1}{2}-S(\eta)\right)\right)$$

Thus
$$A = a_0\frac{\lambda}{\cancel{2}}\times\cancel{2}\times(1-i)e^{-ikb}\left[\left(\frac{1}{2}-C(\eta)\right)\right.$$

$$\left. - i\left(\frac{1}{2}-S(\eta)\right)\right] + a_0\lambda(1-i)\,e^{-ikb}$$

where we have used

$$\int_0^{\sqrt{b\lambda/2}} e^{-ik\rho^2/2b}\,2\pi\rho\,d\rho = \frac{2\pi i b}{k}(-1-i) = \frac{2\pi b}{k}(1-i) = \lambda b(1-i)$$

Thus the intensity is

$$I = |A|^2 = a_0^2\lambda^2\times 2\left[(3/2-C(\eta))^2 + \left(\frac{1}{2}-S(\eta)\right)^2\right]$$

From Cornu's Spiral,

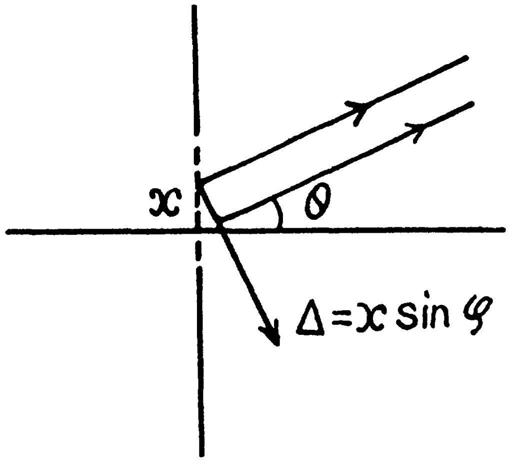
$$C(\eta) = C(1{\cdot}07) = 0{\cdot}76$$

$$S(\eta) = S(1{\cdot}07) = 0{\cdot}50$$

$$I = a_0^2\lambda^2 \times 2 \times (0{\cdot}74)^2 = 1{\cdot}09\, a_0^2\lambda^2$$

As before $$I_0 = a_0^2\lambda^2 \text{ so } I \approx I_0.$$

5.119 If a plane wave is incident normally from the left on a slit of width b and the diffracted wave is observed at a large distance, the resulting pattern is called Fraunhofer diffraction. The condition for this is $b^2 \ll l\lambda$ where l is the distance between the slit and the screen. In practice light may be focussed on the screen with the help of a lens (or a telescope).



Consider an element of the slit which is an infinite strip of width dx. We use the formula of problem 5.103 with the following modifications.

The factor $\frac{1}{r}$ characteristic of spherical waves will be omitted. The factor $K(\varphi)$ will also be dropped if we confine overselves to not too large φ. In the direction defined by the angle φ the extra path difference of the wave emitted from the element at x relative to the wave emitted from the centre is

$$\Delta = -x\sin\varphi$$

Thus the amplitude of the wave is given by

$$\alpha \int_{-b/2}^{+b/2} e^{ik\sin\varphi}\,dx = \left(e^{i\frac{1}{2}kb\sin\varphi} - e^{-i\frac{1}{2}kb\sin\varphi}\right) \Big/ ik\sin\varphi$$

$$= \frac{\sin\left(\frac{\pi b}{\lambda}\sin\varphi\right)}{\frac{\pi b}{\lambda}\sin\varphi}$$

Thus $$I = I_0 \frac{\sin^2\alpha}{\alpha^2}$$

where $$\alpha = \frac{\pi b}{\lambda}\sin\varphi \text{ and}$$

I_0 is a constant

Minima are observed for $\sin\alpha = 0$ but $\alpha \neq 0$

Thus we find minima at angles given by

$$b\sin\varphi = k\lambda,\ k = \pm 1, \pm 2, \pm 3, \ldots$$

5.120 Since $I(\alpha)$ is $+ve$ and vanishes for $b \sin \varphi = k\lambda$ i.e for $\alpha = k\pi$, we expect maxima of $I(\alpha)$ between $\alpha = +\pi$ & $\alpha = +2\pi$, etc. We can get these values by.

$$\frac{d}{d\alpha}(I(\alpha)) = I_0 \, 2 \frac{\sin \alpha}{\alpha} \frac{d}{d\alpha} \frac{\sin \alpha}{\alpha} = 0$$

$$\frac{\alpha \cos \alpha - \sin \alpha}{\alpha^2} = 0 \quad \text{or} \quad \tan \alpha = \alpha$$

Solutions of this transcendental equation can be obtained graphically.

The first three solutions are

$$\alpha_1 = 1{\cdot}43\,\pi, \; \alpha_2 = 2{\cdot}46\,\pi, \; \alpha_3 = 3{\cdot}47\,\pi$$

on the +ve side. (On the negative side the solution are $-\alpha_1, -\alpha_2, -\alpha_3, \dots$)

Thus

$$b \sin \varphi_1 = 1{\cdot}43\,\lambda$$

$$b \sin \varphi_2 = 2{\cdot}46\,\lambda$$

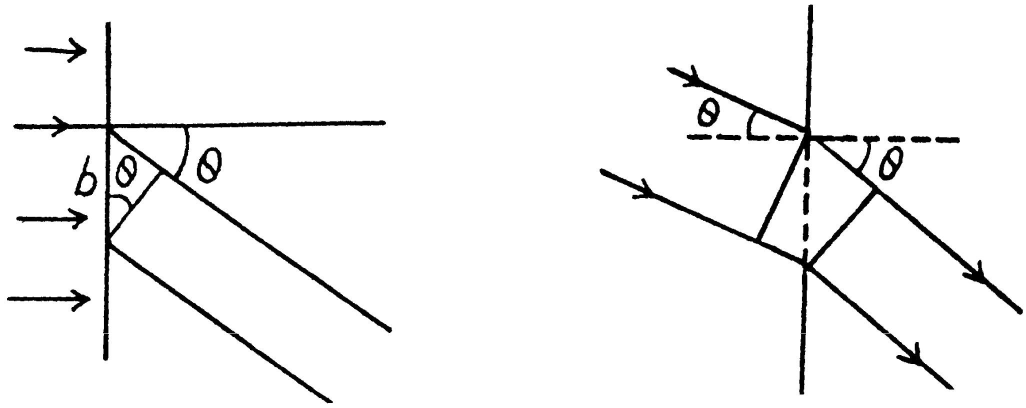
$$b \sin \varphi_3 = 3{\cdot}47\,\lambda$$

Asymptotically the solutions are

$$b \sin \varphi_m \approx \left(M + \frac{1}{2}\right)\lambda$$

5.121 The relation $b \sin \theta = k\lambda$

for minima (when light is incident normally on the slit) has a simple interpretation : $b \sin \theta$ is the path difference between extreme wave normals emitted at angle θ



When light is incident at an angle θ_0 the path difference is

$$b(\sin \theta - \sin \theta_0)$$

and the condition of minima is

$$b(\sin \theta - \sin \theta_0) = k\lambda$$

For the first minima

$$b(\sin \theta - \sin \theta_0) = \pm\lambda \quad \text{or} \quad \sin \theta = \sin \theta_0 \pm \frac{\lambda}{b}$$

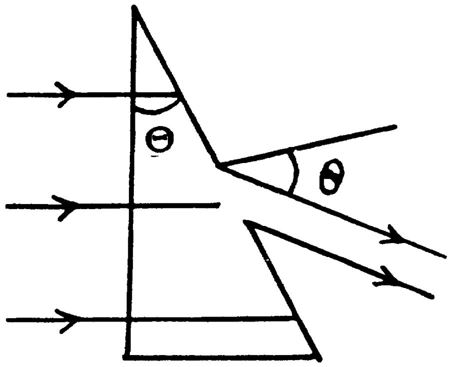
Putting in numbers $\theta_0 = 30°$, $\lambda = 0{\cdot}50\,\mu\text{m}$ $b = 10\,\mu\text{m}$

$$\sin \theta = \frac{1}{2} \pm \frac{1}{20} = 0{\cdot}55 \text{ or } 0{\cdot}45$$

$$\theta_{+1} = 33° - 20' \text{ and } \theta_{-1} = 26°\,44'$$

5.122 (a) This case is analogous to the previous one except that the incident wave moves in glass of RI n. Thus the expression for the path difference for light diffracted at angle θ from the normal to the hypotenuse of the wedge is

$$b(\sin\theta - n\sin\Theta)$$



we write $$\theta = \Theta + \Delta\theta$$

Then for the direction of principal Fraunhofer maximum

$$b(\sin(\Theta + \Delta\theta) - n\sin\Theta) = 0$$

or $$\Delta\theta = \sin^{-1}(n\sin\Theta) - \Theta$$

Using $\Theta = 15°$, $n = 1{\cdot}5$ we get

$$\Delta\theta = 7{\cdot}84°$$

(b) The width of the central maximum is obtained from ($\lambda = 0{\cdot}60\ \mu$ m, $b = 10\ \mu\, m$)

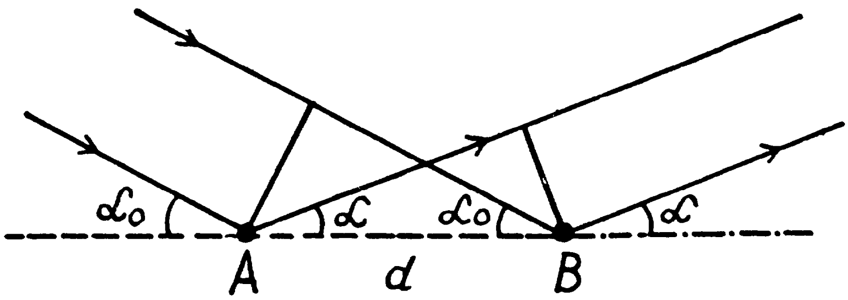
$$b(\sin\theta_1 - n\sin\Theta) = \pm\lambda$$

Thus $$\theta_{+1} = \sin^{-1}\left(n\sin\Theta + \frac{\lambda}{b}\right) = 26{\cdot}63°$$

$$\theta_{-1} = \sin^{-1}\left(n\sin\Theta \frac{\lambda}{b}\right) = 19{\cdot}16°$$

$$\therefore\quad \delta\theta = \theta_{+1} - \theta_{-1} = 7{\cdot}47°$$

5.123



The path difference between waves reflected at A and B is

$$d(\cos\alpha_0 - \cos\alpha)$$

and for maxima

$$d(\cos\alpha_0 - \cos\alpha) = k\lambda,\ k = 0, \pm 1, \pm 2, \ldots$$

In our case, $k = 2$ and α_0, α are small in radiaus. Then

$$2\lambda = d\left(\frac{\alpha^2 - \alpha_0^2}{2}\right)$$

Thus $$\lambda = \frac{(\alpha^2 - \alpha_0^2)\,d}{4} = 0{\cdot}61\ \mu\,\text{m}$$

for $$\alpha = \frac{3\pi}{180},\ \alpha_0 = \frac{\pi}{180},\ d = 10^{-3}\ \text{m}$$

5.124 The general formula for diffraction from N slits is

$$I = I_0 \frac{\sin^2 \alpha}{\alpha^2} \frac{\sin^2 N\beta}{\sin^2 \beta}$$

where
$$\alpha = \frac{\pi a \sin\theta}{\lambda}$$

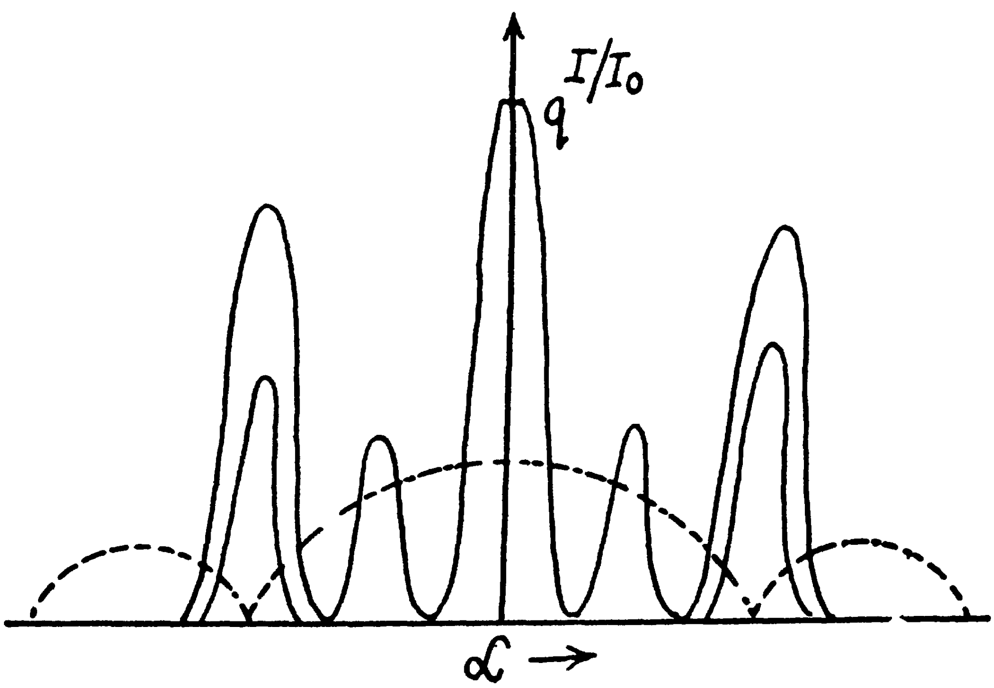
$$\beta = \frac{\pi(a+b)\sin\theta}{\lambda}$$

and $N = 3$ in the cases here.

(a) In this case $a + b = 2a$

so
$$\beta = 2\alpha \quad \text{and} \quad I = I_0 \frac{\sin^2 \alpha}{\alpha^2}(3 - 4\sin^2 2\alpha)^2$$

On plotting we get a curve that qualitatively looks like the one below



(b) In this case $a + b = 3a$

so
$$\beta = 3\alpha$$

and
$$I = I_0 \frac{\sin^2 \alpha}{\alpha^2}(2 - 4\sin^2 3\alpha)^2$$

This has 3 minima between the principal maxima

5.125 From the formula $\quad d\sin\theta = m\lambda$

we have $d \sin 45° = 2\lambda_1 = 2 \times 0{\cdot}65\ \mu\text{m}$

or
$$d = 2\sqrt{2} \times 0{\cdot}65\ \mu\text{m}$$

Then for $\lambda_2 = 0{\cdot}50$ in the third order

$$2\sqrt{2} \times 0{\cdot}65 \sin\theta = 3 \times 0{\cdot}50$$

$$\sin\theta = \frac{1{\cdot}5}{1{\cdot}3 \times \sqrt{2}} = 0{\cdot}81602$$

This gives $\theta = 54{\cdot}68° \approx 55°$

5.126 The diffraction formula is

$$d \sin \theta_0 = n_0 \lambda$$

where $\theta_0 = 35°$ is the angle of diffraction corresponding to order n_0 (which is not yet known).

Thus
$$d = \frac{n_0 \lambda}{\sin \theta_0} = n_0 \times 0{\cdot}9327\ \mu\,m$$

on using $\lambda = 0{\cdot}535\ \mu\,m$

For the n^{th} order we get

$$\sin \theta = \frac{n}{n_0} \sin \theta_0 = \frac{n}{n_0} (0{\cdot}573576)$$

If $n_0 = 1$, then $n > n_0$ is at least 2 and $\sin \theta > 1$ so $n = 1$ is the highest order of diffraction.

If $n_0 = 2$ then $n = 3, 4$, but $\sin \theta > 1$ for $n = 4$ thus the highest order of diffraction is 3.

If $n_0 = 3$,

then
$$n = 4, 5, 6\,.$$

For $n = 6$, $\sin \theta = 2 \times 0{\cdot}57 > 1$, so not allowed while for

$$n = 5,\ \sin \theta = \frac{5}{3} \times 0{\cdot}573576 < 1$$

is allowed. Thus in this case the highest order of diffraction is five as given. Hence

$$n_0 = 3$$

and
$$d = 3 \times 0{\cdot}9327 = 2{\cdot}7981 \approx 2{\cdot}8\ \mu\,m\,.$$

5.127 Given that

$$d \sin \theta_1 = \lambda$$

$$d \sin \theta_2 = d \sin(\theta_1 + \Delta\theta) = 2\lambda$$

Thus
$$\sin \theta_1 \cos \Delta\theta + \cos \theta_1 \sin \Delta\theta = 2 \sin \theta_1$$

or
$$\sin \theta_1 (2 - \cos \Delta\theta) = \cos \theta_1 \sin \Delta\theta$$

or
$$\tan \theta_1 = \frac{\sin \Delta\theta}{2 - \cos \Delta\theta}$$

or
$$\sin \theta_1 = \frac{\sin \Delta\theta}{\sqrt{\sin^2 \Delta\theta + (2 - \cos \Delta\theta)^2}}$$

$$= \frac{\sin \Delta\theta}{\sqrt{5 - 4\cos \Delta\theta}}$$

Finally
$$\lambda = \frac{d \sin \Delta\theta}{\sqrt{5 - 4\cos \Delta\theta}}\,.$$

Substitution gives $\lambda \approx 0{\cdot}534\ \mu\,m$

5.128 (a) Here the simple formula

$$d \sin \theta = m_1 \lambda \text{ holds.}$$

Thus
$$1{\cdot}5 \sin \theta = m \times 0{\cdot}530 \quad \sin \theta = \frac{m \times 0{\cdot}530}{1{\cdot}5}$$

Highest permissible m is $m = 2$ because $\sin\theta > 1$ if $m = 3$. Thus

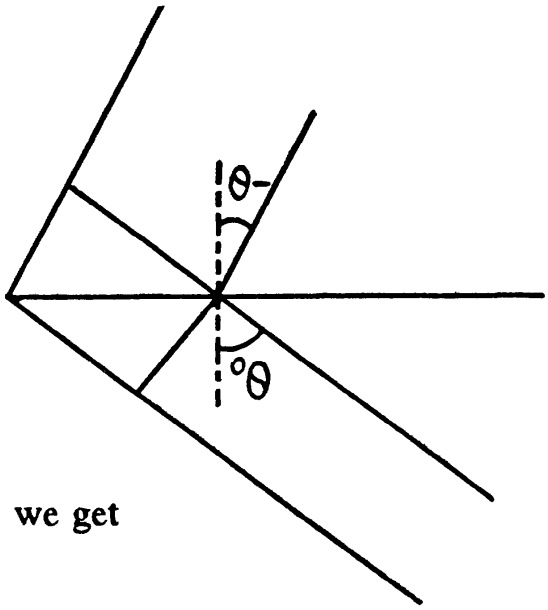
$$\sin\theta = \frac{1{\cdot}06}{1{\cdot}50} \text{ for } m = 2, \text{ This gives } \theta = 45^\circ \text{ nearby.}$$

(b) Here $d(\sin\theta_0 - \sin\theta) = n\lambda$

Thus $\sin\theta = \sin\theta_0 - \dfrac{n\lambda}{d}$

$$= \sin 60^\circ - n \times \frac{0{\cdot}53}{1{\cdot}5}$$

$$= 0{\cdot}86602 - n \times 0{\cdot}353333.$$



For $n = 5$, $\sin\theta = -0{\cdot}900645$

for $n = 6$, $\sin\theta < -1$.

Thus the highest order is $n = 5$ and we get

$\theta = \sin^{-1}(-0{\cdot}900645) \cong -64^\circ$

5.129 For the lens

$$\frac{1}{f} = (n-1)\left(\frac{1}{R} - \frac{1}{\infty}\right) \quad \text{or} \quad f = \frac{R}{n-1}$$

For the grating

$$d\sin\theta_1 = \lambda \quad \text{or} \quad \sin\theta_1 = \frac{\lambda}{d}$$

$$\operatorname{cosec}\theta_1 = \frac{d}{\lambda}, \ \cot\theta_1 = \sqrt{\left(\frac{d}{\lambda}\right)^2 - 1}$$

$$\tan\theta_1 = \frac{1}{\sqrt{\left(\frac{d}{\lambda}\right)^2 - 1}}$$

Hence the distance between the two symmetrically placed first order maxima

$$= 2f\tan\theta_1 = \frac{2R}{(n-1)\sqrt{\left(\frac{d}{\lambda}\right)^2 - 1}}$$

On putting $R = 20$, $n = 1{\cdot}5$, $d = 6{\cdot}0\,\mu\text{m}$

$\lambda = 0{\cdot}60\,\mu\text{m}$ we get 8·04 cm.

5.130 The diffraction formula is easily obtained on taking account of the fact that the optical path in the glass wedge acquires a factor n (refractive index). We get

$$d(n\sin\Theta - \sin(\Theta\ \theta_k)) = k\lambda$$

Since $n > 0$, $\Theta - \theta_0 > \Theta$ and so θ_0 must be negative. We get, using $\Theta = 30^\circ$

$$\frac{3}{2} \times \frac{1}{2} = \sin(30^\circ - \theta_0) = \sin 48{\cdot}6^\circ$$

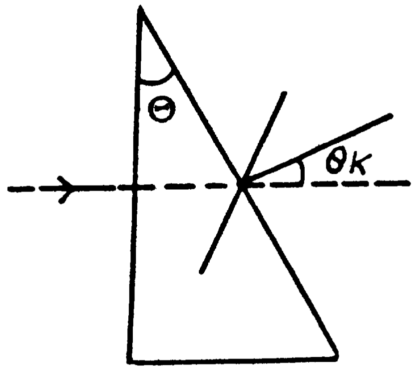
Thus $\theta_0 = -18{\cdot}6°$

Also for $k = 1$

$$\frac{3}{4} - \sin(30° - \theta_{+1}) = \frac{\lambda}{d} = \frac{0{\cdot}5}{2{\cdot}0} = \frac{1}{4}$$

Thus $\theta_{+1} = 0°$

We calculate θ_k for various k by the above formula. For $k = 6$.



$$\sin(\theta_k - 30°) = \frac{3}{4} \Rightarrow \theta_k = 78{\cdot}6°$$

For $k = 7$

$$\sin(\theta_k - 30°) = +1 \Rightarrow \theta_k = 120°$$

This is in admissible. Thus the highest order that can be observed is

$$k = 6$$

corresponding to $\theta_k = 78{\cdot}6°$

(for $k = 7$ the diffracted ray will be grazing the wedge).

5.131 The intensity of the central Fraunhofer maximum will be zero if the waves from successive grooves (not in the same plane) differ in phase by an odd multiple of π. Then since the phase difference is

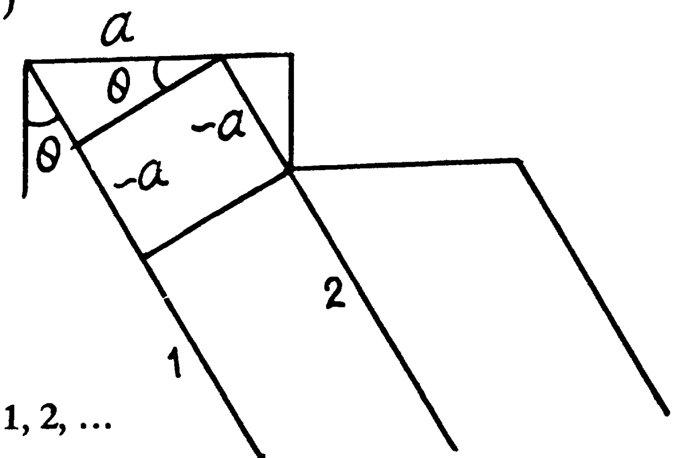
$$\delta = \frac{2\pi}{\lambda}(n-1)h$$

for the central ray we have

$$\frac{2\pi}{\lambda}(n-1)h = \left(k - \frac{1}{2}\right)2\pi, \; k = 1, 2, 3, \ldots$$

or $$h = \frac{\lambda}{n-1}\left(k - \frac{1}{2}\right).$$

The path difference between the rays 1 & 2 is approximately (neglecting terms of order θ^2)



$a \sin\theta + a - na$

$= a\sin\theta - (n-1)a$

Thus for a maximum

$$a\sin\theta - \left(k' + \frac{1}{2}\right)\lambda = m\lambda$$

or $$a\sin\theta = \left(m + k' + \frac{1}{2}\right)\lambda,$$

$$k' = 0, 1, 2, \ldots$$

$$m = 0, \pm 1, \pm 2, \ldots$$

The first maximum after the central minimum is obtained when $m + k' = 0$

We get $$a\sin\theta_1 = \frac{1}{2}\lambda$$

5.132 When standing ultra sonic waves are sustained in the tank it behaves like a grating whose grating element is

$$d = \frac{v}{\nu} = \text{wavelength of the ultrasonic}$$

v = velocity of ultrasonic. Thus for maxima

$$\frac{v}{\nu} \sin \theta_m = m \lambda$$

On the other hand

$$f \tan \theta_m = m \Delta x$$

Assuming θ_m to be small $\left(\text{because } \lambda << \frac{v}{\nu}\right)$

we get $$\Delta x = \frac{f \tan \theta_m}{m} = \frac{f \tan \theta_m}{\frac{v}{\nu \lambda} \sin \theta_m} = \frac{\lambda \nu f}{v}$$

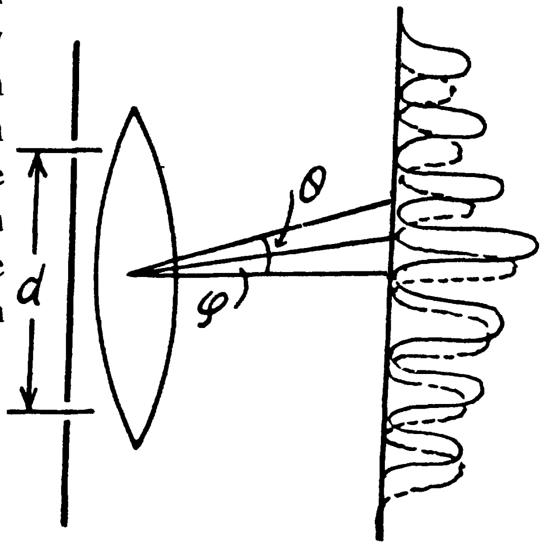
or $$v = \frac{\lambda \nu f}{\Delta x}$$

Putting the values $\lambda = 0{\cdot}55\ \mu\text{m}$, $\nu = 4{\cdot}7\ \text{MHz}$

$f = 0{\cdot}35\ \text{m}$ and $\Delta x = 0{\cdot}60 \times 10^{-3}\ \text{m}$ we easily get

$$v = 1{\cdot}51\ \text{km/sec}.$$

5.133 Each star produces its own diffraction pattern in the focal plane of the objective and these patterns are separated by angle ψ. As the distance d decreases the angle θ between the neighbouring maxima in either diffraction pattern increases ($\sin \theta = \lambda/d$). When θ becomes equal to 2ψ the first deterioration of visibility occurs because the maxima of one system of fringes coincide with the minima of the other system. Thus from the condition $\theta = 2\psi$ and $\sin \theta = \frac{\lambda}{d}$ we get



$$\psi = \frac{1}{2}\theta \approx \frac{\lambda}{2d} \text{ (radians)}$$

Putting the values we get $\psi = 0{\cdot}06''$

5.134 (a) For normal incidence, the maxima are given by

$$d \sin \theta = n \lambda$$

so $$\sin \theta = n \frac{\lambda}{d} = n \times \frac{0{\cdot}530}{1{\cdot}500}$$

Clearly $n \leq 2$ as $\sin \theta > 1$ for $n = 3$.

Thus the highest order is $n = 2$. Then

$$D = \frac{d\theta}{d\lambda} = \frac{k}{d\cos\theta} = \frac{k}{d}\frac{1}{\sqrt{1-\left(\frac{k\lambda}{d}\right)^2}}$$

Putting $k = 2$, $\lambda = 0{\cdot}53\,\mu\text{m}$, $d = 1{\cdot}5\,\mu\text{m} = 1500\,\text{nm}$

we get $D = \dfrac{2}{1500}\dfrac{1}{\sqrt{1-\left(\frac{1{\cdot}06}{1{\cdot}5}\right)^2}} \times \dfrac{180}{\pi} \times 60 = 6{\cdot}47$ ang. min/nm.

(b) We write the diffraction formula as

$$d(\sin\theta_0 + \sin\theta) = k\lambda$$

so $$\sin\theta_0 + \sin\theta = k\frac{\lambda}{d}$$

Here $$\theta_0 = 45° \text{ and } \sin\theta_0 = 0{\cdot}707$$

so $$\sin\theta_0 + \sin\theta \le 1{\cdot}707\text{. Since}$$

$$\frac{\lambda}{d} = \frac{0{\cdot}53}{1{\cdot}5} = 0{\cdot}353333\text{, we see that}$$

$$k \le 4$$

Thus highest order corresponds to $k = 4$.

Now as before $D = \dfrac{d\theta}{d\lambda}$ so

$$D = \frac{k}{d\cos\theta} = \frac{k/d}{\sqrt{1-\left(\frac{k\lambda}{d}-\sin\theta_0\right)^2}}$$

$$= 12{\cdot}948 \text{ ang. min/nm},$$

5.135 We have $$d\sin\theta = k\lambda$$

so $$\frac{d\theta}{d\lambda} = D = \frac{k}{d\cos\theta} = \frac{\tan\theta}{\lambda}$$

5.136 For the second order principal maximum

$$d\sin\theta_2 = 2\lambda = k\lambda$$

or $$\frac{N\pi}{\lambda}d\sin\theta_2 = 2N\pi$$

minima adjacent to this maximum occur at

$$\frac{N\pi}{\lambda}d\sin(\theta_2 \pm \Delta\theta) = (2N \pm 1)\pi$$

or $$d\cos\theta_2\,\Delta\theta = \frac{\lambda}{N}$$

Finally angular width of the 2nd principal maximum is

$$2\,\Delta\theta = \frac{2\lambda}{N d \cos\theta_2} = \frac{2\lambda}{N d \sqrt{1-(k\lambda/d)^2}} = \frac{\tan\theta_2}{N}$$

On putting the values we get 11·019″ of arc

5.137 Using

$$R = \frac{\lambda}{\delta\lambda} = kN = \frac{N d \sin\theta}{\lambda}$$

$$= \frac{l \sin\theta}{\lambda} \leq \frac{l}{\lambda}.$$

5.138 For the just resolved waves the frequency difference

$$\delta\nu = \frac{c\,\delta\lambda}{\lambda} = \frac{c}{\lambda R} = \frac{c}{\lambda k N}$$

$$= \frac{c}{N d \sin\theta} = \frac{1}{\delta t}$$

since $N d \sin\theta$ is the path difference between waves emitted by the extremities of the grating.

5.139 $\delta\lambda = \cdot 050$ nm

$$R = \frac{\lambda}{\delta\lambda} = \frac{600}{\cdot 05} = 12000 \text{ (nearly)}$$

$$= kN$$

On the other hand

$$d \sin\theta = k\lambda$$

Thus

$$\frac{l}{kN} \sin\theta = \lambda$$

where $l = 10^{-2}$ metre is the width of the grating

Hence

$$\sin\theta = 12000 \times \frac{\lambda}{l}$$

$$= 12000 \times 600 \times 10^{-7} = 0{\cdot}72$$

or

$$\theta = 46°\,.$$

5.140 (a) We see that

$$N = 6{\cdot}5 \times 10 \times 200 = 13000$$

Now to resolve lines with $\delta\lambda = 0{\cdot}015$ nm and $\lambda \approx 670{\cdot}8$ nm we must have

$$R = \frac{670{\cdot}8}{0{\cdot}015} = 44720$$

Since $3N < R < 4N$ one must go to the fourth order to resolve the said components.

(b) we have $d = \dfrac{1}{200}$ mm $= 5\,\mu$m

so

$$\sin\theta = \frac{k\lambda}{d} = \frac{k \times 0{\cdot}670}{5}$$

since $|\sin\theta| \le 1$ we must have $\quad k \le 7{\cdot}46$

so
$$k_{max} = 7 \approx \frac{d}{\lambda}$$

Thus
$$R_{max} = k_{max} N = 91000 \approx \frac{Nd}{\lambda} = \frac{l}{\lambda}$$

where $l = 6{\cdot}5$ cm is the grating width.

Finally
$$\delta\lambda_{min} = \frac{\lambda}{R_{max}} = \frac{670}{91000} = {\cdot}007\,\text{nm} = 7\,\text{pm} \approx \frac{\lambda^2}{l}.$$

5.141 Here

$$R = \frac{\lambda}{\delta\lambda} = \frac{589{\cdot}3}{0{\cdot}6} = \text{kN} = 5\,\text{N}$$

so
$$N = \frac{589{\cdot}3}{3} = \frac{10^{-2}}{d}$$

$$d = \frac{3\times 10^{-2}}{589{\cdot}3}\text{m} = {\cdot}0509\,\text{mm}$$

(b) To resolve a doublet with $\lambda = 460{\cdot}0$ nm and $\delta\lambda = 0{\cdot}13$ nm in the third order we must have

$$N = \frac{R}{3} = \frac{460}{3\times 0{\cdot}13} = 1179$$

This means that the grating is
$$Nd = 1179 \times 0{\cdot}0509 = 60{\cdot}03\,\text{mm}$$
wide = 6 cm wide.

5.142 (a) From $d\sin\theta = k\lambda$

we get
$$\delta\theta = \frac{k\,\delta\lambda}{d\cos\theta}$$

On the other hand
$$x = f\sin\theta$$

so
$$\delta x = f\cos\theta\,\delta\theta = \frac{kf}{d}\delta\lambda$$

For
$$f = 0{\cdot}80\,\text{m},\ \delta\lambda = 0{\cdot}03\,\text{nm and}$$
$$d = \frac{1}{250}\text{mm}$$

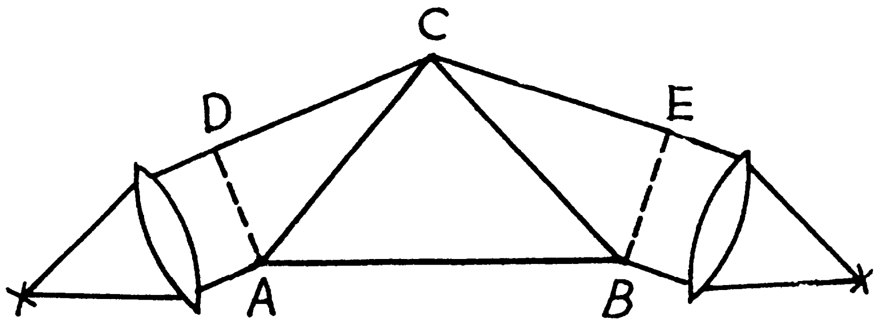
we get
$$\delta x = \begin{cases} 6\,\mu\text{m} & \text{if } k = 1 \\ 12\,\mu\text{m} & \text{if } k = 2 \end{cases}$$

(b) Here $N = 25 \times 250 = 6250$

and
$$\frac{\lambda}{\delta\lambda} = \frac{310{\cdot}169}{0{\cdot}03} = 10339\cdots > N$$

and so to resolve we need $k = 2$ For $k = 1$ gives an R.P. of only 6250.

5.143



Suppose the incident light consists of two wavelengths λ and $\lambda + \delta\lambda$ which are just resolved by the prism. Then by Rayleigh's criterion, the maximum of the line of wavelength λ must coincide with the first minimum of the line of wavelength $\lambda + \delta\lambda$. Let us write both conditions in terms of the optical path differences for the extreme rays :

For the light of wavelength λ

$$bn - (DC + CE) = 0$$

For the light of wavelength $\lambda + \delta\lambda$

$$b(n + \delta n) - (DC + CE) = \lambda + \delta\lambda$$

because the path difference between extreme rays equals λ for the first minimum in a single slit diffraction (from the formula $a \sin\theta = \lambda$).

Hence $$b\,\delta n \approx \lambda$$

and $$R = \frac{\lambda}{\delta\lambda} = b\left|\frac{\delta n}{\delta\lambda}\right| = b\left|\frac{dn}{d\lambda}\right|$$

5.144 (a) $\frac{\lambda}{\delta\lambda} = R = b\left|\frac{dn}{d\lambda}\right| = 2Bb/\lambda^3$

For $b = 5$ cm, $B = 0{\cdot}01\ \mu\text{m}^2$ $\lambda_1 = 0{\cdot}434\ \mu\text{m} = 5 \times 10^4\ \mu\text{m}$

$$R_1 = 1{\cdot}223 \times 10^4$$

for $$\lambda_2 = 0{\cdot}656\ \mu\text{m}$$

$$R_2 = 0{\cdot}3542 \times 10^4$$

(b) To resolve the D-lines we require

$$R = \frac{5893}{6} = 982$$

Thus $$982 = \frac{0{\cdot}02 \times b}{(0{\cdot}5893)^3}$$

$$b = \frac{982 \times (0{\cdot}5893)^3}{0{\cdot}02}\ \mu\text{m} = 1{\cdot}005 \times 10^4\ \mu\text{m} = 1{\cdot}005\ \text{cm}$$

5.145 $b\left|\frac{dn}{d\lambda}\right| = kN = 2 \times 10{,}000$

$$b \times 0{\cdot}10\ \mu\text{m}^{-1} = 2 \times 10^4$$

$$b = 2 \times 10^5\ \mu\text{m} = 0{\cdot}2\,m = 20\ \text{cm}.$$

5.146 Resolving power of the objective

$$= \frac{D}{1{\cdot}22\,\lambda} = \frac{5 \times 10^{-2}}{1{\cdot}22 \times 0{\cdot}55 \times 10^{-6}} = 7{\cdot}45 \times 10^4$$

Let $(\Delta y)_{min}$ be the minimum distance between two points at a distance of 3·0 km which the telescope can resolve. Then

$$\frac{(\Delta y)_{min}}{3 \times 10^3} = \frac{1{\cdot}22\,\lambda}{D} = \frac{1}{7{\cdot}45 \times 10^4}$$

or $$(\Delta y)_{min} = \frac{3 \times 10^3}{7{\cdot}45 \times 10^4} = 0{\cdot}04026 \text{ m} = 4{\cdot}03 \text{ cm}.$$

5.147 The limit of resolution of a reflecting telescope is determined by diffraction from the mirror and obeys a formula similar to that from a refracting telescope. The limit of resolution is

$$\frac{1}{R} = \frac{1{\cdot}22\,\lambda}{D} = \frac{(\Delta y)_{min}}{L}$$

where L = distance between the earth and the moon = 384000 km

Then putting the values $\lambda = 0{\cdot}55\ \mu$m, $D = 5$ m

we get $$(\Delta y)_{min} = 51{\cdot}6 \text{ metre}$$

5.148 By definition, the magnification

$$\Gamma = \frac{\text{angle subtended by the image at the eye}}{\text{angle subtended by the object at the eye}} = \frac{\psi'}{\psi}$$

At the limit of resolution $$\psi = \frac{1{\cdot}22\,\lambda}{D}$$

where D = diameter of the objective

On the other hand to be visible to the eye $\psi' \geq \dfrac{1{\cdot}22\,\lambda}{d_0}$

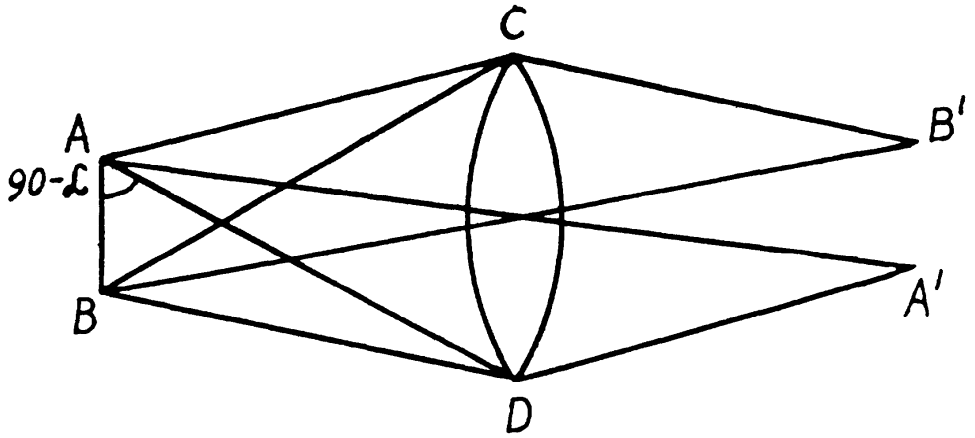
where d_0 = diameter of the pupil

Thus to avail of the resolution offered by the telescope we must have

$$\Gamma \geq \frac{1{\cdot}22\,\lambda}{d_0} \Big/ \frac{1{\cdot}22\,\lambda}{D} = \frac{D}{d_0}$$

Hence $$\Gamma_{min} = \frac{D}{d_0} = \frac{50 \text{ mm}}{4 \text{ mm}} = 12{\cdot}5$$

5.149



Let A and B be two points in the field of a microscope which is represented by the lens C D. Let A', B' be their image points which are at equal distances from the axis of the lens CD. Then all paths from A to A' are equal and the extreme difference of paths from A to B' is equal to

$$\begin{aligned}&ADB' - ACB'\\ &= AD + DB' - (AC + CB')\\ &= AD + DB' - BD - DB'\\ &\quad + BC + CB' - AC - CB'\\ &(\text{as } BD + DB' = BC + CB')\\ &= AD - BD + BC - AC\\ &= 2AB\cos(90° - \alpha) = 2AB\sin\alpha\end{aligned}$$

From the theory of diffraction by circular apertures this distance must be equal to $1{\cdot}22\,\lambda$

when B' coincides with the minimum of the diffraction due to A and A' with the minimum of the diffraction due to B. Thus

$$AB = \frac{1{\cdot}22\,\lambda}{2\sin\alpha} = 0{\cdot}61\frac{\lambda}{\sin\alpha}$$

Here 2α is the angle subtended by the objective of the microscope at the object.

Substituting the values

$$AB = \frac{0{\cdot}61\times 0{\cdot}55}{0{\cdot}24}\,\mu\text{m} = 1{\cdot}40\,\mu\text{m}.$$

5.150 Suppose d_{min} = minimum separation resolved by the microscope

ψ = angle subtended at the eye by this object when the object is at the least distance of distinct vision l_0 (= 25 cm).

$$\psi' = \text{minimum angular separation resolved by the eye} = \frac{1{\cdot}22\,\lambda}{d_0}$$

From the previous problem

$$d_{min} = \frac{0{\cdot}61\,\lambda}{\sin\alpha}$$

and

$$\psi = \frac{d_{min}}{l_0} = \frac{0{\cdot}61\,\lambda}{l_0\sin\alpha}$$

Now

$$\Gamma = \text{magnifying power} = \frac{\text{angle subtended at the eye by the image}}{\text{angle subtended at the eye by the object}}$$

when the object is at the least distance of distinct vision

$$\geq \frac{\psi'}{\psi} = 2\left(\frac{l_0}{d_0}\right)\sin\alpha$$

Thus

$$\Gamma_{min} = 2\left(\frac{l_0}{d_0}\right)\sin\alpha = 2\times\frac{25}{0{\cdot}4}\times 0{\cdot}24 = 30$$

5.151 Path difference

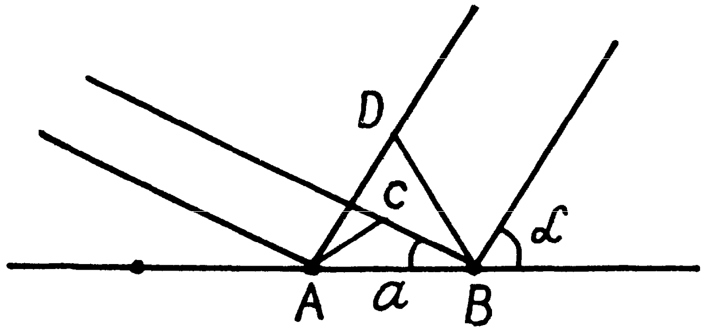
$= BC - AD$

$= a(\cos 60° - \cos\alpha)$

For diffraction maxima

$a(\cos 60° - \cos\alpha) = k\lambda$,

since $\lambda = \frac{2}{5}a$, we get



$$\cos\alpha = \frac{1}{2} - \frac{2}{5}k$$

and we get

$$k = -1,\ \cos\alpha = \frac{1}{2} + \frac{2}{5} = 0{\cdot}9,\ \alpha = 26°$$

$$k = 0,\ \cos\alpha = \frac{1}{2} = 0{\cdot}5,\ \alpha = 60°$$

$$k = 1,\ \cos\alpha = \frac{1}{2} - \frac{2}{5} = 0{\cdot}1,\ \alpha = 84°$$

$$k = 2,\ \cos\alpha = \frac{1}{2} - \frac{4}{5} = -0{\cdot}3,\ \alpha = 107{\cdot}5°$$

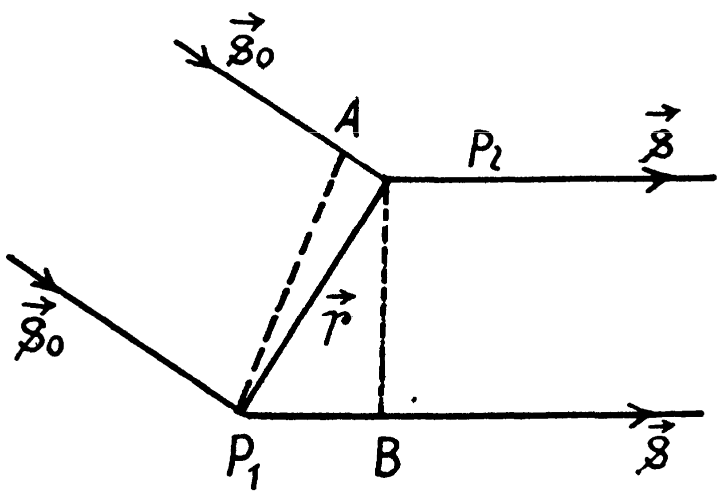
$$k = 3,\ \cos\alpha = \frac{1}{2} - \frac{6}{5} = -0{\cdot}7,\ \alpha = 134{\cdot}4°$$

Other values of k are not allowed as they lead to $|\cos\alpha| > 1$.

5.152 We give here a simple derivation of the condition for diffraction maxima, known as Laue equations. It is easy to see form the above figure that the path difference between waves scattered by nearby scattering centres P_1 and P_2 is

$P_2A - P_1B = \vec{r}\cdot\vec{s}_0 - \vec{r}\cdot\vec{s}$

$= \vec{r}\cdot(\vec{s}_0 - \vec{s}) = \vec{r}\cdot\vec{S}.$



Here $\vec{r}$ is the radius vector $\overrightarrow{P_1P_2}$. For maxima this path difference must be an integer multiple of λ for any two neighbouring atoms. In the present case of two dimensional lattice with $X-$ rays incident normally $\vec{r}\cdot\vec{s} = 0$. Taking successively · nearest neighbours in the $x-$ & $y-$ directions

We get the equations

$$a\cos\alpha = h\lambda$$

$$b\cos\beta = k\lambda$$

Here $\cos\alpha$ and $\cos\beta$ are the direction cosines of the ray with respect to the x & y axes of the two dimensional crystal.

$$\cos\alpha = \frac{\Delta x}{\sqrt{(\Delta x)^2 + 4l^2}} = \sin\left(\tan^{-1}\frac{\Delta x}{2l}\right) = 0{\cdot}28735$$

so using $h = k = 2$ we get

$$a = \frac{40 \times 2}{\cdot 28735}\,\text{pm} = 0{\cdot}278\,\text{nm}$$

Similarly $$\cos\beta = \frac{\Delta y}{\sqrt{(\Delta y)^2 + 4l^2}} = \sin\left(\tan^{-1}\frac{\Delta y}{2l}\right) = 0{\cdot}19612$$

$$b = \frac{80}{\cos\beta}\,\text{p m} = 0{\cdot}408\,\text{nm}$$

5.153 Suppose α, β, and γ are the angles between the direction to the diffraction maximum and the directions of the array along the periods a, b, and c respectively (call them x, y, & z axes). Then the value of these angles can be found from the following familiar conditions

$$a(1 - \cos\alpha) = k_1 \lambda$$

$$b\cos\beta = k_2\lambda \quad \text{and} \quad c\cos\gamma = k_3\lambda$$

where k_1, k_2, k_3 are whole numbers (+, −, or 0)

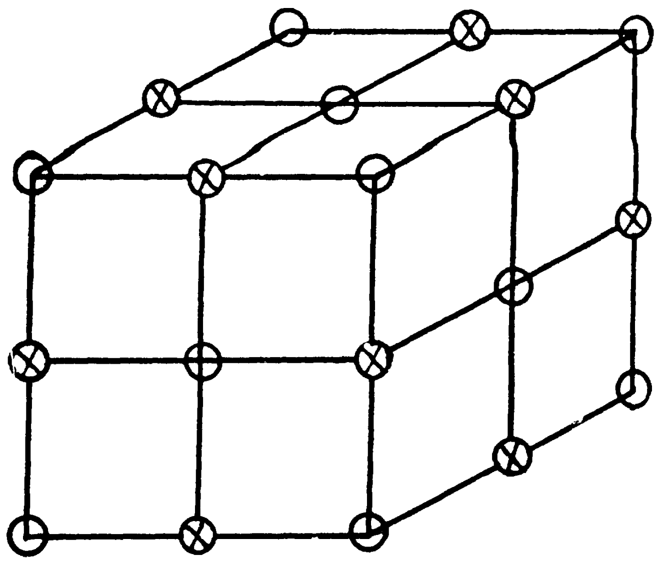
(These formulas are, in effect, Laue equations, see any text book on modern physics). Squaring and adding we get on using $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$2 - 2\cos\alpha = \left[\left(\frac{k_1}{a}\right)^2 + \left(\frac{k_2}{b}\right)^2 + \left(\frac{k_3}{c}\right)^2\right]\lambda^2 = \frac{2k_1\lambda}{a}$$

Thus $$\lambda = \frac{2k_1/a}{\left[(k_1/a)^2 + (k_2/a)^2 + (k_3/a)^2\right]}.$$

Knowing a, b, c and the integer k_1, k_2, k_3 we can find α, β, γ as well as λ.

5.154 The unit cell of $NaCl$ is shown below. In an infinite crystal, there are four Na^+ and four Cl^- ions per unit cell. (Each ion on the middle of the edge is shared by four unit cells; each ion on the face centre by two unit cells, the ion in the middle of the cell by one cell only and finally each ion on the corner by eight unit cells.) Thus



$$4\frac{M}{N_A} = \rho \cdot a^3$$

where M = molecular weight of NaCl in gms = 58·5 gms

N_A = Avogadro number = $6{\cdot}023 \times 10^{23}$

Thus $$\frac{1}{2}a = \sqrt[3]{\frac{M}{2N_A\rho}} = 2{\cdot}822\,\text{Å}$$

The natural facet of the crystal is one of the faces of the unit cell. The interplanar distance

$$d = \frac{1}{2}a = 2{\cdot}822\,\text{Å}$$

Thus $$2d\sin\alpha = 2\lambda$$

So $$\lambda = d\sin\alpha = 2{\cdot}822\ \text{Å} \times \frac{\sqrt{3}}{2} = 244\ \text{pm}.$$

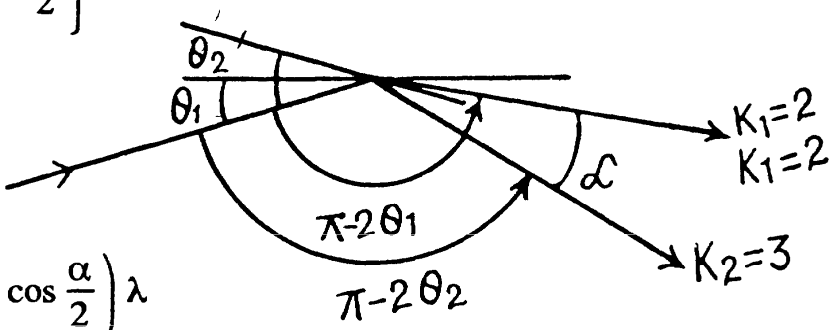
5.155 When the crystal is rotated, the incident monochromatic beam is diffracted from a given crystal plane of interplanar spacing d whenever in the course of rotation the value of θ satisfies the Bragg equation.

We have the equations $2d\sin\theta_1 = k_1\lambda$ and $2d\sin\theta_2 = k_2\lambda$

But $\pi - 2\theta_1 = \pi - 2\theta_2 + \alpha$ or $2\theta_1 = 2\theta_2 - \alpha$

so $$\theta_2 = \theta_1 + \frac{\alpha}{2}.$$

Thus $$2d\left\{\sin\theta_1\cos\frac{\alpha}{2} + \cos\theta_1\sin\frac{\alpha}{2}\right\} = k_2\lambda$$



Hence $$2d\sin\frac{\alpha}{2}\cos\theta_1 = \left(k_2 - k_1\cos\frac{\alpha}{2}\right)\lambda$$

also $$2d\sin\frac{\alpha}{2}\sin\theta_1 = k_1\lambda\sin\frac{\alpha}{2}$$

Squaring and adding $$2d\sin\frac{\alpha}{2} = \left(k_1^2 + k_2^2 - 2k_1k_2\cos\frac{\alpha}{2}\right)^{1/2}\lambda$$

Hence $$d = \frac{\lambda}{2\sin\frac{\alpha}{2}}\left[k_1^2 + k_2^2 - 2k_1k_2\cos\frac{\alpha}{2}\right]^{1/2}$$

Substituting $\alpha = 60°,\ k_1 = 2,\ k_2 = 3,\ \lambda = 174\ \text{pm}$

we get $$d = 281\ \text{pm} = 2{\cdot}81\ \text{Å}$$

(and not $0{\cdot}281$ p m as given in the book.)

(Lattice parameters are typically in Å$'s$ and not in fractions of a pm.)

5.156 In a polycrystalline specimen, microcrystals are oriented at various angles with respect to one another. The microcrystals which are oriented at certain special angles with respect to the incident beam produce diffraction maxima that appear as rings.

The radial of these rings are given by

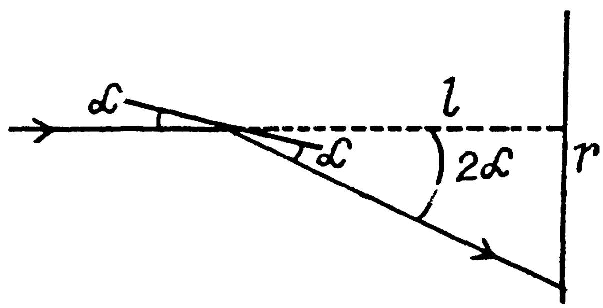
$$r = l\tan 2\alpha$$

where the Bragg's law gives

$$2d\sin\alpha = k\lambda$$

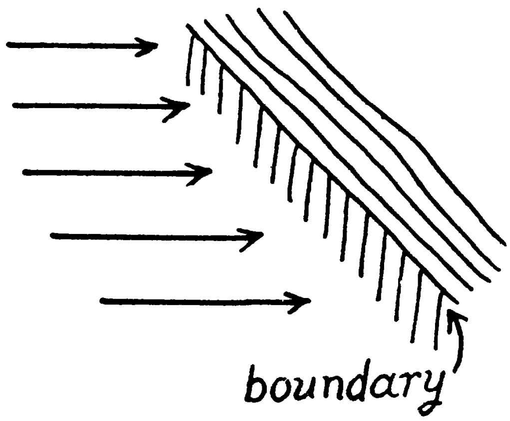
In our case $k = 2,\ d = 155\ \text{pm},\ \lambda = 17{\cdot}8\ \text{pm}$

so $\alpha = \sin^{-1}\dfrac{17{\cdot}8}{155} = 6{\cdot}6°$ and $r = 3{\cdot}52$ cm.



5.4 POLARIZATION OF LIGHT

5.157 Natural light can be considered to be an incoherent mixture of two plane polarized light of intensity $I_0 / 2$ with mutually perpendicular planes of vibration. The screen consisting of the two polaroid half-planes acts as an opaque half-screen for one or the other of these light waves. The resulting diffraction pattern has the alterations in intensity (in the illuminated region) characteristic of a straight edge on both sides of the boundary.

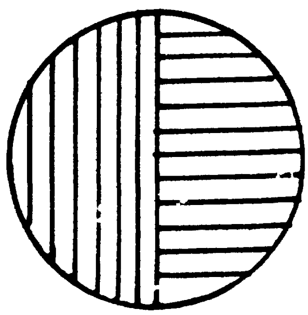


At the boundary the intensity due to either component is

$$\frac{(I_0 / 2)}{4}$$

and the total intensity is $\frac{I_0}{4}$. (Recall that when light of intensity I_0 is incident on a straight edge, the illuminance in front of the edge is $I_0 / 4$).

5.158 (a) Assume first that there is no polaroid and the amplitude due to the entire hole which extends over the first Fresnel zone is A_1



Then, we know, as usual, $I_0 = \frac{A_1^2}{4}$,

When the polaroid is introduced as shown above, each half transmits only the corresponding polarized light. If the full hole were covered by one polaroid the amplitude transmitted will be $(A_1 / \sqrt{2})$.

Therefore the amplitude transmitted in the present case will be $\frac{A_1}{2\sqrt{2}}$ through either half.

Since these transmitted waves are polarized in mutually perpendicular planes, the total intensity will be

$$\left(\frac{A_1}{2\sqrt{2}}\right)^2 + \left(\frac{A_1}{2\sqrt{2}}\right)^2 = \frac{A_1^2}{4} = I_0 .$$

(b) We interpret the problem to mean that the two polaroid pieces are separated along the circumference of the circle limiting the first half of the Fresnel zone. (This however is inconsistent with the polaroids being identical in shape; however no other interpretation makes sense.)

From (5.103) and the previous problems we see that the amplitudes of the waves transmitted through the two parts is

$$\frac{A_1}{2\sqrt{2}}(1+i) \text{ and } \frac{A_1}{2\sqrt{2}}(1-i)$$

and the intensity is

$$\left|\frac{A_1^2}{2\sqrt{2}}(1+i)\right|^2 + \left|\frac{A_1}{2\sqrt{2}}(1-i)\right|^2$$

$$= \frac{A_1^2}{2} = 2I_0$$

5.159 When the polarizer rotates with angular velocity ω its instantaneous principal direction makes angle ωt from a reference direction which we choose to be along the direction of vibration of the plane polarized incident light. The transmitted flux at this instant is

$$\Phi_0 \cos^2 \omega t$$

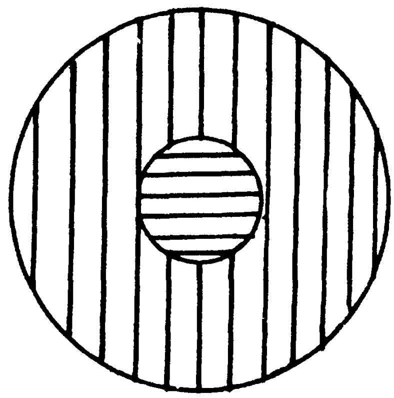
and the total energy passing through the polarizer per revolution is

$$\int_0^T \Phi_0 \cos^2 \omega t \, dt\,, \quad T = 2\pi/\omega$$

$$= \Phi_0 \frac{\pi}{\omega} = 0{\cdot}6 \text{ mJ}.$$

5.160 Let I_0 = intensity of the incident beam.
Then the intensity of the beam transmitted through the first Nicol prism is

$$I_1 = \frac{1}{2} I_0$$



and through the 2[nd] prism is

$$I_2 = \left(\frac{1}{2} I_0\right) \cos^2 \varphi$$

Through the N^{th} prism it will be

$$I_N = I_{N-1} \cos^2 \varphi$$

$$= \frac{1}{2} I_0 \cos^{2(N-1)} \varphi$$

Hence fraction transmitted

$$= \frac{I_N}{I_0} = \eta = \frac{1}{2} \cos^{(2N-1)} \varphi = 0{\cdot}12 \text{ for } N = 6.$$

and $\varphi .= 30°$

5.161 When natural light is incident on the first polaroid, the fraction transmitted will be $\frac{1}{2}\tau$ (only the component polarized parallel to the principal direction of the polaroid will go).

The emergent light will be plane polarized and on passing through the second polaroid will be polarized in a different direction (corresponding to the principal direction of the 2nd polaroid) and the intensity will have decreased further by $\tau \cos^2 \varphi$.

In the third polaroid the direction of polarization will again have to change by φ thus only a fraction $\tau \cos^2 \varphi$ will go through.

Finally $$I = I_0 \times \frac{1}{2} \tau^3 \cos^4 \varphi$$

Thus the intensity will have decreased

$$\frac{I_o}{I} = \frac{2}{\tau^3 \cos^4 \varphi} = 60{\cdot}2 \text{ times}$$

for $$\tau = 0{\cdot}81, \; \varphi = 60° .$$

5.162 Suppose the partially polarized light consists of natural light of intensity I_1 and plane polarized light of intensity I_2 with direction of vibration parallel to, say, x – axis.

Then when a polaroid is used to transmit it, the light transmitted will have a maximum intensity

$$\frac{1}{2} I_1 + I_2 ,$$

when the principal direction of the polaroid is parallel to x – axis, and will have a minimum intensity $\frac{1}{2} I_1$ when the principal direction is $\perp^r$ to x – axis.

Thus $$P = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{I_2}{I_1 + I_2}$$

so $$\frac{I_2}{I_1} = \frac{P}{1 - P} = \frac{0{\cdot}25}{0{\cdot}75} = \frac{1}{3} .$$

5.163 If, as above,

I_1 = intensity of natural component

I_2 = intensity of plane polarized component

then $$I_{\max} = \frac{1}{2} I_1 + I_2$$

and $$I = \frac{I_{\max}}{\eta} = \frac{1}{2} I_1 + I_2 \cos^2 \varphi$$

so $$I_2 = I_{\max} \left(1 - \frac{1}{\eta} \right) \operatorname{cosec}^2 \varphi$$

$$I_1 = 2 I_{\max} \left[1 - \left(1 - \frac{1}{\eta} \right) \operatorname{cosec}^2 \varphi \right] = \frac{2 I_{\max}}{\sin^2 \varphi} \left[\frac{1}{\eta} - \cos^2 \varphi \right]$$

Then $$P = \frac{I_2}{I_1 + I_2} = \frac{1 - \frac{1}{\eta}}{2 \left(\frac{1}{\eta} - \cos^2 \varphi \right) + 1 - \frac{1}{\eta}} = \frac{\eta - 1}{1 - \eta \cos 2 \varphi}$$

On putting $\eta = 3{\cdot}0,\ \varphi = 60°$

we get $$P = \frac{2}{1 + 3 \times \frac{1}{2}} = \frac{4}{5} = 0{\cdot}8$$

5.164 Let us represent the natural light as a sum of two mutually perpendicular components, both with intensity I_0 . Suppose that each polarizer transmits a fraction α_1 of the light with oscillation plane parallel to the prinicipal direction of the polarizer and a fraction α_2 with oscillation plane perpendicular to the principal direction of the polaizer. Then the intensity of light transmitted through the two polarizers is equal to

$$I_{\parallel} = \alpha_1^2 I_0 + \alpha_2^2 I_0$$

when their principal direction are parallel and

$$I_{\perp} = \alpha_1 \alpha_2 I_0 + \alpha_2 \alpha_1 I_0 = 2\,\alpha_1 \alpha_2 I_0$$

when they are crossed. But

$$\frac{I_{\perp}}{I_{\parallel}} = \frac{2\,\alpha_1 \alpha_2}{\alpha_1^2 + \alpha_2^2} = \frac{1}{\eta}$$

so $$\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} = \sqrt{\frac{\eta - 1}{\eta + 1}}$$

(a) Now the degree of polarization produced by either polarizer when used singly is

$$P_0 = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}$$

(assuming, of course, $\alpha_1 > \alpha_2$)

Thus $$P_0 = \sqrt{\frac{\eta - 1}{\eta + 1}} = \sqrt{\frac{9}{11}} = 0{\cdot}905$$

(b) When both polarizer are used with their principal directions parallel, the transmitted light, when analysed, has

maximum intensity, $I_{max} = \alpha_1^2 I_0$ and minimum intensity, $I_{min} = \alpha_2^2 I_0$

so $$P = \frac{\alpha_1^2 - \alpha_2^2}{\alpha_1^2 + \alpha_2^2} = \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} \cdot \frac{(\alpha_1 + \alpha_2)^2}{\alpha_1^2 + \alpha_2^2}$$

$$= \sqrt{\frac{\eta - 1}{\eta + 1}} \cdot \left(1 + \frac{2\,\alpha_1 \alpha_2}{\alpha_1^2 + \alpha_2^2}\right)$$

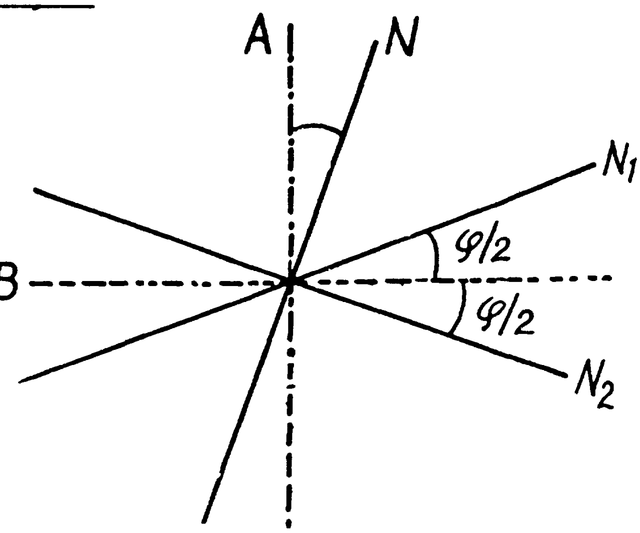
$$= \sqrt{\frac{\eta - 1}{\eta + 1}} \left(1 + \frac{1}{\eta}\right) = \frac{\sqrt{\eta^2 - 1}}{\eta} = \sqrt{1 - \frac{1}{\eta^2}} = 0.995.$$

5.165 If the principal direction N of the Nicol is along A or B, the intensity of light transmitted is the same whether the light incident is one with oscillation plane N_1 or one with N_2. If N makes an angle $\delta\varphi$ with A as shown then the fractional difference in intensity transmitted (when the light incident is N_1 or N_2) is

$$\left(\frac{\Delta I}{I}\right)_A = \frac{\cos^2\left(90° - \frac{\varphi}{2} - \delta\varphi\right) - \cos^2\left(90° + \frac{\varphi}{2} - \delta\varphi\right)}{\cos^2\left(90° - \frac{\varphi}{2}\right)}$$

$$= \frac{\sin^2\left(\frac{\varphi}{2} + \delta\varphi\right) - \sin^2\left(\frac{\varphi}{2} - \delta\varphi\right)}{\sin^2\frac{\varphi}{2}}$$

$$= \frac{2\sin\frac{\varphi}{2}\cdot 2\cos\frac{\varphi}{2}\,\delta\varphi}{\sin^2\frac{\varphi}{2}} = 4\cot\frac{\varphi}{2}\,\delta\varphi$$



If N makes an angle $\delta\varphi$ ($<<\varphi$) with B then

$$\left(\frac{\Delta I}{I}\right)_B = \frac{\cos^2(\varphi/2 - \delta\varphi) - \cos^2(\varphi/2 + \delta\varphi)}{\cos^2\varphi/2} = \frac{2\cos\frac{\varphi}{2}\cdot 2\sin\varphi/2\,\delta\varphi}{\cos^2\varphi/2} = 4\tan\varphi/2\,\delta\varphi$$

Thus
$$\eta = \left(\frac{\Delta I}{I}\right)_A \Big/ \left(\frac{\Delta I}{I}\right)_B = \cot^2\varphi/2$$

or
$$\varphi = 2\tan^{-1}\frac{1}{\sqrt{\eta}}$$

This gives $\varphi = 11{\cdot}4°$ for $\eta = 100$.

5.166 Fresnel equations read

$$I'_\perp = I_\perp\frac{\sin^2(\theta_1 - \theta_2)}{\sin^2(\theta_1 + \theta_2)} \text{ and } I'_{||} = I_{||}\frac{\tan^2(\theta_1 - \theta_2)}{\tan^2(\theta_1 + \theta_2)}$$

At the boundary between vacuum and a dielectric $\theta_1 \neq \theta_2$ since by Snell's law

$$\sin\theta_1 = n\sin\theta_2$$

Thus $I'_\perp / I_\perp$ cannot be zero. However, if $\theta_1 + \theta_2 = 90°$, $I'_{||} = 0$ and the reflected light is polarized in this case. The condition for this is

$$\sin\theta_1 = n\sin\theta_2, \quad = n\sin(90° - \theta_1)$$

or
$$\tan\theta_1 = n \quad \theta_1 \text{ is called Brewsta's angle.}$$

The angle between reflected light and refracted light is 90° in this case.

5.167 (a) From Fresnel's equations

$$\left.\begin{aligned} I'_\perp &= I_\perp \frac{\sin^2(\theta_1-\theta_2)}{\sin^2(\theta_1+\theta_2)} \\ I'_{||} &= 0 \end{aligned}\right\} \text{at Brewste's angle}$$

$$I'_\perp = I_\perp \sin^2(\theta_1-\theta_2)$$

$$= \frac{1}{2} I (\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)^2$$

Now $\tan\theta_1 = n, \ \sin\theta_1 = \dfrac{n}{\sqrt{n^2+1}}$

$$\cos\theta_1 = \frac{1}{\sqrt{n^2+1}}, \ \sin\theta_2 = \cos\theta_1$$

$$\cos\theta_2 = \sin\theta_1$$

$$I'_\perp = \frac{1}{2} I \left(\frac{n^2-1}{n^2+1}\right)^2$$

Thus reflection coefficient $= \rho = \dfrac{I'_\perp}{I}$

$$= \frac{1}{2}\left(\frac{n^2-1}{n^2+1}\right)^2 = 0{\cdot}074$$

on putting $n = 1{\cdot}5$

(b) For the refracted light

$$I''_\perp = I_\perp - I'_\perp = \frac{1}{2} I \left\{1 - \left(\frac{n^2-1}{n^2+1}\right)^2\right\}$$

$$= \frac{1}{2} I \frac{4n^2}{(n^2+1)^2}$$

$$I'_{||} = \frac{1}{2} I$$

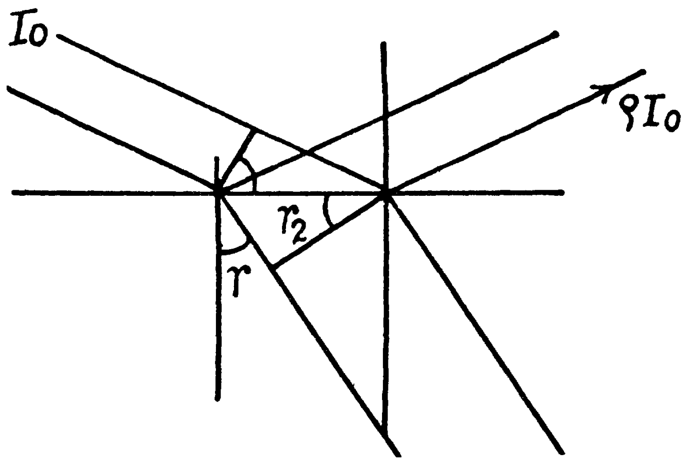
at the Brewster's angle.

Thus the degree of polarization of the refracted light is

$$P = \frac{I''_{||} - I''_\perp}{I''_{||} + I''_\perp} = \frac{(n^2+1)^2 - 4n^2}{(n^2+1)^2 + 4n^2}$$

$$= \frac{(n^2-1)^2}{2(n^2+1)^2 - (n^2-1)^2} = \frac{\rho}{1-\rho}$$

On putting $\rho = 0{\cdot}074$ we get $P = 0{\cdot}080$.



5.168 The energy transmitted is, by conservation of energy, the difference between incident energy and the reflected energy. However the intensity is affected by the change of the cross section of the beam by refraction. Let A_i, A_r, A_t be the cross sections of the incident, reflected and transmitted beams. Then

$$A_i = A_r$$

$$A_t = A_i \frac{\cos r}{\cos i}$$

But at Brewster's angle $r = 90 - i$

so $$A_t = A_i \tan i = n A_i$$

Thus $$I_t = \frac{(1-\rho) I_0}{n}$$

5.169 The amplitude of the incident component whose oscillation vector is perpendicular to the plane of incidence is

$$A_\perp = A_0 \sin \varphi$$

and similarly $$A_{||} = A_0 \cos \varphi$$

Then $$I'_\perp = I_0 \frac{\sin^2(\theta_1 - \theta_2)}{\sin^2(\theta_1 + \theta_2)} \sin^2 \varphi$$

$$= I_0 \left[\frac{\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2}{\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2} \right]^2 \sin^2 \varphi$$

$$= I_0 \left[\frac{n^2 - 1}{n^2 + 1} \right]^2 \sin^2 \varphi$$

Hence $$\rho = \frac{I'_\perp}{I_0} = \left[\frac{n^2 - 1}{n^2 + 1} \right]^2 \sin^2 \varphi$$

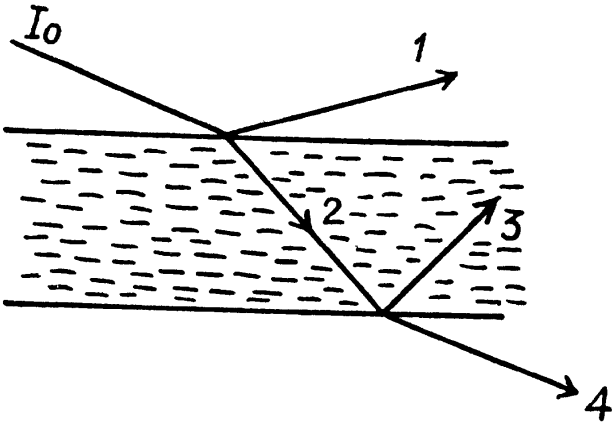
Putting $n = 1{\cdot}33$ for water we get $\rho = 0{\cdot}0386$

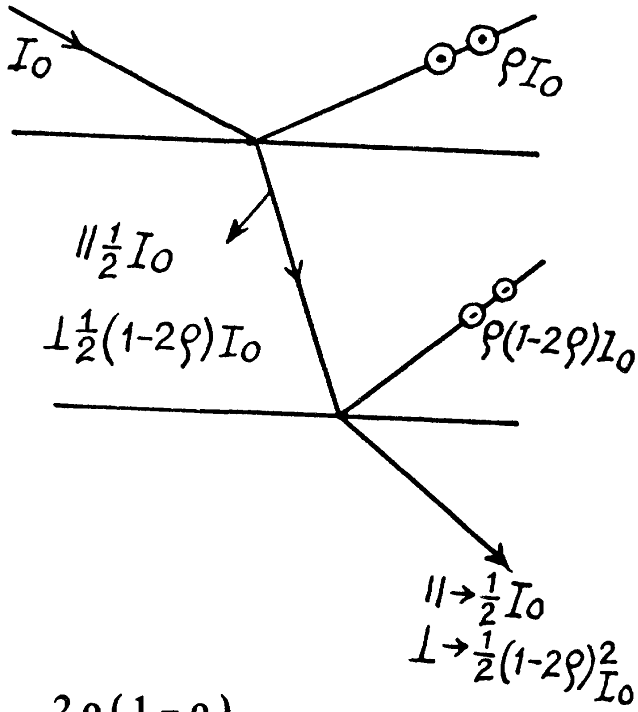
5.170 Since natural light is incident at the Brewster's angle, the reflected light 1 is completely polarized and $P_1 = 1$.

Similarly the ray 2 is incident on glass air surface at Brewster's angle $\left(\tan^{-1} \frac{1}{n} \right)$ sc 3 is also completely polarized. Thus $P_3 = 1$

Now as in 5.167 (b)

$$P_2 = \frac{\rho}{1 - \rho} = 0{\cdot}087 \text{ if } \rho = 0{\cdot}080$$





Finally as shown in the figure

$$P_4 = \frac{\frac{1}{2} - \frac{1}{2}(1 - 2\rho)^2}{\frac{1}{2} + \frac{1}{2}(1 - 2\rho)^2} = \frac{2\rho(1-\rho)}{1 - 2\rho(1-\rho)} = 0{\cdot}173$$

5.171 (a) In this case from Fresnel's equations

$$I'_\perp = I_\perp \frac{\sin^2(\theta_1 - \theta_2)}{\sin^2(\theta_1 + \theta_2)}$$

we get
$$I_1 = \left(\frac{n^2 - 1}{n^2 + 1}\right)^2 I_0 = \rho I_0 \text{ say}$$

then
$$I_2 = (1 - \rho) I_0, \; I_3 = \rho(1 - \rho) I_0$$

(ρ is invariant under the substitution $n \rightarrow \frac{1}{n}$)

finally
$$I_4 = (1 - \rho)^2 I_0 = \frac{16 n^4}{(n^2 + 1)^4} I_0 = 0{\cdot}726 \, I_0 .$$

(b) Suppose ρ' = coefficient of reflection for the component of light whose electric vector oscillates at right angles to the incidence plane.

From Fresnel's equations
$$\rho' = \left(\frac{n^2 - 1}{n^2 + 1}\right)^2$$

Then in the transmitted beam we have a partially polarized beam which is a superposition of two ($\|$ & $\perp$) components with intensities

$$\frac{1}{2} I_0 \;\&\; \frac{1}{2} I_0 (1 - \rho')^2$$

Thus
$$P = \frac{1 - (1 - \rho')^2}{1 + (1 - \rho')^2} = \frac{(n^2 + 1)^4 - 16 n^4}{(n^2 + 1)^4 + 16 n^4} = \frac{1 - 0{\cdot}726}{1 + 0{\cdot}726} \simeq 0{\cdot}158$$

5.172 (a) When natural light is incident on a glass plate at Brewster's angle, the transmitted light has

$$I''_{||} = I_0/2 \text{ and } I''_{\perp} = \frac{16\,n^4}{(n^2+1)^4} I_0/2 = \alpha^4 I_0/2$$

where I_0 is the incident intensity (see 5.171 a)

After passing through the 2^{nd} plate we find

$$I_{||}'''' = \frac{1}{2} I_0 \quad \text{and} \quad I_{|}'''' = (\alpha^4)^2 \frac{1}{2} I_0$$

Thus after N plates
$$I_{||}^{trans} = \frac{1}{2} I_0$$
$$I_{\perp}^{trans} = \alpha^{4N} \frac{1}{2} I_0$$

Hence
$$P = \frac{1-\alpha^{4N}}{1+\alpha^{4N}} \quad \text{where} \quad \alpha = \frac{2n}{1+n^2}$$

(b) $\alpha^4 = 0{\cdot}726$ for $n = \frac{3}{2}$.

Thus
$$P(N=1) = 0{\cdot}158,\ P(N=2) = 0{\cdot}310$$
$$P(N=5) = 0{\cdot}663,\ P(N=10) = 0{\cdot}922.$$

5.173 (a) We decompose the natural light into two components with intensity $I_{||} = \frac{1}{2} I_0 = I_{\perp}$ where $||$ has its electric vector oscillating parallel to the plane of incidence and $\perp$ has the same $\perp^r$ to it.

By Fresnel's equations for normal incidence

$$\frac{I'_{\perp}}{I_{\perp}} = \lim_{\theta_1 \to 0} \frac{\sin^2(\theta_1-\theta_2)}{\sin^2(\theta_1+\theta_2)} = \lim_{\theta_1 \to 0}\left(\frac{\theta_1-\theta_2}{\theta_1+\theta_2}\right)^2 = \left(\frac{n-1}{n+1}\right)^2 = \rho$$

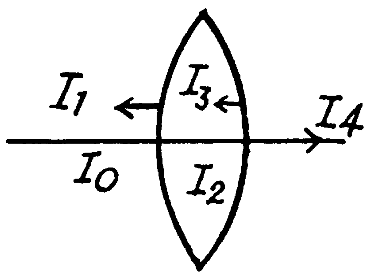
similarly
$$\frac{I'_{||}}{I_{||}} = \rho = \left(\frac{n-1}{n+1}\right)^2$$

Thus
$$\frac{I'}{I} = \rho = \left(\frac{0{\cdot}5}{2{\cdot}5}\right)^2 = \frac{1}{25} = 0{\cdot}04$$

(b) The reflected light at the first surface has the intensity $I_1 = \rho I_0$

Then the transmitted light has the intensity $I_2 = (1-\rho) I_0$

At the second surface where light emerges from glass into air, the reflection coefficient is again ρ because ρ is invariant under the substitution $n \to \frac{1}{n}$.

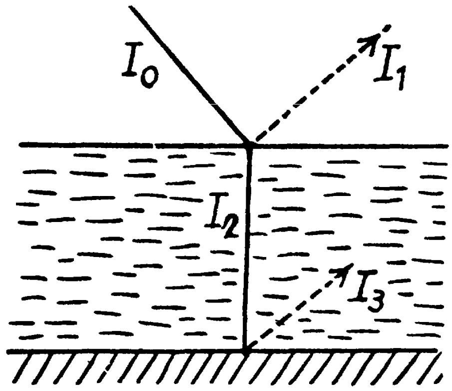


Thus $I_3 = \rho(1-\rho) I_0$ and $I_4 = (1-\rho)^2 I_0$.

For N lenses the loss in luminous flux is then

$$\frac{\Delta \Phi}{\Phi} = 1-(1-\rho)^{2N} = 0.335 \text{ for } N = 5$$

5.174 Suppose the incident light can be decomposed into waves with intensity $I_{||}$ & $I_{\perp}$ with oscillations of the electric vectors parallel and perpendicular to the plane of incidence.



For normal incidence we have from Fresnel equations

$$I_{\perp}' = I_{\perp}\left(\frac{\theta_1-\theta_2}{\theta_1+\theta_2}\right)^2 \longrightarrow I_{\perp}\left(\frac{n-1}{n+1}\right)^2$$

where we have used $\sin\theta \approx \theta$ for small θ.

Similarly $$I'_{||} = I_{||}\left(\frac{n'-1}{n'+1}\right)^2$$

Then the refracted wave will be

$$I''_{||} = I_{||}\frac{4n'}{(n'+1)^2} \text{ and } I_{\perp}'' = I_{\perp}\frac{4n'}{(n'+1)^2}$$

At the interface with glass

$$I_{\perp}''' = I_{\perp}''\left(\frac{n'-n}{n'+n}\right)^2, \text{ similarly for } I_{||}'''$$

we see that

$$\frac{I_{\perp}'}{I_{\perp}} = \frac{I_{\perp}'''}{I_{\perp}''} \quad \text{if} \quad n' = \sqrt{n}, \text{ similarly for } || \text{ component.}$$

This shows that the light reflected as a fraction of the incident light is the same on the two surfaces if $n' = \sqrt{n}$.

Note:- The statement of the problem given in the book is incorrect. Actual amplitudes are not equal; only the reflectance is equal.

5.175 Here $\theta_1 = 45°$

$$\sin\theta_2 = \frac{1}{\sqrt{2}}\times\frac{1}{n} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3} = 0{\cdot}4714$$

$$\theta_2 = \sin^{-1} 0{\cdot}4714 = 28{\cdot}1°$$

Hence $$I'_{\perp} = I_{\perp}\frac{\sin^2(\theta_1-\theta_2)}{\sin^2(\theta_1+\theta_2)}$$

$$= \frac{1}{2}I_0\left(\frac{\sin 16{\cdot}9°}{\sin 73{\cdot}1°}\right)^2 = \frac{1}{2}I_0\times 0{\cdot}0923$$

$$I'_{||} = \frac{1}{2}I_0\left(\frac{\tan 16{\cdot}9}{\tan 73{\cdot}1}\right)^2 = \frac{1}{2}I_0 \times 0{\cdot}0085$$

Thus

(a) Degree of polarization P of the reflected light

$$= \frac{0{\cdot}0838}{0{\cdot}1008} = 0{\cdot}831$$

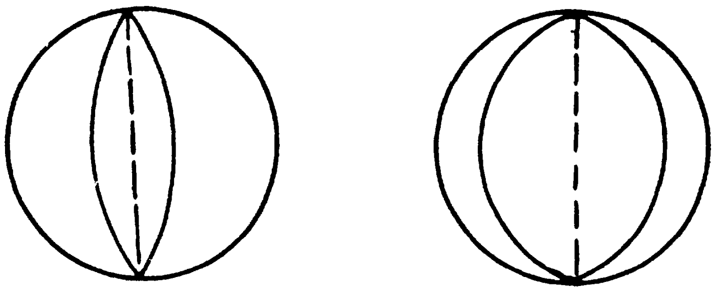
(b) By conservation of energy

$$I''_{\perp} = \frac{1}{2}I_0 \times 0{\cdot}9077$$

$$I''_{||} = \frac{1}{2}I_0 \times 0{\cdot}9915$$

Thus $$P = \frac{0{\cdot}0838}{1{\cdot}8982} = 0{\cdot}044$$

5.176 The wave surface of a uniaxial crystal consists of two sheets of which one is a sphere while the other is an ellipsoid of revolution.

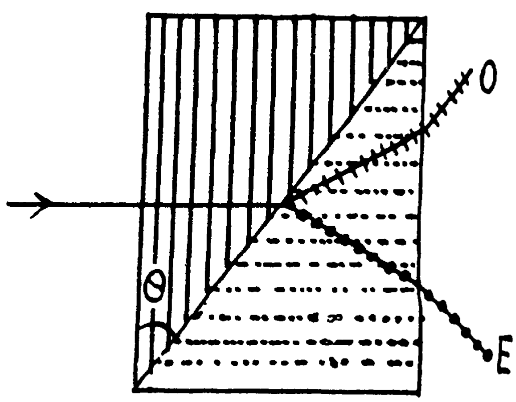


The optic axis is the line joining the points of contact.

To makes the appropriate Huyghen's construction we must draw the relevant section of the wave surface inside the crystal and determine the directions of the ordinary and extraordinary rays. The result is as shown in Fig. 42 (a, b & c) of the answers

5.177 In a uniaxial crystal, an unpolarized beam of light (or even a polarized one) splits up into O (for ordinary) and E (for extraordinary) light waves. The direction of vibration in the O and E waves are most easily specified in terms of the O and E principal planes. The principal plane of the ordinary wave is defined as the plane containing the O ray and the optic axis. Similarly the principal plane of the E wave is the plane containing the E ray and the optic axis. In terms of these planes the following is true : The O vibrations are perpendicular to the principal plane of the O ray while the E vibrations are in the principal plane of the E ray.

When we apply this definition to the wollaston prism we find the following :



(exaggerated.)

When unpolarized light enters from the left the O and E waves travel in the same direction but with different speeds. The O ray on the left has its vibrations normal to the plane of the paper and it becomes E ray on crossing the diagonal boundary of the two prism similarly the E ray on the left becomes O ray on the right. In this case Snell's law is applicable only approximately. The two rays are incident on the boundary at an angle θ and in the right prism the ray which we have called O ray on the right emerges at

$$\sin^{-1}\frac{n_e}{n_0}\sin\theta = \sin^{-1}\frac{1{\cdot}658}{1{\cdot}486}\times\frac{1}{2} = 33{\cdot}91°$$

where we have used

$$n_e = 1{\cdot}1658\,,\ n_0 = 1{\cdot}486 \text{ and } \theta = 30°\,.$$

Similarly the E ray on the right emerges within the prism at

$$\sin^{-1}\frac{n_0}{n_e}\sin\theta = 26{\cdot}62°$$

This means that the O ray is incident at the boundary between the prism and air at

$$33{\cdot}91 - 30° = 3{\cdot}91°$$

and will emerge into air with a deviation of

$$\sin^{-1} n_0 \sin 3{\cdot}91°$$

$$= \sin^{-1}(1{\cdot}658 \sin 3{\cdot}91°) = 6{\cdot}49°$$

The E ray will emerge with an opposite deviation of

$$\sin^{-1}(n_e \sin(30° - 26{\cdot}62°))$$

$$= \sin^{-1}(1{\cdot}486 \sin 3{\cdot}38°) = 5{\cdot}03°$$

Hence $\qquad \delta \approx 6{\cdot}49° + 5{\cdot}03° = 11{\cdot}52°$

This result is accurate to first order in $(n_e - n_0)$ because Snell's law holds when $n_e = n_0$.

5.178 The wave is moving in the direction of $z-$ axis

(a) Here $E_x = E\cos(\omega t - kz)$, $E_y = E\sin(\omega t - kz)$

$$\frac{E_x^2}{E^2} + \frac{E_y^2}{E^2} = 1$$

so the tip of the electric vector moves along a circle. For the right handed coordinate system this represents circular anticlockwise polarization when observed towards the in-coming wave.

(b) $E_x = E\cos(\omega t - kz)$, $E_y = E\cos\left(\omega t - kz + \frac{\pi}{4}\right)$

so $$\frac{E_y}{E} = \frac{1}{\sqrt{2}}\cos(\omega t - kz) - \frac{1}{\sqrt{2}}\sin(\omega t - kz)$$

or $$\left(\frac{E_y}{E} - \frac{1}{\sqrt{2}}\frac{E_x}{E}\right)^2 = \frac{1}{2}\left(1 - \frac{E_x^2}{E^2}\right)$$

or $$\frac{E_y^2}{E^2} + \frac{E_x^2}{E^2} - \sqrt{2}\,\frac{E_y E_x}{E^2} = \frac{1}{2}$$

This is clearly an ellipse. By comparing with the previous case (compare the phase of E_y in the two cases) we see this represents elliptical clockwise polarization when viewed towards the incoming wave.

We write the equations as

$$E_x + E_y = 2E\cos\left(\omega t - kz + \frac{\pi}{8}\right)\cos\frac{\pi}{8}$$

$$E_x - E_y = +2E\sin\left(\omega t - kz + \frac{\pi}{8}\right)\sin\frac{\pi}{8}$$

Thus

$$\left(\frac{E_x + E_y}{2E\cos\frac{\pi}{8}}\right)^2 + \left(\frac{E_x - E_y}{2E\sin\frac{\pi}{8}}\right)^2 = 1$$

Since $\cos\frac{\pi}{8} > \sin\frac{\pi}{8}$, the major axis is in the direction of the straight line $y = x$.

(c) $E_x = E\cos(\omega t - kz)$

$E_y = E\cos(\omega t - kz + \pi) = -E\cos(\omega t - kz)$

Thus the top of the electric vector traces the curve

$$E_y = -E_x$$

which is a straight line ($y = -x$). It corresponds to plane polarization.

5.179 For quartz

$$\left.\begin{aligned} n_e &= 1{\cdot}553 \\ n_0 &= 1{\cdot}544 \end{aligned}\right\} \text{ for } \lambda = 589\text{ n m}.$$

In a quartz plate cut parallel to its optic axis, plane polarized light incident normally from the left divides itself into O and E waves which move in the same direction with different speeds and as a result acquire a phase difference. This phase difference is

$$\delta = \frac{2\pi}{\lambda}(n_e - n_0)\,d$$

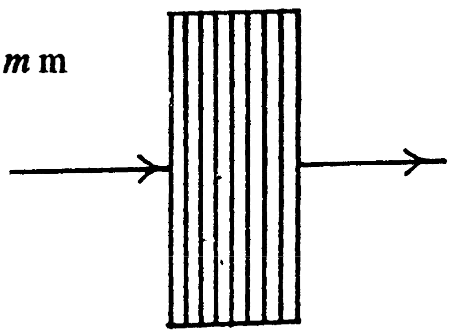
where d = thickness of the plate. In general this makes the emergent light elliptically polarized.

(a) For emergent light to experience only rotation of polarization plane

$\delta = (2k+1)\pi,\ k = 0, 1, 2, 3 \ldots$

For this $d = (2k+1)\dfrac{\lambda}{2(n_e - n_0)}$.

$$= (2k+1)\frac{{\cdot}589}{2\times{\cdot}009}\,\mu\text{m} = (2k+1)\frac{{\cdot}589}{18}\,m\text{m}$$



The maximum value of $(2k+1)$ for which this is less than 0·50 is obtained from

$$\frac{0{\cdot}50\times 18}{0{\cdot}589} = 15{\cdot}28$$

Then we must take $k = 7$ and $d = 15\times\dfrac{{\cdot}589}{18} = 0{\cdot}4908$ mm

(b) For circular polarization $\delta = \frac{\pi}{2}$

modulo 2π i.e. $\delta = (4k+1)\frac{\pi}{2}$

so $$d = (4k+1)\frac{\lambda}{4(n_e - n_0)} = (4k+1)\frac{0{\cdot}589}{36}$$

Now $$\frac{0{\cdot}50 \times 36}{0{\cdot}589} = 30{\cdot}56$$

The nearest integer less than this which is of the form $4k+1$ is 29 for $k = 7$. For this

$$d = 0{\cdot}4749 \text{ mm}$$

5.180 As in the previous problem the quartz plate introduces a phase difference δ between the O & E components. When $\delta = \pi/2$ (modulo π) the resultant wave is circularly polarized. In this case intensity is independent of the rotation of the rear prism. Now

$$\delta = \frac{2\pi}{\lambda}(n_e - n_0)\,d$$

$$= \frac{2\pi}{\lambda}\,0{\cdot}009 \times 0{\cdot}5 \times 10^{-3}\text{ m}$$

$$= \frac{9\pi}{\lambda},\ \lambda \text{ in } \mu\text{m}$$

For $\lambda = 0{\cdot}50\ \mu m$. $\delta = 18\pi$. The relevant values of δ have to be chosen in the form

$$\left(k + \frac{1}{2}\right)\pi. \text{ For } k = 17, 16, 15 \text{ we get}$$

$$\lambda = 0{\cdot}5143\ \mu m,\ 0{\cdot}5435\ \mu\text{m} \text{ and } 0{\cdot}5806\ \mu\text{m}$$

These are the values of λ which lie between $0{\cdot}50\ \mu$m and $0{\cdot}60\ \mu$m.

5.181 As in the previous two problems the quartz plate will introduce a phase difference δ. The light on passing through the plate will remain plane polarized only for $\delta = 2k\pi$ or $(2k+1)\pi$. In the latter case the plane of polarization of the light incident on the plate will be rotated by 90° by it so light passing through the analyser (which was originally crossed) will be a maximum. Thus dark bands will be observed only for those λ for which

$$\delta = 2k\pi$$

Now $$\delta = \frac{2\pi}{\lambda}(n_e - n_0)\,d \doteq \frac{2\pi}{\lambda} \times {\cdot}009 \times 1{\cdot}5 \times 10^{-3}\text{ m}$$

$$= \frac{27\pi}{\lambda}(\lambda \text{ in } \mu\text{m})$$

For $$\lambda = 0{\cdot}55 \text{ we get } \delta = 49{\cdot}09\,\pi$$

Choosing $\delta = 48\pi, 46\pi, 44\pi, 42\pi$ we get $\lambda = 0{\cdot}5625\ \mu$m, $\lambda = 0{\cdot}5870\ \mu$m, $\lambda = 0{\cdot}6136\ \mu$m and $\lambda = 0{\cdot}6429\ \mu$m. These are the only values between $0{\cdot}55\ \mu$m and $0{\cdot}66\ \mu$m. Thus there are four bands.

5.182 Here

$$\delta = \frac{2\pi}{\lambda} \times 0{\cdot}009 \times 0{\cdot}25 \text{ m}$$

$$= \frac{4{\cdot}5\,\pi}{\lambda},\ \lambda \text{ in } \mu\text{m}.$$

We check that for $\lambda = 428{\cdot}6$ nm $\delta = 10{\cdot}5\,\pi$

$\lambda = 529{\cdot}4$ nm $\delta = 8{\cdot}5\,\pi$

$\lambda = 692{\cdot}3$ nm $\delta = 6{\cdot}5\,\pi$

These are the only values of λ for which the plate acts as a quarter wave plate.

5.183 Between crossed Nicols, a quartz plate, whose optic axis makes 45° with the principal directions of the Nicols, must introduce a phase difference of $(2k+1)\pi$ so as to transmit the incident light (of that wavelength) with maximum intensity. For in this case the plane of polarization of the light emerging from the polarizer will be rotated by 90° and will go through the analyser undiminished. Thus we write for light of wavelengths 643 nm

$$\delta = \frac{2\pi \times 0{\cdot}009}{0{\cdot}643 \times 10^{-6}} \times d\,(\text{mm}) \times 10^{-3}$$

$$= \frac{18\,\pi\,d}{0{\cdot}643} = (2k+1)\pi \tag{1}$$

To nearly block light of wavelength 564 nm we require

$$\frac{18\,\pi\,d}{0{\cdot}564} = (2k')\pi \tag{2}$$

We must have $2k' > 2k+1$. For the smallest value of d we take $2k' = 2k+2$.

Thus $0{\cdot}643\,(2k+1) = 0{\cdot}564 \times (2k+2)$

so $0{\cdot}079 \times 2k = 0{\cdot}564 \times 2 - 0{\cdot}643$

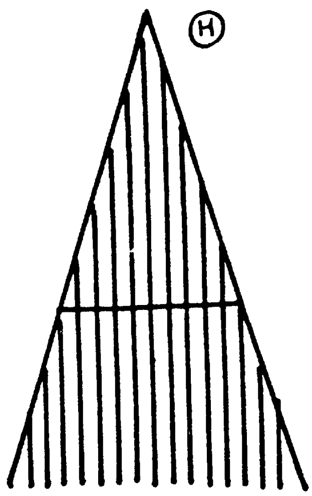
or $2k = 6{\cdot}139$

This is not quite an integer but is close to one. This means that if we take $2k = 6$ equations (1) can be satisfied exactly while equation (2) will hold approximately. Thus

$$d = \frac{7 \times 0{\cdot}643}{18} = 0{\cdot}250 \text{ mm}$$

5.184 If a ray traverses the wedge at a distance x below the joint, then the distance that the ray moves in the wedge is $2x \tan\frac{\Theta}{2}$ and this cause a phase difference

$$\delta = \frac{2\pi}{\lambda}(n_e - n_0)\, 2x \tan\frac{\Theta}{2}$$



between the E and O wave components of the ray. For a general x the resulting light is elliptically polarized and is not completely quenched by the analyser polaroid. The condition for complete quenching is

$$\delta = 2k\pi \text{— dark fringe}$$

That for maximum brightness is

$$\delta = (2k+1)\pi - \text{bright fringe.}$$

The fringe width is given by

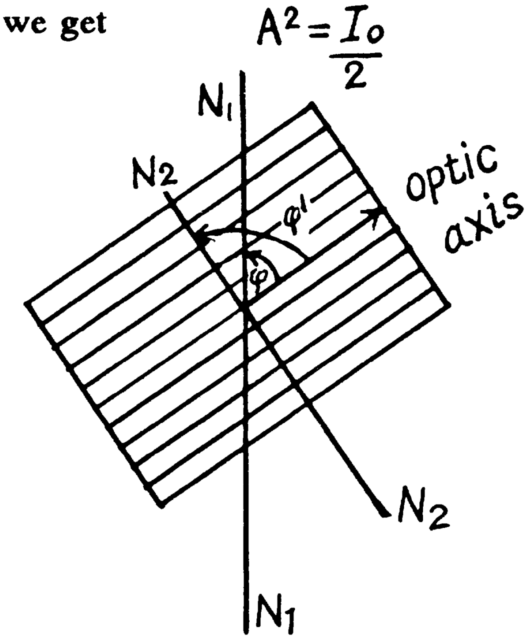
$$\Delta x = \frac{\lambda}{2(n_e - n_0)\tan\frac{\Theta}{2}}$$

Hence $$(n_e - n_0) = \frac{\lambda}{2\,\Delta x \tan \Theta/2}$$

using $$\tan(\Theta/2) = \tan 175° = 0{\cdot}03055\,,$$

$$\lambda = 0{\cdot}55\,\mu m \text{ and } \Delta x = 1\,mm\text{, we get}$$

$$n_e - n_0 = 9{\cdot}001 \times 10^{-3}$$



5.185 Light emerging from the first polaroid is plane polarized with amplitude A where N_1 is the principal direction of the polaroid and a vibration of amplitude can be resolved into two vibration : E wave with vibration along the optic axis of amplitude $A\cos\varphi$ and the O wave with vibration perpendicular to the optic axis and having an amplitude $A\sin\varphi$. These acquire a phase difference δ on passing through the plate. The second polaroid transmits the components :

$$A\cos\varphi\cos\varphi'$$

and $$A\sin\varphi\sin\varphi'$$

What emerges from the second polaroid is a set of two plane polarized waves in the same direction and same plane of polarization but phase difference δ. They interfere and produce a wave of amplitude squared

$$R^2 = A^2\left[\cos^2\varphi\cos^2\varphi' + \sin^2\varphi\sin^2\varphi' + 2\cos\varphi\cos\varphi'\sin\varphi\sin\varphi'\cos\delta\right],$$

using $$\cos^2(\varphi - \varphi') = (\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi')^2$$

$$= \cos^2\varphi\cos^2\varphi' + \sin^2\varphi\sin^2\varphi' + 2\cos\varphi\cos\varphi'\sin\varphi\sin\varphi'$$

we easily find

$$R^2 = A^2\left[\cos^2(\varphi - \varphi') - \sin 2\varphi\sin 2\varphi'\sin^2\frac{\delta}{2}\right]$$

Now $$A^2 = I_0/2 \text{ and } R^2 = I \text{ so the result is}$$

$$I = \frac{1}{2}I_0\left[\cos^2(\varphi - \varphi') - \sin 2\varphi\sin 2\varphi'\sin^2\frac{\delta}{2}\right]$$

Special cases :- Crossed polaroids : Here $\varphi - \varphi' = 90°$ or $\varphi' = \varphi - 90°$ and $2\varphi' = 2\varphi - 180°$

Thus in this case

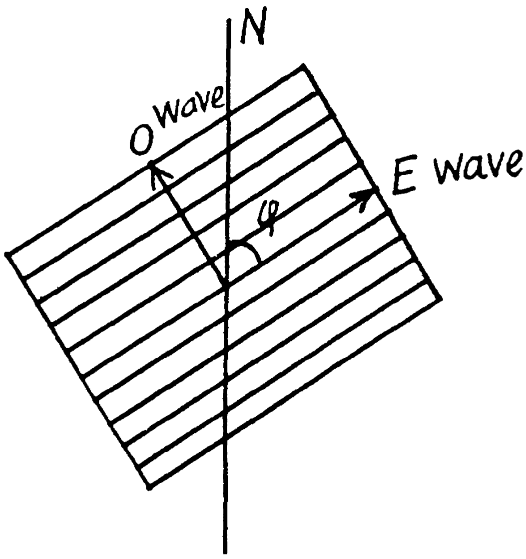
$$I = I_\perp = \frac{1}{2}I_0\sin^2 2\varphi\sin^2\frac{\delta}{2}$$

Parallel polaroids : Here $\varphi = \varphi'$ and

$$I = I_{||} = \frac{1}{2} I_0 \left(1 - \sin^2 2\varphi \sin^2 \frac{\delta}{2} \right)$$

With $\delta = \dfrac{2\pi}{\lambda}\Delta$, the conditions for the maximum and minimum are easily found to be that shown in the answer.

5.186 Let the circularly polarized light be resolved into plane polarized components of amplitude A_0 with a phase difference $\frac{\pi}{2}$ between then.



On passing through the crystal the phase difference becomes $\delta + \frac{\pi}{2}$ and the components of the E and O wave in the direction N are respectively
$A_0 \cos\varphi$ and $A_0 \sin\varphi$

They interfere to produce the amplitude squared

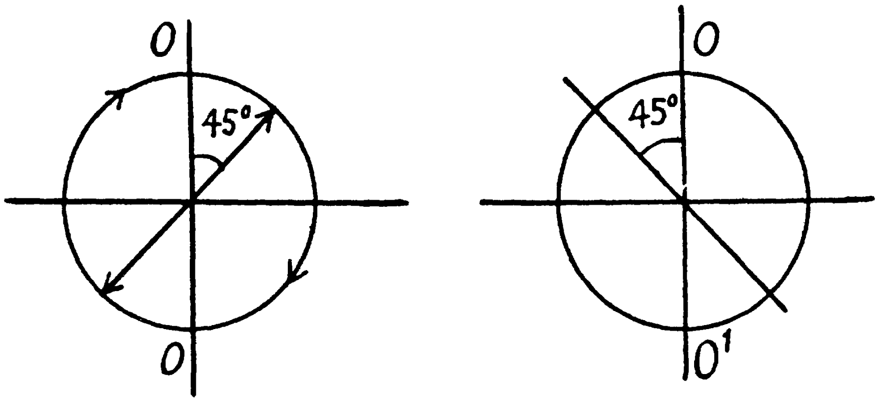
$$R^2 = A_0^2 \cos^2\varphi + A_0^2 \sin^2\varphi + 2A_0^2 \cos\varphi \sin\varphi \cos\left(\delta + \frac{\pi}{2}\right)$$

$$= A_0^2 (1 + \sin 2\varphi \sin\delta)$$

Hence $\qquad I = I_0 (1 + \sin 2\varphi \sin\delta)$

Here I_0 is the intensity of the light transmitted by the polaroid when there is no crystal plate.

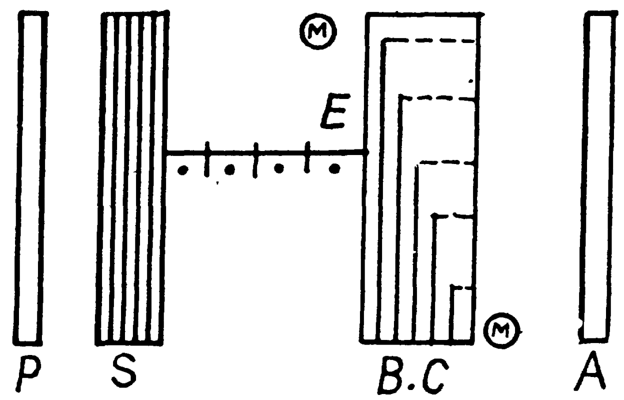
5.187 (a) The light with right circular polarization (viewed against the oncoming light, this means that the light vector is moving clock wise.) becomes plane polarized on passing through a quarter-wave plate. In this case the direction of oscillations of the electric vector of the electromagnetic wave forms an angle of + 45° with the axis of the crystal OO′ (see Fig (a) below). In the case of left hand circular polarizations, this angle will be − 45° (Fig (b)).



(b) If for any position of the plate the rotation of the polaroid (located behind the plate) does not bring about any variation in the intensity of the transmitted light, the incident light

is unpolarized (i.e. natural). If the intensity of the transmitted light can drop to zero on rotating the analyzer polaroid for some position of the quarter wave plate, the incident light is circularly polarized. If it varies but does not drop to zero, it must be a mixture of natural and circularly polarized light.

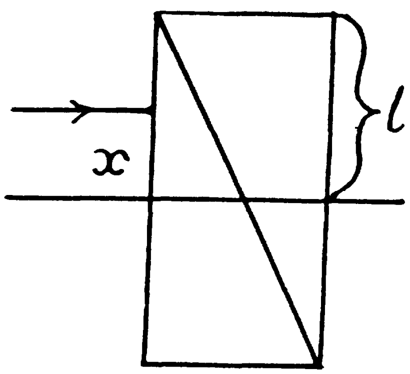
5.188 The light from P is plane polarized with its electric vector vibrating at 45° with the plane of the paper. At first the sample S is absent. Light from P can be resolved into components vibrating in and perpendicular to the plane of the paper. The former is the E ray in the left half of the Babinet compensator and the latter is the O ray. In the right half the nomenclature is the opposite. In the compensator the two components acquire a pahse difference which depends on the relative position of the ray. If the ray is incident at a distance x above the central line through the compensator then the E ray acquires a phase



$$\frac{2\pi}{\lambda}\left(n_E(l-x)+n_0(l+x)\right)\tan\Theta$$

while the O ray acquires

$$\frac{2\pi}{\lambda}\left(n_o(l-x)+n_E(l+x)\right)\tan\Theta$$



so the phase difference between the two reays is

$$\frac{2\pi}{\lambda}(n_o-n_E)\,2x\tan\Theta = \delta$$

we get dark fringes when ever $\delta = 2k\pi$

because then the emergent light is the same as that coming from the polarizer and is quenched by the analyser. {If $\delta = (2k+1)\pi$, we get bright fringes because in this case, the plane of polarizaton of the emergent hight has rotated by 90° and is therefore fully transmitted by the analyser.}

If follows that the fringe width Δx is given by

$$\Delta x = \frac{\lambda}{2\,|n_0-n_E|\tan\Theta}$$

(b) If the fringes are displaced upwards by δx, then the path difference introduced by the sample between the O and the E rays must be such as to be exactly cancelled by the compensator. Thus

$$\frac{2\pi}{\lambda}\left[d(n'_O-n'_E)+(n_E-n_O)\,2\,\delta x\tan\Theta\right] = 0$$

or $$d(n'_O-n'_E) = -2(n_E-n_O)\,\delta x\,\Theta$$

using $\tan\Theta \approx \Theta$.

5.189 Light polarized along the x-direction (i.e. one whose electric vector has only an x component) and propagating along the z-direction can be decomposed into left and right circularly polarized light in accordance with the formula

$$E_x = \frac{1}{2}(E_x + iE_y) + \frac{1}{2}(E_x - iE_y)$$

On passing through a distance l of an active medium these acquire the phases $\delta_R = \frac{2\pi}{\lambda} n_R l$ and $\delta_l = \frac{2\pi}{\lambda} n_l l$ so we get for the complex amplitude

$$E' = \frac{1}{2}(E_x + iE_y)e^{i\delta_R} + \frac{1}{2}(E_x - iE_y)e^{i\delta_L}$$

$$= e^{i\frac{\delta_R + \delta_l}{2}}\left[\frac{1}{2}(E_x + iE_y)e^{i\delta/2} + \frac{1}{2}(E_x - iE_y)e^{-i\delta/2}\right]$$

$$= e^{i\frac{\delta_R + \delta_l}{2}}\left[E_x \cos\frac{\delta}{2} - E_y \sin\frac{\delta}{2}\right], \quad \delta = \delta_R - \delta_L.$$

Apart from an over all phase $(\delta_R + \delta_l)/2$ (which is irrelevant) this represents a wave whose plane of polarization has rotated by

$$\frac{\delta}{2} = \frac{\pi}{\lambda}(\Delta n)l, \quad \Delta n = |n_R - n_l|$$

By definition this equals $\alpha\, l$ so

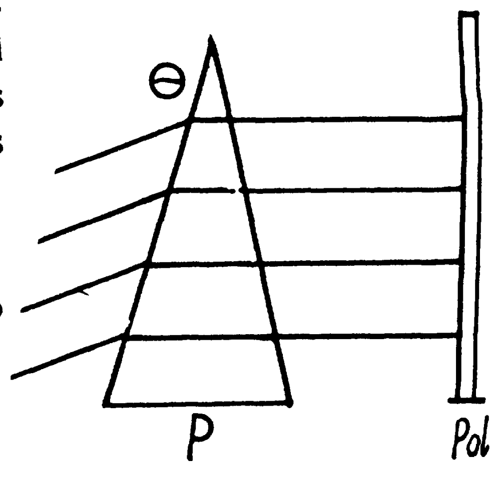
$$\Delta n = \frac{\alpha\lambda}{\pi}.$$

$$= \frac{589{\cdot}5 \times 10^{-6}\ \text{m m} \times 21{\cdot}72\ \text{deg/m m}}{\pi} \times \frac{\pi}{180}\ (\text{rad})$$

$$= \frac{{\cdot}5895 \times 21{\cdot}72}{180} \times 10^{-3}$$

$$= 0{\cdot}71 \times 10^{-4}$$

5.190 Plane polarized light on entering the wedge decomposes into right and left circularly polarized light which travel with different speeds in P and the emergent light gets its plane of polarization rotated by an angle which depends on the distance travelled.



Given that Δx = fringe width

$\Delta x \tan\theta$ = difference in the path length traversed by two rays which form successive bright or dark fringes.

Thus $\quad \frac{2\pi}{\lambda}|n_R - n_l|\Delta x \tan\theta = 2\pi$

Thus $\quad \alpha = \frac{\pi\Delta n}{\lambda} = \pi/\Delta x \tan\theta$

$= 20{\cdot}8$ ang deg/m m

Let x = distance on the polaroid Pol as measured from a maximum. Then a ray that falls at this distance traverses an extra distance equal to

$$\pm\, x \tan\theta$$

and hence a rotation of $$\pm\, \alpha x \tan\theta = \pm\, \frac{\pi x}{\Delta x}$$

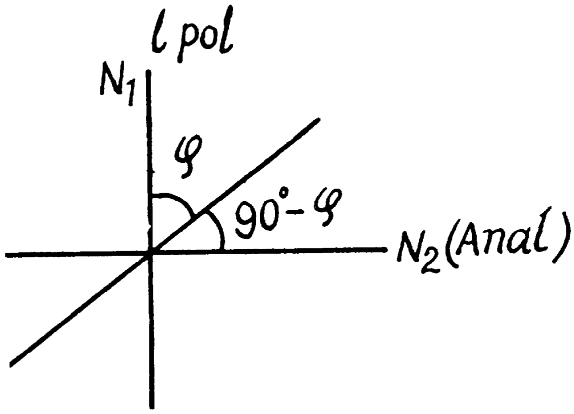
By Malus' law the intensity at this point will be $\sim \cos^2\left(\frac{\pi x}{\Delta x}\right)$.

5.191 If I_0 = intensity of natural light then

$\frac{1}{2} I_0$ = intensity of light emerging from the polarizer nicol.

Suppose the quartz plate rotates this light by φ, then the analyser will transmit

$$\frac{1}{2} I_0 \cos^2(90 - \varphi)$$
$$= \frac{1}{2} I_0 \sin^2\varphi$$



of this intensity. Hence $$\eta I_0 = \frac{1}{2} I_0 \sin^2\varphi$$

or $$\varphi = \sin^{-1}\sqrt{2\eta}$$

But $$\varphi = \alpha d \text{ so}$$

$$d_{min} = \frac{1}{\alpha}\sin^{-1}\sqrt{2\eta}$$

For minimum d we must take the principal value of inverse sine. Thus using $\alpha = 17$ ang deg/m m.

$$d_{min} = 2{\cdot}99 \text{ m m}.$$

5.192 For light of wavelength 436 nm

$$41{\cdot}5^\circ \times d = k \times 180^\circ = 2k \times 90^\circ$$

(Light will be completely cut off when the quartz plate rotates the plane of polarization by a multiple of 180°.) Here d = thickness of quartz plate in mm.

For natural incident light, half the light will be transmitted when the quartz rotates light by an odd multiple of 90°. Thus

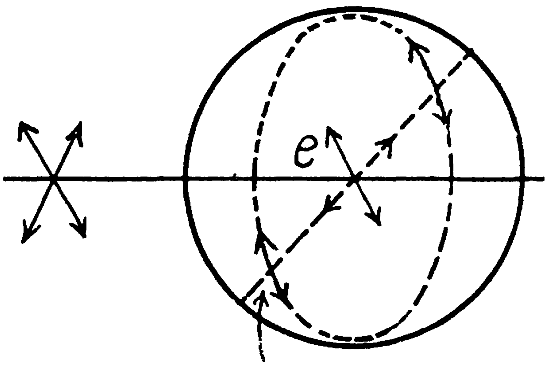
$$31{\cdot}1^\circ \times d = (2k' + 1) \times 90^\circ$$

Now $$\frac{41{\cdot}5}{31{\cdot}1} = 1{\cdot}3344 \approx \frac{4}{3}$$

Thus $$k = 2 \text{ and } k' = 1 \text{ and}$$

$$d = \frac{4 \times 90}{41{\cdot}5} = 8{\cdot}67 \text{ m m}.$$

5.193 Two effects are involved here : rotation of plane of polarizatin by sugar solution and the effect of that rotation on the scattering of light in the transverse direction. The latter is shown in the figure given below. It is easy to see from the figure that there will be no scattering of light in this transverse direction if the incident light has its electric vector parallel to the line of sight. In such a situation, we expect fringes to occur in the given experiment.



From the given data we see that in a distance of 50 cm, the rotation of plane of polarization must be 180°. Thus the specific rotation constant of sugar

$$= \frac{\text{rotation constant}}{\text{concentration}}$$

$$= \frac{180/50}{500\,\frac{g}{l}}\ \text{ang/deg/cm} = \frac{180}{5{\cdot}0\ \text{d m} \times (\cdot 500\ \text{gm/cc})}$$

$$= 72^\circ\ \text{ang deg/(dm} \cdot \text{gm/cc)}\ \ (1\ \text{d m} = 10\ \text{cm})$$

5.194 (a) In passing through the Kerr cell the two perpendicular components of the electric field will acquire a phase difference. When this phase difference equals 90° the emergent light will be circularly polarized because the two perpendicular components O & E have the same magnitude since it is given that the direction of electric field E in the capacitor forms an angle of 45° with the principal directions of then icols. In this case the intensity of light that emerges from this system will be independent of the rotation of the analyser prism.

Now the phase difference introduced is given by

$$\delta = \frac{2\pi}{\lambda}(n_e - n_0)\,l$$

In the present case $\delta = \frac{\pi}{2}$ (for minimum electric field)

$$n_e - n_0 = \frac{\lambda}{4l}$$

Now $$n_e - n_0 = B\lambda E^2$$

so $$E_{min} = \sqrt{\frac{1}{4Bl}}\ . = 10^5/\sqrt{88} = 10{\cdot}66\ \text{kV/cm}\,.$$

(b) If the applied electric field is

$$E = E_m \sin\omega t,\ \omega = 2\pi\nu$$

than the Kerr cell introduces a time varying phase difference

$$\delta = 2\pi B\,l\,E_m^2 \sin^2\omega t$$

$$= 2\pi \times 2{\cdot}2 \times 10^{-10} \times 10 \times (50 \times 10^3)^2 \sin^2\omega t$$

$$= 11\,\pi \sin^2\omega t$$

In one half-cycle $\left(\text{i.e. in time } \dfrac{\pi}{\omega} = T/2 = \dfrac{1}{2\nu}\right)$

this reaches the value $2k\pi$ when

$$\sin^2 \omega t = 0, \frac{2}{11}, \frac{4}{11}, \frac{6}{11}, \frac{8}{11}, \frac{10}{11}$$

$$\frac{2}{11}, \frac{4}{11}, \frac{6}{11}, \frac{8}{11}, \frac{10}{11}$$

i.e. 11 times. On each of these occasions light will be interrupted. Thus light will be interrupted

$$2\nu \times 11 = 2{\cdot}2 \times 10^8 \text{ times per second}$$

(Light will be interrupted when the Kerr cell (placed between crossed Nicols) introduces a phase difference of $2k\pi$ and in no other case.)

5.195 From problem 189, we know that

$$\Delta n = \frac{\alpha\lambda}{\pi}$$

where α is the rotation constant. Thus

$$\Delta n = \frac{2\alpha}{2\pi/\lambda} = \frac{2\alpha c}{\omega}$$

On the other hand $\qquad \alpha_{mag} = VH$

Thus for the magnetic rotations $\qquad \Delta n = \dfrac{2cVH}{\omega}.$

5.196 Part of the rotations is due to Faraday effect and part of it is ordinary optical rotation. The latter does not change sign when magnetic field is reversed. Thus

$$\varphi_1 = \alpha l + VlH$$

$$\varphi_2 = \alpha l - VlH$$

Hence $\qquad 2VlH = (\varphi_1 - \varphi_2)$

or $\qquad V = \left(\dfrac{\varphi_1 - \varphi_2}{2}\right) \Big/ lH$

Putting the values

$$V = \frac{510 \text{ ang min}}{2 \times {\cdot}3 \times 56{\cdot}5} \times 10^{-3} \text{ per } A = 0{\cdot}015 \text{ ang min/A}$$

5.197 We write

$$\varphi = \varphi_{chemical} + \varphi_{magnetic}$$

We look against the transmitted beam and count the positive direction clockwise. The chemical part of the rotation is annulled by reversal of wave vector upon reflection.

Thus $\qquad \varphi_{chemical} = \alpha l$

Since in effect there is a single transmission.

On the other hand

$$\varphi_{mag} = -NHVl$$

To get the signs right recall that dextro rotatory compounds rotate the plane of vibration in a clockwise direction on looking against the oncoming beam. The sense of rotation of light vibration in Faraday effect is defined in terms of the direction of the field, positive rotation being that of a right handed screw advancing in the direction of the field. This is the opposite of the definition of $\varphi_{chemical}$ for the present case. Finally

$$\varphi = (\alpha - VNH)l$$

(Note : If plane polarized light is reflected back & forth through the same active medium in a magnetic field, the Faraday rotation increases with each traveresal.)

5.198 There must be a Faraday rotation by 45° in the opposite direction so that light could pass through the second polaroid. Thus

$$VlH_{min} = \pi/4$$

or

$$H_{min} = \frac{\pi/4}{Vl} = \frac{45 \times 60}{2{\cdot}59 \times 0{\cdot}26}\frac{A}{m}$$

$$= 4{\cdot}01\,\frac{\text{kA}}{\text{m}}$$

If the direction of magnetic field is changed then the sense of rotation will also change. Light will be completely quenched in the above case.

5.199 Let r = radius of the disc

then its moment of inertia about its axis = $\frac{1}{2}mr^2$

In time t the disc will acquire an angular momentum

$$t \cdot \pi r^2 \cdot \frac{I}{\omega}$$

when circularly polarized light of intensity I falls on it. By conservation of angular momentum this must equal

$$\frac{1}{2}mr^2 \cdot \omega_0$$

where ω_0 = final angular velocity.

Equating $$t = \frac{m\,\omega\,\omega_0}{2\pi I}.$$

But $$\frac{\omega}{2\pi} = \nu = \frac{c}{\lambda} \quad \text{so } t = \frac{m\,c\,\omega_0}{I\lambda}.$$

Substituting the values of the various quantities we get

$$t = 11{\cdot}9 \text{ hours}$$

5.5 DISPERSION AND ABSORPTION OF LIGHT

5.200 In a travelling plane electromagnetic wave the intensity is simply the time averaged magnitude of the Poynting vector :-

$$I = \left\langle |\vec{E}\times\vec{H}| \right\rangle = \left\langle \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E^2 \right\rangle = \left\langle c\,\varepsilon_0 E^2 \right\rangle$$

on using $$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}},\quad E\sqrt{\varepsilon_0} = H\sqrt{\mu_0}\,.$$

(see chapter 4.4 of the book).

Now time averaged value of E^2 is $E_0^2/2$ so

$$I = \frac{1}{2} c\,\varepsilon_0 E_0^2 \quad \text{or} \quad E_0 = \sqrt{\frac{2I}{c\,\varepsilon_0}},$$

(a) Represent the electric field at any point by $E = E_0 \sin\omega t$. Then for the electron we have the equation.

$$m\ddot{x} = e E_0 \sin\omega t$$

so $$x = -\frac{e E_0}{m\omega^2}\sin\omega t$$

The ampitude of the forced oscillation is

$$\frac{e E_0}{m\omega^2} = \frac{e}{m\omega^2}\sqrt{\frac{2I}{c\,\varepsilon_0}} = 5{\cdot}1\times 10^{-16}\ \text{cm}$$

The velocity amplitude is clearly

$$\frac{e E_0}{m\omega} = 5{\cdot}1\times 10^{-16}\times 3{\cdot}4\times 10^{15} = 1{\cdot}73\ \text{cm/sec}$$

(b) For the electric force

$F_e =$ amplitude of the electric force

$= e E_0$

For the magnetic force (which we have neglected above), it is

$$(e\,\mathrm{v}\,B) = (e\,\mathrm{v}\,\mu_0 H)$$

$$= e\,\mathrm{v}\,E\sqrt{\varepsilon_0\mu_0} = e\,\mathrm{v}\,\frac{E}{c}$$

writing $\mathrm{v} = -\mathrm{v}_0\cos\omega t$

where $$\mathrm{v}_0 = \frac{e E_0}{m\omega}$$

we see that the magnetic force is apart from a sign

$$\frac{e\,\mathrm{v}_0 E_0}{2c}\sin 2\omega t$$

Hence $\frac{F_m}{F_e}$ = Ratio of amplitudes of the two forces

$$= \frac{v_0}{2c} = 2{\cdot}9 \times 10^{-11}$$

This is negligible and justifies the neglect of magnetic field of the electromagnetic wave in calculating v_0.

5.201 (a) It turns out that one can neglect the spatial dependence of the electric field as well as the magnetic field. Thus for a typical electron

$$m\ddot{\vec{r}} = e\vec{E}_0 \sin\omega t$$

so $\vec{r} = -\frac{e\vec{E}_0}{m\omega^2}\sin\omega t$ (neglecting any nonsinusoidal part).

The ions will be practically unaffected. Then

$$\vec{P} = n_o e\vec{r} = -\frac{n_0 e^2}{m\omega^2}\vec{E}$$

and
$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\left(1 - \frac{n_0 e^2}{\varepsilon_0 m\omega^2}\right)\vec{E}$$

Hence the permittivity
$$\varepsilon = 1 - \frac{n_0 e^2}{\varepsilon_0 m\omega^2}.$$

(b) The phase velocity is given by

$$v = \omega/K = \frac{c}{\sqrt{\varepsilon}}$$

So
$$ck = \omega\sqrt{1 - \frac{\omega_P^2}{\omega^2}}, \quad \omega_P^2 = \frac{n_0 e^2}{\varepsilon_0 m}$$

$$\omega^2 = c^2k^2 + \omega_P^2$$

Thus
$$v = c\sqrt{1 + \frac{\omega_P^2}{c^2k^2}} = c\sqrt{1 + \left(\frac{n_0 e^2}{4\pi^2 m c^2 \varepsilon_0}\right)\lambda^2}$$

5.202 From the previous problem

$$n^2 = 1 - \frac{n_0 e^2}{\varepsilon_0 m\omega^2}$$

$$= 1 - \frac{n_0 e^2}{4\pi^2\varepsilon_0 m v^2}$$

Thus $n_0 = (4\pi^2 v^2 m\varepsilon_0/e^2)(1 - n^2) = 2{\cdot}36 \times 10^7\ \text{cm}^{-3}$

5.203 For hard x– rays, the electrons in graphite will behave as if nearly free and the formula of previous problem can be applied. Thus

$$n^2 = 1 - \frac{n_0 e^2}{\varepsilon_0 m \omega^2}$$

and
$$n \approx 1 - \frac{n_0 e^2}{2 \varepsilon_0 m \omega^2}$$

on taking square root and neglecting higher order terms.

So
$$n - 1 = -\frac{n_0 e^2}{2 \varepsilon_0 m \omega^2} = -\frac{n_0 e^2 \lambda^2}{8 \pi^2 \varepsilon_0 m e^2}$$

We calculate n_0 as follows : There are $6 \times 6{\cdot}023 \times 10^{23}$ electrons in 12 gms of graphite of density $1{\cdot}6$ gm/c.c. Thus

$$n_0 = \frac{6 \times 6{\cdot}023 \times 10^{23}}{(12/1{\cdot}6)} \text{ per c.c}$$

Using the values of other constants and $\lambda = 50 \times 10^{-12}$ metre we get

$$n - 1 = -5{\cdot}4 \times 10^{-7}$$

5.204 (a) The equation of the electron can (under the stated conditions) be written as

$$m \ddot{x} + \gamma \dot{x} + k x = e E_0 \cos \omega t$$

To solve this equation we shall find it convenient to use complex displacements. Consider the equation

$$m \ddot{z} + \gamma \dot{z} + k z = e E_0 e^{-i \omega t}$$

Its solution is

$$z = \frac{e E_0 e^{-i \omega t}}{-m \omega^2 - i \gamma \omega + k}$$

(we ignore transients.)

Writing
$$\beta = \frac{\gamma}{2m}, \quad \omega_0^2 = \frac{k}{m}$$

we find
$$z = \frac{e E_0}{m} e^{-i \omega t} \Big/ (\omega_o^2 - \omega^2 - 2 i \beta \omega)$$

Now x = Real part of z

$$= \frac{e E_0}{m} \cdot \frac{\cos(\omega t + \varphi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4 \beta^2 \omega^2}} = a \cos(\omega t + \varphi)$$

where
$$\tan \varphi = \frac{2 \beta \omega}{\omega^2 - \omega_0^2}$$

$$\left(\sin \varphi = -\frac{2 \beta \omega}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4 \beta^2 \omega^2}} \right).$$

(b) We calculate the power absorbed as

$$P = \langle F\dot{x} \rangle = \langle e E_0 \cos\omega t\,(-\omega a \sin(\omega t + \varphi)) \rangle$$

$$= e E_0 \cdot \frac{e E_0}{m} \frac{1}{2} \cdot \frac{2\beta\omega}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \cdot \omega = \left(\frac{e E_0}{m}\right)^2 \frac{\beta m \omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

This is clearly maximum when $\omega_0 = \omega$ because P can be written as

$$P = \left(\frac{e E_0}{m}\right)^2 \frac{\beta m}{\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + 4\beta^2}$$

and $$P_{\max} = \frac{m}{4\beta}\left(\frac{e E_0}{m}\right)^2 \quad \text{for } \omega = \omega_0 .$$

P can also be calculated from $P = \langle \gamma \dot{x} \cdot \dot{x} \rangle$

$$= (\gamma\omega^2 a^2/2) = \frac{\beta m \omega^2 (e E_0/m)^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} .$$

5.205 Let us write the solutions of the wave equation in the form

$$A = A_0 e^{i(\omega t - kx)}$$

where $k = \dfrac{2\pi}{\lambda}$ and λ is the wavelength in the medium. If $n' = n + i\chi$, then

$$k = \frac{2\pi}{\lambda_0} n'$$

(λ_0 is the wavelength in vacuum) and the equation becomes

$$A = A_0 e^{\chi' x} \exp(i(\omega t_1 - k' x))$$

where $\chi' = \dfrac{2\pi}{\lambda_0}\chi$ and $k' = \dfrac{2\pi}{\lambda_0} n$. In real form,

$$A = A_0 e^{\chi' x} \cos(\omega t - k' x)$$

This represents a plane wave whose amplitude diminishes as it propagates to the right (provided $\chi' < 0$).

when $n' = i\chi$, then similarly

$$A = A_0 e^{\chi' x} \cos\omega t$$

(on putting $n = 0$ in the above equation).

This represents a standing wave whose amplitude diminishes as one goes to the right (if $\chi' < 0$). The wavelength of the wave is infinite ($k' = 0$).

Waves of the former type are realized inside metals as well as inside dielectrics when there is total reflection. (penetration of wave).

5.206 In the plasma radio waves with wavelengths exceeding λ_0 are not propagated. We interpret this to mean that the permittivity becomes negative for such waves. Thus

$$0 = 1 - \frac{n_0 e^2}{\varepsilon_0 m \omega^2} \quad \text{if} \quad \omega = \frac{2\pi c}{\lambda_0}$$

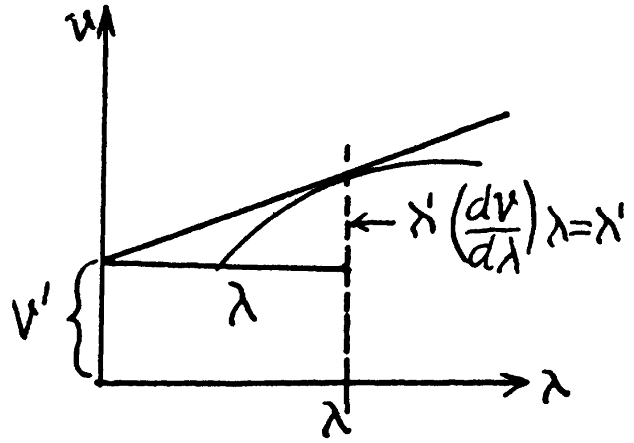
Hence
$$\frac{n_0 e^2 \lambda_0^2}{4\pi^2 \varepsilon_0 m c^2} = 1$$

or
$$n_0 = \frac{4\pi^2 \varepsilon_0 m c^2}{e^2 \lambda_0^2} = 1{\cdot}984 \times 10^9 \text{ per c.c}$$

5.207 By definition

$$u = \frac{d\omega}{dk} = \frac{d}{dk}(\mathrm{v}\,k) \quad \text{as}\ \omega = \mathrm{v}\,k \quad = \mathrm{v} + k\frac{d\mathrm{v}}{dk}$$

Now $\quad k = \dfrac{2\pi}{\lambda} \quad$ so $\quad dk = -\dfrac{2\pi}{\lambda^2}\,d\lambda$



Thus $\quad u = \mathrm{v} - \lambda\dfrac{d\mathrm{v}}{d\lambda}.$

Its interpretation is the following :

$\left(\dfrac{d\mathrm{v}}{d\lambda}\right)_{\lambda = \lambda'}$ is the slope of the $\mathrm{v} - \lambda$ curve at $\lambda = \lambda'$.

Thus as is obvious from the diagram
$\mathrm{v}' = \mathrm{v}(\lambda') - \lambda'\left(\dfrac{d\mathrm{v}}{d\lambda}\right)_{\lambda = \lambda'}$ is the group velocity for $\lambda = \lambda'$.

5.208 (a) $\mathrm{v} = a/\sqrt{\lambda}$, a = constant

Then
$$u = \mathrm{v} - \lambda\frac{d\mathrm{v}}{d\lambda}$$
$$= \frac{a}{\sqrt{\lambda}} - \lambda\left(-\frac{1}{2}a\lambda^{-3/2}\right) = \frac{3}{2}\cdot\frac{a}{\sqrt{\lambda}} = \frac{3}{2}\mathrm{v}.$$

(b) $\mathrm{v} = bk = \omega k$, b = constant

so
$$\omega = bk^2 \quad \text{and} \quad u = \frac{d\omega}{dk} = 2bk = 2\mathrm{v}.$$

(c) $\mathrm{v} = \dfrac{c}{\omega^2}$, c = constant $= \dfrac{\omega}{k}$.

so
$$\omega^3 = ck \quad \text{or} \quad \omega = c^{1/3}k^{1/3}$$

Thus
$$u = \frac{d\omega}{dk} = c^{1/3}\frac{1}{3}k^{-2/3} = \frac{1}{3}\frac{\omega}{k} = \frac{1}{3}\mathrm{v}$$

5.209 We have

$$u\,v = \frac{\omega}{k}\frac{d\omega}{dk} = c^2$$

Integrating we find

$$\omega^2 = A + c^2 k^2, \ A \text{ is a constant.}$$

so
$$k = \frac{\sqrt{\omega^2 - A}}{c}$$

and
$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{A}{\omega^2}}}$$

writing this as $c/\sqrt{\varepsilon(\omega)}$ we get $\varepsilon(\omega) = 1 - \frac{A}{\omega^2}$

(A can be +ve or negative)

5.210 The phase velocity of light in the vicinity of $\lambda = 534\,nm = \lambda_0$ is obtained as

$$v(\lambda_0) = \frac{c}{n(\lambda_0)} = \frac{3 \times 10^8}{1{\cdot}640} = 1{\cdot}829 \times 10^8 \text{ m/s}$$

To get the group velocity we need to calculate

$\left(\frac{dn}{d\lambda}\right)_{\lambda = \lambda_0}$. We shall use linear

interpolation in the two intervals. Thus

$$\left(\frac{dn}{d\lambda}\right)_{\lambda = 521\cdot5} = -\frac{\cdot007}{25} = -28 \times 10^{-5} \text{ per nm}$$

$$\left(\frac{dn}{d\lambda}\right)_{\lambda = 561\cdot5} = -\frac{\cdot01}{55} = -18{\cdot}2 \times 10^{-5} \text{ per nm}$$

There $(dn/d\lambda)$ values have been assigned to the mid-points of the two intervals. Interpolating again we get

$$\left(\frac{dn}{d\lambda}\right)_{\lambda = 534} = \left[-28 + \frac{9{\cdot}8}{40} \times 12{\cdot}5\right] \times 10^{-5} \text{ per mm} = -24{\cdot}9 \times 10^{-5} \text{ per n m}.$$

Finally

$$u = \frac{c}{n} - \lambda\frac{d}{d\lambda}\left(\frac{c}{n}\right) = \frac{c}{n}\left[1 + \frac{\lambda}{n}\left(\frac{dn}{d\lambda}\right)\right]$$

At $\lambda = 534$

$$u = \frac{3 \times 10^8}{1{\cdot}640}\left[1 - \frac{534}{1{\cdot}640} \times 24{\cdot}9 \times 10^{-5}\right] \text{m/s} = 1{\cdot}59 \times 10^8 \text{ m/s}$$

5.211 We write

$$v = \frac{\omega}{k} = a + b\lambda$$

so $$\omega = k(a + b\lambda) = 2\pi b + ak.$$

$\left(\text{since } k = \dfrac{2\pi}{\lambda}\right)$. Suppose a wavetrain at time $t = 0$ has the form

$$F(x, 0) = \int f(k) e^{ikx} dk$$

Then at time t it will have the form

$$F(x, t) = \int f(k) e^{ikx - i\omega t} dk$$

$$= \int f(k) e^{ikx - i(2\pi b + ak)t} = \int f(k)\, e^{ik(x-at)}\, e^{-i2\pi bt} dk$$

At $t = \dfrac{1}{b} = \tau$

$$F(x, \tau) = F(x - a\tau, 0)$$

so at time $t = \tau$ the wave train has regained its shape though it has advanced by $a\tau$.

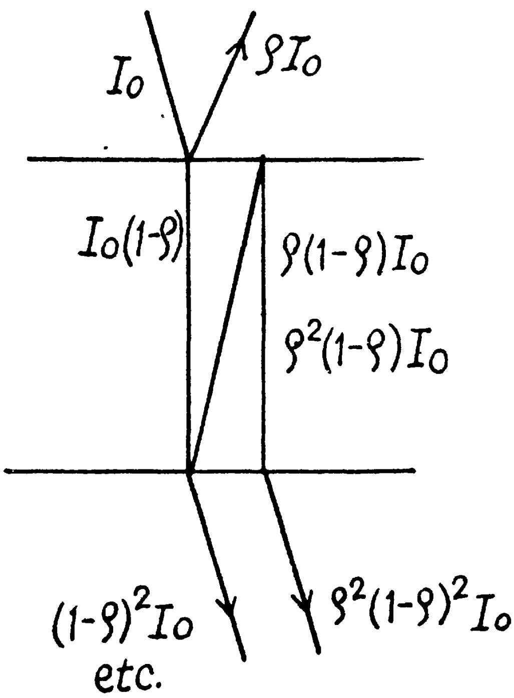
5.212 On passing through the first (polarizer) Nicol the intensity of light becomes $\frac{1}{2}I_0$ because one of the components has been cut off. On passing through the solution the plane of polarization of the light beam will rotate by

$\varphi = VlH$

and its intensity will also decrease by a factor $e^{-\chi l}$. The plane of vibraton of the light wave wll then make an angle $90° - \varphi$ with the principal direction of the analyzer Nicol. Thus by Malus' law the intensity of light coming out of the second Nicol will be

$$\frac{1}{2}I_0 \cdot e^{-\chi l} \cdot \cos^2(90° - \varphi)$$

$$= \frac{1}{2}I_0\, e^{-\chi l} \sin^2\varphi.$$



5.213 (a) The multiple reflections are shown below. Transmission gives a factor $(1 - \rho)$ while reflections give factors of ρ. Thus the transmitted intensity assuming incoheren light is

$$(1-\rho)^2 I_0 + (1-\rho)^2 \rho^2 I_0 + (1-\rho)^2 \rho^4 I_0 + \ldots$$

$$= (1-\rho)^2 I_0 (1 + \rho^2 + \rho^4 + \rho^6 + \ldots)$$

$$= (1-\rho)^2 I_0 \times \frac{1}{1-\rho^2} = I_0 \frac{1-\rho}{1+\rho}.$$

(b) When there is absorption, we pick up a factor $\sigma = e^{-\chi d}$ in each traversal of the plate. Thus we get

$$(1-\rho)^2 \sigma I_0 + (1-\rho)^2 \sigma^3 \rho^2 I_0 + (1-\rho)^2 \sigma^5 \rho^4 I_0 + \ldots$$
$$= (1-\rho)^2 \sigma I_0 (1 + \sigma^2 \rho^2 + \sigma^4 \rho^4 + \ldots)$$
$$= I_0 \frac{\sigma(1-\rho)^2}{1-\sigma^2 \rho^2}$$

5.214 We have

$$\tau_1 = e^{-\chi d_1} (1-\rho)^2$$
$$\tau_2 = e^{-\chi d_2} (1-\rho)^2$$

where ρ is the reflectivity; see previous problem, multiple reflection have been ignored.

Thus
$$\frac{\tau_1}{\tau_2} = e^{\chi(d_2 - d_1)}$$

or
$$\chi = \frac{\ln\left(\frac{\tau_1}{\tau_2}\right)}{d_2 - d_1} = 0.35 \text{ cm}^{-1}.$$

5.215 On each surface we pick up a factor $(1-\rho)$ from reflection and a factor $e^{-\chi l}$ due to absorption in each plate.

Thus
$$\tau = (1-\rho)^{2N} e^{-\chi N l}$$

Thus
$$\chi = \frac{1}{Nl} \ln \frac{(1-\rho)^{2N}}{\tau} = 0.034 \text{ cm}^{-1}.$$

5.216 Apart from the factor $(1-\rho)$ on each end face of the plate, we shall get a factor due to absorptions. This factor can be calculated by assuming the plate to consist of a large number of very thin slab within each of which the absorption coefficient can be assumed to be constant. Thus we shall get a product like

$$\ldots\ldots e^{-\chi(x)dx}\, e^{-\chi(x+dx)dx}\, e^{-\chi(x+2dx)dx} \ldots\ldots$$

This product is nothing but

$$e^{-\int_0^l \chi(x)\,dx}$$

Now $\chi(0) = \chi_1$, $\chi(l) = \chi_2$ and variation

with x is linear so $\chi(x) = \chi_1 + \frac{x}{l}(\chi_2 - \chi_1)$

Thus the factor becomes

$$e^{-\int_0^l \left[\chi_1 + \frac{x}{l}(\chi_2 - \chi_1)\right] dx} = e^{-\frac{1}{2}(\chi_1 + \chi_2) l}$$

5.217 The spectral density of the incident beam (i.e. intensity of the components whose wave length lies in the interval λ & $\lambda + d\lambda$) is

$$\frac{I_0}{\lambda_2 - \lambda_1} d\lambda, \ \lambda_1 \le \lambda \le \lambda_2$$

The absorption factor for this component is

$$e^{-\left[\chi_1 + \frac{\lambda + \lambda_1}{\lambda_2 - \lambda_1}(\chi_2 - \chi_1)\right] l}$$

and the transmission factor due to reflection at the surface is $(1 - \rho)^2$. Thus the intensity of the transmitted beam is

$$(1-\rho)^2 \frac{I_0}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} d\lambda \ e^{-l\left[\chi_1 + \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1}(\chi_2 - \chi_1)\right]}$$

$$= (1-\rho)^2 \frac{I_0}{\lambda_2 - \lambda_1} e^{-\chi_1 l}\left(\frac{1 - e^{-(\chi_2 - \chi_1) l}}{(\chi_2 - \chi_1)\text{'}me}\right) \chi(\lambda_2 - \lambda_1) = (1-\rho)^2 I_0 \frac{e^{-\chi_1 l} - e^{-\chi_2 l}}{(\chi_2 - \chi_1) l}$$

5.218 At the wavelength λ_0, the absorption coefficient vanishes and loss in transmission is entirely due to reflection. This factor is the same at all wavelengths and therefore cancels out in calculating the pass band and we need not worry about it. Now

$$T_0 = (\text{transmissivity at } \lambda = \lambda_0) = (1 - \rho)^2$$

$$T = \text{transmissivity at } \lambda = (1-\rho)^2 e^{-\chi(\lambda) d}$$

The edges of the passband are $\lambda_0 \pm \frac{\Delta\lambda}{2}$ and at the edge

$$\frac{T}{T_0} = e^{-\alpha d\left(\frac{\Delta\lambda}{2\lambda_0}\right)^2} = \eta$$

Thus
$$\frac{\Delta\lambda}{2\lambda_0} = \sqrt{\left(\ln\frac{1}{\eta}\right) / \alpha d}$$

or
$$\Delta\lambda = 2\lambda_0 \sqrt{\frac{1}{\alpha d}\left(\ln\frac{1}{\eta}\right)}$$

5.219 We have to derive the law of decrease of intensity in an absorbing medium taking in to account the natural geometrical fall-off (inverse sequare law) as well as absorption.
Consider a thin spherical shell of thickness dx and internal radius x. Let $I(x)$ and $I(x+dx)$ be the intersities at the inner and outer surfaces of this shell.

Then
$$4\pi x^2 I(x)\ e^{-\chi dx} = 4\pi (x + dx)^2 I\ (x + dx)$$

Except for the factor $e^{-\chi dx}$ this is the usual equation. We rewrite this as

$$x^2 I(x) = I(x+dx)(x+dx)^2(1 + \chi dx)$$

$$= \left(I + \frac{dI}{dx} dx \right) (x^2 + 2x\,dx)(1 + \chi\,dx)$$

or
$$x^2 \frac{dI}{dx} + \chi x^2 I + 2xI = 0$$

Hence
$$\frac{d}{dx}(x^2 I) + \chi (x^2 I) = 0$$

so
$$x^2 I = C e^{-\chi x}$$

where C is a constant of integration.

In our case we apply this equation for $a \le x \le b$

For $x \le a$ the usual inverse square law gives

$$I(a) = \frac{\Phi}{4\pi a^2}$$

Hence
$$C = \frac{\Phi}{4\pi} e^{\chi a}$$

and
$$I(b) = \frac{\Phi}{4\pi b^2} e^{-\chi(b-a)}$$

This does not take into account reflections. When we do that we get

$$I(b) = \frac{\Phi}{4\pi b^2} (1-\rho)^2 e^{-\chi(b-a)}$$

5.220 The transmission factor is $e^{-\mu d}$ and so the intensity will decrease

$$e^{\mu d}$$

$$= e^{3\cdot6 \times 11\cdot3 \times 0\cdot1} = 58\cdot4 \text{ timestimes}$$

(we have used $\mu = (\mu/\rho) \times \rho$ and used the known value of density of lead).

5.221 We require $\mu_{Pb}\, d_{Pb} = \mu_{Al}\, d_{Al}$

or
$$\left(\frac{\mu_{Pb}}{\rho_{Pb}}\right) \rho_{Pb}\, d_{Pb} = \left(\frac{\mu_{Al}}{\rho_{Al}}\right) \rho_{Al}\, d_{Al}$$

$$72\cdot0 \times 11\cdot3 \times d_{Pb} = 3\cdot48 \times 2\cdot7 \times 2\cdot6$$

$$d_{Pb} = 0\cdot3 \text{ mm}$$

5.222 $\frac{1}{2} = e^{-\mu d}$

or
$$d = \frac{\ln 2}{\mu} = \frac{\ln 2}{\left(\frac{\mu}{\rho}\right)\rho} = 0\cdot80 \text{ cm}$$

5.223 We require N plates where

$$\left(\frac{1}{2}\right)^N = \frac{1}{50} \quad \text{So } N = \frac{\ln 50}{\ln 2} = 5\cdot6$$

5.6 OPTICS OF MOVING SOURCES

5.224 In the Fizean experiment, light disappears when the wheel rotates to bring a tooth in the position formerly occupied by a gap in the time taken by light to go from the wheel to the mirror and back. Thusdistance travelled = $2l$ Suppose the m^{th} tooth after the gap has come in place of the latter. Then time taken

$$= \frac{2(m-1)+1}{2zn_1} \text{ sec. in the first case}$$

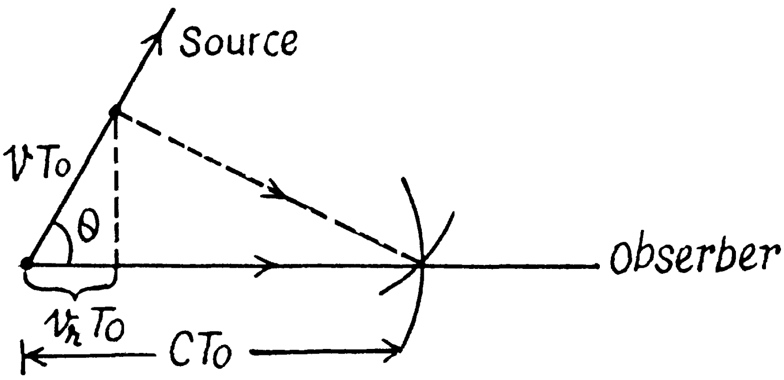
$$= \frac{2m+1}{2zn_2} \text{ sec in the second case } = \frac{1}{z(n_2-n_1)}$$

Then $$c = 2lz(n_2-n_1) = 3{\cdot}024 \times 10^8 \text{ m/sec}$$

5.225 When $v << c$ time dilation effect of relativity can be neglected (i.e. $t' \approx t$) and we can use time in the reference frame fixed to the observer. Suppose the source emits short pulses with intervals T_0. Then in the reference frame fixed to the receiver the distance between two seccessive pulses is $\lambda = cT_0 - v_r T_0$ when measured along the observation line.

Here $v_r = v\cos\theta$ is the projection of the source velocity on the observation line. The frequency of the pulses received by the observer is

$$\nu = \frac{c}{\lambda} = \frac{\nu_0}{1-\frac{v_r}{c}} \approx \nu_0\left(1+\frac{v_r}{c}\right)$$

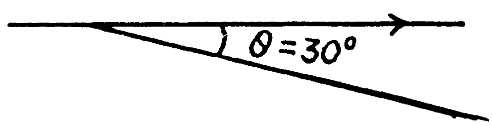


(The formula is accurate to first order only)

Thus $$\frac{\nu-\nu_0}{\nu_0} = \frac{v_r}{c} = \frac{v\cos\theta}{c}$$

The frequency increases when the source is moving towards the observer.

5.226 $\frac{\Delta\nu}{\nu} = \frac{v}{c}\cos\theta$ from the previous problem



But $\nu\lambda = c$ gives an differentiation

$$\frac{\Delta\nu}{\nu} = -\frac{\Delta\lambda}{\lambda}$$

So $$\Delta\lambda = -\lambda\sqrt{\frac{v^2}{c^2}}\cos\theta = -\lambda\sqrt{\frac{2T}{mc^2}}\cos\theta$$

on using $$T = \frac{1}{2}mv^2,\ m = \text{mass of } He^+ \text{ ion}$$

We use $mc^2 = 4 \times 938$ MeV. Putting other values

$$\Delta\lambda = -26 \text{ nm}$$

5.227 One end of the solar disc is moving towards us while the other end is moving away from us. The angle θ between the direction in which the edges of the disc are moving and the line of observation is small ($\cos\theta \approx 1$). Thus

$$\frac{\Delta\lambda}{\lambda} = \frac{2\omega R}{c}$$

where $\omega = \frac{2\pi}{T}$ is the angular velocity of the Sun. Thus

$$\omega = \frac{c\,\Delta\lambda}{2R\lambda}$$

So
$$T = \frac{4\pi R\lambda}{c\,\Delta\lambda}$$

Putting the values ($R = 6{\cdot}95 \times 10^8$ m)

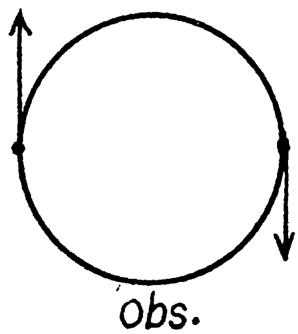
we get
$$T = 24{\cdot}85 \text{ days}$$

5.228 Maximum splitting of the spectral lines will occur when both of the stars are moving in the direction of line of observation as shown. We then have the equations

$$\left(\frac{\Delta\lambda}{\lambda}\right)_m = \frac{2v}{c}$$

$$\frac{m v^2}{R} = \frac{\gamma m^2}{4R^2}$$

$$\tau = \frac{\pi R}{v}$$



From these we get

$$d = 2R = \left(\frac{\Delta\lambda}{\lambda}\right)_m c\tau/\pi = 2{\cdot}97 \times 10^7 \text{ km}$$

$$m = \left(\frac{\Delta\lambda}{\lambda}\right)_m^3 c^3\tau/2\pi\gamma = 2{\cdot}9 \times 10^{29} \text{ kg}$$

5.229 We define the frame S (the lab frame) by the condition of the problem. In this frame the mirror is moving with velocity v (along say $x-$ axis) towards left and light of frequency ω_0 is approaching it from the left. We introduce the frame S' whose axes are parallel to those of S but which is moving with velocity v along x axis towards left (so that the mirror is at rest in S'). In S' the frequency of the incident light is

$$\omega_1 = \omega_0\left(\frac{1 + v/c}{1 - v/c}\right)^{1/2}$$

In S' the reflected light still has frequency ω_1 but it is now moving towards left. When we transform back to S this reflected light has the frequency

$$\omega = \omega_1\left(\frac{1+v/c}{1-v/c}\right)^{1/2} = \omega_0\left(\frac{1+v/c}{1-v/c}\right),$$

In the nonrelativistic limit

$$\omega \approx \omega_0\left(1+\frac{2v}{c}\right)$$

5.230 From the previous problem, the beat frequency is clearly

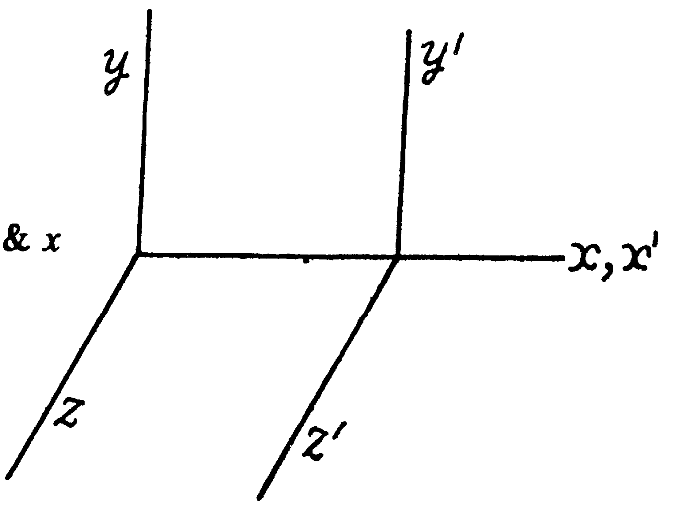
$$\Delta\nu = \nu_0\frac{2v}{c} = 2v/(c/\nu_0) = 2v/\lambda_9$$

Hence
$$v = \frac{1}{2}\lambda\,\Delta\nu = \frac{10^{+3}}{2}\times 50\ \text{cm/sec} = 900\ \text{km/hour}$$

5.231 From the invariance of phase under Lorentz transformations we get

$$\omega t - kx = \omega' t' - k' x'$$

Here $\omega = ck$. The primed coordinates refer to the frame S' which is moving to the right with velocity v :-



Then $x' = \gamma(x - vt)$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

where $\gamma = \left(\sqrt{1-v^2/c^2}\right)^{-1}$

Substituting and equating the coefficients of t & x

$$\omega = \gamma\omega' + \gamma k' v = \omega'\frac{1+v/c}{\sqrt{1-v^2/c^2}}$$

$$k = \gamma\frac{\omega' v}{c^2} + \gamma k' = k'\frac{1+v/c}{\sqrt{1-v^2/c^2}}$$

5.232 From the previous problem using

$$k = \frac{2\pi}{\lambda}$$

we get
$$\lambda' = \lambda\sqrt{\frac{1+v/c}{1-v/c}}$$

Thus
$$\frac{1+\frac{v}{c}}{1-v/c} = \frac{\lambda'^2}{\lambda^2}$$

or
$$v/c = \frac{\lambda'^2-\lambda^2}{\lambda'^2+\lambda^2} = \frac{564^2-434^2}{564^2+434^2} = 0{\cdot}256$$

5.233 As in the previous problem

$$\frac{v}{c} = \frac{\lambda^2 - \lambda'^2}{\lambda^2 + \lambda'^2}$$

so
$$v = c\,\frac{\left(\dfrac{\lambda}{\lambda'}\right)^2 - 1}{\left(\dfrac{\lambda}{\lambda'}\right)^2 + 1} = 7{\cdot}1 \times 10^4\ \mathrm{k\,m/s}$$

5.233 We go to the frame in which the observer is at rest. In this frame the velocity of the source of light is, by relativistic velocity addition foumula,

$$v = \frac{\frac{3}{4}c - \frac{1}{2}c}{1 - \left(\frac{3}{4}c \cdot \frac{1}{2}c/c^2\right)} = \frac{2c}{5}.$$

When this source emits light of proper frequency ω_0, the frequency recorded by observer will be

$$\omega = \omega_0 \sqrt{\frac{1 - v/c}{1 + v/c} + v/c} = \sqrt{\frac{3}{7}}\ \omega_0$$

Note that $\omega < \omega_0$ as the source is moving away from the observer (red shift).

5.235 In transverse Doppler effect.

$$\omega = \omega_0 \sqrt{1 - \beta^2} \approx \omega_0 \left(1 - \frac{1}{2}\beta^2\right)$$

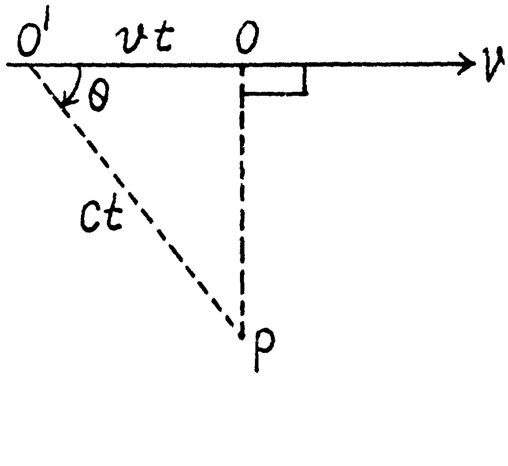
So
$$\lambda = \frac{c}{\omega} = \frac{c}{\omega_0}\left(1 + \frac{1}{2}\beta^2\right) = \lambda_0\left(1 + \frac{1}{2}\beta^2\right)$$

Hence
$$\Delta\lambda = \frac{1}{2}\beta^2\lambda$$

Using $\beta^2 = \dfrac{v^2}{c^2} = \dfrac{2T}{mc^2}$, where T = KE of H atoms

$$\Delta\lambda = \frac{T}{mc^2}\lambda = \frac{1}{938} \times 656{\cdot}3\ \mathrm{nm} = 0{\cdot}70\ \mathrm{nm}$$

.236 (a) If light is received by the observer at P at the moment when the source is at O, it must have been emitted by the source when it was at O' and travelled along $O'P$. Then if $O'P = ct$, $O'O = vt$



and $\cos\theta = \dfrac{v}{c} = \beta$

In the frame of the observer, the frequency of the light is ω while its wave vector is

$$\frac{\omega}{c}(\cos\theta, \sin\theta, 0)$$

we can calculate the value of ω by relating it to proper frequency ω. The relation is

$$\omega_0 = \frac{\omega}{\sqrt{1-\beta^2}}(1-\beta\cos\theta)$$

(To derive the formula is this form it is easiest to note that $\frac{\omega}{\sqrt{1-v/c^2}} - \frac{\vec{k}\cdot\vec{v}}{\sqrt{1-v^2/c^2}}$ is an invariant which takes the value ω_0 in the rest frame of the source).

Thus $$\omega = \frac{\omega_0\sqrt{1-\beta^2}}{1-\beta^2} = \frac{\omega_0}{\sqrt{1-\beta^2}} = 5\times10^{10}\,\text{sec}^{-1}$$

(b) For the light to be received at the instant observer sees the source at O, light must be emitted when the observer is at O at $90^\circ = \theta$

$$\cos\theta = 0$$

Then as before $\omega_0 = \frac{\omega}{\sqrt{1-\beta^2}}$ or $\omega = \omega_0\sqrt{1-\beta^2} = 1{\cdot}8\times10^{10}\,\text{sec}^{-1}$

In this case the observer will receive light along OP and he will "see" that the source is at O even though the source will have moved ahead at the instant the light is received.

5.237 An electron moving in front of a metal mirror sees an image charge of equal and opposite type. The two together constitute a dipole. Let us look at the problem in the rest frame of the electron. In this frame the grating period is Lorenz contracted to

$$d' = d\sqrt{1-v^2/c^2}$$

Because the metal has etchings the dipole moment of electron-image pair is periodically disturbed with a period $\frac{d'}{v}$

The corresponding frequency is $\frac{v}{d'}$ which is also the proper frequency of radiation emitted. Due to Doppler effect the frequency observed at an angle θ is

$$\nu = \nu'\frac{\sqrt{1-(v/c)^2}}{1-\frac{v}{c}\cos\theta} = \frac{v/d}{1-\frac{v}{c}\cos\theta}$$

The corresponding wave length is $\lambda = \frac{c}{\nu} = d\left(\frac{c}{v}-\cos\theta\right)$

Putting $$c \approx v,\ \theta = 45^\circ,\ d = 2\,\mu m \text{ we get}$$

$$\lambda = 0{\cdot}586\,\mu\text{m}$$

5.238 (a) Let v_x be the projection of the velocity vector of the radiating atom on the observation direction. The number of atoms with projections falling within the interval v_x and $v_x + d\,v_x$ is

$$n(v_x)\,d\,v_x \sim \exp(-m\,v_x^2/2\,k\,T)\,d\,v_x$$

The frequency of light emitted by the atoms moving with velocity v_x is $\omega = \omega_0\left(1+\frac{v_x}{c}\right)$. From the expressions the frequency distribution of atoms can be found : $n(\omega)\,d\,\omega = n(v_x)\,d\,v_x$. Now using

$$v_x = c\,\frac{\omega-\omega_0}{\omega_0}$$

we get $$n(\omega)\,d\,\omega \sim \exp\left(-\frac{m\,c^2}{2\,k\,T}\left(1-\frac{\omega}{\omega_0}\right)^2\right)\frac{c\,d\,\omega}{\omega_0}$$

Now the spectral radiation density $I_\omega \propto n_\omega$

Hence $$I_\omega = I_0\,e^{-a\left(1-\frac{\omega}{\omega_0}\right)^2},\quad a = \frac{m\,c^2}{2\,k\,T}.$$

(The constant of proportionality is fixed by I_0.)

(b) On putting $\omega = \omega_0 \pm \frac{1}{2}\Delta\,\omega$ and demanding

$$I_\omega = I_0/2$$

we get $$\frac{1}{2} = e^{-a\left(\frac{\Delta\,\omega}{2\,\omega_0}\right)^2}$$

so $$a\left(\frac{\Delta\,\omega}{2\,\omega_0}\right)^2 = \ln 2$$

Hence $$\frac{\Delta\,\omega}{2\,\omega_0} = \sqrt{(2\ln 2)\,k\,T/m\,c^2}$$

and $$\frac{\Delta\,\omega}{\omega} = 2\sqrt{(2\ln 2)\,k\,T/m\,c^2}$$

5.239 In vacuum inertial frames are all equivalent; the velocity of light is c in any frame. This equivalence of inertial frames does not hold in material media and here the frame in which the medium is at rest is singled out. It is in this frame that the velocity of light is $\frac{c}{n}$ where n is the refractive in desc of light for that medium.

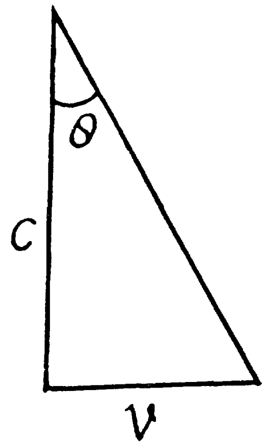
The velocity of light in the frame in which the medium is moving is then by the law of addition of velocities

$$\frac{\frac{c}{n}+\mathrm{v}}{1+\frac{c}{n}\cdot \mathrm{v}/c^2} = \frac{\frac{c}{n}+\mathrm{v}}{1+\frac{\mathrm{v}}{cn}} = \left(\frac{c}{n}+\mathrm{v}\right)\left(1-\frac{\mathrm{v}}{cn}+\dots\right)$$

$$= \frac{c}{n}+\mathrm{v}-\frac{\mathrm{v}}{n^2}+\dots = \frac{c}{n}+\mathrm{v}\left(1-\frac{1}{n^2}\right)$$

This is the velocity of light in the medium in a frame in which the medium is moving with velocity $\mathrm{v} << c$.

5.240 Although speed of light is the same in all inertial frames of reference according to the principles of relativity, the direction of a light ray can appear different in different frames. This phenomenon is called aberration and to first order in $\frac{\mathrm{v}}{c}$, can be calculated by the elementary law of addition of velocities applied to light waves.



The angle of aberration is

$$\tan^{-1}\frac{\mathrm{v}}{c}.$$

and in the present case it equals $\frac{1}{2}\delta\theta$ on either side. Thus equating

$\frac{v}{c} = \tan\frac{1}{2}\delta\theta \approx \frac{1}{2}\delta\theta\ (\delta\theta \text{ radians})$

or $$v = \frac{c}{2}\delta\theta = \frac{3\times 10^8}{2}\times\frac{41}{3600}\times\frac{\pi}{180}$$

$$= \frac{3\times 4{\cdot}1\times\pi}{3{\cdot}6\times 3{\cdot}6}\times 10^4\ \mathrm{m/s} = 29{\cdot}8\ \mathrm{k\,m/sec}$$

5.241 We consider the invariance of the phase of a wave moving in the $x-y$ plane.
We write

$$\omega' t' - k'_x \chi' - k'_y y' = \omega t - k_x \chi - k_y y$$

From Lorentz transformations, L.H.S.

$$= \omega'\gamma\left(t-\frac{\mathrm{v}x}{c^2}\right) - k'_x(x-\mathrm{v}t)\gamma - k'_y y$$

so equating $$\omega = \gamma(\omega' + \mathrm{v}k'_x)$$

$$k_x = \gamma\left(k'_x + \frac{\mathrm{v}\omega'}{c^2}\right) \quad \text{and} \quad k_y = k'_y$$

so inverting $$\omega' = \gamma(\omega - \mathrm{v}k_x)$$

$$k'_x = \gamma\left(k_x - \frac{\mathrm{v}\omega}{c^2}\right)$$

$$k'_y = k_y$$

writing $$k'_x = k'\cos\theta',\ k_x = k\cos\theta$$

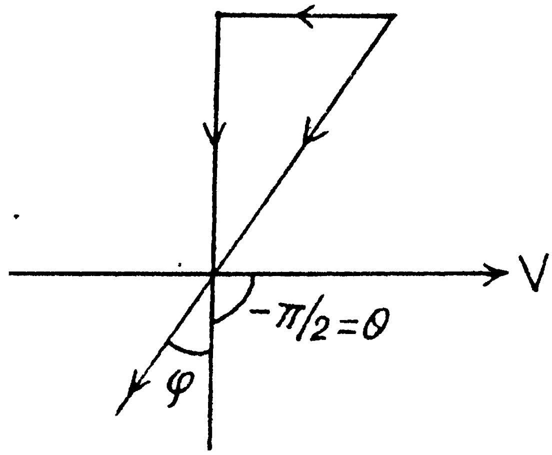
$$k'_y = k'\sin\theta',\ k_y = k\sin\theta$$

we get on using $ck' = \omega'$, $ck = \omega$

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}$$

where $\beta = \frac{v}{c}$ and the primed frame is moving with velocity v in the $x-$ direction w.r.t the unprimed frame.

For small $\beta << 1$, the situation is as shown.



We see that $\cos\theta' = -\beta$

if $\theta = -\pi/2$.

Then $\theta' = -\left(\frac{\pi}{2} + \sin^{-1}\beta\right)$

This is exactly what we get from elementary nonrelativistic law of addition of velocities,

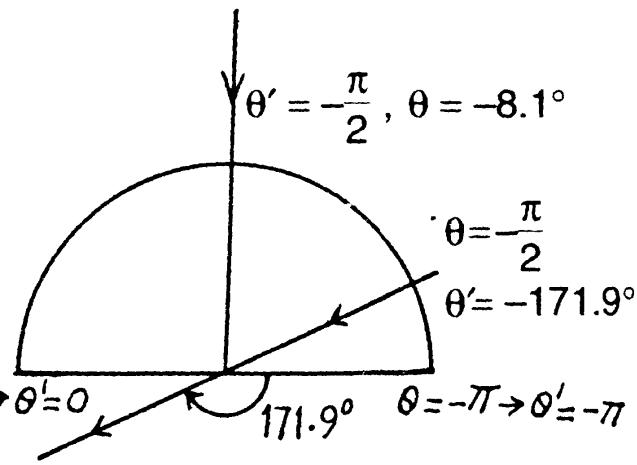
5.242 The statement of the problem is not quite properly worked and is in fact misleading. The correct situation is described below. We consider, for simplicity, stars in the $x-z$ plane. Then the previous formula is applicable,

and we have

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta} = \frac{\cos\theta - 0{\cdot}99}{1 - 0{\cdot}99\cos\theta}$$

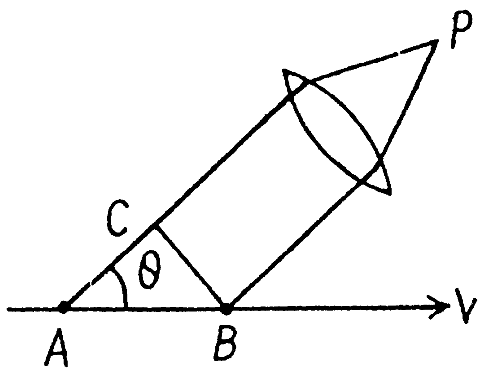
The distribution of θ' is given in the diagram below

The light that appears to come from the forward quadrant in the frame K ($\theta = -\pi$ to $\theta = -\pi/2$) is compressed into an angle of magnitude $+8{\cdot}1°$ in the forward direction while the remaining stars are spread out.



The three dimensional distribution can also be found out from the three dimensional generalization of the formula in the previous problems.

243 The field induced by a charged particle moving with velocity V excites the atoms of the medium turning them into sources of light waves. Let us consider two arbitrary points A and B along the path of the particle. The light waves emitted from these points when the particle passes them reach the point P simultaneously and reinforce each other provided they are in phase which is the case is general if the time taken by the light wave to propagate from the point A to the point C is equal to that taken by the particle to fly over the distance AB. Hence we obtain



$$\cos\theta = \frac{v}{V}$$

where $v = \frac{c}{n}$ is the phase velocity of light. It is evident that the radiation is possible only if $V > v$ i.e. when the velocity of the particle exceeds the phase velocity of light in the medium.

5.244 We must have

$$V \geq \frac{c}{n} = \frac{3}{1 \cdot 6} \times 10^8 \text{ m/s} \quad \text{or} \quad \frac{V}{c} \geq \frac{1}{1 \cdot 6}$$

For electrons this means a K.E. greater than

$$T_e \geq \frac{m_e c^2}{\sqrt{1 - \left(\frac{1}{1 \cdot 6}\right)^2}} - m_e c^2 = m_e c^2 \left[\frac{n}{\sqrt{n^2 - 1}} - 1\right]$$

$$= 0 \cdot 511 \left[\frac{1}{\sqrt{1\left(\frac{1}{1 \cdot 6}\right)^2}} - 1\right] \quad \text{using } m_e c^2 = 0 \cdot 511 \text{ MeV} = 0 \cdot 144 \text{ MeV}$$

For protons with $m_p c^2 = 938$ MeV

$$T_p \geq 938 \left[\frac{1}{\sqrt{1 - \left(\frac{1}{1 \cdot 6}\right)^2}} - 1\right] = 264 \text{ MeV} = 0 \cdot 264 \text{ GeV}$$

Also $$T_{\min} = 29 \cdot 6 \text{ MeV} = m c^2 \left[\frac{1}{\sqrt{1 - \left(\frac{1}{1 \cdot 6}\right)^2}} - 1\right]$$

Then $m c^2 = 105 \cdot 3$ MeV. This is very nearly the mass of means.

5.245 From $\cos \theta = \frac{v}{V}$

we get $$V = v \sec \theta$$

so $$\frac{V}{c} = \frac{v}{c} \sec \theta = \frac{\sec \theta}{n} = \frac{\sec 30°}{1 \cdot 5} = \frac{2/\sqrt{3}}{3/2} = \frac{4}{3\sqrt{3}}$$

Thus for electorns

$$T_e = 0 \cdot 511 \left[\frac{1}{\sqrt{1 - \frac{16}{27}}} - 1\right] = 0 \cdot 511 \left[\sqrt{\frac{27}{11}} - 1\right] = 0 \cdot 289 \text{ MeV}$$

Generally $$T = m c^2 \left[\frac{1}{\sqrt{1 - \frac{1}{n^2 \cos^2 \theta}}} - 1\right]$$

5.7 THERMAL RADIATION. QUANTUM NATURE OF LIGHT

5.246 (a) The most probable radiation frequency ω_{pr} is the frequency for which

$$\frac{d}{d\omega}u_\omega = 3\,\omega^2 F(\omega/T) + \frac{\omega^3}{T}F'(\omega/T) = 0$$

The maximum frequency is the root other than $\omega = 0$ of this equation. It is

$$\omega = -\frac{3\,T F(\omega/T)}{F'(\omega/T)}$$

or $\omega_{pr} = x_0 T$ where x_0 is the solution of the transcendental equation

$$3F(x_0) + x_0 F'(x_0) = 0$$

(b) The maximum spectral density is the density corresponding to most probable frequency. It is

$$(u_\omega)_{max} = x_0^3 F(x_0)\,T^3 \;\alpha\; T^3$$

where x_0 is defined above.

(c) The radiosity is

$$M_e = \frac{c}{4}\int_0^\infty \omega^3 F\left(\frac{\omega}{T}\right) d\omega = T^4\left[\frac{c}{4}\int_0^\infty x^3 F(x)\,dx\right] \alpha\; T^4$$

5.247 For the first black body

$$(\lambda_m)_1 = \frac{b}{T_1}$$

Then $$(\lambda_m)_2 = \frac{b}{T_1} + \Delta\lambda = \frac{b}{T_2}$$

Hence $$T_2 = \frac{b}{\dfrac{b}{T_1} + \Delta\lambda} = \frac{b\,T_1}{b + T_1\,\Delta\lambda} = 1{\cdot}747\,kK$$

5.248 From the radiosity we get the temperature of the black body. It is

$$T = \left(\frac{M_e}{\sigma}\right)^{1/4} = \left(\frac{3{\cdot}0\times 10^4}{5{\cdot}67\times 10^{-8}}\right)^{1/4} = 852{\cdot}9\,K$$

Hence the wavelength corresponding to the maximum emissive capacity of the body is

$$\frac{b}{T} = \frac{0{\cdot}29}{852{\cdot}9}\ \text{cm} = 3{\cdot}4\times 10^{-4}\ \text{cm} = 3{\cdot}4\ \mu\text{m}$$

(Note that $3{\cdot}0\ \text{W/cm}^2 = 3{\cdot}0\times 10^4\ \text{W/m}^2$.)

5.249 The black body temperature of the sun may be taken as

$$T_\odot = \frac{0{\cdot}29}{0{\cdot}48\times 10^{-4}} = 6042\ \text{K}$$

Thus the radiosity is

$$M_{e\odot} = 5{\cdot}67 \times 10^{-8}\,(6042)^4 = 0{\cdot}7555 \times 10^8\ \mathrm{W/m^2}$$

Energy lost by sun is

$$4\pi\,(6{\cdot}95)^2 \times 10^{16} \times 0{\cdot}7555 \times 10^8 = 4{\cdot}5855 \times 10^{26}\ \text{watt}$$

This corresponds to a mass loss of

$$\frac{4{\cdot}5855 \times 10^{26}}{9 \times 10^{16}}\ \mathrm{kg/sec} = 5{\cdot}1 \times 10^9\ \mathrm{kg/sec}$$

The sun loses 1 % of its mass in

$$\frac{1{\cdot}97 \times 10^{30} \times 10^{-2}}{5{\cdot}1 \times 10^9}\ \mathrm{sec} \approx 1{\cdot}22 \times 10^{11}\ \text{years}.$$

5.250 For an ideal gas $p = n\,k\,T$ where n = number density of the particles and $k = \frac{R}{N_A}$ is Boltzman constant. In a fully ionized hydrogen plasma, both H ions (protons) and electrons contribute to pressure but since the mass of electrons is quite small ($\approx m_p/1836$), only protons contribute to mass density. Thus

$$n = \frac{2\rho}{m_H}$$

and

$$p = \frac{2\rho R}{N_A\, m_H}\, T$$

where $m_H \approx m_p$ is the proton or hydrogen mass.

Equating this to thermal radiation pressure

$$\frac{2\rho R}{N_A\, m_H}\, T = \frac{u}{3} = \frac{M_e}{3} \times \frac{4}{c} = \frac{4\sigma T^4}{3c}$$

Then

$$T^3 = \frac{3c\rho R}{2\sigma N_A\, m_H} = \frac{3c\rho R}{\sigma M}$$

where $M = 2N_A\, m_H$ = molecular weight of hydrogen = 2×10^{-3} kg.

Thus

$$T = \left(\frac{3c\rho R}{\sigma M}\right)^{1/3} \approx 1{\cdot}89 \times 10^7\ \mathrm{K}$$

5.251 In time dt after the instant t when the temperature of the ball is T, it loses

$$\pi d^2 \sigma T^4\, dt$$

Joules of energy. As a result its temperature falls by $-dT$ and

$$\pi d^2 \sigma T^4\, dt = -\frac{\pi}{6} d^3 \rho\, C\, dT$$

where ρ = density of copper, C = its sp.heat

Thus $$dt = -\frac{C\rho d}{6\sigma}\frac{dT}{T^4}$$

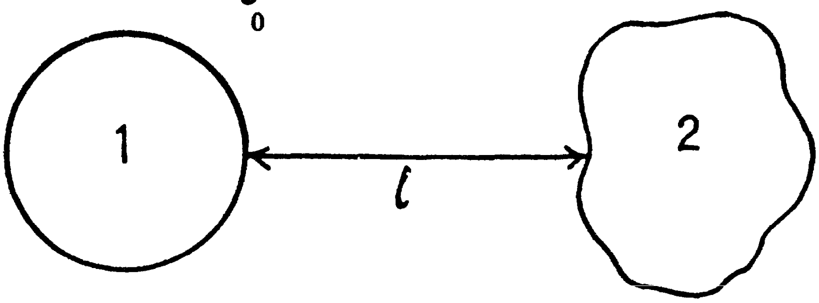
or $$t_0 = \frac{C\rho d}{6\sigma}\int_{T_0}^{T_0/\eta} -\frac{dT}{T^4} = \frac{C\rho d}{18\sigma T_0^3}(\eta^3 - 1) = 2{\cdot}94 \text{ hours}.$$

5.252 Taking account of cosine low of emission we write for the energy radiated per second by the hole in cavity # 1 as

$$dI(\Omega) = A\cos\theta\, d\Omega$$

where A is an constant, $d\Omega$ is an element of solid angle around some direciton defined by the symbol Ω . Integrating over the whole forward hemisphere we get

$$I = A\int_0^{\pi/2} \cos\theta\, 2\pi \sin\theta\, d\theta = \pi A$$



We find A by equating this to the quantity $\sigma T_1^4 \cdot \frac{\pi d^2}{4}$ σ is stefan-Boltzman constant and d is the diameter of th hole.

Then $$A = \frac{1}{4}\sigma d^2 T_1^4$$

Now energy reaching 2 from 1 is ($\cos\theta \approx 1$)

$$\frac{1}{4}\sigma d^2 T_1^4 \cdot \Delta\Omega$$

where $\Delta\Omega = \frac{(\pi d^2/4)}{l^2}$ is the solid angle subtended by the hole of 2 at 1. {We are assuming $d << l$ so $\Delta\Omega$ = area of hole / (distance $)^2$ }.

This must equal $$\sigma T_2^4 \pi d^2/4$$

which is the energy emitted by 2. Thus equating

$$\frac{1}{4}\sigma d^2 T_1^4 \frac{\pi d^2}{4 l^2} = \sigma T_2^4 \frac{\pi d^2}{4}$$

or $$T_2 = T_1\sqrt{\frac{d}{2l}}$$

Substituting we get $$T_2 = 0{\cdot}380\, kK = 380\text{ K}.$$

5.253 (a) The total internal energy of the cavity is

$$U = \frac{4\sigma}{c} T^4 V$$

Hence
$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{16\sigma}{C} T^3 V$$

$$= \frac{16 \times 5{\cdot}67 \times 10^8}{3 \times 10^8} \times 10^9 \times 10^{-3} \text{ Joule/ °K}$$

$$= \frac{1{\cdot}6 \times 5{\cdot}67}{3} n\,\text{J/K} = 3{\cdot}024\, n\,\text{J/K}$$

(b) From first law

$$T\,dS = dU + p\,dV$$

$$= V\,dU + U\,dV + \frac{u}{3} dV \quad \left(p = \frac{U}{3}\right)$$

$$= V\,dU + \frac{4U}{3} dV$$

$$= \frac{16\sigma}{C} V T^3\, dT + \frac{16\sigma}{3C} T^4\, dV$$

so
$$dS = \frac{16\sigma}{C} V T^2\, dT + \frac{16\sigma}{3C} T^3\, dV$$

$$= d\left(\frac{16\sigma}{3C} V T^3\right)$$

Hence
$$S = \frac{16\sigma}{3C} V T^3 = \frac{1}{3} C_V = 1{\cdot}008\, n\,\text{J/K}.$$

5.254 We are given

$$u(\omega, T) = A\,\omega^3 \exp(-a\,\omega/T)$$

(a) Then
$$\frac{du}{d\omega} = \left(\frac{3}{\omega} - \frac{a}{T}\right) u = 0$$

so
$$\omega_{pr} = \frac{3T}{a} = \frac{6000}{7{\cdot}64} \times 10^{12}\ \text{s}^{-1}$$

(b) We determine the spectral distribution in wavelength.

$$-\tilde{u}(\lambda, T)\, d\lambda = u(\omega, T)\, d\omega$$

But
$$\omega = \frac{2\pi c}{\lambda} \quad \text{or} \quad \lambda = \frac{2\pi c}{\omega} = \frac{C'}{\omega}$$

so
$$d\lambda = -\frac{C'}{\omega^2} d\omega,\ d\omega = -\frac{C'}{\lambda^2} d\lambda$$

(we have put a minus sign before $d\lambda$ to subsume just this fact $d\lambda$ is -ve where $d\omega$ is +ve.)

$$\tilde{u}(\lambda, T) = \frac{C'}{\lambda^2} u\left(\frac{C'}{\lambda}, T\right) = \frac{C'^4 A}{\lambda^5} \exp\left(-\frac{a C'}{\lambda T}\right)$$

This is maximum when

$$\frac{\partial \tilde{u}}{\partial \lambda} = 6 = \tilde{u}\left[\frac{-5}{\lambda} + \frac{a C'}{\lambda^2 T}\right]$$

or

$$\lambda_{pr} = \frac{a C'}{5 T} = \frac{2 \pi c a}{5 T} = 1{\cdot}44\ \mu\,\text{m}$$

5.255 From Planek's formula

$$u_\omega = \frac{\hbar \omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar \omega / k T} - 1}$$

(a) In a range $\hbar \omega << k T$ (long wavelength or high temperature).

$$u_\omega \rightarrow \frac{\hbar \omega^3}{\pi^2 c^3} \frac{1}{\frac{\hbar \omega}{k T}}$$

$$= \frac{\omega^2}{\pi^2 c^3} k T, \text{ using } e^x = 1 + x \qquad \text{for small } x.$$

(b) In the range $\hbar \omega >> T$ (high frequency or low temperature) :

$$\frac{\hbar \omega}{k T} >> 1 \text{ so } e^{\frac{\hbar \omega}{k T}} >> 1$$

and

$$u_\omega = \frac{\hbar \omega^3}{\pi^2 c^3} e^{-\hbar \omega / k T}$$

5.256 We write

$$u_\omega\, d\omega = \tilde{u}_\nu\, d\nu \quad \text{where} \quad \omega = 2 \pi \nu$$

Then

$$\tilde{u}_\nu = \frac{2 \pi \hbar (2 \pi \nu)^3}{\pi^2 c^3} \frac{1}{e^{2 \pi \hbar \nu / k T} - 1} = \frac{16 \pi^2 \hbar \nu^3}{c^3} \frac{1}{e^{2 \pi \hbar \nu / k T} - 1}$$

Also

$$-\tilde{u}(\lambda, T)\, d\lambda = u_\omega\, d\omega \quad \text{where} \quad \lambda = \frac{2 \pi c}{\omega},$$

$$d\omega = -\frac{2 \pi c}{\lambda^2} d\lambda$$

$$\tilde{u}(\lambda, T) = \frac{2 \pi c}{\lambda^2} u\left(\frac{2 \pi c}{\lambda}, T\right)$$

$$= \frac{2 \pi c}{\lambda^2} \left(\frac{2 \pi c}{\lambda}\right)^3 \frac{\hbar}{\pi^2 c^3} \frac{1}{e^{2 \pi \hbar c / \lambda k T} - 1} = \frac{16 \pi^2 c \hbar}{\lambda^5} \frac{1}{e^{2 \pi \hbar c / \lambda k T} - 1}$$

5.257 We write the required power in terms of the radiosity by considering only the energy radiated in the given range. Then from the previous problem

$$\Delta P = \frac{c}{4}\tilde{u}(\lambda_m, T)\,\Delta\lambda$$

$$= \frac{4\pi^2 c^2 \hbar}{\lambda_m^5}\,\frac{\Delta\lambda}{e^{2\pi c\hbar/k\lambda_m T} - 1}$$

But

$$\lambda_m T = b$$

so

$$\Delta P = \frac{4\pi^2 c^2 \hbar}{\lambda_m^5}\,\frac{\Delta\lambda}{e^{2\pi c\hbar/kb} - 1}\,\Delta\lambda$$

Using the data

$$\frac{2\pi c\hbar}{kb} = \frac{2\pi \times 3\times 10^8 \times 1{\cdot}05 \times 0^{-34}}{1{\cdot}38\times 10^{-23}\times 2{\cdot}9\times 10^{-3}} = 4.9643$$

$$\frac{1}{e^{2\pi c\hbar/kb} - 1} = 7{\cdot}03\times 10^{-3}$$

and

$$\Delta P = 0{\cdot}312\ \text{W/cm}^2$$

5.258 (a) From the curve of the function $y(x)$ we see that $y = 0{\cdot}5$ when $x = 1{\cdot}41$

Thus

$$\lambda = 1{\cdot}41 \times \frac{0{\cdot}29}{3700}\ \text{cm} = 1{\cdot}105\ \mu\text{m}.$$

(b) At 5000 K

$$\lambda = \frac{0{\cdot}29}{{\cdot}5}\times 10^{-6}\ \text{m} \doteq 0{\cdot}58\ \mu\text{m}$$

So the visible range ($0{\cdot}40$ to $0{\cdot}70$) μm corresponds to a range ($0{\cdot}69$ to $1{\cdot}31$) of x.

From the curve

$$y(0{\cdot}69) = 0{\cdot}07$$
$$y(1{\cdot}31) = 0{\cdot}44$$

so the fraction is 0·37

(c) The value of x corresponding to 0·76 are

$$x_1 = 0{\cdot}76 \Big/ \frac{0{\cdot}29}{0{\cdot}3} = 0{\cdot}786 \text{ at } 3000\,\text{K}$$

$$x_2 = 0{\cdot}76 \Big/ \frac{0{\cdot}29}{0{\cdot}5} = 1{\cdot}31 \text{ at } 5000\,\text{K}$$

The requisite fraction is then

$$\left(\frac{P_2}{P_1}\right) = \underbrace{\left(\frac{T_2}{T_1}\right)^4}_{\text{ratio of total power}} \times \underbrace{\frac{1-y_2}{1-y_1}}_{\text{ratio of the fraction of required wavelengths in the radiated power}}$$

$$= \left(\frac{5}{3}\right)^4 \times \frac{1-0{\cdot}44}{1-0{\cdot}12} = 4{\cdot}91$$

5.259 We use the formula $\varepsilon = \hbar\,\omega$

Then the number of photons in the spectral interval $(\omega,\ \omega + d\,\omega)$ is

$$n(\omega)\,d\,\omega = \frac{u(\omega,\ T)\,d\,\omega}{\hbar\,\omega} = \frac{\omega^2}{\pi^2 c^3}\,\frac{1}{e^{\hbar\,\omega/kT}-1}\,d\,\omega$$

using $$n(\omega)\,d\,\omega = -\tilde{n}(\lambda)\,d\,\lambda$$

we get $$d\,\lambda\,\tilde{n}(\lambda) = n\left(\frac{2\,\pi\,c}{\lambda}\right)\frac{2\,\pi\,c}{\lambda^2}\,d\,\lambda$$

$$= \frac{(2\,\pi\,c)^3}{\pi^2 c^3 \lambda^4}\,\frac{1}{e^{2\pi\hbar c/k\lambda T}-1}\,d\,\lambda$$

$$= \frac{8\,\pi}{\lambda^4}\,\frac{d\,\lambda}{e^{2\pi\hbar c/\lambda kT}-1}$$

5.260 (a) The mean density of the flow of photons at a distance r is

$$<j> = \frac{P}{4\,\pi\,r^2}\bigg/\frac{2\,\pi\,\hbar\,c}{\lambda} = \frac{P\,\lambda}{8\,\pi^2\hbar\,c\,r^2}\ \mathrm{m}^{-2}\,\mathrm{s}^{-2}$$

$$= \frac{10 \times {\cdot}589 \times 10^{-6}}{8\,\pi^2 \times 1{\cdot}054 \times 10^{-34} \times 10^8 \times 4}\ \mathrm{m}^{-2}\,\mathrm{s}^{-1}$$

$$= \frac{10 \times {\cdot}589}{8\,\pi^2 \times 1{\cdot}054 \times 12} \times 10^{16}\ \mathrm{cm}^{-2}\,\mathrm{s}^{-1}$$

$$= 5{\cdot}9 \times 10^{13}\ \mathrm{cm}^{-2}\,\mathrm{s}^{-1}$$

(b) If $n(r)$ is the mean concentration (number per unit volume) of photons at a distance r form the source, then, since all photons are moving outwards with a velocity c, there is an outward flux of cn which is balanced by the flux from the source. In steady state, the two are equal and so

$$n(r) = \frac{<j>}{c} = \frac{P\lambda}{8\pi^2\hbar c^2 r^2} = n$$

so

$$r = \frac{1}{2\pi c}\sqrt{\frac{P\lambda}{2\hbar n}}$$

$$= \frac{1}{6\pi \times 10^8}\sqrt{\frac{10 \times \cdot 589 \times 10^{-6}}{2 \times 1{\cdot}054 \times 10^{-34} \times 10^2 \times 10^6}}$$

$$= \frac{10^2}{6\pi}\sqrt{\frac{5{\cdot}89}{2{\cdot}108}} = 8{\cdot}87 \text{ metre}$$

5.261 The statement made in the question is not always correct. However it is correct in certain cases, for example, when light is incident on a perfect reflector or perfect absorber.

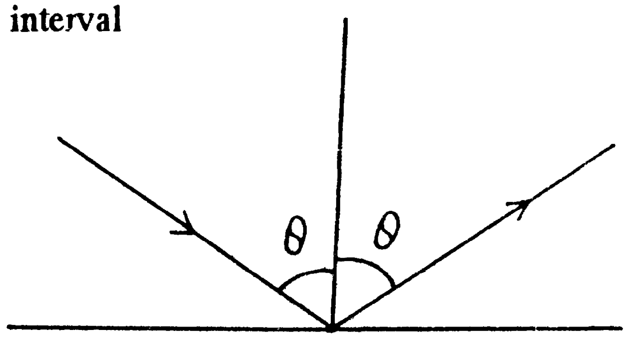
Consider the former case. If light is incident at an angle θ and reflected at the angle θ, then momentum transfered by each photon is

$$2\frac{h\nu}{c}\cos\theta$$

If there are $n(\nu)\,d\nu$ photons in frequency interval $(\nu, \nu + d\nu)$, then total momentum transfered is

$$\int_0^\infty 2\,n(\nu)\frac{h\nu}{c}\cos\theta\,d\nu$$

$$= \frac{2\Phi_e}{c}\cos\theta$$



5.262 The mean pressure $<p>$ is related to the force Fexerted by the beam by

$$<p> \times \frac{\pi d^2}{4} = F$$

The force F equals momentum transfered per second. This is (assuming that photons, not reflected, are absorbed)

$$2\rho\frac{E}{c\tau} + (1-\rho)\frac{E}{c\tau} = (1+\rho)\frac{E}{c\tau}.$$

The first term is the momentum transfered on reflection (see problem (261)); the second on absorption.

$$<p> = \frac{4(1+\rho)E}{\pi d^2 c\tau}$$

Substituting the values we get

$$<p> = 48{\cdot}3 \text{ atmosphere.}$$

5.263 The momentum transfered to the plate is

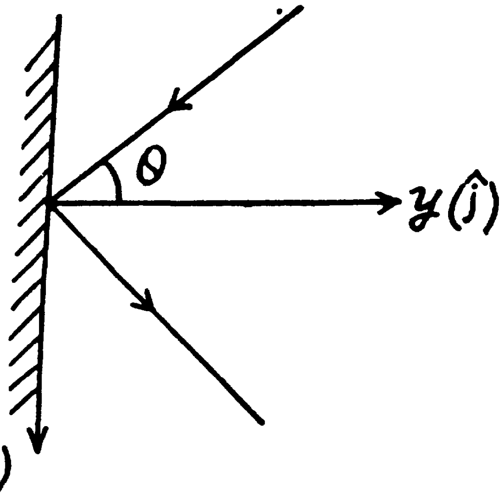
$$= \frac{E}{c}(1-\rho)\{\sin\theta\,\hat{i} - \cos\theta\,\hat{j}\}$$

$\downarrow$

(momentum transfered on absorption) $+\ \frac{E}{c}\rho\{-2\cos\theta\,\hat{j}\}$

$\downarrow$

(momentum transtered on reflection)



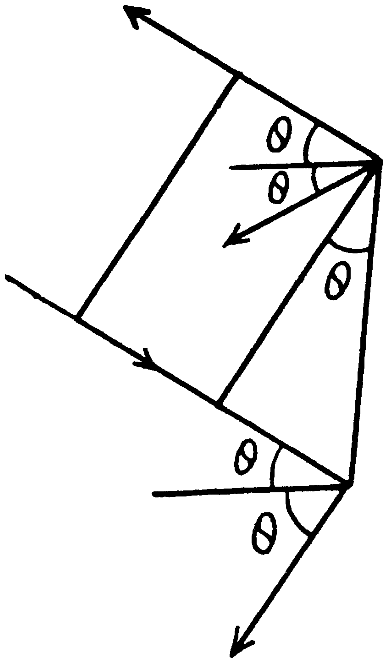
$$= \frac{E}{c}(1-\rho)\sin\theta\,\hat{j} - \frac{E}{c}(1+P)\cos\theta\,\hat{j}$$

Its magnitude is

$$\frac{E}{c}\sqrt{(1-\rho)^2\sin^2\theta + (1+\rho)^2\cos^2\theta} = \frac{E}{c}\sqrt{1+\rho^2+2\rho\cos 2\theta}$$

Substitution gives 35 n N.s as the answer.

5.264 Suppose the mirror has a surtace area A.



The incident bean then has a cross section of $A\cos\theta$ and the incident energy is $IA\cos\theta$: Then the momentum transfered per second (= Force) is from the last problem

$$-\frac{IA\cos\theta}{c}(1+\rho)\cos\theta\,\hat{j} + \frac{IA\cos\theta}{c}(1-\rho)\sin\theta\,\hat{i}$$

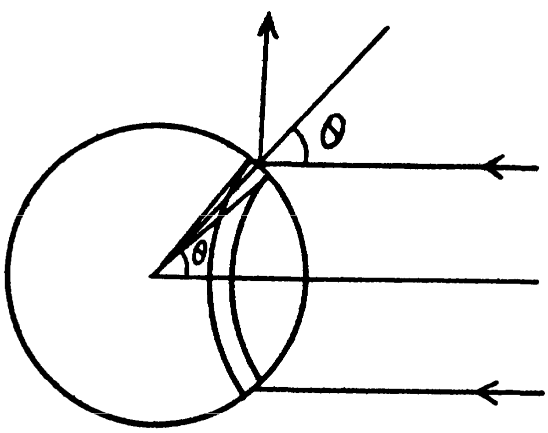
The normal pressure is then $p = \frac{I}{c}(1+\rho)\cos^2\theta$

($\hat{j}$ is the unit vector $\perp^r$ to the plane mirror.)

Putting in the values

$$p = \frac{0{\cdot}20\times10^4}{3\times10^8}\times1{\cdot}8\times\frac{1}{2} = 0{\cdot}6\,n\,\mathrm{N\,cm^{-2}}$$

5.265 We consider a strip defined by the angular range $(\theta, \theta+d\theta)$. From the previous problem the normal pressure exerted on this strip is



$$\frac{2I}{c}\cos^2\theta$$

This pressure gives rise to a force whose resultant, by symmetry is in the direction of the incident light. Thus

$$F = \frac{2I}{c}\int_0^{\pi/2}\cos^2\theta\cdot\cos\theta\cdot2\pi R^2\sin\theta\,d\theta = \pi R^2\frac{I}{c}$$

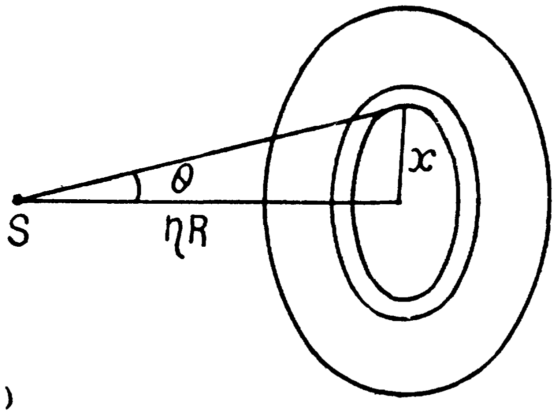
Putting in the values

$$F = \pi \times 25 \times 10^{-4} \frac{0{\cdot}70 \times 10^4}{3 \times 10^8} \text{N} = 0{\cdot}183\ \mu\text{N}$$

5.266 Consider a ring of radius x on the plate. The normal pressure on this ring is, by problem (264),

$$\frac{2}{c} \frac{P}{4\pi(x^2+\eta^2 R^2)} \cdot \cos^2\theta$$

$$= \frac{P}{2\pi c} \frac{\eta^2 R^2}{(x^2+\eta^2 R^2)^2}$$



The total force is then

$$\int_0^R \frac{P}{2\pi c} \frac{\eta^2 R^2}{(x^2+\eta^2 R^2)^2} 2\pi x\, dx$$

$$= \frac{P\eta^2 R^2}{2c} \int_{\eta^2 R^2}^{R^2(1+\eta^2)} \frac{dy}{y^2}$$

$$= \frac{P\eta^2 R^2}{2c} \left[\frac{1}{\eta^2 R^2} - \frac{1}{R^2(1+\eta^2)} \right] = \frac{P}{2c(1+\eta^2)}$$

5.267 (a) In the reference frame fixed to the mirror, the frequency of the photon is, by the Doppler shift formula

$$\overline{\omega} = \omega \sqrt{\frac{1+\beta}{1-\beta}} \left(= \omega \frac{\sqrt{1-\beta^2}}{1-\beta} \right).$$

(see Eqn. (5.6b) of the book.)

In this frame momentum imparted to the mirror is

$$\frac{2\hbar\overline{\omega}}{c} = \frac{2\hbar\omega}{c} \sqrt{\frac{1+\beta}{1-\beta}},$$

(b) In the K frame, the incident particle carries a momentum of $\hbar\omega/c$ and returns with momentum

$$\frac{\hbar\omega}{c} \frac{1+\beta}{1-\beta}$$

(see problem 229). The momentum imparted to the mirror, then, has the magnitude

$$\frac{\hbar\omega}{c} \left[\frac{1+\beta}{1-\beta} + 1 \right] = \frac{2\hbar\omega}{c} \frac{1}{1-\beta}$$

Here $$\beta = \frac{V}{c}.$$

5.268 When light falls on a small mirror and is reflected by it, the mirror recoils. The energy of recoil is obtained from the incident beam photon and the frequency of reflected photons is less than the frequency of the incident photons. This shift of frequency can however be neglected in calculating quantities related to recoil (to a first approximation.)

Thus, the momentum acquired by the mirror as a result of the laser pulse is

$$\left|\vec{p_f} - \vec{p_i}\right| = \frac{2E}{c}$$

Or assuming $\vec{p_i} = 0$, we get

$$\left|\vec{p_f}\right| = \frac{2E}{c}$$

Hence the kinetic energy of the mirror is

$$\frac{p_f^2}{2m} = \frac{2E^2}{mc^2}$$

Suppose the mirror is deflected by an angle θ. Then by conservation of energy

final P.E. $= mgl(1 - \cos\theta) =$ Initial K. E. $= \dfrac{2E^2}{mc^2}$

or $$mgl\,2\sin^2\frac{\theta}{2} = \frac{2E^2}{mc^2}$$

or $$\sin\frac{\theta}{2} = \left(\frac{E}{mc}\right)\frac{1}{\sqrt{gl}}$$

Using the data. $\sin\dfrac{\theta}{2} = \dfrac{13}{10^{-5} \times 3 \times 10^8 \sqrt{9 \cdot 8 \times \cdot 1}} = 4{\cdot}377 \times 10^{-3}$

This gives $\theta = 0{\cdot}502$ degrees.

5.269 We shall only consider stars which are not too compact so that the gravitational field at their surface is weak :

$$\frac{\gamma M}{c^2 R} << 1$$

We shall also clarify the problem by making clear the meaning of the (slightly changed) notation.

Suppose the photon is emitted by some atom whose total relativistic energies (including the rest mass) are E_1 & E_2 with $E_1 < E_2$. These energies are defined in the absence of gravitational field and we have

$$\omega_0 = \frac{E_2 - E_1}{\hbar}$$

as the frequency at infinity of the photon that is emitted in $2 \rightarrow 1$ transition. On the surface of the star, the energies have the values

$$E'_2 = E_2 - \frac{E_2}{c^2} \cdot \frac{\gamma M}{R} = E_2 \left(1 - \frac{\gamma M}{c^2 R} \right)$$

$$E'_1 = E_1 \left(1 - \frac{\gamma M}{c^2 R} \right)$$

Thus, from $\hbar \omega = E'_2 - E'_2$ we get

$$\omega = \omega_0 \left(1 - \frac{\gamma M}{c^2 R} \right)$$

Here ω is the frequency of the photon emitted in the transition $2 \rightarrow 1$ when the atom is on the surface of the star. In shows that the frequency of spectral lines emitted by atoms on the surface of some star is less than the frequency of lines emitted by atoms here on earth (where the gravitational effect is quite small).

Finally $$\frac{\Delta \omega}{\omega_0} = -\frac{\gamma M}{c^2 R}.$$

The answer given in the book is incorrect in general though it agrees with the above result for $\frac{\gamma M}{c^2 R} << 1$.

5.270 The general formula is

$$\frac{2 \pi \hbar c}{\lambda} = e V$$

Thus $$\lambda = \frac{2 \pi \hbar c}{e V}$$

Now $$\Delta \lambda = \frac{2 \pi \hbar c}{e V} \left(1 - \frac{1}{\eta} \right)$$

Hence $$V = \frac{2 \pi \hbar c}{e \Delta \lambda} \left(\frac{\eta - 1}{\eta} \right) = 15{\cdot}9 \text{ kV}$$

5.271 We have as in the above problem

$$\frac{2 \pi \hbar c}{\lambda} = e V$$

On the other hand, from Bragg's law

$$2 d \sin \alpha = k \lambda = \lambda$$

since $k = 1$ when α takes its smallest value.

Thus $$V = \frac{\pi \hbar c}{e d \sin \alpha} = 30{\cdot}974 \, k V \approx 31 \text{ kV}.$$

5.272 The wavelength of X- rays is the least when all the K.E. of the electrons approaching the anticathode is converted into the energy of X- rays.

But the K.E. of electron is

$$T_m = m c^2 \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right]$$

($m c^2$ = rest mass energy of electrons = 0·511 MeV)

Thus
$$\frac{2 \pi \hbar c}{\lambda} = T_m$$

or
$$\lambda = \frac{2 \pi \hbar c}{T_m} = \frac{2 \pi \hbar}{m c} \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right]^{-1}$$

$$= \frac{2 \pi \hbar}{m c (\gamma - 1)}, \ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = 2{\cdot}70 \text{ pm}.$$

5.273 The work function of zinc is

$$A = 3{\cdot}74 \, romane = 3{\cdot}74 \times 1{\cdot}602 \times 10^{-19} \text{ Joule}$$

The threshold wavelength for photoelectric effect is given by

$$\frac{2 \pi \hbar c}{\lambda_0} = A$$

or
$$\lambda_0 = \frac{2 \pi \hbar c}{A} = 331{\cdot}6 \text{ nm}$$

The maximum velocity of photoelectrons liberated by light of wavelength λ is given by

$$\frac{1}{2} m v_{max}^2 = 2 \pi \hbar c \left(\frac{1}{\lambda} - \frac{1}{\lambda_0} \right)$$

So
$$v_{max} = \sqrt{\frac{4 \pi \hbar c}{m} \left(\frac{1}{\lambda} - \frac{1}{\lambda_0} \right)} = 6{\cdot}55 \times 10^5 \text{ m/s}$$

5.274 From the last equation of the previous problem, we find

$$\eta = \frac{(v_1)_{max}}{(v_2)_{max}} = \sqrt{\frac{\frac{1}{\lambda_1} - \frac{1}{\lambda_0}}{\frac{1}{\lambda_2} - \frac{1}{\lambda_0}}}$$

Thus
$$\eta^2 \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_0} \right) = \frac{1}{\lambda_1} - \frac{1}{\lambda_0}$$

or
$$\frac{1}{\lambda_0} (\eta^2 - 1) = \frac{\eta^2}{\lambda_2} - \frac{1}{\lambda_1}$$

and $$\frac{1}{\lambda_0} = \left(\frac{\eta^2}{\lambda_2} - \frac{1}{\lambda_1}\right) \Big/ (\eta^2 - 1)$$

So $$A = \frac{2\pi\hbar c}{\lambda_0} = \frac{2\pi\hbar c}{\lambda_2} \frac{\eta^2 - \frac{\lambda_2}{\lambda_1}}{\eta^2 - 1} = 1{\cdot}88 \text{ eV}$$

5.275 When light of sufficiently short wavelength falls on the ball, photoelectrons are ejected and the copper ball gains positive change. The charged ball tends to resist further emission of electrons by attracting them. When the copper ball has enough charge even the most energetic electrons are unable to leave it. We can calculate this final maximum potential of the copper ball. It is obviously equal in magnitude (in volt) to the maximum K.E of electrons (in electron volts) initially emitted. Hence

$$\varphi_{\max} = \frac{2\pi\hbar c}{\lambda e} - A_{cu}$$
$$= 8{\cdot}86 - 4{\cdot}47 = 4{\cdot}39 \text{ volts}$$

(A_{cu} is the work function of copper.)

5.276 We write

$$E = a(1 + \cos\omega t)\cos\omega_0 t$$
$$= a\cos\omega_0 t + \frac{a}{2}\left[\cos(\omega_0 - \omega)t + \cos(\omega_0 + \omega)t\right]$$

It is obvious that light has three frequencies and the maximum K.E. of photo electrons ejected is

$$h(\omega + \omega_0) - A_{Li}$$

where $A_{Li} = 2{\cdot}39$ eV. Substituting we get $0{\cdot}37$ eV.

5.277 Suppose N photons fall on the photocell per sec. Then the power incident is

$$N\frac{2\pi\hbar c}{\lambda}$$

This will give rise to a photocurrent of $N\frac{2\pi\hbar c}{\lambda}\cdot J$

which means that $$N\frac{2\pi\hbar c}{e\lambda}\cdot J$$

electrons have been emitted. Thus the number of photoelectrons produced by each photon is

$$w = \frac{2\pi\hbar c J}{e\lambda} = 0{\cdot}0198 \approx 0{\cdot}02$$

5.278 A simple application of Einstein's equation

$$\frac{1}{2}m v_{\max}^2 = h\nu - h\nu_0 = \frac{2\pi\hbar c}{\lambda} - A_{cs}$$

gives incorrect result in this case because the photoelectrons emitted by the Cesium electrode are retarded by the small electric field that exists between the cesium electrode and the Copper electrode even in the absence of external emf. This small electric field is caused by the contact potential difference whose magnitude equals the difference of work functions

$$\frac{1}{e}(A_{cu}-A_{cs})\text{ volts}.$$

Its physical origin is explained below.

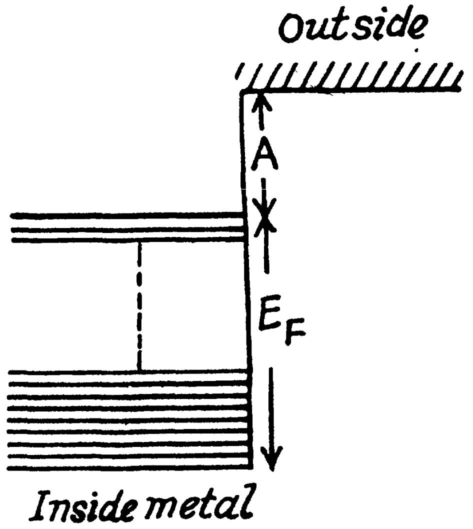
The maximum velocity of the photoelectrons reaching the copper electrode is then

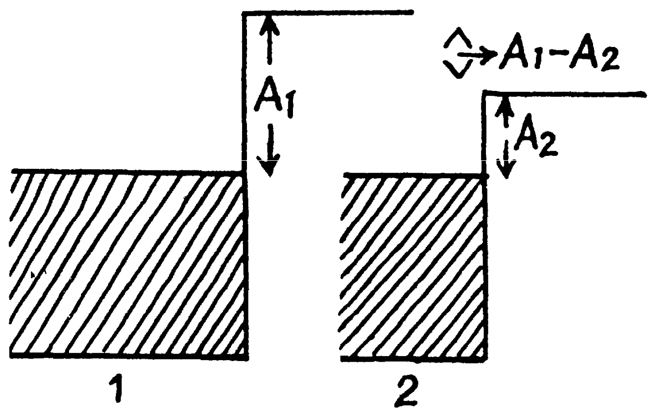
$$\frac{1}{2}m\,v_m^2 = \frac{1}{2}m\,v_0^2-(A_{cu}-A_{cs}) = \frac{2\pi\hbar c}{\lambda}-A_{cu}$$

Here v_0 is the maximum velocity of the photoelectrons immediately after emission. Putting the values we get, on using $A_{cu} = 4{\cdot}47\text{ eV}$, $\lambda = 0{\cdot}22\,\mu\text{m}$,

$$v_m = 6{\cdot}41\times10^5\text{ m/s}$$

The origin of contact potential difference is the following. Inside the metals free electrons can be thought of as a Fermi gas which occupy energy levels upto a maximum called the Fermi energy E_F. The work function A measures the depth of the Fermi level.





When two metals 1 & 2 are in contact, electrons flow from one to the other till their Fermi levels are the same. This requires the appearance of contact potential difference of A_1-A_2 between the two metals externally.

5.279 The maximum K.E. of the photoelectrons emitted by the Zn cathode is

$$E_{\max} = \frac{2\pi\hbar c}{\lambda}-A_{zn}$$

On calculating this comes out to be $0{\cdot}993\text{ eV} \approx 1{\cdot}0\text{ eV}$

Since an external decelerating voltage of $1{\cdot}5\ V$ is required to cancel this current, we infer that a contact potential difference of $1{\cdot}5-1{\cdot}0 = 0{\cdot}5\ V$ exists in the circuit whose polarity is opposite of the decelerating voltage.

5.280 The unit of $\hbar$ is Joule-sec. Since $m\,c^2$ is the rest mass energy, $\frac{\hbar}{m\,c^2}$ has the dimension of time and multiplying by c we get a quantity

$$\lambda\!\!\!^{-}_c = \frac{\hbar}{m\,c}.$$

whose dimension is length. This quantity is called reduced compton wavelength.

(The name compton wavelength is traditionally reserved for $\left.\frac{2\pi\hbar}{m\,c}\right)$.

5.281 We consider the collision in the rest frame of the initial electron. Then the reaction is

$$\gamma + e\,(\text{rest}) \longrightarrow e\,(\text{moving})$$

Energy momentum conservation gives

$$\hbar\,\omega + m_0\,c^2 = m_0\,c^2 \Big/ \sqrt{1-\beta^2}$$

$$\frac{\hbar\,\omega}{c} = \frac{m_0\,c\,\beta}{\sqrt{1-\beta^2}}$$

where ω is the angular frequency of the photon.

Eliminating $h\,\omega$ we get

$$m_0\,c^2 = m_0\,c^2\,\frac{1-\beta}{\sqrt{1-\beta^2}} = m_0\,c^2\sqrt{\frac{1-\beta}{1+\beta}}$$

This gives $\beta = 0$ which implies $\hbar\,\omega = 0$.

But a zero energy photon means no photon.

5.282 (a) Compton scattering is the scattering of light by free electrons. (The free electrons are the electrons whose binding is much smaller than the typical energy transfer to the electrons). For this reason the increase in wavelength $\Delta\,\lambda$ is independent of the nature of the scattered substance.

(b) This is because the effective number of free electrons increases in both cases. With increasing angle of scattering, the energy transfered to electrons increases. With diminishing atomic number of the substance the binding energy of the electrons decreases.

(c) The presence of a non-displaced component in the scattered radiation is due to scattering from strongy bound (inner) electrons as well as nuclei. For scattering by these the atom essentially recoils as a whole and there is very little energy transfer.

5.283 Let λ_0 = wavelength of the incident radiation.
Then
wavelength of the radiation scattened at $\theta_1 = 60°$

$$= \lambda_1 = \lambda_0 + 2\pi\lambda\!\!\!^{-}_c(1-\cos\theta_1) \quad \text{where } \lambda\!\!\!^{-}_c = \frac{\hbar}{mc}.$$

and similarly

$$\lambda_2 = \lambda_0 + 2\pi\lambda\!\!\!^{-}_c(1-\cos\theta_2)$$

From the data $\theta_1 = 60°$, $\theta_2 = 120°$ and

$$\lambda_2 = \eta\lambda_1$$

Thus
$$(\eta-1)\lambda_0 = 2\pi\lambda\!\!\!^{-}_c\left[1-\cos\theta_2-\eta(1-\cos\theta_1)\right]$$

$$= 2\pi\lambda\!\!\!^{-}_c\left[1-\eta+\eta\cos\theta_1-\cos\theta_2\right]$$

Hence
$$\lambda_0 = 2\pi\lambda\!\!\!^{-}_c\left[\frac{\eta\cos\theta_1-\cos\theta_2}{\eta-1}-1\right]$$

$$= 4\pi\lambda\!\!\!^{-}_c\left[\frac{\sin^2\theta_2/2-\eta\sin^2\theta_1/2}{\eta-1}\right] = 1{\cdot}21\text{ pm}.$$

The expression λ_0 given in the book contains misprints.

5.284 The wave lengths of the photon has increased by a fraction η so its final wavelength is

$$\lambda_f = (2+\eta)\lambda_i$$

and its energy is
$$\frac{\hbar\omega}{1+\eta}$$

The K.E. of the compton electron is the energy lost by the photon and is

$$T = \hbar\omega\left(1-\frac{1}{1+\eta}\right) = \hbar\omega\frac{\eta}{1+\eta}$$

5.285 (a) From the Compton formula

$$\lambda' = 2\pi\lambda\!\!\!^{-}_c(1-\cos 90)+\lambda$$

Thus
$$\omega' = \frac{2\pi c}{\lambda'} = \frac{2\pi c}{\lambda+2\pi\lambda\!\!\!^{-}_c} \quad \text{where } 2\pi\lambda\!\!\!^{-}_c = \frac{h}{mc}.$$

Substituting the values. we get $\omega' = 2{\cdot}24\times10^{20}$ rad/sec

(b) The kinetic energy of the scattered electron (in the frame in which the initial electron was stationary) is simply

$$T = \hbar\omega - \hbar\omega'$$

$$= \frac{2\pi\hbar c}{\lambda} - \frac{2\pi\hbar c}{\lambda + 2\pi \lambda\!\!\!^{-}_c}$$

$$= \frac{4\pi^2 \hbar c \lambda\!\!\!^{-}_c}{\lambda(\lambda + 2\pi\lambda_c)} = \frac{2\pi\hbar c/\lambda}{1 + \lambda/2\pi\lambda\!\!\!^{-}_c} = 59{\cdot}5 \text{ kV}$$

5.286 The wave length of the incident photon is

$$\lambda_0 = \frac{2\pi c}{\omega}$$

Then the wavelength of the final photon is

$$\frac{2\pi c}{\omega} + 2\pi\lambda\!\!\!^{-}_c(1 - \cos\theta)$$

and the energy of the final photon is

$$\hbar\omega' = \frac{2\pi\hbar c}{\dfrac{2\pi c}{\omega} + 2\pi\lambda_c(1 - \cos\theta)} = \frac{h\omega}{1 + \dfrac{h\omega}{mc^2}(1 - \cos\theta)}$$

$$= \frac{h\omega}{1 + 2\left(\dfrac{h\omega}{mc^2}\right)\sin^2(\theta/2)} = 144{\cdot}2 \text{ kV}$$

5.287 We use the equation $\lambda = \dfrac{h}{p} = \dfrac{2\pi\hbar}{p}$.

Then from Compton formula

$$\frac{2\pi\hbar}{p'} = \frac{2\pi\hbar}{p} + 2\pi\frac{\hbar}{mc}(1 - \cos\theta)$$

so $$\frac{1}{p'} = \frac{1}{p} + \frac{1}{mc}\cdot 2\sin^2\theta/2$$

Hence $$\sin^2\frac{\theta}{2} = \frac{mc}{2}\left(\frac{1}{p'} - \frac{1}{p}\right)$$

$$= \frac{mc(p - p')}{2pp'}$$

or $$\sin\frac{\theta}{2} = \sqrt{\frac{mc(p - p')}{2pp'}},$$

Substituting from the data

$$\sin\frac{\theta}{2} = \sqrt{\frac{mc^2(cp - cp')}{2cp \cdot cp'}} = \sqrt{\frac{0{\cdot}511\,(1{\cdot}02 - 0{\cdot}255)}{2 \times 1{\cdot}02 \times 0{\cdot}255}}$$

This gives $\theta = 120{\cdot}2$ degrees.

5.288 From the Compton formula

$$\lambda = \lambda_0 + \frac{2\pi\hbar}{mc}(1-\cos\theta)$$

From conservation of energy

$$\frac{2\pi\hbar c}{\lambda_0} = \frac{2\pi\hbar c}{\lambda} + T = \frac{2\pi\hbar c}{\lambda_0 + \frac{2\pi\hbar}{mc}(1-\cos\theta)} + T$$

or

$$\frac{4\pi\hbar}{mc}\sin^2\frac{\theta}{2} = \frac{T}{2\pi\hbar c}\lambda_0\left(\lambda_0 + \frac{4\pi\hbar}{mc}\sin^2\frac{\theta}{2}\right)$$

or introducing $\hbar\omega_0 = 2\pi\hbar c/\lambda_0$

$$\frac{2\sin^2\theta/2}{mc^2} = \frac{T}{\hbar\omega_0}\left(\frac{1}{\hbar\omega_0} + \frac{2}{mc^2}\sin^2\frac{\theta}{2}\right)$$

Hence

$$\left(\frac{1}{\hbar\omega_0}\right)^2 + 2\frac{1}{\hbar\omega_0}\frac{\sin^2\frac{\theta}{2}}{mc^2} - \frac{2\sin^2\frac{\theta}{2}}{mc^2 T} = 0$$

$$\left(\frac{1}{\hbar\omega_0} + \frac{\sin^2\frac{\theta}{2}}{mc^2}\right)^2 = \frac{2\sin^2\frac{\theta}{2}}{mc^2 T} + \left(\frac{\sin^2\frac{\theta}{2}}{mc^2}\right)^2$$

$$\frac{1}{\hbar\omega_0} = \frac{\sin^2\frac{\theta}{2}}{mc^2}\left[\sqrt{1 + \frac{2mc^2}{T\sin^2\theta/2}} - 1\right]$$

or

$$\hbar\omega_0 = \frac{mc^2/\sin^2\theta/2}{\sqrt{1 + \frac{2mc^2}{T\sin^2\frac{\theta}{2}}} - 1}$$

$$= \frac{T}{2}\left[\sqrt{1 + \frac{2mc^2}{T\sin^2\theta/2}} + 1\right]$$

Substituting we get $\hbar\omega_0 = 0{\cdot}677\,\text{MeV}$

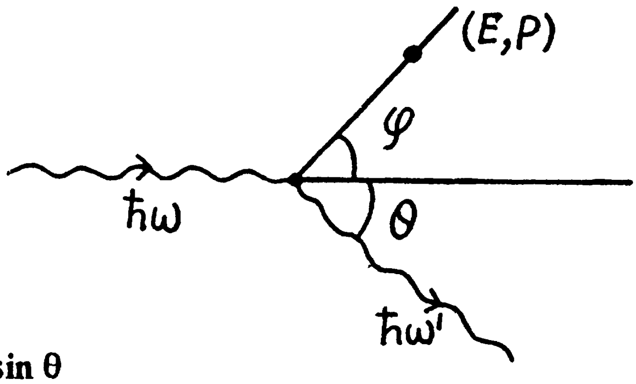
5.289 We see from the previous problem that the electron gains the maximum K.E. when the photon is scattered backwards $\theta = 180°$. Then

$$\omega_0 = \frac{m c^2/\hbar}{\sqrt{1 + \frac{2 m c^2}{T_{max}}} - 1}$$

Hence
$$\lambda_0 = \frac{2\pi c}{\omega_0} = \frac{2\pi\hbar}{m c}\left[\sqrt{1 + \frac{2 m c^2}{T_{max}}} - 1\right]$$

Substituting the values we get $\lambda_0 = 3{\cdot}695\ \text{pm}$.

5.290 Refer to the diagram. Energy momentum conservation gives



$$\frac{\hbar\omega'}{c} - \frac{\hbar\omega'}{c}\cos\theta = p\cos\varphi$$

$$\frac{h\omega'}{c}\sin\theta = p\sin\varphi$$

$$h\omega + m c^2 = h\omega' + E$$

where $E^2 = c^2 p^2 + m^2 c^4$. we see

$$\tan\varphi = \frac{\omega'\sin\theta}{\omega - \omega'\cos\theta} = \frac{\frac{1}{\lambda'}\sin\theta}{\frac{1}{\lambda} - \frac{1}{\lambda'}\cos\theta}$$

$$= \frac{\lambda\sin\theta}{\lambda' - \lambda\cos\theta} = \frac{\sin\theta}{\frac{\Delta\lambda}{\lambda} + 2\sin^2\frac{\theta}{2}}$$

where
$$\Delta\lambda = \lambda' - \lambda = 2\pi\lambda_c(1 - \cos\theta) = 4\pi\lambda_c\sin^2\frac{\theta}{2}$$

Hence
$$\tan\varphi = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\frac{\Delta\lambda}{\lambda} + \frac{\Delta\lambda}{2\pi\lambda_c}}$$

But
$$\sin\theta = 2\sqrt{\frac{\Delta\lambda}{4\pi\bar{\lambda}_c}}\sqrt{1 - \frac{\Delta\lambda}{4\pi\bar{\lambda}_c}} = \frac{\Delta\lambda}{2\pi\bar{\lambda}_c}\sqrt{\frac{4\pi\bar{\lambda}_c}{\Delta\lambda} - 1}$$

Thus
$$\tan\varphi = \frac{\sqrt{\frac{4\pi\hbar}{m c\,\Delta\lambda} - 1}}{1 + \frac{2\pi\hbar}{m c\lambda}} = \frac{\sqrt{\frac{4\pi\hbar}{m c\,\Delta\lambda} - 1}}{1 + \frac{\hbar\omega}{m c^2}} = 31{\cdot}3°$$

5.291 By head on collision we understand that the electron moves on in the direction of the incident photon after the collision and the photon is scaltered backwards. Then, let us write

$$\hbar\omega = \eta m c^2$$

$$\hbar\omega' = \sigma m c^2$$

$$(E, p) = (\varepsilon m c^2, \mu m c) \text{ of the electron.}$$

Then by energy momentum conservation (cancelling factors of mc^2 and mc)

$$1 + \eta = \sigma + \varepsilon$$

$$\eta = \mu - \sigma$$

$$\varepsilon^2 = 1 + \mu^2$$

So eliminating σ & ε $\quad 1 + \eta = -\eta + \mu + \sqrt{\mu^2 + 1}$

or $\quad (1 + 2\eta - \mu) = \sqrt{\mu^2 + 1}$

Squaring $\quad (1 + 2\eta)^2 - 2\mu(1 + 2\eta) = 1$

$$4\eta + 4\eta^2 = 2\mu(1 + 2\eta)$$

or $\quad \mu = \dfrac{2\eta(1+\eta)}{1 + 2\eta}$

Thus the momentum of the Compton electron is

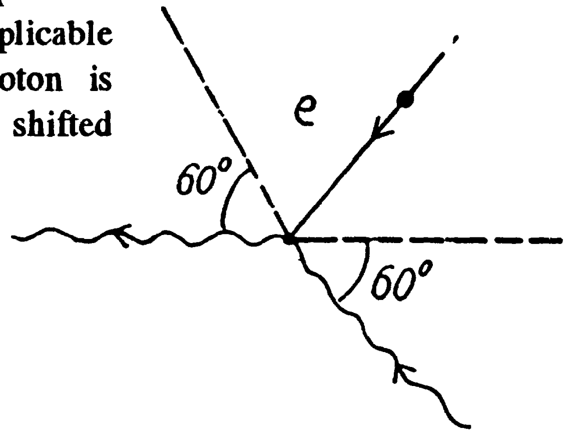
$$p = \mu m c = \frac{2\eta(1+\eta) m c}{1 + 2\eta}.$$

Now in a magnetic field $\quad p = B e \rho$

Thus $\quad \rho = 2\eta(1+\eta) \Big/ (1 + 2\eta) \dfrac{mc}{Be}.$

Substituting the values $\quad \rho = 3{\cdot}412 \text{ cm}.$

5.292 This is the inverse of usual compton scattering.
When we write down the energy-momentum conservation equation for this process we find that they are the same for the inverse process as they arc for the usual process. If follows that the formula for compton shift is applicable except that the energy (frequency) of the photon is increased on scattering and the wavelength is shifted downward. With this understanding, we write



$$\Delta\lambda = 2\pi \frac{\hbar}{mc}(1 - \cos\theta)$$

$$= 4\pi\left(\frac{\hbar}{mc}\right)\sin^2\frac{\theta}{2} = 1{\cdot}21 \text{ pm}$$

☆☆☆

PART SIX

ATOMIC AND NUCLEAR PHYSICS

In this chapter the formulas in the book are given in the CGS units. Since most students are familar only with MKS units, we shall do the problems in MKS units. However, where needed, we shall also write the formulas in the Gaussian units.

6.1 SCATTERING OF PARTICLES. RUTHERFORD-BOHR ATOM

6.1 The Thomson model consists of a uniformly charged nucleus in which the electrons are at rest at certain equilibrium points (the plum in the pudding model). For the hydrogen nucleus the charge on the nucleus is +e while the charge on the electron is -e. The electron by symmetry must be at the centre of the nuclear charge where the potential (from problem (3.38a)) is

$$\varphi_0 = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{3e}{2R}$$

where R is the radius of the nucleur charge distribution. The potential energy of the electron is $-e\varphi_0$ and since the electron is at rest, this is also the total energy. To ionize such an electron will require an energy of $E = e\varphi_0$

From this we find $$R = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{3e^2}{2E}$$

In Gaussian system the factor $\frac{1}{4\pi\varepsilon_0}$ is missing.

Putting the values we get $R = 0{\cdot}159\text{ nm}$.

Light is emitted when the electron vibrates. If we displace the electron slightly inside the nucleus by giving it a push r in some radial direction and an energy δE of oscillation then since the potential at a distance r in the nucleus is

$$\varphi(r) = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{e}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right).$$

the total energy of the nucleus becomes

$$\frac{1}{2}m\dot{r}^2 - \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{e^2}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right) = -e\varphi_0 + \delta E$$

or $$\delta E = \frac{1}{2}m\dot{r}^2 + \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{e^2}{2R^3}r^2$$

This is the energy of a harmonic oscillator whose frequency is : $$\omega^2 = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{e^2}{mR^3}$$

The vibrating electron emits radiation of frequency ω whose wavelength is

$$\lambda = \frac{2\pi c}{\omega} = \frac{2\pi c}{e}\sqrt{mR^3}\,(4\pi\varepsilon_0)^{1/2}$$

In Gaussian units the factor $(4\pi\varepsilon_0)^{1/2}$ is missing.

Putting the values we get $\lambda = 0{\cdot}237\,\mu\,\mathrm{m}$.

6.2 Equation (6.1a) of the book reads in MKS units

$$\tan\theta/2 = \left(\frac{q_1 q_2}{4\pi\varepsilon_0}\right)\Big/ 2bT$$

Thus
$$b = \left(\frac{q_1 q_2}{4\pi\varepsilon_0}\right)\frac{\cot\theta/2}{2T}$$

For α particle $q_1 = 2e$, for gold $q_2 = 79e$

(In Gaussian units there is no factor $\left(\frac{1}{4\pi\varepsilon_0}\right)$.)

Substituting we get $b = 0{\cdot}731\,pm$.

6.3 (a) In the Pb case we shall ignore the recoil of the nucleus both because Pb is quite heavy ($A_{Pb} = 208 = 52\times A_{He}$) as well as because Pb in not free. Then for a head on collision, at the distance of closest approach, the K.E. of the α – particle must become zero (because α – particle will turn back at this point). Then

$$\frac{2Z\ e^2}{(4\pi\varepsilon_0)\,r_{\min}} = T$$

(No $(4\pi\varepsilon_0)$ in Gaussian units.). Thus putting the values

$$r_{\min} = 0{\cdot}591\ \mathrm{pm}.$$

(b) Here we have to take account of the fact that part of the energy is spent in the recoil of Li nucleus. Suppose x_1 = coordinate of the α – particle from some arbitrary point on the line joining it to the Li nucleus, x_2 = coordinate of the Li nucleus with respect to the same point. Then we have the energy momentum equations

$$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{2\times 3\,e^2}{(4\pi\varepsilon_0)\,|x_1 - x_2|} = T$$

$$m_1\dot{x}_1 + m_2\dot{x}_2 = \sqrt{2m_1 T}$$

Here m_1 = mass of He^{++} nucleus, m_2 = mass of Li nucleus. Eliminating $\dot{x}_2$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2m_2}\left(\sqrt{2m_1 T} - m_1\dot{x}_1\right)^2 + \frac{6\,e^2}{(4\pi\varepsilon_0)(x_1 - x_2)}$$

We complete the square on the right hand side and rewrite the above equation as

$$\frac{m_2}{m_1+m_2}T = \frac{1}{2m_2}\left[\sqrt{m_1(m_1+m_2)}\,\dot{x}_1 - \sqrt{\frac{m_1}{m_1+m_2}}\sqrt{2m_1T}\right]^2 + \frac{6e^2}{(4\pi\varepsilon_0)|x_1-x_2|}$$

For the least distance of apporach, the second term on the right must be greatest which implies that the first term must vanish.

Thus $$|x_1-x_2|_{\min} = \frac{6e^2}{(4\pi\varepsilon_0)T}\left(1+\frac{m_1}{m_2}\right)$$

Using $\frac{m_1}{m_2} = \frac{4}{7}$ and other values we get

$$|x_1-x_2|_{\min} = \cdot 034\ \text{pm}.$$

(In Gaussian units the factor $4\pi\varepsilon_0$ is absent).

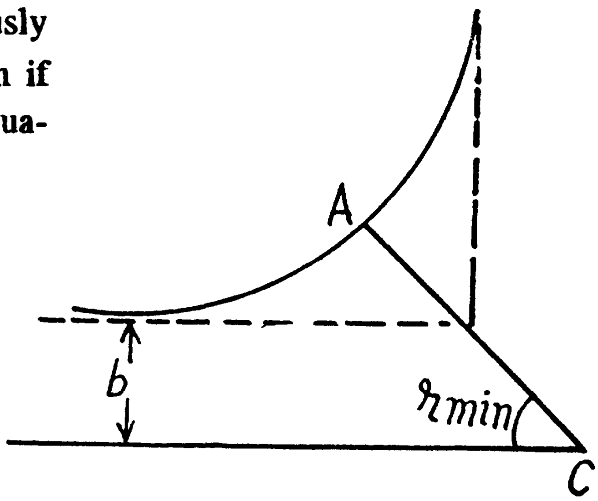
6.4 We shall ignore the recoil of Hg nucleus.

(a) Let A be the point of closest approach to the centre C, $AC = r_{\min}$. At A the motion is instantaneously circular because the radial velocity vanishes. Then if v_0 is the speed of the particle at A, the following equations hold

$$\Gamma = \frac{1}{2}mv_0^2 + \frac{Z_1Z_2e^2}{(4\pi\varepsilon_0)r_{\min}} \tag{1}$$

$$mv_0r_{\min} = \sqrt{2mT}\,b \tag{2}$$

$$\frac{mv_0^2}{\rho_{\min}} = \frac{Z_1Z_2e^2}{(4\pi\varepsilon_0)r_{\min}^2} \tag{3}$$



(This is Newton's law. Here $\rho = \rho_{\min}$ is the radius of curvature of the path at A and ρ is minimum at A by symmetry.) Finally we have Eqn. (6.1 a) in the form

$$b = \frac{Z_1Z_2e^2}{(4\pi\varepsilon_0)2T}\cot\frac{\theta}{2} \tag{4}$$

From (2) and (3) $$\frac{2Tb^2}{\rho_{\min}} = \frac{Z_1Z_2e^2}{(4\pi\varepsilon_0)}$$

or $$\rho_{\min} = \frac{Z_1Z_2e^2}{(4\pi\varepsilon_0)2T}\cot^2\frac{\theta}{2},$$

with $z_1 = 2$, $z_2 = 80$ we get

$$\rho_{\min} = 0{\cdot}231\ \text{pm}.$$

(b) From (2) and (4) we write

$$r_{\min} = \frac{Z_1 \; Z_2 \; e^2}{(4\pi\varepsilon_0)\sqrt{2mT}} \; \frac{\cot\theta/2}{v_0},$$

Substituting in (1) $\quad T = \frac{1}{2} m v_0^2 + \sqrt{2mT}\, v_0 \tan\theta/2$

Solving for v_0 we get $\quad v_0 = \sqrt{\frac{2T}{m}}\left(\sec\frac{\theta}{2} - \tan\frac{\theta}{2}\right)$

Then

$$r_{\min} = \frac{Z_1 \; Z_2 \; e^2}{(4\pi\varepsilon_0)\,2T} \; \frac{\cot\frac{\theta}{2}}{\sec\frac{\theta}{2} - \tan\frac{\theta}{2}}$$

$$= \frac{Z_1 \; Z_2 \; e^2}{(4\pi\varepsilon_0)\,2T} \cot\frac{\theta}{2}\left(\sec\frac{\theta}{2} + \tan\frac{\theta}{2}\right)$$

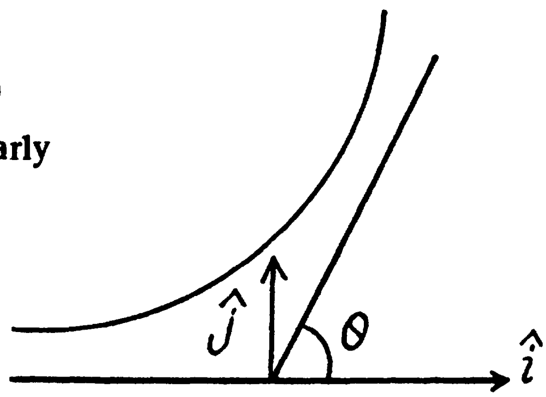
$$= \frac{Z_1 \; Z_2 \; e^2}{(4\pi\varepsilon_0)\,2T}\left(1 + \operatorname{cosec}\frac{\theta}{2}\right) = 0.557 \text{ pm}.$$

6.5 By momentum conservation

$$\underset{(\text{proton})}{\vec{P}} + \underset{(\text{Au})}{\vec{P_i}} = \underset{(\text{proton})}{\vec{P'}} + \underset{(\text{Au})}{\vec{P_f}}$$

Thus the momentum transfered tó the gold nucleus is clearly

$$\Delta\vec{P} = \vec{P_f} - \vec{P_i} = \vec{p} - \vec{p'}$$



Although the momentum transfered to the Au nucleus is not small, the energy associated with this recoil is quite small and its effect back on the motion of the proton can be neglected to a first approximation. Then

$$\Delta\vec{P} \approx \sqrt{2mT}\,(1 - \cos\theta)\,\hat{i}\;\sqrt{2mT}\,\sin\theta\,\hat{j}$$

Here $\hat{i}$ is the unit vector in the direction of the incident proton and $\hat{j}$ is normal to it on the side on which it is scattered. Thus

$$|\Delta\vec{P}| \approx 2\sqrt{2mT}\,\sin\frac{\theta}{2}$$

Or using $\tan\theta/2 = \dfrac{Z\,e^2}{(4\pi\varepsilon_0)\,2bT}$ for the proton we get

$$|\Delta\vec{P}| \approx 2\sqrt{2mT \Big/ \left(1 + \left(\frac{2bT(4\pi\varepsilon_0)}{Z\,e^2}\right)^2\right)}$$

6.6 The proton moving by the electron first accelerates and then decelerates and it not easy to calculate the energy lost by the proton so energy conservation does not do the trick. Rather

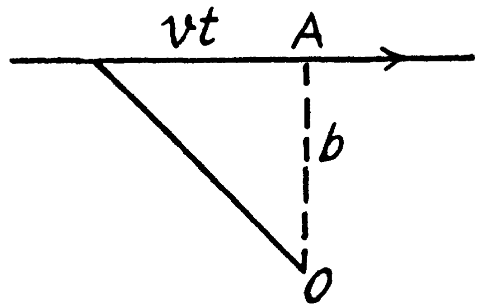
we must directly calculate the momentum acquire by the electron. By symmetry that momentum is along $\vec{OA}$ and its magnitude is

$$P_d = \int F_\perp \, dt$$

where $F_\perp$ is the component along OA of the force on electron. Thus

$$P_d = \int_{-\infty}^{\infty} \frac{e^2}{4\pi\varepsilon_0} \frac{b}{\sqrt{b^2+v^2t^2}} \cdot \frac{1}{b^2+v^2t^2} dt$$

$$= \frac{e^2 b}{4\pi\varepsilon_0 v} \cdot \int_{-\infty}^{\infty} \frac{dx}{(b^2+x^2)^{3/2}}$$



Evaluate the integral by substituting

$$x = b\tan\theta$$

Then
$$P_e = \frac{2e^2}{(4\pi\varepsilon_0)vb}.$$

Then
$$T_e = \frac{P_e^2}{2m_e} = \frac{m_p e^4}{(4\pi\varepsilon_0)^2 T b^2 m_e}$$

In Gaussian units there is no factor $(4\pi\varepsilon_0)^2$. Substituting the values we get

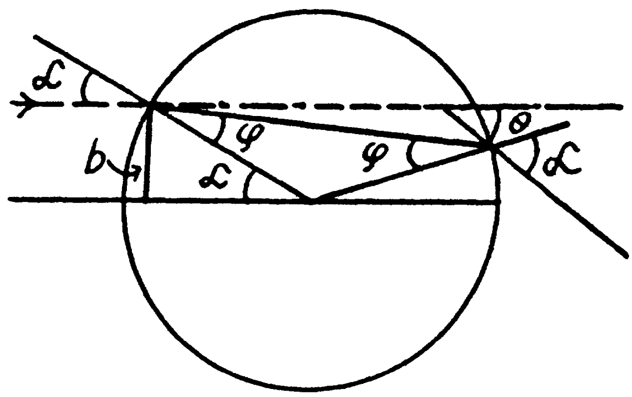
$$T_e = 3{\cdot}82\text{ eV}.$$

6.7 See the diagram on the next page. In the region where potential is nonzero, the kinetic energy of the particle is, by energy conservation,

$T + U_0$ and the momentum of the particle has the magnitude $\sqrt{2m(T+U_0)}$. On the boundary the force is radial, so the tangential component of the momentum does not change :

$\sqrt{2mT}\sin\alpha = \sqrt{2m(T+U_0)}\sin\varphi$

so $\sin\varphi = \sqrt{\frac{T}{T+U_0}}\sin\alpha = \frac{\sin\alpha}{n}$



where $n = \sqrt{1+\frac{U_0}{T}}$. We also have

$$\theta = 2(\alpha - \varphi)$$

Therefore

$$\sin\frac{\theta}{2} = \sin(\alpha-\varphi) = \sin\alpha\cos\varphi - \cos\alpha\sin\varphi$$

$$= \sin\alpha\left(\cos\varphi - \frac{\cos\alpha}{n}\right)$$

or
$$\frac{n\sin\theta/2}{\sin\alpha} = \sqrt{n^2-\sin^2\alpha} - \cos\alpha$$

or
$$\left(\frac{n\sin\theta/2}{\sin\alpha} + \cos\alpha\right)^2 = n^2 - \sin^2\alpha$$

or
$$n^2\sin^2\frac{\theta}{2}\cot^2\alpha + 2n\sin\frac{\theta}{2}\cot\alpha + 1 = n^2\cos^2\frac{\theta}{2}$$

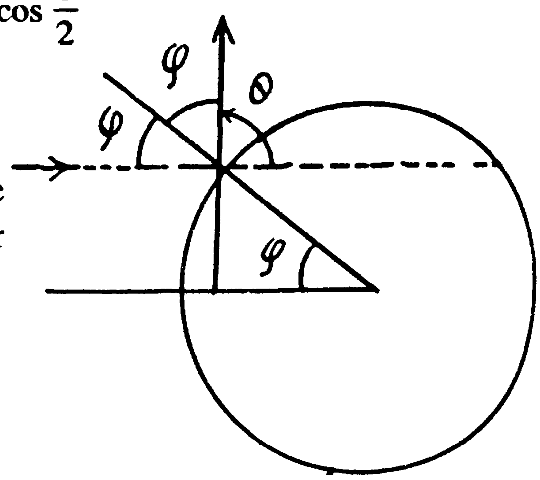
or
$$\cot\alpha = \frac{n\cos\frac{\theta}{2} - 1}{n\sin\frac{\theta}{2}}$$

Hence
$$\sin\alpha = \frac{n\sin\frac{\theta}{2}}{\sqrt{1+n^2-2n\cos\frac{\theta}{2}}}$$

Finally, the impact parameter is

$$b = R\sin\alpha = \frac{nR\sin\frac{\theta}{2}}{\sqrt{1+n^2-2n\cos\frac{\theta}{2}}}.$$

6.8 It is implied that the ball is too heavy to recoil.



(a) The trajectory of the particle is symmetrical about the radius vector through the point of impact. It is clear from the diagram that

$$\theta = \pi - 2\varphi \quad \text{or} \quad \varphi = \frac{\pi}{2} - \frac{\theta}{2}.$$

Also $\quad b = (R+r)\sin\varphi = (R+r)\cos\frac{\theta}{2}.$

(b) With b defined above, the fraction of particles scattered between θ and $\theta + d\theta$ (or the probability of the same) is

$$dP = \frac{|2\pi b\,db|}{\pi(R+r)^2} = \frac{1}{2}\sin\theta\,d\theta$$

(c) This is

$$P = \int_0^{\pi/2}\frac{1}{2}\sin\theta\,d\theta = \frac{1}{2}\int_{-1}^{0} d(-\cos\theta) = \frac{1}{2}$$

6.9 From the formula (6.1 b) of the book

$$\frac{dN}{N} = n\left(\frac{Ze^2}{(4\pi\varepsilon_0)2T}\right)^2 \frac{d\Omega}{\sin^4\frac{\theta}{2}}.$$

We have put $q_1 = 2e$, $q_2 = Ze$ here. Also n = no. of Pt nuclei in the foil per unit area

$$= \underbrace{(At\rho)}_{\text{mass of the foil}} \cdot \underbrace{\frac{N_A}{A_{Pt}} \cdot \frac{1}{A}}_{\text{no. of nuclei per unit mass}} = \frac{N_A\,\rho\,t}{A_{Pt}}$$

Using the values $A_{Pt} = 195$, $\rho = 21{\cdot}5 \times 10^3\,\text{kg/m}^3$

$$N_A = 6{\cdot}023 \times 10^{26}/\text{kilo mole}$$

we get $\quad n = 6{\cdot}641 \times 10^{22}$ per m^2

Also $\quad d\Omega = \frac{dS_n}{r^2} = 10^{-2}\,\text{Sr}$

Substituting we get $\quad \frac{dN}{N} = 3{\cdot}36 \times 10^{-5}$

6.10 A scattered flux density of J (perticles per unit area per second) equals $J\Big/\frac{1}{r^2} = r^2 J$ particles scattered per unit time per steradian in the given direction. Let n = concentration of the gold nuclei in the foil. Then

$$n = \frac{N_A\,\rho}{A_{Au}}$$

and the number of Au nuclei per unit area of the foil is nd where d = thickness of the foil Then from Eqn. (6.1 b) (note that $n \to nd$ here)

$$r^2 J = dN = n\,d\,I\left(\frac{Ze^2}{(4\pi\varepsilon_0)2T}\right)^2 \operatorname{cosec}^4\frac{\theta}{2}$$

Here I is the number of α – particles falling on the foil per second

Hence $\quad d = \dfrac{4T^2 r^2 J}{nI\left(\dfrac{Ze^2}{4\pi\varepsilon_0}\right)^2}\sin^4\theta/2$

using $Z = 79$, $A_{Au} = 197$, $\rho = 19{\cdot}3 \times 10^3\,\text{kg/m}^3$ $N_A = 6{\cdot}023 \times 10^{26}$ / kilo mole and other data from the problem we get

$$d = 1{\cdot}47\,\mu\,\text{m}$$

6.11 From the formula (6.1 b) of the book, we find

$$\frac{dN_{Pt}}{dN_{Ag}} = \frac{n_{Pt}}{n_{Ag}} \cdot \frac{Z_{Pt}^2}{Z_{Ag}^2} = \eta$$

But since the foils have the same mass thickness ($= \rho d$), we have

$$\frac{n_{Pt}}{n_{Ag}} = \frac{A_{Ag}}{A_{Pt}}$$

see the problem (6.9). Hence

$$Z_{Pt} = Z_{Ag} \cdot \sqrt{\frac{\eta A_{Ag}}{A_{Pt}}}$$

Substituting $Z_{Ag} = 47$, $A_{Ag} = 108$, $A_{Pt} = 195$ and $\eta = 1{\cdot}52$ we get

$$Z_{Pt} = 77{\cdot}86 \approx 78$$

6.12 (a) From Eqn. (6.1 b) we get

$$dN = I_0 \tau \frac{\rho \cdot d \cdot N_A}{A_{Au}} \left(\frac{Z e^2}{(4 \pi \varepsilon_0) 2 T} \right)^2 \frac{2 \pi \sin \theta \, d \theta}{\sin^4 \theta/2}$$

(we have used $d\Omega = 2 \pi \sin \theta \, d \theta$ and $N = I_0 I$)

From the data

$$d\theta = 2° = \frac{2}{57{\cdot}3} \text{ radian}$$

Also $Z_{Au} = 79$, $A_{Au} = 197$. Putting the values we get

$$dN = 1{\cdot}63 \times 10^6$$

(b) This number is

$$N(\theta_0) = I_0 \tau \left(\frac{\rho \, d N_A}{A_{Au}} \right) \left(\frac{Z e^2}{(4 \pi \varepsilon_0) 2 T} \right)^2 4 \pi \int_{\theta_0}^{\pi} \frac{\cos \frac{\theta}{2} d \theta}{\sin^3 \frac{\theta}{2}}$$

The integral is

$$2 \int_{\sin \frac{\theta_0}{2}}^{1} \frac{dx}{x^2} = \frac{1}{2} \left[\frac{-1}{2x^2} \right]_{\sin \frac{\theta_0}{2}}^{1} = \cot^2 \frac{\theta_0}{2}$$

Thus
$$N(\theta_0) = \pi n d \left(\frac{Z e^2}{(4 \pi \varepsilon_0) T} \right)^2 I_0 \tau \cot^2 \frac{\theta_0}{2}$$

where n is the concentration of nuclei in the foil. ($n = \rho N_1 / A_{Au}$)
Substitution gives

$$N(\theta_0) = 2{\cdot}02 \times 10^7$$

6.13 The requisite probability can be written easily by analogy with (*b*) of the previous problem. It is

$$P = \frac{N(\pi/2)}{I_0 \tau} = n d \left(\frac{Z e^2}{(4 \pi \varepsilon_0) 2 m v^2} \right)^2 4 \pi \int_{\pi/2}^{\pi} \frac{\cos \theta/2 \, d\theta}{\sin^3 \frac{\theta}{2}}$$

The integral is unity. Thus

$$P = \pi n d \left(\frac{Z e^2}{(4 \pi \varepsilon_0) m v^2} \right)^2$$

Substitution gives using

$$n = \frac{\rho_{Ag} N_A}{A_{Ag}} = \frac{10{\cdot}5 \times 10^3 \times 6.023 \times 10^{26}}{108}, \; P = {\cdot}006$$

6.14 Because of the $\operatorname{cosec}^4 \frac{\theta}{2}$ dependence of the scattering, the number of particles (or fraction) scattered through $\theta < \theta_0$ cannot be calculated directly. But we can write this fraction as

$$P(\theta_0) = 1 - Q(\theta_0)$$

where $Q(\theta_0)$ is the fraction of particles scattered through $\theta \geq \theta_0$. This fraction has been calculated before and is (see the results of 6.12 (b))

$$Q(\theta_0) = \pi n \left(\frac{Z e^2}{(4 \pi \varepsilon_0) T} \right)^2 \cot^2 \frac{\theta_0}{2}$$

where n here is number of nuclei/cm^2. Using the data we get

$$Q = 0{\cdot}4$$

Thus $$P(\theta_0) = 0{\cdot}6$$

6.15 The relevant fraction can be immediately written down (see 6.12 (b)) (Note that the projectiles are protons)

$$\frac{\Delta N}{N} = \left(\frac{e^2}{(4 \pi \varepsilon_0) 2 T} \right)^2 \pi \cot^2 \frac{\theta_0}{2} \cdot (n_1 Z_1^2 + n_2 Z_2^2)$$

Here $n_1 (n_2)$ is the number of $Z_n (Cu)$ nuclei per cm^2 of the foil and $Z_1 (Z_2)$ is the atomic number of $Z_n (Cu)$. Now

$$n_1 = \frac{\rho d N_A}{M_1} = 0{\cdot}7 , \; n_2 = \frac{\rho d N_A}{M_2} = 0{\cdot}3$$

Here M_1, M_2 are the mass numbers of Z_n and Cu.

Then, substituting the values $Z_1 = 30$, $Z_2 = 29$, $M_1 = 65{\cdot}4$, $M_2 = 63{\cdot}5$, we get

$$\frac{\Delta N}{N} = 1{\cdot}43 \times 10^{-3}$$

6.16 From the Rutherford scattering formula

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{(4\pi\varepsilon_0)2T}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

or

$$d\sigma = \left(\frac{Ze^2}{(4\pi\varepsilon_0)2T}\right)^2 \frac{2\pi\sin\theta\, d\theta}{\sin^4\frac{\theta}{2}}$$

$$= \left(\frac{Ze^2}{(4\pi\varepsilon_0)T}\right)^2 \pi\frac{\cos\theta/2\, d\theta}{\sin^3\theta/2}$$

Then integrating from $\theta = \theta_0$ to $\theta = \pi$ we get the required cross section

$$\Delta\sigma = \left(\frac{Ze^2}{(4\pi\varepsilon_0)T}\right)^2 \pi\int_{\theta_0}^{\pi} \frac{\cos\theta/2\, d\theta}{\sin^3\frac{\theta}{2}}$$

$$= \left(\frac{Ze^2}{(4\pi\varepsilon_0)2T}\right)^2 \cot^2\frac{\theta_0}{2}.$$

For U nucleus $Z = 92$ and we get on putting the values

$$\Delta\sigma = 737\,b = 0{\cdot}737\,\text{kb}.$$

$(1b = 1 \text{ barn} = 10^{-28}\ \text{m}^2)$.

6.17 (a) From the previous formula

$$\Delta\sigma = \left(\frac{Ze^2}{(4\pi\varepsilon_0)2T}\right)^2 \pi\cot^2\frac{\theta_0}{2}$$

or

$$T = \frac{Ze^2}{4\pi\varepsilon_0}\cot\frac{\theta_0}{2}\sqrt{\frac{\pi}{\Delta\sigma}}$$

Substituting the values with $Z = 79$ we get ($\theta_0 = 90°$)

$$T = 0{\cdot}903\ \text{MeV}$$

(b) The differential scattering cross section is

$$\frac{d\sigma}{d\Omega} = C\operatorname{cosec}^4\frac{\theta}{2}$$

where

$$\Delta\sigma(\theta > \theta_0) = 4\pi C\cot^2\frac{\theta_0}{2}$$

Thus from the given data

$$C = \frac{500}{4\pi}\,b = 39{\cdot}79\ \text{b/sr}$$

So $$\frac{d\sigma}{d\Omega}(\theta = 60°) = 39{\cdot}79 \times 16 \text{ b/sr} = 0{\cdot}637 \text{ kb/sr}.$$

6.18 The formula in MKS units is

$$\frac{dE}{dt} = -\frac{\mu_0 e^2}{6\pi c}\vec{w}^2$$

For an electron performing (linear) harmonic vibrations $\vec{w}$ is in some definite directions with

$$w_x = -\omega^2 x \text{ say.}$$

Thus $$\frac{dE}{dt} = -\frac{\mu_0 e^2 \omega^4}{6\pi c}x^2$$

If the radiation loss is small (i.e. if ω is not too large), then the motion of the electron is always close to simple harmonic with slowly decreasing amplitude. Then we can write

$$E = \frac{1}{2} m\omega^2 a^2$$

and $$x = a\cos\omega t$$

and average the above equation ignoring the variation of a in any cycle. Thus we get the equation, on using $< x^2 > = \frac{1}{2}a^2$

$$\frac{dE}{dt} = -\frac{\mu_0 e^2 \omega^4}{6\pi c}\cdot\frac{1}{2}a^2 = -\frac{\mu_0 e^2 \omega^2}{6\pi m c}E$$

since $E = \frac{1}{2} m\omega^2 a^2$ for a harmonic oscillator.

This equation integrates to

$$E = E_o e^{-t/T}$$

where $$T = 6\pi m c/e^2\omega^2\mu_0.$$

It is then seen that energy decreases η times in

$$t_0 = T\ln\eta = \frac{6\pi m c}{e^2\omega^2\mu_0}\ln\eta = 14{\cdot}7 \text{ ns.}$$

6.19 Moving around the nucleus, the electron radiates and its energy decreases. This means that the electron gets nearer the nucleus. By the statement of the problem we can assume that the electron is always moving in a circular orbit and the radial acceleration by Newton's law is

$$w = \frac{e^2}{(4\pi\varepsilon_0)m r^2}$$

directed inwards. Thus

$$\frac{dE}{dt} = -\frac{\mu_0 e^6}{6\pi c}\frac{1}{(4\pi\varepsilon_0)^2 m^2 r^4}$$

On the other hand in a circular orbit

$$E = -\frac{e^2}{(4\pi\varepsilon_0)2r}$$

so

$$\frac{e^2}{(4\pi\varepsilon_0)2r^2}\frac{dr}{dt} = -\frac{\mu_0 e^6}{(4\pi\varepsilon_0)^2 6\pi c m^2 r^4}$$

or

$$\frac{dr}{dt} = -\frac{\mu_0 e^4}{(4\pi\varepsilon_0)3\pi c m^2 r^2}$$

Integrating

$$r^3 = r_0^3 - \frac{\mu_0 e^4}{4\pi^2\varepsilon_0 c m^2}t$$

and the radius falls to zero in

$$t_0 = \frac{4\pi^2\varepsilon_0 c m^2 r_0^3}{\mu_0 e^4}\text{sec.} = 13{\cdot}1\text{ ps}.$$

6.20 In a circular orbit we have the following formula

$$\frac{m v^2}{r} = \frac{Z e'^2}{(4\pi\varepsilon_0)r^2}$$

$$m v r = n\hbar$$

Then

$$v = \frac{Z e^2}{(4\pi\varepsilon_0)n\hbar}$$

$$r = \frac{n^2}{Z}\frac{\hbar(4\pi\varepsilon_0)}{m e^2}$$

The energy E is $\quad E_n = \frac{1}{2}m v^2 - \frac{Z e^2}{(4\pi\varepsilon_0)r}$

$$= \left(\frac{Z e^2}{4\pi\varepsilon_0}\right)^2 \frac{m}{2\hbar^2 n^2} - \left(\frac{Z e^2}{4\pi\varepsilon_0}\right)^2 \frac{m}{\hbar^2 n^2} = m\left(\frac{Z e^2}{4\pi\varepsilon_0}\right)^2 \Big/ 2\hbar^2 n^2$$

and the circular frequency of this orbit is

$$\omega_n = \frac{v}{r} = \left(\frac{Z e^2}{4\pi\varepsilon_0}\right)^2 m \Big/ \hbar^3 n^3$$

On the other hand the frequency ω of the light emitted when the electron makes a transition $n+1 \rightarrow n$ is

$$\omega = \left(\frac{Z e^2}{4\pi\varepsilon_0}\right)^2 \frac{m}{2\hbar^2}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$$

Thus the inequality

$$\omega_n > \omega > \omega_{n+1}$$

will result if

$$\frac{1}{n^3} > \frac{1}{2}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) > \frac{1}{(n+1)^3}$$

Or multiplying by $n^2(n+1)^2$ we have to prove

$$\frac{(n+1)^2}{n} > \frac{1}{2}(2n+1) > \frac{n^2}{n+1}$$

This can be written as

$$n+2 + \frac{1}{n} > n + \frac{1}{2} > n+1-2+\frac{1}{n+1}$$

This is obvious because $-1 + \frac{1}{n+1} < -\frac{1}{2}$ since $n \geq 1$

For large n

$$\frac{\omega_n}{\omega_{n+1}} = \left(\frac{n+1}{n}\right)^3 = 1 + \frac{3}{n}$$

so $$\frac{\omega_n}{\omega_{n+1}} \longrightarrow 1 \text{ and we may say } \frac{\omega}{\omega_n} \longrightarrow 1$$

6.21 We have the following equation (we ignore reduced mass effects)

$$\frac{m\mathrm{v}^2}{r} = kr$$

$$m\mathrm{v}r = n\hbar$$

so $$m\mathrm{v} = \sqrt{mk}\,r$$

and $$r = \sqrt{\frac{n\hbar}{\sqrt{mk}}},$$

and $$\mathrm{v} = \sqrt{n\hbar\sqrt{mk}}\,/\,m$$

The energy levels are

$$E_n = \frac{1}{2}m\mathrm{v}^2 + \frac{1}{2}kr^2$$

$$= \frac{1}{2}\frac{n\hbar\sqrt{mk}}{m} + \frac{1}{2}k\frac{n\hbar}{\sqrt{mk}}$$

$$= n\hbar\sqrt{\frac{k}{m}}$$

6.22 The basic equations have been derived in the problem (6.20). We rewrite them here and determine the the required values.

(a) $r_1 = \dfrac{\hbar^2}{m\,(Z e^2/4\pi\varepsilon_0)}$, $Z = 1$ for H, $Z = 2$ for He^+

Thus $r_1 = 52{\cdot}8$ pm, for H atom

$r_1 = 26{\cdot}4$ pm, for He^+ ion

$$v_1 = \frac{Z e^2}{(4\pi\varepsilon_0)\hbar}$$

$v_1 = 2{\cdot}191 \times 10^6$ m/s for H atom

$= 4{\cdot}382 \times 10^6$ m/s for He^+ ion

(b) $T = \dfrac{1}{2} m v_1^2 = \dfrac{m\,(Z e^2)^2}{(4\pi\varepsilon_0)^2\, 2\hbar^2}$

$T = 13{\cdot}65$ eV for H atom

$T = 54{\cdot}6$ eV for He^+ ion

In both cases $E_b = T$ because $E_b = -E$ and $E = -T$ (Recall that for coulomb force $V = -2T$)

(c) The ionization potential φ_i is given by

$$e\,\varphi_i = E_b$$

so $\varphi_i = 13{\cdot}65$ volts for H atom

$\varphi_i = 54{\cdot}6$ volts for He^+ ion

The energy levels are $E_n = -\dfrac{13{\cdot}65}{n^2}$ eV for H atom

and $E_n = -\dfrac{54{\cdot}6}{n^2}$ eV for He^+ ion

Thus $\varphi_1 = 13{\cdot}65\left(1 - \dfrac{1}{4}\right)$ volts $= 10{\cdot}23$ volts for H atom

$\varphi_1 = 4 \times 10{\cdot}23 = 40{\cdot}9$ volts for He^+ ion

The wavelength of the resonance line

$(n' = 2 \rightarrow n = 1)$ is given by

$$\frac{2\pi\hbar c}{\lambda} = -\frac{13{\cdot}6}{4} + \frac{13{\cdot}6}{1} = 10{\cdot}23 \text{ eV for H atom}$$

so $\lambda = 121{\cdot}2$ nm for H atom

For He^+ ion $\lambda = \dfrac{121{\cdot}2}{4} = 30{\cdot}3$ nm.

6.23 This has been calculated before in problem (6.20). It is

$$\omega = \frac{m\,(Z e^2/4\pi\varepsilon_0)^2}{\hbar^3 n^3} = 2{\cdot}08 \times 10^{16} \text{ rad/sec}$$

6.24 An electron moving in a circle with a time period T constitutes a current

$$I = \frac{e}{T}$$

and forms a current loop of area πr^2. This is equivalent to magnetic moment,

$$\mu = I\pi r^2 = \frac{e\pi r^2}{T} = \frac{e\,v\,r}{2}$$

on using $v = 2\pi r/T$. Thus

$$\mu_n = \frac{e\,m\,v\,r}{2m} = \frac{n\,e\hbar}{2m}$$

for the n^{th} orbit. (In Gaussian units

$$\mu_n = n\,e\hbar/2mc)$$

We see that

$$\mu_n = \frac{e}{2m} M_n$$

where $M_n = n\hbar = m\,v\,r$ is the angular momentum

Thus
$$\frac{\mu_n}{M_n} = \frac{e}{2m}$$

$$\mu_1 = \frac{e\hbar}{2m} = \mu_B = 9{\cdot}27 \times 10^{-24}\,\text{A·m}^2$$

$$(\text{In CGS units } \mu_1 = \mu_B = 9{\cdot}27 \times 10^{-21} \text{ erg/gauss})$$

6.25 The revolving electron is equivalent to a circular current

$$I = \frac{e}{T} = \frac{e}{2\pi r/v} = \frac{e\,v}{2\pi r}$$

The magnetic induction

$$B = \frac{\mu_0 I}{2r} = \frac{\mu_0 e\,v}{4\pi r^2} = \frac{\mu_0}{4\pi}\cdot e\cdot\frac{e^2}{(4\pi\varepsilon_0)\hbar}\cdot\left[\frac{m e^2}{\hbar^2 (4\pi\varepsilon_0)}\right]^2$$

$$= \frac{\mu_0 m^2 e^7}{256\,\pi^4 \varepsilon_0^3 \hbar^5}$$

Substitution gives $B = 12{\cdot}56$ T at the centre.

(In Gaussian units

$$B = \frac{m^2 e^7}{c\hbar^5} = 125{\cdot}6 \text{ k G}.\,)$$

6.26 From the general formula for the transition $n_2 \to n_1$

$$\hbar\omega = E_H\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

where $$E_H = 13{\cdot}65 \text{ eV}. \text{ Then}$$

(1) Lyman, $n_1 = 1$, $n_2 = 2, 3$. Thus

$$\hbar\omega \geq \frac{3}{4}E_H = 10{\cdot}238 \text{ eV}$$

This corresponds to $\lambda = \dfrac{2\pi c\hbar}{\hbar\omega} = 0{\cdot}121\ \mu\text{m}$

and Lyman lines have $\lambda \leq 0{\cdot}121\ \mu\text{m}$ with the series limit at $\cdot 0909\ \mu\text{m}$

(2) Balmer : $n_2 = 2$, $n_3 = 3, 4,$

$$\hbar\omega \geq E_H\left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5}{36}E_H = 1{\cdot}876 \text{ eV}$$

This corresponds to

$$\lambda = 0{\cdot}65\ \mu\text{m}$$

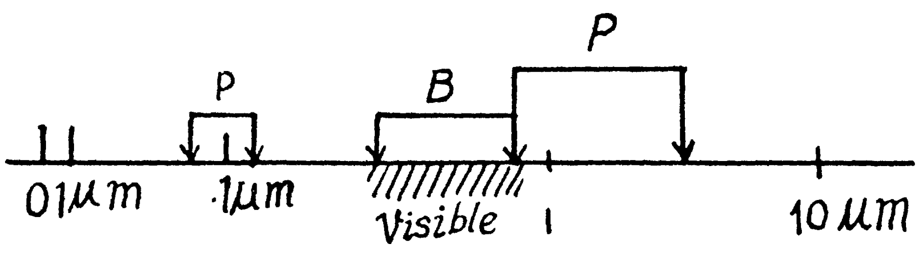
and Balmer series has $\lambda \leq 0{\cdot}65\ \mu\text{m}$ with the series limit at $\lambda = 0{\cdot}363\ \mu\text{m}$.

(3) Paschen : $n_2 = 3$, $n_1' = 4, 5, \ldots$

$$h\omega \geq E_H\left(\frac{1}{9} - \frac{1}{16}\right) = \frac{7}{144}E_H = 0{\cdot}6635 \text{ eV}$$

This corresponds to $\lambda = 1{\cdot}869\ \mu\text{m}$

with the series limit at $\lambda = 0{\cdot}818\ \mu\text{m}$



6.27 The Balmer line of wavelength 486·1 nm is due to the transition $4 \to 2$ while the Balmer line of wavelingth 410·2 nm is due to the transition $6 \to 2$. The line whose wave number corresponds to the difference in wave numbers of these two lines is due to the transition $6 \to 4$. That line belongs to the Brackett series. The wavelength of this line is

$$\lambda = \frac{1}{\dfrac{1}{\lambda_2} - \dfrac{1}{\lambda_1}} = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} = 2{\cdot}627\ \mu\text{m}$$

6.28 The energies are

$$E_H\left(\frac{1}{4}-\frac{1}{9}\right) = \frac{5}{36}E_H,\ E_H\left(\frac{1}{4}-\frac{1}{16}\right) = \frac{3}{16}E_H$$

$$E_H\left(\frac{1}{4}-\frac{1}{25}\right) = \frac{21}{100}E_H$$

They correspond to wavelengths

$$654{\cdot}2\ \text{n m}\ ,\ 484{\cdot}6\ \text{n m}\ \text{and}\ 433\ \text{nm}$$

The n^{th} line of the Balmer series has the energy

$$E_H\left(\frac{1}{4}-\frac{1}{(n+2)^2}\right)$$

For $n = 19$, we get the wavelength $366{\cdot}7450$ nm

For $n = 20$ we get the wavelength $366{\cdot}4470$ nm

To resolve these lines we require a resolving power of

$$R \approx \frac{\lambda}{\delta\lambda} = \frac{366{\cdot}6}{0{\cdot}298} = 1{\cdot}23\times 10^3$$

6.29 For the Balmer series

$$\hbar\,\omega_n = \hbar R\left(\frac{1}{4}-\frac{2}{n^2}\right),\ n \geq 3\ ,$$

where $\hbar R = E_H = 13{\cdot}65$ eV. Thus

$$\frac{2\pi\hbar c}{\lambda_n} = \hbar R\left(\frac{1}{4}-\frac{1}{n^2}\right)$$

or

$$\frac{2\pi\hbar c}{\lambda_{n+1}} - \frac{2\pi\hbar c}{\lambda_n} = \hbar R\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)$$

$$= \hbar R\left(\frac{2n+1}{n^2(n+1)^2}\right) \approx \frac{2R}{n^3}\ \text{for}\ n \gg 1$$

Thus

$$\frac{2\pi\hbar c}{\lambda_n^2}\,\delta\lambda \approx \frac{2R\hbar}{n^3}$$

or

$$\frac{\lambda_n}{\delta\lambda} \approx \frac{\pi\hbar c n^3}{\lambda_n R\hbar} = \frac{\pi c n^3}{\lambda_n R}$$

On the other hand for just resolution in a diffraction grating

$$\frac{\lambda}{\delta\lambda} = kN = k\frac{l}{d} = \frac{l}{\lambda d}k\lambda = \frac{l}{\lambda d}d\sin\theta = \frac{l}{\lambda}\sin\theta$$

Hence

$$\sin\theta = \frac{\pi c n^3}{lR}$$

Substitution gives $\theta \approx 59{\cdot}4°$.

6.30 If all wavelengths are four times shorter but otherwise similar to the hydrogen atom spectrum then the energy levels of the given atom must be four times greater.

This means
$$E_n = -\frac{4E_H}{n^2}$$

compared to $E_n = -\frac{E_H}{n^2}$ for hydrogen atom. Therefore the spectrum is that of He^+ ion $(Z = 2)$.

6.31 Because of cascading all possible transitions are seen. Thus we look for the number of ways in which we can select upper and lower levels. The number of ways we can do this is
$$\frac{1}{2}n(n-1)$$

where the factor $\frac{1}{2}$ takes account of the fact that the photon emission always arises from upper $\to$ lower transition.

6.32 These are the Lyman lines
$$\hbar\omega = E_H\left(\frac{1}{1} - \frac{1}{n^2}\right) \quad n = 2, 3, 4, \ldots$$

For $n = 2$ we get $\lambda = 121{\cdot}1$ nm

For $n = 3$ we get $\lambda = 102{\cdot}2$ nm

For $n = 4$ we get $\lambda = 96{\cdot}9$ nm

For $n = 5$ we get $\lambda = 94{\cdot}64$ nm

For $n = 6$ we get $\lambda = 93{\cdot}45$ nm

Thus at the level of accuracy of our calculation, there are four lines

121·1 nm, 102·2 nm, 96·9 nm and 94·64 nm.

6.33 If the wavelengths are λ_1, λ_2 then the total energy of the excited start must be
$$E_n = E_1 + \frac{2\pi c\hbar}{\lambda_1} + \frac{2\pi c\hbar}{\lambda_2}$$

But $E_1 = -4E_H$ and $E_n = -\frac{4E_H}{n^2}$ where we are ignoring reduced mass effects.

Then
$$4E_H = \frac{4E_H}{n^2} + \frac{2\pi c\hbar}{\lambda_1} + \frac{2\pi c\hbar}{\lambda_2}$$

Substituting the values we get $n^2 = 23$

which we take to mean $n = 5$. (The result is sensitive to the values of the various quantities and small differences get multiplied because difference of two large quantities is involved :
$$n^2 = \frac{E_H}{E_H - \frac{\pi c\hbar}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)}.$$

6.34 For the longest wavelength (first) line of the Balmer series we have on using the generalized Balmer formula

$$\omega = Z^2 R\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

the result
$$\lambda_{1\,Lyman} = \frac{2\pi c}{Z^2 R\left(1 - \frac{1}{4}\right)} = \frac{8\pi c}{3 Z^2 R}$$

Then
$$\Delta\lambda = \lambda_{1\,\text{Balmar}} - \lambda_{1\,\text{Lyman}} = \frac{176\,\pi c}{15\,Z^2 R}$$

so
$$R = \frac{176\,\pi c}{15\,Z^2\,\Delta\lambda} = 2{\cdot}07 \times 10^{16}\ \text{sec}^{-1}$$

6.35 From the formula of the previous problem

$$\Delta\lambda = \frac{176\,\pi c}{15\,Z^2 R}$$

or
$$Z = \sqrt{\frac{176\,\pi c}{15\,R\,\Delta\lambda}}$$

Substitution of $\Delta\lambda = 59{\cdot}3$ nm and R and the previous problem gives $Z = 3$

This identifies the ion as Li^{++}

6.36 We start from the generalized Balmer formula

$$\omega = R Z^2\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

Here
$$m = n+1, n+2, \ldots \infty$$

The interval between extreme lines of this series (series n) is

$$\Delta\omega = R Z^2\left(\frac{1}{n^2} - \frac{1}{(\infty)^2}\right) - R Z^2\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = R Z^2/(n+1)^2$$

Hence
$$n = Z\sqrt{\frac{R}{\Delta\omega}} - 1$$

Then the angular frequency of the first line of this series (series n) is

$$\omega_1 = R Z^2\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = \Delta\omega\left(\left(\frac{n+1}{n}\right)^2 - 1\right)$$

$$= \Delta\omega\left[\left\{\frac{Z\sqrt{\frac{R}{\Delta\omega}}}{Z\sqrt{\frac{R}{\Delta\omega}} - 1}\right\}^2 - 1\right] = \Delta\omega\,\frac{2Z\sqrt{\frac{R}{\Delta\omega}} - 1}{\left(Z\sqrt{\frac{R}{\Delta\omega}} - 1\right)^2}$$

Then the wavelength will be

$$\lambda_1 = \frac{2\pi c}{\omega_1} = \frac{2\pi c}{\Delta\omega} \frac{\left(Z\sqrt{\frac{R}{\Delta\omega}} - 1\right)^2}{2Z\sqrt{\frac{R}{\Delta\omega}} - 1}$$

Substitution (with the value of R from problem 6.34 which is also the correct value determined directly) gives

$$\lambda_1 = 0{\cdot}468\ \mu\,\text{m}.$$

6.37 For the third line of of Balmer series

$$\omega = R Z^2 \left(\frac{1}{2^2} - \frac{1}{5^2}\right) = \frac{21}{100} R Z^2$$

Hence
$$\lambda = \frac{2\pi c}{\omega} = \frac{200\pi c}{21 R Z^2}$$

or
$$Z = \sqrt{\frac{200\pi c}{21 R \lambda}}$$

Substitution gives $Z = 2$. Hence the binding energy of the electron in the ground state of this ion is

$$E_b = 4 E_H = 4 \times 13{\cdot}65 = 54{\cdot}6\ \text{eV}$$

The ion is He^+.

6.38 To remove one electron requires 24·6 eV.

The ion that is left is He^+ which in its ground start has a binding energy of $4E_H = 4\hbar R$.

The complete binding energy of both electrons is then

$$E = E_0 + 4\hbar R$$

Substitution gives
$$E = 79{\cdot}1\ \text{eV}$$

6.39 By conservation of energy

$$\frac{1}{2} m v^2 = \frac{2\pi\hbar c}{\lambda} - E_b$$

where $E_b = 4\hbar R$ is the binding energy of the electron in the ground state of He^+. (Recoil of He^{++} nucleus is neglected). Then

$$v = \sqrt{\frac{2}{m}\left(\frac{2\pi\hbar c}{\lambda} - E_b\right)}$$

Substitution gives

$$v = 2{\cdot}25 \times 10^6\ \text{m/s}$$

6.40 Photon can be emitted in $H-H$ collision only if one of the H is excited to an $n = 2$ state which then dexcites to $n = 1$ state by emitting a photon. Let v_1 and v_2 be the velocities of the two Hydrogen atoms after the collision and M their masses. Then, energy momentum conservation

$$M v_1 + M v_2 = \sqrt{2MT}$$

(in the frame of the stationary H atom)

$$\frac{1}{2}M v_1^2 + \frac{1}{2}M v_2^2 + \frac{3}{4}\hbar R = T$$

$\frac{3}{4}\hbar R = \hbar R\left(1 - \frac{1}{4}\right)$ is the excitation energy of the $n = 2$ state from the ground state.

Eliminating v_2
$$\frac{1}{2}M\left\{v_1^2 + \left(\sqrt{\frac{2T}{M}} - v_1\right)^2\right\} + \frac{3}{4}\hbar R = T$$

or
$$\frac{1}{2}M\left\{2v_1^2 - 2\sqrt{\frac{2T}{M}}\, v_1 + \frac{2T}{M}\right\} + \frac{3}{4}\hbar R = T$$

$$M\left\{\left(v_1 - \frac{1}{2}\sqrt{\frac{2T}{M}}\right)^2\right\} + \frac{1}{2}T + \frac{3}{4}\hbar R = T$$

or
$$M\left\{\left(v_1 - \frac{1}{2}\sqrt{\frac{2T}{M}}\right)^2\right\} + \frac{3}{4}\hbar R = \frac{1}{2}T$$

For minimum T, the square on the left should vanish. Thus $T = \frac{3}{2}\hbar R = 20{\cdot}4$ eV

6.41 In the rest frame of the original excited nucleus we have the equations $O = \vec{p}_\gamma + \vec{p}_H$

$$\frac{3}{4}\hbar R = c\,|\vec{p}_\gamma| + p_H^2/2M$$

$\left(\frac{3}{4}\hbar R\right.$ is the energy available in $n = 2 \rightarrow = 1$ transition corresponding to the first Lyman line.)

Then
$$p_H^2 + 2Mc\,p_H - \frac{3\hbar RM}{2} = 0$$

or
$$(p_H + Mc)^2 = M^2c^2 + \frac{3}{2}\hbar RM$$

$$p_H = -Mc + \sqrt{M^2c^2 + \frac{3}{2}\hbar RM} = -Mc + Mc\left(1 + \frac{3\hbar R}{2Mc^2}\right)^{1/2} \approx \frac{3\hbar R}{4c}$$

(We could have written this directly by noting that $p_H^2/2M \ll c\,p_\gamma$.) Then

$$v_H = \frac{3\hbar R}{4Mc} = 3{\cdot}3 \text{ m/s}$$

6.42 We have

$$\varepsilon = \frac{3}{4}\hbar R \quad \text{and} \quad \varepsilon' \approx \frac{3}{4}\hbar R - \frac{1}{2M}\left(\frac{3}{4}\hbar R/c\right)^2$$

Then
$$\frac{\varepsilon - \varepsilon'}{\varepsilon} = \frac{3\hbar R}{8Mc^2} = \frac{v_H}{2c} = 5{\cdot}5 \times 10^{-9} = 0{\cdot}55 \times 10^{-6}\,\%$$

6.43 We neglect recoil effects. The energy of the first Lyman line photon emitted by He^+ is

$$4\hbar R\left(1 - \frac{1}{4}\right) = 3\hbar R$$

The velocity v of the photoelectron that this photon liberates is given by

$$3\hbar R = \frac{1}{2}mv^2 + \hbar R$$

where $\hbar R$ on the right is the binding energy of the $n = 1$ electron in H atom. Thus

$$v = \sqrt{\frac{4\hbar R}{m}} = 2\sqrt{\frac{\hbar R}{m}} = 3{\cdot}1 \times 10^6 \text{ m/s}$$

Here m is the mass of the electron.

6.44 Since $\Delta\lambda\,(= 0{\cdot}20 \text{ nm}) << \lambda\,(= 121 \text{ nm})$ of the first Lyman line of H atom, we need not worry about v^2/c^2 effects. Then

$$\omega' = \frac{\omega}{1 - \beta\cos\theta}, \quad \beta = \frac{v}{c}$$

Hence
$$1 - \beta\cos\theta = \frac{\omega}{\omega'} = \frac{\lambda'}{\lambda}$$

or
$$\beta\cos\theta = 1 - \frac{\lambda'}{\lambda} = \frac{\Delta\lambda}{\lambda}$$

But
$$\omega = \frac{3}{4}R \quad \text{so} \quad \lambda = \frac{2\pi c}{\omega} = \frac{8\pi c}{3R}$$

Hence
$$v = c\beta = \frac{3R\,\Delta\lambda}{8\pi\cos\theta}$$

Substitution gives $\left(\cos\theta = \frac{1}{\sqrt{2}}\right)$

$$v = 7{\cdot}0 \times 10^5 \text{ m/s}$$

6.45 (a) If we measure energy from the bottom of the well, then $V(x) = 0$ inside the walls. Then the quantization conditon reads $\oint p\,dx = 2lp = 2\pi n\hbar$

or $p = \pi n\hbar/l$

Hence
$$E_n = \frac{p^2}{2m} = \frac{\pi^2 n^2 \hbar}{2ml^2}\,.$$

$\oint p\,dx = 2lp$ because we have to consider the integral form $-\frac{l}{2}$ to $\frac{l}{2}$ and then back to $-\frac{l}{2}$.)

(b) Here $\oint p\,dx = 2\pi r p = 2\pi n\hbar$

or $$p = \frac{n\hbar}{r}$$

Hence $$E_n = \frac{n^2\hbar^2}{2mr^2}$$

(c) By energy conservation $$\frac{p^2}{2m} + \frac{1}{2}\alpha x^2 = E$$

so $$p = \sqrt{2mE - m\alpha x^2}$$

Then $$\oint p\,dx = \oint \sqrt{2mE - m\alpha x^2}\,dx$$

$$= 2\sqrt{m\alpha}\int_{-\sqrt{\frac{2E}{\alpha}}}^{\sqrt{\frac{2E}{\alpha}}} \sqrt{\frac{2E}{\alpha} - x^2}\,dx$$

The integral is $$\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = a^2\int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\,d\theta = a^2\frac{\pi}{2}.$$

Thus $$\oint p\,dx = \pi\sqrt{m\alpha}\cdot\frac{2E}{\alpha} = E\cdot 2\pi\cdot\sqrt{\frac{m}{\alpha}} = 2\pi n\hbar$$

Hence $$E_n = n\hbar\sqrt{\frac{\alpha}{m}}.$$

(b) It is required to find the energy levels of the circular orbait for the rotential $U(r) = -\frac{\alpha}{r}$

In a circular orbit, the particle only has tangential velocity and the quantization condition reads $\oint p\,dx = m\text{v}\cdot 2\pi r = 2\pi n\hbar$

so $$m\text{v}r = M = n\hbar$$

The energy of the particle is

$$E = \frac{n^2\hbar^2}{2m\,r^2} - \frac{\alpha}{r}$$

Equilibrium requires that the energy as a function of r be minimum. Thus

$$\frac{n^2\hbar^2}{m\,r^3} = \frac{\alpha}{r^2} \quad \text{or } r = \frac{n^2\hbar^2}{m\,\alpha}.$$

Hence $$E_n = -\frac{m\,\alpha^2}{2\,n^2\hbar^2}.$$

6.46 The total energy of the H-atom in an arbitrary frame is

$$E = \frac{1}{2}m\vec{V}_1^2 + \frac{1}{2}M\vec{V}_2^2 - \frac{e^2}{(4\pi\varepsilon_0)\,|\,r_1 - \vec{r}_2|}$$

Here $\vec{V}_1 = \dot{\vec{r}}_1$, $\vec{V}_2 = \dot{\vec{r}}_2$, $\vec{r}_1$ & $\vec{r}_2$ are the coordinates of the electron and protons.

We define

$$\vec{R} = \frac{m\,\vec{r}_1 + M\,\vec{r}_2}{M+m}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

Then $$\vec{V} = \frac{m\,\vec{V}_1 + M\,\vec{V}_2}{m+M}$$

$$\vec{\text{v}} = \vec{V}_1 - \vec{V}_2$$

or $$\vec{V}_1 = \vec{V} + \frac{M}{m+M}\vec{\text{v}}$$

$$\vec{V}_2 = \vec{V} - \frac{m}{m+M}\vec{\text{v}}$$

and we get $$E = \frac{1}{2}(m+M)\vec{V}^2 + \frac{1}{2}\frac{mM}{m+M}\text{v}^2 - \frac{e^2}{4\pi\varepsilon_0 r}$$

In the frame $\vec{V} = 0$, this reduces to the energy of a particle of mass

$$\mu = \frac{mM}{m+M}$$

μ is called the reduced mass.

Then $$E_b = \frac{\mu e^4}{2\hbar^2} \quad \text{and} \quad R = \frac{\mu e^4}{2\hbar^3}$$

Since
$$\mu = \frac{m}{1+\frac{m}{M}} \approx m\left(1-\frac{m}{M}\right)$$

these values differ by $\frac{m}{M}$ ($\approx$ 0·54 %) from the values obtained without considering nuclear motion. (M = 1837 m)

6.47 The difference between the binding energies is

$$\Delta E_b = E_b(D) - E_b(H)$$
$$= \Delta \frac{m}{1+\frac{m}{M}} \frac{e^4}{2\hbar^2} + \frac{m}{1+\frac{m}{2M}} \frac{e^4}{2\hbar^2}$$
$$= \frac{m e^4}{2\hbar^2}\left(\frac{m}{2M}\right)$$

Substitution gives $\Delta E_b = 3{\cdot}7$ meV.

For the first line of the Lyman series

$$; \frac{2\pi\hbar c}{\lambda} = \hbar R\left(\frac{1}{1}-\frac{1}{4}\right) = \frac{3}{4}\hbar R$$

or
$$\lambda = \frac{8\pi c}{3R} = \frac{8\pi\hbar c}{3E_b}$$

Hence

$$\lambda_H - \lambda_D = \frac{8\pi\hbar c}{3}\left(\frac{1}{E_b(H)} - \frac{1}{E_b(D)}\right)$$
$$= \frac{8\pi\hbar c}{3}\cdot\left(\frac{m e^4}{2\hbar^2}\right)^{-1}\left(1+\frac{m}{M}-1-\frac{m}{2M}\right)$$
$$= \frac{8\pi\hbar c}{3\left(\frac{m e^4}{2\hbar^2}\right)}\cdot\frac{m}{2M}$$
$$= \frac{m}{2M}\times\lambda_1$$

(where λ_1 is the wavelength of the first line of Lyman series without considering nuclear motion).

Substitution gives (see 6.21 for λ_1) using λ_1 = 121 nm

$$\Delta\lambda = 33\,pm$$

6.48 (a) In the mesonic system, the reduced mass of the system is related to the masses of the meson (m_μ) and proton (m_p) by

$$\mu = \frac{m_\mu m_p}{m_\mu + M_p} = 186{\cdot}04\, m_e$$

Then,

$$\text{separation between the particles in the ground state} = \frac{\hbar^2}{\mu e^2}$$

$$= \frac{1}{186}\frac{\hbar^2}{m e^2}$$

$$= 0{\cdot}284 \text{ pm}$$

$$E_b(\text{meson}) = \frac{\mu e^4}{2\hbar^2} = 186 \times 13{\cdot}65 \text{ eV}$$

$$= 2{\cdot}54 \text{ keV}$$

$$\lambda_1 = \frac{8\pi\hbar c}{3E_b(\text{meson})} = \frac{\lambda_1(\text{Hydrogen})}{186} = 0{\cdot}65 \text{ nm}$$

(on using λ_1 (Hydrogen) = 121 nm).

(b) In the positronium

$$\mu = \frac{m_e^2}{2m_e} = \frac{m_e}{2}$$

Thus separation between the particles is the ground state

$$= 2\frac{\hbar^2}{m_e e^2} = 105{\cdot}8 \text{ pm}$$

$$E_b(\text{positronium}) = \frac{m_e}{2}\cdot\frac{e^4}{2\hbar^2} = \frac{1}{2}E_b(H) = 6{\cdot}8 \text{ eV}$$

$$\lambda_1(\text{positronium}) = 2\lambda_1(\text{Hydrogen}) = 0{\cdot}243 \text{ nm}$$

6.2 WAVE PROPERTIES OF PARTICLES. SCHRODINGER EQUATION

6.49 The kinetic energy is nonrelativistic in all three cases. Now

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{\sqrt{2mT}}$$

using $T = 1{\cdot}602 \times 10^{-17}$ Joules, we get

$$\lambda_e = 122{\cdot}6 \text{ pm}$$

$$\lambda_p = 2{\cdot}86 \text{ pm}$$

$$\lambda_U = \frac{\lambda_p}{\sqrt{238}} = 0{\cdot}185 \text{ pm}.$$

(where we have used a mass number of 238 for the U nucleus).

6.50 From $\lambda = \dfrac{2\pi\hbar}{p} = \dfrac{2\pi\hbar}{\sqrt{2mT}}$

we find
$$T = \frac{4\pi^2\hbar^2}{2m\lambda^2} = \frac{2\pi^2\hbar^2}{m\lambda^2}$$

Thus
$$T_2 - T_1 = \frac{2\pi^2\hbar^2}{m}\left(\frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2}\right)$$

Substitution gives $\Delta T = 451$ eV $= 0{\cdot}451$ keV.

6.51 We shall use $M_0 \approx 2M_n$. The CM is moving with velocity

$$V = \frac{\sqrt{2M_nT}}{3M_n} = \sqrt{\frac{2T}{9M_n}}$$

with respect to the Lab frame. In the CM frame the velocity of neutron is

$$v'_n = v_n - V = \sqrt{\frac{2T}{M_n}} - \sqrt{\frac{2T}{9M_n}} = \sqrt{\frac{2T}{M_n}} \cdot \frac{2}{3}$$

and
$$\lambda'_n = \frac{2\pi\hbar}{M_n v'_n} = \frac{3\pi\hbar}{\sqrt{2M_nT}}$$

Substitution gives $\lambda'_n = 8{\cdot}6$ pm

Since the momenta are equal in the CM frame the de Broglie wavelengths will also be equal.
If we do not assume $M_d \approx 2M_n$ we shall get

$$\lambda'_n = \frac{2\pi\hbar(1 + M_n/M_d)}{\sqrt{2M_nT}}$$

6.52 If $\vec{p_1}$, $\vec{p_2}$ are the momenta of the two particles then their momenta in the CM frame will be $\pm\ (\vec{p_1}-\vec{p_2})/2$ as the particle are identical.

Hence their de Broglie wavelength will be

$$\tilde{\lambda} = \frac{2\pi\hbar}{\frac{1}{2}|\vec{p_1}-\vec{p_2}|} = \frac{4\pi\hbar}{\sqrt{p_1^2+p_2^2}} \quad (\text{because } \vec{p_1}\perp\vec{p_2})$$

$$= \frac{2}{\sqrt{\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}}} = \frac{2\lambda_1\lambda_2}{\sqrt{\lambda_1^2+\lambda_2^2}}$$

6.53 In thermodynamic equilibrium, Maxwell's velocity distribution law holds :

$$dN(\mathrm{v}) = \Phi(\mathrm{v})\,d\mathrm{v} = A\mathrm{v}^2 e^{-m\mathrm{v}^2/2kT}\,d\mathrm{v}$$

$\Phi(\mathrm{v})$ is maximum when

$$\Phi'(\mathrm{v}) = \Phi(\mathrm{v})\left[\frac{2}{\mathrm{v}}-\frac{m\mathrm{v}}{kT}\right] = 0.$$

The difines the most probable velocity.

$$\mathrm{v}_{pr} = \sqrt{\frac{2kT}{m}}.$$

The de Broglie wavelength of H molecules with the most probable velocity is

$$\lambda = \frac{2\pi\hbar}{m\mathrm{v}_{pr}} = \frac{2\pi\hbar}{\sqrt{2mkT}}$$

Substituting the appropriate value especially

$$m = m_{H_2} = 2m_H,\ T = 300\,K, \text{ we get}$$

$$\lambda = 126\ \text{pm}$$

6.54 To find the most probable de Broglie wavelength of a gas in thermodynamic equilibrium we determine the distribution is λ corresponding to Maxwellian velocity distribution.
It is given by

$$\psi(\lambda)\,d\lambda = -\Phi(\mathrm{v})\,d\mathrm{v}$$

(where – sign takes account of the fact that λ decreaes as v increases). Now

$$\lambda = \frac{2\pi\hbar}{m\mathrm{v}} \quad \text{or} \quad \mathrm{v} = \frac{2\pi\hbar}{m\lambda}$$

$$d\mathrm{v} = -\frac{2\pi\hbar}{m\lambda^2}\,d\lambda$$

Thus

$$\Psi(\lambda) = +A\mathrm{v}^2 e^{-m\mathrm{v}^2/2kT}\left(-\frac{d\mathrm{v}}{d\lambda}\right)$$

$$= A\left(\frac{2\pi\hbar}{m\lambda}\right)^2\left(\frac{2\pi\hbar}{m\lambda^2}\right)e^{-\frac{m}{2kT}\cdot\left(\frac{2\pi\hbar}{m\lambda}\right)^2}$$

$$= \text{Const}\cdot\lambda^{-4}e^{-a/\lambda^2}$$

where $$a = \frac{2\pi^2\hbar^2}{mkT}$$

This is maximum when

$$\psi'(\lambda) = 0 = \psi(\lambda)\left[\frac{-4}{\lambda}+\frac{2a}{\lambda^3}\right]$$

or $$\lambda_{pr} = \sqrt{a/2} = \pi\hbar\Big/\sqrt{mkT}$$

Using the result of the previous problem it is

$$\lambda_{pr} = \frac{126}{\sqrt{2}}\text{ pm} = 89{\cdot}1\text{ pm}.$$

6.55 For a relativistic particle

$$T+mc^2 = \text{total energy} = \sqrt{c^2p^2+m^2c^4}$$

Squaring

$$\sqrt{T(T+2mc^2)} = cp$$

Hence $$\lambda = \frac{2\pi\hbar c}{\sqrt{T(T+2mc^2)}}$$

$$= \frac{2\pi\hbar}{\sqrt{2mT\left(1+\dfrac{T}{2mc^2}\right)}}$$

If we use nonrelativistic formula,

$$\lambda_{NR} = \frac{2\pi\hbar}{\sqrt{2mT}}$$

so $$\frac{\Delta\lambda}{\lambda} = \frac{\lambda_{NR}-\lambda}{\lambda_{NR}} \simeq \frac{T}{4mc^2}$$

$$\left(\text{If } T/2mc^2 \ll 1\text{, we can write }\left(1+\frac{T}{2mc^2}\right)^{-1/2} \simeq 1-\frac{T}{4mc^2}\right)$$

Thus $T \le \dfrac{4mc^2\Delta\lambda}{\lambda}$ if the error is less than $\Delta\lambda$

For electron the error is not more than 1 % if

$$T \le 4 \times 0{\cdot}511 \times {\cdot}01 \text{ MeV}$$

$$\le 20{\cdot}4 \text{ keV}$$

For a proton, the error is not more than 1 % if

$$T \le 4 \times 938 \times 0{\cdot}01 \text{ MeV}$$

i.e. $$T \le 37{\cdot}5 \text{ MeV}.$$

6.56 The de Broglie wavelength is

$$\lambda_{dB} = \frac{\dfrac{2\pi\hbar}{m_0 v}}{\sqrt{1 - v^2/c^2}} = \frac{2\pi\hbar}{m_0 v}\sqrt{1 - v^2/c^2}$$

and the Compton wavelength is

$$\lambda_c = \frac{2\pi\hbar}{m_0 c}$$

The two are equal if $\beta = \sqrt{1-\beta^2}$, where $\beta = \frac{v}{c}$

or $$\beta = \frac{1}{\sqrt{2}}$$

The corresponding kinetic energy is

$$T = \frac{m_0 c^2}{\sqrt{1-\beta^2}} - m_0 c^2 = \left(\sqrt{2} - 1\right) m_0 c^2$$

Here m_0 is th rest mass of the particle (here an electron).

6.57 For relativistic electrons, the formula for the short wavelength limit of $X-$ rays will be

$$\frac{2\pi\hbar c}{\lambda_{sh}} = m_0 c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) = c\sqrt{p^2 + m^2 c^2} - m c^2$$

or $$\left(\frac{2\pi\hbar}{\lambda_{sh}} + m c\right)^2 = p^2 + m^2 c^2$$

or $$\left(\frac{2\pi\hbar}{\lambda_{sh}}\right)\left(\frac{2\pi\hbar}{\lambda_{sh}} + 2 m c\right) = p^2$$

or $$p = \frac{2\pi\hbar}{\lambda_{sh}}\sqrt{1 + \frac{m c \lambda_{sh}}{\pi\hbar}}$$

Hence $$\lambda_{dB} = \lambda_{sh} \Big/ \sqrt{1 + \frac{m c \lambda_{sh}}{\pi\hbar}} = 3{\cdot}29 \text{ pm}$$

6.58 The first minimum in a Fraunhofer diffraction is given by (b is the width of the slit) $b \sin\theta = \lambda$

Here
$$\sin\theta = \frac{\Delta x/2}{\sqrt{l^2 + \left(\frac{\Delta x}{2}\right)^2}} \approx \frac{\Delta x}{2l}$$

Thus
$$\lambda = \frac{b\,\Delta x}{2l} = \frac{2\pi\hbar}{m\mathrm{v}}$$

so
$$\mathrm{v} = \frac{4\pi\hbar l}{m b \Delta x} = 2{\cdot}02 \times 10^6 \text{ m/s}$$

6.59 From the Young slit formula
$$\Delta x = \frac{l\lambda}{d} = \frac{l}{d} \cdot \frac{2\pi\hbar}{\sqrt{2meV}}$$

Substitution gives
$$\Delta x = 4{\cdot}90\ \mu\text{m}.$$

6.60 From Bragg's law, for the first case
$$2d\sin\theta = n_0\lambda = n_0 \frac{2\pi\hbar}{\sqrt{2meV_0}}$$

where n_0 is an unknown integer. For the next higher voltage
$$2d\sin\theta = (n_0 + 1)\frac{2\pi\hbar}{\sqrt{2me\eta V_0}}$$

Thus
$$n_0 = \frac{n_0 + 1}{\sqrt{\eta}}$$

or
$$n_0\left(1 - \frac{1}{\sqrt{\eta}}\right) = \frac{1}{\sqrt{\eta}} \quad \text{or} \quad n_0 = \frac{1}{\sqrt{\eta} - 1}$$

Going back we get
$$V_0 = \frac{\pi^2\hbar^2}{2med^2\sin^2\theta} \frac{1}{\left(\sqrt{\eta} - 1\right)^2} = 0{\cdot}150 \text{ keV}$$

Note :- In the Bragg's formula, θ is the glancing angle and not the angle of incidence. We have obtained correct result by taking θ to be the glancing angle. If θ is the angle of incidence, then the glancing angle will be $90 - \theta$. Then the final answer will be smaller by a factor $\tan^2\theta = \frac{1}{3}$.

6.61 Path difference is

$d + d\cos\theta = 2\,d\cos^2\dfrac{\theta}{2}.$

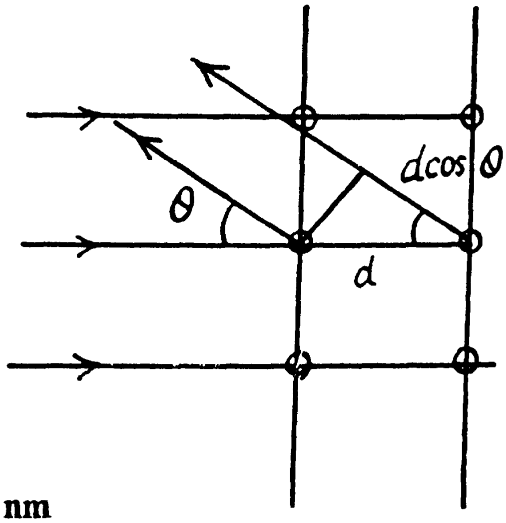
Thus for reflection maximum of the k^{th} order

$$2\,d\cos^2\frac{\theta}{2} = k\lambda = k\frac{2\pi\hbar}{\sqrt{2mT}}$$

Hence $\quad d = \dfrac{k\pi\hbar}{\sqrt{2mT}}\,\text{Sec}^2\dfrac{\theta}{2}.$

Substitution with $k = 4$ gives

$$d = 0{\cdot}232\ \text{nm}$$



6.62 See the analogous problem with $X-$ rays (5.156)
The glancing angle is obtained from

$$\tan 2\theta = \frac{D}{2l}$$

where D = diameter of the ring, l = distance from the foil to the screen.

Then for the third order Bragg reflection

$$2\,d\sin\theta = k\lambda = k\frac{2\pi\hbar}{\sqrt{2mT}}, (k = 3)$$

Thus

$$d = \frac{\pi\hbar k}{\sqrt{2mT}\sin\theta} = 0{\cdot}232\ \text{nm}$$

6.63 Inside the metal, there is a negative potential energy of $-eV_i$. (This potential energy prevents electrons from leaking out and can be measured in photoelectric effect etc.) An electron whose K.E. is eV outside the metal will find its K.E. increased to $e(V + V_i)$ in the metal. Then

(a) de Broglie wavelength in the metal

$$= \lambda_m = \frac{2\pi\hbar}{\sqrt{2me(V+V_i)}}$$

Also de Broglie wavelength in vacuum

$$= \lambda_0 = \frac{2\pi\hbar}{\sqrt{2mVe}}$$

Hence refractive index $n = \dfrac{\lambda_0}{\lambda_m} = \sqrt{1 + \dfrac{V_i}{V}}$

Substituting we get

$$n = \sqrt{1 + \frac{1}{10}} \simeq 1{\cdot}05$$

(b) $n - 1 = \sqrt{1 + \frac{V_i}{V}} - 1 \leq \eta$

then $$1 + \frac{V_i}{V} \leq (1+\eta)^2$$

or $$V_i \leq \eta(2+\eta)V$$

or $$\frac{V}{V_i} \geq \frac{1}{\eta(2+\eta)}$$

For $$\eta = 1\% = 0{\cdot}01$$

we get $$\frac{V}{V_i} \geq 50$$

6.64 The energy inside the well is all kinetic if energy is measured from the value inside. We require

$$l = n\lambda/2 = n\frac{\pi\hbar}{\sqrt{2mE}}$$

or $$E_n = \frac{n^2\pi^2\hbar^2}{2ml^2}, \quad n = 1, 2, \ldots$$

6.65 The Bohr condition

$$\oint p\,dx = \oint \frac{2\pi\hbar}{\lambda}\,dx = 2\pi n\hbar$$

For the case when λ is constant (for example in circular orbits) this means

$$2\pi r = n\lambda$$

Here r is the radius of the circular orbit.

6.66 From the uncertainty principle (Eqn. (6.2b))

$$\Delta x\,\Delta p_x \gtrsim \hbar$$

Thus $$\Delta p_x = m\,\Delta v_x \gtrsim \frac{\hbar}{\Delta x}$$

or $$\Delta v_x \gtrsim \frac{\hbar}{m\,\Delta x}$$

For an electron this means an uncertainty in velocity of 116 m/s if $\Delta x = 10^{-6}\,\text{m} = 1\,\mu\text{m}$

For a proton

$$\Delta v_x = 6{\cdot}3\ \text{cm/s}$$

For a ball

$$\Delta v_x = 1 \times 10^{-20}\ \text{cm/s}$$

6.67 As in the previous problem

$$\Delta v \gtrsim \frac{\hbar}{m\,l} = 1{\cdot}16 \times 10^6 \text{ m/s}$$

The actual velocity v_1 has been calculated in problem 6.21. It is

$$v_1 = 2{\cdot}21 \times 10^6 \text{ m/s}$$

Thus $\Delta v \sim v_1$ (They are of the same order of magnitude)

6.68 If $\Delta x = \lambda/2\pi = \frac{2\pi\hbar}{p}\cdot\frac{1}{2\pi} = \frac{\hbar}{p} = \frac{\hbar}{m\,v}$

Thus
$$\Delta v \gtrsim \frac{\hbar}{m\,\Delta x} = v$$

Thus Δv is of the same order as v.

6.69 Initial uncertainty $\Delta v \gtrsim \frac{\hbar}{m\,l}$. With this incertainty the wave train will spread out to a distance $\eta\, l$ long in time

$$t_0 \approx \eta\, l \Big/ \frac{\hbar}{m\,l} \approx \frac{\eta\, m\, l^2}{\hbar}\,\text{sec.} = 8{\cdot}6 \times 10^{16}\,\text{sec.} \sim 10^{-15}\,\text{sec.}$$

6.70 Clearly $\Delta x \le l$ so $\Delta p_x \ge \frac{\hbar}{l}$

Now $p_x \ge \Delta p_x$ and so

$$T = \frac{p_x^2}{2\,m} \ge \frac{\hbar^2}{2\,m\,l^2}$$

Thus
$$T_{\min} = \frac{\hbar^2}{2\,m\,l^2} \simeq 0{\cdot}95 \text{ eV}.$$

6.71 The momentum the electron is $\Delta p_x = \sqrt{2\,m\,T}$
Uncertainty in its momentum is

$$\Delta p_x \ge \hbar / \Delta x = \hbar/l$$

Hence relative uncertainty

$$\frac{\Delta p_x}{p_x} = \frac{\hbar}{l\sqrt{2\,m\,T}} = \sqrt{\frac{\hbar^2}{2\,m\,l^2}\Big/\,T} = \frac{\Delta v}{v}$$

Substitution gives

$$\frac{\Delta v}{v} = \frac{\Delta p}{p} = 9{\cdot}75 \times 10^{-5} \approx 10^{-4}$$

6.72 By uncertainty principle, the uncertainty in momentum

$$\Delta p \gtrsim \frac{\hbar}{l}$$

For the ground state, we expect $\Delta p \sim p$ so

$$E \sim \frac{\hbar^2}{2ml^2}$$

The force excerted on the wall can be obtained most simply from

$$F = -\frac{\partial U}{\partial l} = \frac{\hbar^2}{ml^3}.$$

6.73 We write

$$p \sim \Delta p \sim \frac{\hbar}{\Delta x} \sim \frac{\hbar}{x}$$

i.e. all four quantities are of the same order of magnitude. Then

$$E \approx \frac{\hbar^2}{2mx^2} + \frac{1}{2}kx^2 = \frac{1}{2m}\left(\frac{\hbar}{x} - \sqrt{mk}\,x\right)^2 + \hbar\sqrt{\frac{k}{m}}$$

Thus we get an equilibrium situation (E = minimum) when

$$x = x_0 = \sqrt{\frac{\hbar}{\sqrt{mk}}}$$

and then
$$E = E_0 \sim \hbar\sqrt{\frac{k}{m}} = \hbar\omega$$

Quantum mechanics gives

$$E_0 = \hbar\omega/2$$

6.74 Hence we write

$$r \sim \Delta r, \; p \sim \Delta p \sim \hbar/\Delta r$$

Then
$$E = \frac{\hbar^2}{2mr^2} - \frac{e^2}{r}$$

$$= \frac{1}{2m}\left(\frac{\hbar}{r} - \frac{me^2}{\hbar}\right)^2 - \frac{me^4}{2\hbar^2}$$

Hence $r_{eff} = \dfrac{\hbar^2}{me^2} = 53$ pm for the equilibrium state.

and then
$$E = -\frac{me^4}{2\hbar^2} = -13{\cdot}6\text{ eV}.$$

6.75 Suppose the width of the slit (its extension along the $y-$ axis) is δ. Then each electron has an uncertainty $\Delta y \sim \delta$. This translates to an uncertainty $\Delta p_y \sim \hbar/\delta$. We must therefore have

$p_y \gtrsim \hbar/\delta$.

For the image, brodening has two sources.

We write

$$\Delta(\delta) = \delta + \Delta'(\delta)$$

where Δ' is the width caused by the spreading of electrons due to their transverse momentum. We have

$$\Delta' = v_y \frac{l}{v_x} = p_y \frac{l}{p} = \frac{l\hbar}{m v \delta}$$

Thus
$$\Delta(\delta) = \delta + \frac{l\hbar}{m v \delta}$$

For large δ, $\Delta(\delta) \sim \delta$ and quantum effect is unimportant. For small δ, quantum effects are large. But $\Delta(\delta)$ is minimum when

$$\delta = \sqrt{\frac{l\hbar}{m v}}$$

as we see by completing the square. Substitution gives

$$\delta = 1{\cdot}025 \times 10^{-5}\,\text{m} = 0{\cdot}01\,\text{mm}$$

6.76 The Schrodinger equation in one dimension for a free particle is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}$$

we write $\psi(x,t) = \varphi(x)\,\chi(t)$. Then

$$\frac{i\hbar}{\chi} \frac{d\chi}{dt} = -\frac{\hbar^2}{2m} \frac{1}{\varphi} \frac{d^2\varphi}{dx^2} = E, \text{ say}$$

Then
$$\chi(t) \sim \exp\left(-\frac{iEt}{h}\right)$$

$$\varphi(x) \sim \exp\left(i \frac{\sqrt{2mE}}{\hbar} x\right)$$

E must be real and positive if $\varphi(x)$ is to be bounded everywhere. Then

$$\psi(x,t) = \text{Const} \exp\left(\frac{i}{\hbar}\left(\sqrt{2mE}\,x - Et\right)\right)$$

This particular solution describes plane waves.

6.77 We look for the solution of Schrodinger eqn. with

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi,\ 0 \le x \le l \tag{1}$$

The boundary condition of impenetrable walls means

$$\psi(x) = 0 \text{ for } x = 0 \text{ and } x = l$$

$$(\text{as } \psi(x) = 0 \text{ for } x < 0 \text{ and } x > l,)$$

The solution of (1) is

$$\psi(x) = A\sin\frac{\sqrt{2mE}}{\hbar}x + B\cos\frac{\sqrt{2mE}}{\hbar}\cdot x$$

Then

$$\psi(0) = 0 \Rightarrow B = 0$$

$$\psi(l) = 0 \Rightarrow A\sin\frac{\sqrt{2mE}}{\hbar}l = 0$$

$A \ne 0$ so

$$\frac{\sqrt{2mE}}{\hbar}l = n\pi$$

Hence

$$E_n = \frac{n^2\pi^2\hbar^2}{2ml^2},\ n = 1, 2, 3, \ldots$$

Thus the ground state wave function is

$$\psi(x) = A\sin\frac{\pi x}{l}.$$

We evaluate A by nomalization

$$1 = A^2\int_0^l \sin^2\frac{\pi x}{l}\,dx = A^2\frac{l}{\pi}\int_0^\pi \sin^2\theta\,d\theta = A^2\frac{l}{\pi}\cdot\frac{\pi}{2}$$

Thus

$$A = \sqrt{\frac{2}{l}}$$

Finally, the probability P for the particle to lie in $\frac{l}{2} \le x \le \frac{2l}{3}$ is

$$P = P\left(\frac{l}{3} \le x \le \frac{2l}{3}\right) = \frac{2}{l}\int_{\frac{l}{3}}^{2l/3} \sin^2\frac{\pi x}{l}\,dx$$

$$= \frac{2}{\pi}\int_{\pi/3}^{2\pi/3} \sin^2\theta\,d\theta = \frac{1}{\pi}\int_{\pi/3}^{2\pi/3}(1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{\pi}\left(\theta - \frac{1}{2}\sin 2\theta\right)_{\pi/3}^{2\pi/3} = \frac{1}{\pi}\left(\frac{2\pi}{3} - \frac{\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right)$$

$$= \frac{1}{\pi}\left(\frac{\pi}{3} + \frac{1}{2}\frac{\sqrt{3}}{2} + \frac{1}{2}\frac{\sqrt{3}}{2}\right) = \frac{1}{3} + \frac{\sqrt{3}}{2\pi} = 0{\cdot}609$$

6.78 Here $-\frac{l}{2} \le x \le \frac{l}{2}$. Again we have

$$\psi(x) = B\cos\frac{\sqrt{2mE}\,x}{\hbar} + A\sin\frac{\sqrt{2mE}\,x}{\hbar}$$

Then the boundary condition $\psi\left(\pm\frac{l}{2}\right) = 0$

gives
$$B\cos\frac{\sqrt{2mE}\,l}{2\hbar} \pm A\sin\frac{\sqrt{2mE}\,l}{2\hbar} = 0$$

There are two cases.

(1) $A = 0,\ \frac{\sqrt{2mE}\,l}{2\hbar} = n\pi + \frac{\pi}{2}$

gives even solution. Here

$$\sqrt{2mE} = (2n+1)\frac{\pi\hbar}{l}$$

and
$$E_n = (2n+1)^2\frac{\pi^2\hbar^2}{2ml^2}$$

$$\psi_n^e(x) = \sqrt{\frac{2}{l}}\cos(2n+1)\frac{\pi x}{l}$$

$$n = 0, 1, 2, 3, \ldots$$

This solution is even under $x \to -x$.

(2) $B = 0$,

$$\frac{\sqrt{2mE}\,l}{2\hbar} = n\pi,\ n = 1, 2, \ldots$$

$$E_n = (2n\pi)^2\frac{\hbar^2}{2ml^2}$$

$$\psi_n^0 = \sqrt{\frac{2}{l}}\sin\frac{2n\pi x}{l},\ n = 1, 2, \ldots$$ This solution is odd.

6.79 The wave function is given in 6.77. We see that

$$\int_0^l \psi_n(x)\,\psi_{n'}(x)\,dx = \frac{2}{l}\int_0^l \sin\frac{n\pi x}{l}\sin\frac{n'\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^l \left[\cos(n-n')\frac{\pi x}{l} - \cos(n+n')\frac{\pi x}{l}\right] dx$$

$$= \frac{1}{l}\left[\frac{\sin(n-n')\pi x/l}{(n-n')\frac{\pi}{l}} - \frac{\sin(n+n')\frac{\pi x}{l}}{(n+n')\frac{\pi x}{l}}\right]_0^l e.$$

If $n \neq n'$, this is zero as n and n' are integers.

6.80 We have found that

$$E_n = \frac{n^2\pi^2\hbar^2}{2ml^2}$$

Let $N(E)$ = number of states upto E. This number is n. The number of states upto $E+dE$ is $N(E+dE) = N(E)+dN(E)$. Then $dN(E) = 1$ and

$$\frac{dN(E)}{dE} = \frac{1}{\Delta E}$$

where ΔE = difference in energies between the n^{th} & $(n+1)^{th}$ level

$$= \frac{(n+1)^2 - n^2}{2ml^2}\pi^2\hbar^2 = \frac{2n+1}{2ml^2}\pi^2\hbar^2$$

$$\approx \frac{\pi^2\hbar^2}{2ml^2}2n, \quad \text{(neglecting } 1 << n\text{)}$$

$$= \frac{\pi^2\hbar^2}{2ml^2}\times\sqrt{\frac{2ml^2}{\pi^2\hbar^2}}\sqrt{E}\times 2$$

$$= \frac{\pi\hbar}{l}\sqrt{\frac{2}{m}}\sqrt{E}$$

Thus
$$\frac{dN(E)}{dE} = \frac{l}{\pi\hbar}\sqrt{\frac{m}{2E}}.$$

For the given case this gives $\dfrac{dN(E)}{dE} = 0{\cdot}816\times 10^7$ levels per eV

6.81 (a) Here the schrodinger equation is

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi = E\psi$$

we take the origin at one of the corners of the rectangle where the particle can lie. Then the wave function must vanish for

$$x = 0 \quad \text{or} \quad x = l_1$$

or $$y = 0 \quad \text{or } y = l_2 .$$

we look for a solution in the form

$$\psi = A \sin k_1 x \sin k_2 y$$

cosines are not permitted by the boundary condition. Then

$$k_1 = \frac{n_i \pi}{l_1}, \quad k_2 = n_2 \frac{\pi}{l_2}$$

and $$E = \frac{k_1^2 + k_2^2}{2m} \hbar^2 = \frac{\pi^2 \hbar^2}{2m} \left(\frac{n_1^2}{l_1^2} + \frac{n_2^2}{l_2^2} \right)$$

Here n_1, n_2 are nonzero integers.

(b) If $l_1 = l_2 = l$ then

$$\frac{E}{\hbar^2 / m l^2} = \frac{n_1^2 + n_2^2}{2} \pi^2$$

1st level : $$n_1 = n_2 = 1 \rightarrow \pi^2 = 9{\cdot}87$$

2nd level : $$\left. \begin{array}{r} n_1 = 1, n_2 = 2 \\ \text{or} \quad n_1 = 2, n_2 = 1 \end{array} \right\} \rightarrow \frac{5}{2} \pi^2 = 24{\cdot}7$$

3st level : $$n_1 = 2, = n_2 = 2 \rightarrow 4 \pi^2 = 39{\cdot}5$$

4nd level : $$\left. \begin{array}{r} n_1 = 1, n_2 = 3 \\ n_1 = 3, n_2 = 1 \end{array} \right\} \rightarrow 5 \pi^2 = 49{\cdot}3$$

6.82 The wave function for the ground state is

$$\psi_{11}(x, y) = A \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

we find A by normalization

$$1 = A^2 \int_0^a dx \int_0^d dy \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b} = A^2 \frac{a b}{4}$$

Thus $$A = \frac{2}{\sqrt{a b}} .$$

Then the requisite probability is

$$P = \int_0^{a/3} dx \int_0^b dy \frac{4}{a b} \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b}$$

$$= \frac{2}{a} \int_0^{a/3} dx \sin^2 \frac{\pi x}{a} \quad \text{on doing the } y \text{ integral}$$

$$= \frac{1}{a}\int_0^{a/3} d\left(1 - \cos\frac{2\pi x}{a}\right) = \frac{1}{a}\left(\frac{a}{3} - \frac{\sin\frac{2\pi}{3}}{2\pi/a}\right)$$

$$= \frac{1}{3} - \frac{\sqrt{3}}{4\pi} = 0{\cdot}196 = 19{\cdot}6\,\%.$$

6.83 We proceed axactly as in (6.81). The wave function is chosen in the form $\psi(x, y, z) = A \sin k_1 x \sin k_2 y \sin k_3 z$.

(The origin is at one corner of the box and the axes of coordinates are along the edges.) The boundary conditions are that $\psi = 0$ for

$$x = 0,\ x = a,\ y = 0,\ y = a,\ z = 0,\ z = a$$

This gives

$$k_1 = \frac{n_1\pi}{a},\ k_2 = \frac{n_2\pi}{a},\ k_3 = \frac{n_3\pi}{a}$$

The energy eigenvalues are

$$E(n_1, n_2, n_3) = \frac{\pi_2\hbar^2}{2ma^2}(n_1^2 + n_2^2 + n_3)$$

The first level is (1, 1, 1). The second has (1, 1, 2), (1, 2, 1) & (2, 1, 1). The third level is (1, 2, 2) or (2, 1, 2) or (2, 2, 1). Its energy is

$$\frac{9\pi^2\hbar^2}{2ma^2}$$

The fourth energy level is (1, 1, 3) or (1, 3, 1) or (3, 1, 1)

Its energy is $$E = \frac{11\pi^2\hbar^2}{2ma^2}.$$

(b) Thus

$$\Delta = E_4 - E_3 = \frac{\hbar^2\pi^2}{ma^2}.$$

(c) The fifth level is (2, 2, 2). The sixth level is (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)

Its energy is

$$\frac{7\hbar^2\pi^2}{ma^2}$$

and its degree of degeneracy is 6 (six).

6.84 We can for definiteness assume that the discontinuity occurs at the point $x = 0$. Now the schrodinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi(x) = E\psi(x)$$

We integrate this equation around $x = 0$ i.e., from $x = -\varepsilon_1$ to $x = +\varepsilon_2$ where ε_1, ε_2 are small positive numbers. Then

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon_1}^{+\varepsilon_2}\frac{d^2\psi}{dx^2}\,dx = \int_{-\varepsilon_1}^{+\varepsilon_2}(E - U(x)\psi(x)\,dx$$

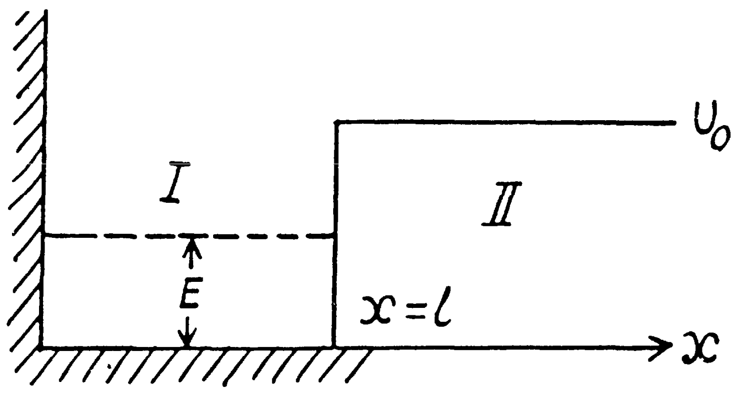
or
$$\left(\frac{d\psi}{dx}\right)_{+\varepsilon_2} - \left(\frac{d\psi}{dx}\right)_{-\varepsilon_1} = -\frac{2m}{\hbar^2}\int_{-\varepsilon_1}^{\varepsilon_2}(E - U(x))_{dx}\,\psi(x)$$

Since the potential and the energy E are finite and $\psi(x)$ is bounded by assumption, the integral on the right exists and $\to 0$ as $\varepsilon_1, \varepsilon_2 \to 0$

Thus
$$\left(\frac{d\psi}{dx}\right)_{+\varepsilon_2} = \left(\frac{d\psi}{dx}\right)_{-\varepsilon_1} \quad \text{as } \varepsilon_1, \varepsilon_2 \to 0$$

So $\left(\frac{d\psi}{dx}\right)$ is continuous at $x = 0$ (the point where $U(x)$ has a finite jump discontinuity.)

6.85



(a) Starting from the Schrodinger equation in the regions I & II

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0 \quad x \text{ in } I \tag{1}$$

$$\frac{d^2\psi}{dx^2} - \frac{2mE(U_0 - E)}{\hbar^2}\psi = 0 \quad x \text{ in } II \tag{2}$$

where $U_0 > E > 0$, we easily derive the solutions in I & II

$$\Psi_I(x) = A\sin kx + B\cos kx \tag{3}$$

$$\Psi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x} \tag{4}$$

where $$k^2 = \frac{2mE}{\hbar^2}, \ \alpha^2 = \frac{2m(U_0 - E)}{\hbar^2}.$$

The boundary conditions are

$$\Psi(o) = 0 \tag{5}$$

and Ψ & $\left(\frac{d\psi}{dx}\right)$ are continuous at $x = l$, and Ψ must vanish at $x = +\infty$.

Then $$\psi_I = A \sin kx$$

and $$\Psi_{II} = D e^{-\alpha x}$$

so $$A \sin kl = D e^{-\alpha l}$$

$$kA \cos kl = -\alpha D e^{-\alpha l}$$

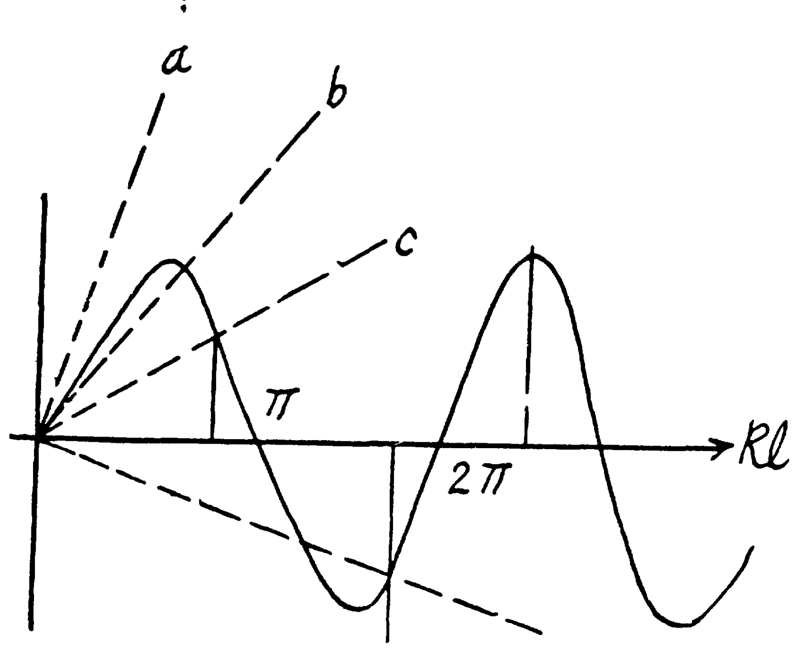
From this we get

$$\tan kl = -\frac{k}{\alpha}$$

or $$\sin kl = \pm kl \Big/ \sqrt{k^2 l^2 + \alpha^2 l^2}$$

$$= \pm kl \Big/ \sqrt{\frac{2mU_0 l^2}{\hbar^2}}$$

$$= \pm kl \sqrt{\hbar^2 / 2mU_0 l^2} \tag{6}$$



Plotting the left and right sides of this equation we can find the points at which the straight lines cross the sine curve. The roots of the equation corresponding to the eigen values of energy E_i and found from the inter section points $(kl)_i$, for which $\tan(kl)_i < 0$ (i.e. 2nd & 4th and other even quadrants). It is seen that bound states do not always exist. For the first bound state to appear (refer to the line (b) above)

$$(kl)_{1,\min} = \frac{\pi}{2}$$

(b) Substituting, we get $(l^2 U_0)_{1,\min} = \dfrac{\pi^2 \hbar^2}{8\,m}$

as the condition for the appearance of the first bound state. The second bound state will appear when kl is in the fourth quadrant. The magnitude of the slope of the straight line must then be less than

$$\frac{1}{3\,\pi/2}$$

Corresponding to $\quad (kl)_{2,\min} = \dfrac{3\,\pi}{2} = (3)\dfrac{\pi}{2} = (2 \times 2 - 1)\dfrac{\pi}{2}$

For n bound states, it is easy to convince one self that the slope of the appropriate straight line (upper or lower) must be less than

$$(kl)_{n,\min} = (2n-1)\frac{\pi}{2}$$

Then

$$(l^2 U_0)_{n,\min} = \frac{(2n-1)^2\,\pi^2\,\hbar^2}{8\,m}$$

Do not forget to note that for large n both + and − signs in the Eq. (6) contribute to solutions.

6.86 $U_0\, l^2 = \left(\dfrac{3}{4}\pi\right)^2 \dfrac{\hbar^2}{m}$

and

$$E\,l^2 = \left(\frac{3}{4}\pi\right)^2 \frac{\hbar^2}{2\,m}$$

or

$$k\,l = \frac{3}{4}\pi$$

It is easy to check that the condition of the boud state is satisfied. Also

$$\alpha\, l = \sqrt{\frac{2\,m}{\hbar^2}(\,U_0 - E\,)\,l^2} = \sqrt{\frac{m\,U_0}{\hbar^2}\,l^2} = \frac{3}{4}\pi$$

Then from the previous problem

$$D = A\,e^{\alpha\, l}\sin k\,l = A\,\frac{e^{3\,\pi/4}}{\sqrt{2}}$$

By normalization

$$I = A^2\left[\int_0^l \sin^2 k\,x\,d\,x + \int_l^\infty \frac{e^{3\,\pi/2}}{2}\,e^{-(3\,\pi/2)\,x/l}\,d\,x\right]$$

$$= A^2 \left[\frac{1}{2} \int_0^l (1 - \cos 2kx)\, dx + l \int_0^\infty \frac{1}{2} e^{-\frac{3\pi}{2} y}\, dy \right]$$

$$= A^2 \left[\frac{1}{2} \left[- \frac{\sin 2kl}{2k} \right] + \frac{1}{2} \cdot \frac{\frac{l}{3\pi}}{2} \right] = A^2 l \left[\frac{1}{2} \left[1 + \frac{\frac{1}{3\pi}}{2} \right] + \frac{1}{2} \frac{\frac{1}{3\pi}}{2} \right]$$

$$= A^2 l \left[\frac{1}{2} + \frac{2}{3\pi} \right] = A^2 \frac{l}{2} \left(1 + \frac{4}{3\pi} \right) \quad \text{or} \quad A = \sqrt{\frac{2}{l}} \left(1 + \frac{4}{3\pi} \right)^{-1/2}$$

The probability of the particle to be located in the region $x > l$ is

$$P = \int_l^\infty \psi^2\, dx = \frac{2}{l} \left(1 + \frac{4}{3\pi} \right)^{-1} \int_l^\infty \frac{e^{3\pi/2}}{2} e^{-\frac{3\pi}{2}\frac{x}{l}}\, dx$$

$$= \left(1 + \frac{4}{3\pi} \right)^{-1} \int_l^\infty e^{3\pi/2} e^{-(3\pi/2)y}\, dy = \frac{2}{3\pi} \times \frac{3\pi}{3\pi + 4} = 14{\cdot}9\,\%.$$

6.87 The Schrodinger equation is

$$\nabla^2 \psi + \frac{2m}{\hbar^2} (E - U(r)) \psi = 0$$

when ψ depends on r only, $\quad \nabla^2 \psi = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi}{dr} \right)$

If we put $\quad \psi = \frac{\chi(r)}{r}, \ \frac{d\psi}{dr} = \frac{\chi'}{r} - \frac{\chi}{r^2}$

and $\quad \nabla^2 \psi = \frac{\chi''}{r}$. Thus we get

$$\frac{d^2\chi}{dr^2} + \frac{2m}{\hbar^2} (E - U(r)) \chi = 0$$

The solution is $\quad \chi = A \sin kr, \ r < r_0$

$$k^2 = \frac{2mE}{\hbar^2}$$

and $\quad \chi = 0 \quad r > r_0$

(For $r < r_0$ we have rejected a term $B \cos kr$ as it does not vanish at $r = 0$). Continuity of the wavefunction at $r = r_0$ requires

$$k r_0 = n\pi$$

Hence $\quad E_n = \frac{n^2 \pi^2 \hbar^2}{2 m r_0^2}.$

6.88 (a) The nomalized wave functions are obtained from the normalization

$$1 = \int |\psi|^2 dV = \int |\psi|^2 4\pi r^2 dr$$

$$= \int_0^{r_0} A^2 4\pi \chi^2 dr = 4\pi A^2 \int_0^{r_0} \sin^2 \frac{n\pi r}{r_0} dr$$

$$= 4\pi A^2 \frac{r_0}{n\pi} \int_0^{n\pi} \sin^2 x \, dr = 4\pi A^2 \frac{r_0}{n\pi} \cdot \frac{n\pi}{2} = r_0 \cdot 2\pi A^2$$

Hence $\quad A = \dfrac{1}{\sqrt{2\pi r_0}}$ and $\psi = \dfrac{1}{\sqrt{2\pi \cdot r_0}} \dfrac{\sin \frac{n\pi r}{r_0}}{r}$

(b) The radial probability distribution function is

$$P_n(r) = 4\pi r^2 (\psi)^2 = \frac{2}{r_0} \sin^2 \frac{n\pi r}{r_0}$$

For the ground state $\quad n = 1$

so $\quad P_1(r) = \dfrac{2}{r_0} \sin^2 \dfrac{\pi r}{r_0}$

By inspection this is maximum for $r = \dfrac{r_0}{2}$. Thus $r_{pr} = \dfrac{r_0}{2}$

The probability for the particle to be found in the region $r < r_{pr}$ is clearly 50 % as one can immediately see from a graph of $\sin^2 x$.

6.89 If we put $\psi = \dfrac{\chi(r)}{r}$

the equation for $\chi(r)$ has the form

$$\chi'' + \frac{2m}{\hbar^2} [E - U(r)] \chi(r) = 0$$

which can be written as $\quad \chi'' + k^2 \chi = 0, \; 0 \le r < r_0$

and $\quad \chi'' - \alpha^2 \chi = 0 \quad r_0 < r < \infty$

where $\quad k^2 = \dfrac{2mE}{\hbar^2}, \; \alpha^2 = \dfrac{2m(U_0 - E)}{\hbar^2}.$

The boundary condition is

$$\left.\begin{array}{c} \chi(0) = 0 \\ \text{and } \chi, \chi' \text{ are continuous at } r = r_0 \end{array}\right\}$$

These are exactly same as in the one dimensional problem in problem (6.85)

We therefore omit further details

6.90 The Schrodinger equation is $\frac{d^2\Psi}{dx^2} + \frac{2m}{\hbar^2}(E - \frac{1}{2}kx^2)\Psi = 0$

We are given $$\Psi = A e^{-\alpha x^2/2}$$

Then $$\Psi' = -\alpha x A e^{-\alpha x^2/2}$$

$$\Psi'' = -\alpha A e^{-\alpha x^2/2} + \alpha^2 x^2 A e^{-\alpha x^2/2}$$

Substituting we find that following equation must hold

$$\left[(\alpha^2 x^2 - \alpha) + \frac{2m}{\hbar^2}(E - \frac{1}{2}kx^2)\right]\Psi = 0$$

since $\Psi \neq 0$, the bracket must vanish identicall. This means that the coefficient of x^2 as well the term independent of x must vanish. We get

$$\alpha^2 = \frac{mk}{\hbar^2} \quad \text{and} \quad \alpha = \frac{2mE}{\hbar^2}$$

Putting $k/m = \omega^2$, this leads to $\alpha = \frac{m\omega}{\hbar^2}$ and $E = \frac{\hbar^2 \omega}{2}$

6.91 The Schrödinger equation for the problem in Gaussian units

$$\nabla^2 \psi + \frac{2m}{\hbar^2}\left[E + \frac{e^2}{r}\right]\psi = 0$$

In MKS units we should read $(e^2/4\pi\varepsilon_0)$ for e^2.

we put $\psi = \frac{\chi(r)}{r}$. Then $\chi'' + \frac{2m}{\hbar^2}\left[E + \frac{e^2}{r}\right]\chi = 0$ (1)

We are given that $$\chi = r\psi = A r(1 + ar)e^{-\alpha r}$$

so $$\chi' = A(1 + 2ar)e^{-\alpha r} - \alpha A r(1 + ar)e^{-\alpha r}$$

$$\chi'' = \alpha^2 A r(1 + ar)e^{-\alpha r} - 2\alpha A(1 + 2ar)e^{-\alpha r} + 2aAe^{-\alpha r}$$

Substitution in (1) gives the condition

$$\alpha^2(r + ar^2) - 2\alpha(1 + 2ar) + 2a + \frac{2m}{\hbar^2}(Er + e^2)\times(1 + ar) = 0$$

Equating the coefficients of r^2, r, and constant term to zero we get

$$2a - 2\alpha + \frac{2me^2}{\hbar^2} = 0 \tag{2}$$

$$a\alpha^2 + \frac{2m}{\hbar^2}Ea = 0 \tag{3}$$

$$\alpha^2 - 4a\alpha + \frac{2m}{\hbar^2}(E + e^2 a) = 0 \tag{4}$$

From (3) either $a = 0$, or $E = -\frac{\hbar^2\alpha^2}{2m}$

In the first case $\alpha = \frac{m e^2}{\hbar^2}, E = -\frac{\hbar^2}{2m}\alpha^2 = -\frac{m e^4}{2\hbar^2}$

This state is the ground state.

In the second case $a = \alpha - \frac{m e^2}{\hbar^2}, \quad \alpha = \frac{1}{2}\frac{m e^2}{\hbar^2}$

$$E = -\frac{m e^4}{8\hbar^2} \quad \text{and} \quad a = -\frac{1}{2}\frac{m e^2}{\hbar^2}$$

This state is one with $n = 2$ (2s).

6.92 We first find A by normalization

$$1 = \int_0^\infty 4\pi A^2 e^{-2r/r_1} r^2 dr = \frac{\pi A^2}{2} r_1^3 \int_0^\infty e^{-x} x^2 dx = \pi A^2 r_1^3$$

since the integral has the value 2.

Thus $$A^2 = \frac{1}{\pi r_1^3} \quad \text{or} \quad A = \frac{1}{\sqrt{r_1^3 \pi}}.$$

(a) The most probable distance r_{pr} is that value of r for which

$$P(r) = 4\pi r^2 |\psi(r)|^2 = \frac{4}{r_1^3} r^2 e^{-2r/r_1}$$

is maximum. This requires

$$P'(r) = \frac{4}{r_1^3}\left[2r - \frac{2r^2}{r_1}\right] e^{-2r/r_1} = 0$$

or $$r = r_1 = r_{pr}.$$

(b) The coulomb force being given by $-e^2/r^2$, the mean value of its modulus is

$$\langle F \rangle = \int_0^\infty 4\pi r^2 \frac{1}{\pi r_1^3} e^{-2r/r_1} \frac{e^2}{r^2} dr$$

$$= \int_0^\infty \frac{4e^2}{r_1^3} e^{-2r/r_1} dr = \frac{2e^2}{r_1^2}\int_0^\infty e^{-x} dx = \frac{2e^2}{r_1^2}$$

In MKS units we should read $(e^2/4\pi\varepsilon_0)$ for e^2

(c) $$\langle U \rangle = \int_0^\infty 4\pi r^2 \frac{1}{\pi r_1^3} e^{-2r/r_1} \frac{-e^2}{r} dr = -\frac{e^2}{r_1}\int_0^\infty x e^{-x} dx = -\frac{e^2}{r_1}$$

In MKS units we should read $(e^2/4\pi\varepsilon_0)$ for e^2.

6.93 We find A by normalization as above. We get

$$A = \frac{1}{\sqrt{\pi r_1^3}}$$

Then the electronic charge density is

$$\rho = -e|\psi|^2 = -e\frac{e^{-2r/r_1}}{\pi r_1^3} = \rho(\vec{r})$$

The potential $\psi(\vec{r})$ due to this charge density is

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3r'$$

so at the origin $\varphi(0) = \frac{1}{4\pi\varepsilon_0}\int_0^\infty \frac{\rho(r')}{r'}4\pi r'^2 dr' = \frac{-e}{4\pi\varepsilon_0}\int_0^\infty \frac{4r'}{r_1^3}e^{-2r'/r_1}dr$

$$= -\frac{e}{4\pi\varepsilon_0 r_1}\int_0^\infty x e^{-x}dx = -\frac{e}{(4\pi\varepsilon_0)r_1}$$

6.94 (a) We start from the Schrodinger equation $\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - U(x))\psi = 0$

which we write as $\Psi_I'' + k^2\Psi_I = 0$, $x < 0$

$$k^2 = \frac{2mE}{\hbar^2}$$

and

$$\Psi_{II}'' + \alpha^2\Psi_{II} = 0 \quad x > 0$$

$$\alpha^2 = \frac{2m}{\hbar^2}(E - U_0) > 0$$

It is convenient to look for solutions in the form

$$\psi_I = e^{ikx} + Re^{-ikx} x < 0$$

$$\Psi_{II} = Ae^{i\alpha x} + Be^{-i\alpha x} \quad x > 0$$

In region $I(x < 0)$, the amplitude of e^{ikx} is written as unity by convention. In II we expect only a transmitted wave to the right, $B = 0$ then. So

$$\Psi_{II} = Ae^{ikx} \quad x > 0$$

The boundary conditions follow from the continuity of Ψ & $\frac{d\psi}{dx}$ at $x = 0$.

$$1 + R = A$$

$$iK(1 - R) = i\alpha A$$

Then

$$\frac{1-R}{1+R} = \frac{\alpha}{k} \quad \text{or} \quad R = \frac{k-\alpha}{k+\alpha}$$

The reflection coefficient is the absolute square of R :

$$r = |R|^2 = \left|\frac{k-\alpha}{k+\alpha}\right|^2$$

(b) In this case $E < U_0$, $\alpha^2 = -\beta^2 < 0$. Then Ψ_I is unchanged in form but

$$\Psi_{II} = A\,e^{-\beta x} + B\,e^{+\beta x}$$

we must have $B = 0$ since otherwise $\psi(x)$ will become unbounded as $x \to \infty$.
Finally

$$\Psi_{II} = A\,e^{-\beta x}$$

Inside the barrier, the particle then has a probability density equal to

$$|\Psi_{II}|^2 = |A|^2 e^{-2\beta x}$$

This decreases to $\frac{1}{e}$ of its value in

$$x_{eff} = \frac{1}{2\beta} = \frac{\hbar}{2\sqrt{2m(U_0-E)}}$$

6.95 The formula is

$$D \approx \exp\left[-\frac{2}{\hbar}\int_{x_1}^{x_2}\sqrt{2m(V(x)-E)}\,dx\right]$$

Here $V(x_2) = V(x_1) = E$ and $V(x) > E$ in the region $x_2 > x > x_1$.

(a) For the problem, the integral is trivial

$$D \approx \exp\left[-\frac{2l}{\hbar}\sqrt{2m(U_0-E)}\right]$$

(b) We can without loss of generality take $x = 0$ at the point the potential begins to climb. Then

$$U(x) = \begin{cases} 0 & x < 0 \\ U_0\frac{x}{l} & 0 < x < l \\ 0 & x > l \end{cases}$$

Then
$$D \approx \exp\left[-\frac{2}{\hbar}\int_{l\frac{E}{U_0}}^{l}\sqrt{2m\left(U_0\frac{x}{l}-E\right)}\,dx\right]$$

$$= \exp\left[-\frac{2}{\hbar}\sqrt{\frac{2mU_0}{l}}\int_{x_0}^{l}\sqrt{x-x_0}\,dx\right] \quad x_0 = l\frac{E}{U_0}$$

$$= \exp\left[-\frac{2}{\hbar}\sqrt{\frac{2mU_0}{l}}\ \frac{2}{3}\ (x - x_0)^{3/2} \Bigg|_{x_0}^{l} \right]$$

$$= \exp\left[-\frac{4}{3\hbar}\sqrt{\frac{2mU_0}{l}} \left(l - l\frac{E}{U_0} \right)^{3/2} \right]$$

$$= \exp\left[-\frac{4l}{3\hbar U_0}(U_0 - E)^{3/2}\sqrt{2m} \right]$$

6.96 The potential is $U(x) = U_0\left(1 - \frac{x^2}{l^2}\right)$. The turning points are

$$\frac{E}{U_0} = 1 - \frac{x^2}{l^2} \quad \text{or} \quad x = \pm\, l\sqrt{1 - \frac{E}{U_0}}\ .$$

Then
$$D \approx \exp\left[-\frac{4}{\hbar} \int_0^{l\sqrt{1-(E/U_0)}} \sqrt{2m\left\{ U_0\left(1 - \frac{x^2}{l^2}\right) - E \right\}}\ dx \right]$$

$$= \exp\left[-\frac{4}{\hbar} \int_0^{l\sqrt{1-(E/U_0)}} \sqrt{2mU_0}\sqrt{1 - \frac{E}{U_0} - \frac{x^2}{l^2}}\ dx \right]$$

$$= \exp\left[-\frac{4l}{\hbar}\sqrt{2mV_0} \int_0^{x_0} \sqrt{x_0^2 - x^2}\ dx \right], \quad x_0 = \sqrt{1 - E/V_0}$$

The integral is

$$\int_0^{x_0} \sqrt{x_0^2 - x^2}\ dx = x_0^2 \int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{\pi}{4}x_0^2$$

Thus
$$D \approx \exp\left[-\frac{\pi l}{\hbar}\sqrt{2mU_0}\left(1 - \frac{E}{U_0}\right) \right]$$

$$= \exp\left[-\frac{\pi l}{\hbar}\sqrt{\frac{2m}{U_0}}\,(U_0 - E) \right]$$

6.3 PROPERTIES OF ATOMS. SPECTRA

6.97 From the Rydberg formula we write

$$E_n = -\frac{\hbar R}{(n+\alpha_l)^2}$$

we use $\hbar R = 13{\cdot}6$ eV. Then for $n = 2$ state

$$5{\cdot}39 = -\frac{13{\cdot}6}{(2+\alpha_0)^2}, \quad l = 0\,(S) \text{ state}$$

$$\alpha_0 = -\,0{\cdot}41$$

for p state

$$3{\cdot}54 = -\frac{13{\cdot}6}{(2+\alpha_1)^2}$$

$$\alpha_1 = -\,0{\cdot}039$$

6.98 The energy of the $3p$ state must be $-(E_0 - e\varphi)$ where $-E_0$ is the energy of the $3S$ state. Then

$$E_0 - e\varphi_1 = \frac{\hbar R}{(3+\alpha_1)^2}$$

so

$$\alpha_1 = \sqrt{\frac{\hbar R}{E_0 - e\varphi_1}} - 3 = -\,0{\cdot}885$$

6.99 For the first line of the sharp series $(3S \to 2P)$ in a Li atom

$$\frac{2\pi\hbar c}{\lambda_1} = -\frac{\hbar R}{(3+\alpha_0)^2} + \frac{\hbar R}{(2+\alpha_1)^2}$$

For the short wave cut-off wave-length of the same series

$$\frac{2\pi\hbar c}{\lambda_2} = \frac{\hbar R}{(2+\alpha_1)^2}$$

From these two equations we get on subtraction

$$3 + \alpha_0 = \sqrt{\hbar R \Big/ \frac{2\pi\hbar c(\lambda_1 - \lambda_2)}{\lambda_1\lambda_2}}$$

$$= \sqrt{\frac{R\lambda_1\lambda_2}{2\pi c\,\Delta\lambda}}, \quad \Delta\lambda = \lambda_1 - \lambda_2$$

Thus in the ground state, the binding energy of the electron is

$$E_b = \frac{\hbar R}{(2+\alpha_0)^2} = \hbar R \Big/ \left(\sqrt{\frac{R\lambda_1\lambda_2}{2\pi c\,\Delta\lambda}} - 1\right)^2 = 5{\cdot}32 \text{ eV}$$

6.100 The energy of the 3 S state is

$$E(3S) = -\frac{\hbar R}{(3-0{\cdot}41)^2} = -2{\cdot}03\ \text{eV}$$

The energy of a 2 S state is

$$E(2S) = -\frac{\hbar R}{(2-0{\cdot}41)^2} = -5{\cdot}39\ \text{eV}$$

The energy of a 2 P state is

$$E(2P) = -\frac{\hbar R}{(2-{\cdot}04)^2} = -3{\cdot}55\ \text{eV}$$

We see that

$$E(2S) < E(2P) < E(3S)$$

The transitions are $3S \rightarrow 2P$ and $2P \rightarrow 2S$.

Direct $3S \rightarrow 2S$ transition is forbidden by selection rules. The wavelengths are determined by

$$E_2 - E_1 = \Delta E = \frac{2\pi\hbar c}{\lambda}$$

Substitution gives

$$\lambda = 0{\cdot}816\ \mu\text{m}\ (3S \rightarrow 2P)$$

and

$$\lambda = 0{\cdot}674\ \mu\text{m}\ (2P \rightarrow 2S)$$

6.101 The splitting of the *Na* lines is due to the fine structure splitting of 3 p lines (The 3 s state is nearly single except for possible hyperfine effects.) The splitting of the 3 p level then equals the energy difference

$$\Delta E = \frac{2\pi\hbar c}{\lambda_1} - \frac{2\pi\hbar c}{\lambda_2} = \frac{2\pi\hbar c(\lambda_2-\lambda_1)}{\lambda_1\lambda_2} \approx \frac{2\pi\hbar c\,\Delta\lambda}{\lambda^2}$$

Here $\Delta\lambda$ = wavelength difference & λ = average wavelength. Substitution gives

$$\Delta E = 2{\cdot}0\ \text{meV}$$

6.102 The sharp series arise from the transitions $ns \rightarrow mp$. The s lines are unsplit so the splitting is due entirely to the p level. The frequency difference between sequent lines is $\frac{\Delta E}{\hbar}$ and is the same for all lines of the sharp series. It is

$$\frac{1}{\hbar}\left(\frac{2\pi\hbar c}{\lambda_1} - \frac{2\pi\hbar c}{\lambda_2}\right) = \frac{2\pi c\,\Delta\lambda}{\lambda_1\lambda_2}$$

Evaluation gives

$$1{\cdot}645 \times 10^{14}\ \text{rad/s}$$

6.103 We shall ignore hyperfine interaction. The state with principal quantum number $n = 3$ has orbital angular momentum quantum number

$$l = 0, 1, 2$$

The levels with these terms are $3S, 3P, 3D$. The total angular momentum is obtained by combining spin and angular momentum. For a single electron this leads to

$$J = \frac{1}{2}, \text{ if } L = 0$$

$$J = L - \frac{1}{2} \text{ and } L + \frac{1}{2} \text{ if } L \neq 0$$

We then get the final designations

$$3S_{\frac{1}{2}},\ 3P_{\frac{1}{2}},\ 3P_{3/2},\ 3D_{3/2},\ 3D_{5/2}.$$

6.104 The rule is that if $\vec{J} = \vec{L} + \vec{S}$ then J takes the values

$$|L - S| \text{ to } L + S$$

in step of 1. Thus :

(a) The values are 1, 2, 3, 4, 5

(b) The values are 0, 1, 2, 3, 4, 5, 6

(c) The values are $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$.

6.105 For the state $4p$, $L = 1$, $S = \frac{3}{2}$ (since $2s + 1 = 4$). For the state $5d$, $L = 2, s = 2$. The possible values of J are

$$J : \frac{5}{2}, \frac{3}{2}, \frac{1}{2} \quad \text{for } 4p$$

$$J : 4, 3, 2, 1, 0 \quad \text{for } 5d$$

The value of the magnitude of angular momentum is $\hbar\sqrt{J(J+1)}$. Substitution gives the values

$4P$:

$$\hbar\sqrt{\frac{1}{2}\cdot\frac{3}{2}} = \frac{\hbar\sqrt{3}}{2},\ \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2}} = \frac{\hbar\sqrt{15}}{2}$$

and

$$\hbar\sqrt{\frac{5}{2}\cdot\frac{7}{2}} = \frac{\hbar\sqrt{35}}{2}$$

$$5D : 0,\ \hbar\sqrt{2},\ \hbar\sqrt{6},\ \hbar\sqrt{12},\ \hbar\sqrt{20}$$

6.106 (a) For the Na atoms the valence electron has principal quantem number $n = 4$, and the possible values of orbital angular momentum are $l = 0, 1, 2, 3$ so $l_{max} = 3$. The state is ^2F, maximum value of J is $\frac{7}{2}$.

Thus the state with maximum angular momentum will be

$$^2F_{7/2}$$

For this state $$M_{max} = \hbar\sqrt{\frac{7}{2}\cdot\frac{9}{2}} = \frac{\hbar\sqrt{63}}{2}$$

(b) For the atom with electronic configuration $1s^2\,2p\,3d$. There are two inequivalent valence electrons. The total orbital angular moments will be 1, 2, 3 so we pick $l = 3$. The total spin angular momentum will be $s = 0, 1$ so we pick up $s = 1$. Finally J will be 2, 3, 4 so we pick up 4. Thus maximum angular momentum state is

$$^3F_4$$

For this state $$M_{max} = \hbar\sqrt{4\times 5} = 2\hbar\sqrt{5}.$$

6.107 For the f state $L = 3$, For the d state $L = 2$. Now if the state has spin s the possible angular momentum are

$$|L-S| \text{ to } L+S$$

The number of J angular momentum values is $2S+1$ if $L \geq S$ and $2L+1$ if $L < S$. Since the number of states is 5, we must have $S \geq L = 2$ for D state while $S \leq 3$ and $2S+1 = 5$ in ply $S = 2$ for F state. Thus for the F state total spin angular momentum

$$M_s = \hbar\sqrt{2\cdot 3} = \hbar\sqrt{6}$$

while for D state $$M_s \geq \hbar\sqrt{6}.$$

6.108 Multiplicity is $2S+1$ so $S = 1$.

Total angular momentum is $\hbar\sqrt{J(J+1)}$ so $J = 4$. Then

$$L \text{ must equal } 3, 4, 5$$

in order that $J = 4$ may be included in

$$|L-S| \text{ to } L+S.$$

6.109 (a) Here $J = 2$, $L = 2$. Then $S = 0, 1, 2, 3, 4$ and the multiplicities$(2S-1)$ are

$$1, 3, 5, 7, 9.$$

(b) Here $J = 3/2$, $L = 1$ Then $S = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$

and the mulitiplicities are 6, 4, 2

(c) Here $J = 1$, $L = 3$. Then $S = 2, 3, 4$

and the multiplicities are 5, 7, 9

6.110 The total angular momentum is greatest when L, S are both greatest and add to form J. Now for a triplet of s, p, d electrons

Maximum spin $\rightarrow S = \frac{3}{2}$ corresponding to

$$M_s = \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2}} = \frac{\hbar\sqrt{15}}{2}$$

Maximum orbital angular momentum $\rightarrow$

$$L = 3$$

corresponding to $$M_L = \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2}} = \frac{\hbar\sqrt{15}}{2}$$

Maximum total angular momentum $$J = \frac{9}{2}$$

corresponding to $$M = \frac{\hbar}{2}\sqrt{99}$$

In vector model $$\vec{L} = \vec{J} - \vec{S}$$

or in magnitude squared

$$L(L+1)\hbar^2 = J(J+1)\hbar^2 + S(S+1)\hbar^2 - 2\vec{J}\cdot\vec{S}$$

Thus $$\cos(\sphericalangle \vec{J}, \vec{S}) = \frac{J(J+1)+S(S+1)-L(L+1)}{2\sqrt{J(J+1)}\sqrt{S(S+1)}}$$

Substitution gives $$\sphericalangle(\vec{J}, \vec{S}) = 31{\cdot}1°.$$

6.111 Total angular momentum $\hbar\sqrt{6}$ means $J = 2$. It is gives that $S = 1$. This means that $L = 1, 2,$ or 3. From vector model relation

$$L(L+1)\hbar^2 = 6\hbar^2 + 2\hbar^2 - 2\hbar^2\sqrt{6}\sqrt{2}\cos 73{\cdot}2°$$

$$= 5{\cdot}998\hbar^2 \approx 6\hbar^2$$

Thus $L = 2$ and the spectral symbol of the state is

$$^3D_2.$$

6.112 In a system containing a p electron and a d electron

$$S = 0, 1$$

$$L = 1, 2, 3$$

For $S = 0$ we have the terms

$$^1P_1,\ ^1D_2,\ ^1F_3$$

For $S = 1$ we have the terms

$$^3P_0,\ ^3P_1,\ ^3P_2,\ ^3D_1,\ ^3D_2,\ ^3D_3,\ ^3F_2,\ ^3F_3,\ ^3F_4$$

6.113 The atom has $S_1 = 1/2$, $l_1 = 1$, $j_1 = \frac{3}{2}$

The electron has $S_2 = \frac{1}{2}$, $l_2 = 2$ so the total angular momentum quantum number must be

$$j_2 = \frac{3}{2} \text{ or } \frac{5}{2}$$

In $L-S$ compling we get $S = 0, 1$. $L = 1, 2, 3$ and the terms that can be formed are the same as written in the problem above. The possible values of angular momentum are consistant with the addition $j_1 = \frac{3}{2}$ to $j_2 = \frac{3}{2}$ or $\frac{5}{2}$.

The latter gives us $J = 0, 1, 2, 3;\ 1, 2, 3, 4$

All these values are reached above.

6.114 Selection rules are $\Delta S = 0$

$$\Delta L = \pm 1$$

$$\Delta J = 0, \pm 1 \,(\text{no } 0 \to 0).$$

Thus $^2D_{3/2} \to {}^2P_{1/2}$ is allowed

$^3P_1 \to {}^2S_{1/2}$ not allowed

$^3F_3 \to {}^3P_2$ is not allowed ($\Delta L = 2$)

$^4F_{7/2} \to {}^4D_{5/2}$ is allowed

6.115 For a 3 d state of a Li atom, $S = \frac{1}{2}$ because there is only one electron and $L = 2$.

The total degeneracy is

$$g = (2L+1)(2S+1) = 5 \times 2 = 10.$$

The states are $^2D_{\frac{3}{2}}$ and $^2D_{5/2}$ and we check that

$$g = 4 + 6 = \left(2 \times \frac{3}{2} + 1\right) + \left(2 \times \frac{5}{2} + 1\right)$$

6.116 The state with greatest possible total angular momentum are

For a ^2P state $J = \frac{1}{2} + 1 = \frac{3}{2}$ i.e. $^2P_{3/2}$

Its degeneracy is 4.

For a ^3D state $J = 1 + 2 = 3$ i.e. $^3D_3$

Its degeneracy is $2 \times 3 + 1 = 7$

For a ^4F state $J = \frac{3}{2} + 3 = \frac{9}{2}$ i.e. $^4F_{\frac{4}{2}}$.

Its degeneracy is $2 \times \frac{9}{2} + 1 = 10$.

6.117 The degeneracy is $2J+1$. So we must have $J = 3$. From $L = 3S$, we see that S must be an integer since L is integral and S can be either integral or half integral. If $S = 0$ then $L = 0$ but this is consistent with $J = 3$. For $S \geq 2$, $L \geq 6$ and then $J \neq 3$. Thus the state is

$$^3F_3$$

6.118 The order of filling is
K, L, M shells, then $4s^2$, $3d^{10}$ then $4p^3$. The electronic configuration of the element will be

$$1s^2\,2s^2\,2p^6\,3s^2\,3p^6\,4s^2\,3d^{10}\,4p^3$$

(There must be three $4p$ electrons)

The number of electrons is $Z = 33$ and the element is As. (The $3d$ subshell must be filled before 4p fills up.)

6.119 (a) when the partially filled shell contains three p electrons, the total spin S must equal $S = \frac{1}{2}$ or $\frac{3}{2}$. The state $S = \frac{3}{2}$ has maximum spin and is totally symmetric under exchange of spin lables. By Pauli's exclusion principle this implies that the angular part of the wavefunction must be totally anti symmetric. Since the angular part of the wave function a p electron is vector $\vec{r}$, the total wavefunction of three p electrons is the totally antisymmetric combination of $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$. The only such combination is

$$\vec{r}_1 \cdot \left(\vec{r}_2 \times \vec{r}_3\right) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

This combination is a scalar and hence has $L = 0$. The spectral term of the ground state is then

$$^4S_{\frac{3}{2}} \quad \text{since } J = \frac{3}{2}.$$

(b) We can think of four p electrons as consisting of a full p shell with two p holes. The state of maximum spin S is then $S = 1$. By Pauli's principle the orbital angular momentum part must be antisymmetric and can only have the form

$$\vec{r}_1 \times \vec{r}_2$$

where $\vec{r}_1$, $\vec{r}_2$ are the coordinates of holes. The result is harder to see if we do not use the concept of holes. Four p electrons can have $S = 0, 1, 2$ but the $S = 2$ state is totally symmetric. The corresponding angular wavefunction must be totally antisymmetric. But this is impossible : there is no quantity which is amtisymmetric in four vectors. Thus the maximum allowed S is $S = 1$. We can construct such a state by coupling the spins of electrons 1 & 2 to $S = 1$ and of electrons 3 & 4 to $S = 1$ and then coupling the resultant spin states to $S = 1$. Such a state is symmetric under the exchange of spins of 1 & 2nd 3 and 4 but antisymmetric under the simultaneous exchange of (1, 2) & (3, 4). the con-

jugate angular wavefunction must be antisymmetric under the exchange of (1, 2) and under the exchange of (3, 4) by Pauli principle. It must also be antisymmetric under the simultaneous exchange of (1, 2) and (3, 4). (This is because two exchanges of electrons are involved.) The required angular wavefunction then has the form

$$\left(\vec{r_1}\times\vec{r_2}\right)\times\left(\vec{r_3}\times\vec{r_4}\right)$$

and is a vector, $L = 1$. Thus, using also the fact that the shell is more than half full, we find the spectral term $^3P_2$

$$(J = L + S).$$

6.120 (a) The maximum spin angular momentum of three electrons can be $S = \frac{3}{2}$. This state is totally symmetric and hence the conjugate angular wavefunction must be antisymmetric By Pauli's exclusion principle the totally antisymmetric state must have different magnetic quantum numbers. It is easy to see that for d electrons the maximum value of the magnetic quantum number for orbital angular momentum $|M_{Lz}| = 3$ (from 2 + 1 + 0). Higher values violate Pauli's principle. Thus the state of highest orbital angular momentum consistent with Pauli's principle is $L = 3$.

The state of the atom is then ^4F_J where $J = L - S$ by Hund's rule. Thus we get

$$^4F_{3/2}$$

The magnitude of the angular momentum is

$$\hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2}} = \frac{\hbar}{2}\sqrt{15}\,.$$

(b) Seven d electrons mean three holes. Then $S = \frac{3}{2}$ and $L = 3$ as before. But $J = L + S = \frac{9}{2}$ by Hund's rule for more than half filled shell. Thus the state is

$$^4F_{9/2}$$

Total angular momentum has the magnitude

$$\hbar\sqrt{\frac{9}{2}\cdot\frac{11}{2}} = \frac{3\hbar}{2}\sqrt{11}\,.$$

6.121 (a) $^3F_2$: The maximum value of spin is $S = 1$ here. This means there are 2 electrons. $L = 3$ so s and p electrons are ruled out. Thus the simplest possibility is d electrons. This is the correct choice for if we were considering f electrons, the maximum value of L allowed by Pauli principle will be $L = 5$ (maximum value of the magnitude of magnetic quantum number will be $3 + 2 = 5$.)
Thus the atom has two d electrons in the unfilled shell.

(b) $^2P_{3/2}$ Here $L = 1$, $S = \frac{1}{2}$ and $J = \frac{3}{2}$

Since $J = L + S$, Hund's rule implies the shell is more than half full. This means one electron less than a full shell. On the basis of hole picture it is easy to see that we have p electrons. Thus the atom has 5 p electrons.

(c) ${}^6S_{5/2}$ Here $S = \frac{5}{2}$, $L = 0$. We either have five electrons or five holes. The angular part is antisymmetric. For five d electrons, the maximum value of the quantum number consistent with Pauli exclusion principle is $(2+1+0-1-2) = 0$ so $L = 0$. For f or g electrons $L > 0$ whether the shell has five electrons or five holes. Thus the atom has five d electrons.

6.122 (a) If $S = 1$ is the maximum spin then there must be two electrons (If there are two holes then the shell will be more than half full.). This means that there are 6 electrons in the full shell so it is a p shell. By Paul's principle the only antisymmetric combination of two electrons has $L = 1$ Also $J = L - S$ as the shell is less than half full. Thus the term is ${}^3P_0$

(b) $S = \frac{3}{2}$ means either 3 electrons or 3 holes. As the shell is more than half full the former possibility is ruled out. Thus we must have seven d electrons. Then as in problem 6.120 we get the term ${}^4F_{9/2}$

6.123 With three electrons $S = \frac{3}{2}$ and the spin part is totally symmetric. It is given that the basic term has $L = 3$ so $L = 3$ is the state of highest orbital angular momentum. This is not possible with p electron so we must have d electrons for which $L = 3$ for 3 electrons. For three f, g electrons $L > 3$. Thus we have 3 d electrons. Then as in (6.120) the ground state is

$${}^4F_{\frac{3}{2}}$$

6.124 We have $5d$ electrons in the only unfilled shell. Then $S = \frac{5}{2}$. Maximum value of L consistent with Pauli's principle is $L = 0$. Then $J = \frac{5}{2}$.

So by Lande's formula

$$g = 1 + \frac{\frac{5}{2}\left(\frac{7}{2}\right) + \frac{5}{2}\left(\frac{7}{2}\right) - 0}{2\frac{5}{2}\left(\frac{7}{2}\right)} = 2$$

Thus $$\mu = g\sqrt{J(J+1)}\,\mu_B = 2\frac{\sqrt{35}}{2}\mu_B = 2\sqrt{35}\,\mu_B .$$

The ground state is ${}^6S_{5/2}$.

6.125 By Boltzmann formula

$$\frac{N_2}{N_1} = \frac{g_2}{g_1}\, e^{-\Delta E/KT}$$

Here ΔE = energy difference between $n = 1$ and $n = 2$ states

$$= 13{\cdot}6\left(1 - \frac{1}{4}\right)\text{eV} = 10{\cdot}22\ \text{eV}$$

$g_1 = 2$ and $g_2 = 8$ (counting $2s$ & $2P$ states.) Thus

$$\frac{N_2}{N_1} = 4\, e^{-10{\cdot}22 \times 1{\cdot}602 \times 10^{-19}/1{\cdot}38 \times 10^{-23} \times 3000} = 2{\cdot}7 \times 10^{-17}$$

Explicitly $\qquad \eta = \dfrac{N_2}{N_1} = n^2\, e^{-\Delta E_n/KT},\ \Delta E_n = \hbar R\left(1 - \dfrac{1}{n^2}\right)$

for the nth excited state because the degeneracy of the state with principal quantum number n is $2\, n^2$.

6.126 We have

$$\frac{N}{N_0} = \frac{g}{g_0}\, e^{-\hbar\omega/kT} = \frac{g}{g_0}\, e^{-2\pi\hbar c/\lambda kT}$$

Here g = degeneracy ofthe $3P$ sate = 6, g_0 = degeneracy of the $3S$ state = 2 and λ = wavelength of the $3P \rightarrow 3S$ line $\left(\dfrac{2\pi\hbar c}{\lambda}\right.$ = energy difference between $3P$ & $3S$ levels.

Substitution gives $\qquad \dfrac{N}{N_0} = 1{\cdot}13 \times 10^{-4}$

6.127 Let T = mean life time of the excited atoms. Then the number of excited atoms will decrease with time as $e^{-t/T}$. In time t the atom travels a distance vt so $t = \dfrac{l}{\text{v}}$. Thus the number of excited atoms in a beam that has traversed a distance l has decreased by

$$e^{-l/\text{v}T}$$

The intensity of the line is proportional to the number of excited atoms in the beam. Thus

$$e^{-l/\text{v}\tau} = \frac{1}{\eta} \quad\text{or}\quad \tau = \frac{l}{\text{v}\ \ln\eta} = 1{\cdot}29 \times 10^{-6}\ \text{second.}$$

6.128 As a result of the lighting by the mercury lamp a number of atoms are pumped to the excited state. In equilibrium the number of such atoms is N. Since the mean life time of the atom is T, the number decaying per unit time is $\dfrac{N}{\tau}$. Since a photon of energy $\dfrac{2\pi\hbar c}{\lambda}$ results from each decay, the total radiated power will be $\dfrac{2\pi\hbar c}{\lambda}\cdot\dfrac{N}{\tau}$. This must equal P. Thus

$$N = P\tau \Big/ \frac{2\pi\hbar c}{\lambda} = \frac{P\tau\lambda}{2\pi\hbar} = 6{\cdot}7 \times 10^9$$

6.129 The number of excited atoms per unit volume of the gas in $2P$ state is

$$N = n\frac{g_p}{g_s}e^{-2\pi\hbar c/\lambda kT}$$

Here g_p = degeneracy of the $2p$ state = 6, g_s = degeneacy of the $2s$ state = 2 and λ = wavelength of the resonant line $2p \rightarrow 2s$. The rate of decay of these atoms is $\frac{N}{\tau}$ per sec. per unit volume. Since each such atom emits light of wavelength λ, we must have

$$\frac{1}{\tau}\frac{2\pi\hbar c}{\lambda}n\frac{g_p}{g_s}e^{-2\pi\hbar c/\lambda kT} = P$$

Thus $$\tau = \frac{1}{P}\frac{2\pi\hbar c}{\lambda}n\frac{g_p}{g_s}e^{-2\pi\hbar c/\lambda kT} = 65{\cdot}4 \times 10^{-9}\text{ s} = 65{\cdot}4\text{ ns}$$

6.130 (a) We know that

$$P_{21}^{sp} = A_{21}$$

$$P_{21}^{ind} = B_{21}\,u_{as}$$

$$= \frac{\pi^2 c^3}{\hbar\omega^3}A_{21}\cdot\frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{e^{\hbar/kT}-1} \qquad = \frac{A_{21}}{e^{\hbar\omega/kT}-1}$$

Thus $$\frac{P_{21}^{ind}}{P_{21}^{sp}} = \frac{1}{e^{\hbar\omega/kT}-1}$$

For the transition $2P \rightarrow 1S$ $\hbar\omega = \frac{3}{4}\hbar R$ and

we get $$\frac{P_{21}^{1in}}{P_{21}^{sp}} = e^{-\hbar\omega kT}$$

substitution gives 7×10^{-18}

(b) The two rates become equal when $e^{\hbar\omega/kT} = 2$

or $$T = (\hbar\omega/k\ln 2) = 1{\cdot}71 \times 10^5\text{ K}$$

6.131 Because of the resonant nature of the processes we can ignore nonresonant processes. We also ignore spontaneous emission since it does not contribute to the absorption coefficient and is a small term if the beam is intense enough.

Suppose I is the intensity of the beam at some point. The decrease in the value of this intensity on passing through the layer of the substance of thickness dx is equal to

$$-dI = XI\,dx = (N_1B_{12} - N_2B_{21})\left(\frac{I}{c}\right)\hbar\omega\ dx$$

Here N_1 = No. of atoms in lower level

N_2 = No of atoms in the upper level per unit volume.

B_{12}, B_{21} are Einstein coefficients and I_c = energy density in the beam, c = velocity of light.

A factor $\hbar\,\omega$ arises because each transition result in a loss or gain of energy $\hbar\,\omega$

Hence $$x = \frac{\hbar\,\omega}{c} N_1 B_{12}\left(1 - \frac{N_2 B_{21}}{N_1 B_{12}}\right)$$

But $$g_1 B_{12} = g_2 B_{21} \text{ so}$$

$$x = \frac{\hbar\,\omega}{c} N_1 B_{12}\left(1 - \frac{g_1}{g_2}\frac{N_2}{N_1}\right)$$

By Boltzman factor $$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-\hbar\,\omega/k\,T}$$

When $\hbar\,\omega >> k\,T$ we can put $N_1 = N_0$ the total number of atoms per unit volume.

Then $$x = x_0\left(1 - e^{-\hbar\,\omega/k\,T}\right)$$

where $x_0 = \frac{\hbar\,\omega}{c} N_0 B_{12}$ is the absorption coefficient for $T \rightarrow 0$.

6.132 A short lived state of mean life T has an uncertainty in energy of $\Delta E \sim \frac{\hbar}{T}$ which is transmitted to the photon it emits as natural broadening. Then

$$\Delta\,\omega_{nat} = \frac{1}{T} \quad \text{so} \quad \Delta\,\lambda_{nat} = \frac{\lambda^2}{2\,\pi\,c\,\tau}.$$

The Döppler broadening on the other hand arises from the thermal motion of radiating atoms. The effect is non-relativistic and the maximum broadening can be written as

$$\frac{\Delta\,\lambda_{Dop}}{\lambda} = 2\,\beta = \frac{2\,v_{pr}}{c}$$

Thus $$\frac{\Delta\,\lambda_{Dopp}}{\Delta\,\lambda_{nat}} = \frac{4\,\pi\,v_{pr}\,\tau}{\lambda}$$

Substitution gives using $v_{pr} = \sqrt{\frac{2\,R\,T}{M}} = 157$ m/s,

$$\frac{\Delta\,\lambda_{Dopp}}{\Delta\,\lambda_{nat}} \approx 1{\cdot}2 \times 10^3$$

Note :- Our formula is an order of magnitude estimate.

6.133 From Moseley's law

$$\omega_{K_\alpha} = \frac{3}{4} R\,(Z - 1)^2$$

or $$\lambda_{K_\alpha} = \frac{4}{3\,R}\frac{1}{(Z-1)^2}$$

Thus $$\frac{\lambda_{K_\alpha}(Cu)}{\lambda_{K_\alpha}(Fe)} = \left(\frac{25}{28}\right)^2 = \left(\frac{Z_{Fe} - 1}{Z_{cu} - 1}\right)^2$$

Substitution gives

$$\lambda_{K_\alpha}(Cu) = 153{\cdot}9 \text{ pm}$$

6.134 (a) From Moseley's law

$$\omega_{k\alpha} = \frac{3}{4}R(Z-\sigma)^2$$

or

$$\lambda_{K_\alpha} = \frac{2\pi c}{\omega_{K_\alpha}} = \frac{8\pi c}{3R}\frac{1}{(Z-\sigma)^2}$$

We shall take $\sigma = 1$. For Aluminium ($Z = 13$)

$$\lambda_{K_\alpha}(Al) = 843{\cdot}2 \text{ pm}$$

and for cobalt ($Z = 27$)

$$\lambda_{K_\alpha}(Co) = 179{\cdot}6 \text{ pm}$$

(b) This difference is nearly equal to the energy of the K_α line which by Moseley's law is equal to ($Z = 23$ for vanadium)

$$\Delta E = \hbar\,\omega_{K_\alpha} = \frac{3}{4} \times 13{\cdot}62 \times 22 \times 22 = 4{\cdot}94 \text{ keV}$$

6.135 We calculate the Z values corresponding to the given wavelengths using Moseley's law. See problem (134).
Substitution gives that

$$Z = 23 \text{ corresponding to } \lambda = 250 \text{ pm}$$

and

$$Z = 27 \text{ corresponding to } \lambda = 179 \text{ pm}$$

There are thus three elements in a row between those whose wavelengths of K_α lines are equal to 250 pm and 179 pm.

6.136 From Moseley's law

$$\lambda_{K_\alpha}(Ni) = \frac{8\pi c}{3R}\frac{1}{(Z-1)^2}$$

where $Z = 28$ for Ni. Substitution gives

$$\lambda_{K_\alpha}(Ni) = 166{\cdot}5 \text{ pm}$$

Now the short wave cut of off the continuous spectrum must be more energetic (smaller wavelength) otherwise K_α lines will not emerge. Then since $\Delta\lambda = \lambda_{K_\alpha} - \lambda_0 = 84$ pm we get

$$\lambda_0 = 82{\cdot}5 \text{ pm}$$

This corresponds to a voltage of

$$V = \frac{2\pi\hbar c}{e\lambda_0}$$

Substitution gives $V = 15{\cdot}0\,k\text{V}$

6.137 Since the short wavelength cut off of the continuous spectrum is

$$\lambda_0 = 0{\cdot}50\ \text{nm}$$

the voltage applied must be $$V = \frac{2\pi\hbar c}{e\lambda_0} = 2{\cdot}48\,k\text{V}$$

since this is greater than the excitation potential of the K series of the characteristic spectrum (which is only $1{\cdot}56\,k\text{V}$) the latter will be observed.

6.138 Suppose λ_0 = wavelength of the characteristic X-ray line. Then using the formula for short wavelength limit of continuous radiation

$$\frac{\lambda_0 - \dfrac{2\pi\hbar c}{eV_1}}{\lambda_0 - \dfrac{2\pi\hbar c}{eV_2}} = \frac{1}{n}$$

Hence $$\lambda_0 = \frac{2\pi\hbar c}{eV_1}\,\frac{\left(n - \dfrac{V_1}{V_2}\right)}{n-1}$$

Using also Moseley's law, we get

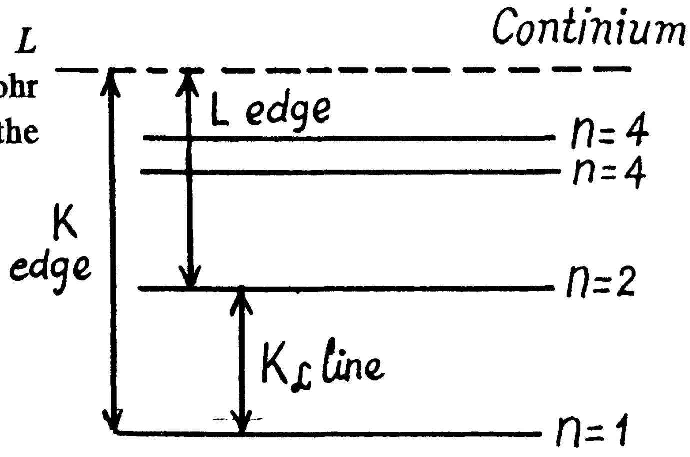
$$Z = 1 + \sqrt{\frac{8\pi c}{3R\lambda}} = 1 + 2\sqrt{\frac{n-1}{3\hbar R}\,\frac{eV_1}{n - \dfrac{V_1}{V_2}}} = 29\,.$$

6.139 The difference in frequencies of the K and L absorption edges is equal, according to the Bohr picture, to the frequency of the K_α line (see the diagram below). Thus by Moseley's formule

$$\Delta\omega = \frac{3}{4}R(Z-1)^2$$

or $$Z = 1 + \sqrt{\frac{4\Delta\omega}{3R}} = 22$$

The metal is titanium.



6.140 From the diagram above we see that the binding energy E_b of a K electron is the sum of the energy of a K_α line and the energy corresponding to the L edge of absorption spectrum

$$E_b = \frac{2\pi\hbar c}{\lambda_L} + \frac{3}{4}\hbar R(Z 1)^2$$

For vanadium $Z = 23$ and the energy of K_α line of vanadium has been calculated in problem 134 (b). Using

$$\frac{2\pi\hbar c}{\lambda_L} = 0{\cdot}51\,k\text{eV} \quad \text{for } \lambda_L = 2{\cdot}4\ \text{nm}$$

we get $$E_b = 5{\cdot}46\,k\text{eV}$$

6.141 By Moseley's law

$$\hbar\omega = \frac{2\pi\hbar c}{\lambda}\ E_K - E_L = \frac{3}{4}\hbar R(Z-1)^2$$

where $-E_K$ is the energy of the K electron and $-E_L$ of the L electron. Also the energy of the line corresponding to the short wave cut off of the K series is

$$E_K = \frac{2\pi\hbar c}{\lambda - \Delta\lambda} = \frac{2\pi\hbar c}{\frac{2\pi c}{\omega} - \Delta\lambda}$$

$$= \frac{\hbar}{\frac{1}{\omega} - \frac{\Delta\lambda}{2\pi c}} = \frac{\hbar\omega}{1 - \frac{\omega\Delta\lambda}{2\pi c}}$$

Hence
$$E_L = \frac{\hbar\omega}{1 - \frac{\omega\Delta\lambda}{2\pi c}} - \hbar\omega = \frac{\hbar\omega}{\frac{2\pi c}{\omega\Delta\lambda} - 1}$$

Substitution gives for titanium ($Z = 22$)

$$\omega = 6{\cdot}85 \times 10^{18}\ s^{-1}$$

and hence $E_L = 0{\cdot}47\ k\mathrm{eV}$

6.142 The energy of the K_α radiation of Zn is

$$\hbar\omega = \frac{3}{4}\hbar R(Z-1)^2$$

where Z = atomic number of Zinc = 30. The binding energy of the K electrons in iron is obtained from the wavelength of K absorption edge as $E_K = 2\pi\hbar c/\lambda_K$

Hence by Einstein equation

$$T = \frac{3}{4}\hbar R(Z-1)^2 - \frac{2\pi\hbar c}{\lambda_K}$$

Substitution gives

$$T = 1{\cdot}463\ k\mathrm{eV}$$

This corresponds to a velocity of the photo electrons of

$$v = 2{\cdot}27 \times 10^6\ \mathrm{m/s}$$

6.143 From the Lande formula

$$g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}$$

(a) For S states $L = 0$. This implies $J = S$. Then, if $S \neq 0$

$$g = 2$$

(For singlet states g is not defined if $L = 0$)

(b) For singlet states, $J = L$

$$g = 1 + \frac{J(J+1) - L(L+1)}{2J(J+1)} = 1$$

6.144 (a) $^6F_{\frac{1}{2}}$ Here $S = \frac{5}{2}$, $L = 3$, $J = \frac{1}{2}$

$$g = 1 + \frac{\frac{3}{4} + \frac{35}{4} - 12}{2 \times \frac{3}{4}} = 1 + \frac{38 - 48}{6} = -\frac{2}{3}$$

(b) $^4D_{1/2}$: Here $S = \frac{3}{2}$, $L = 2$, $J = \frac{1}{2}$

$$g = 1 + \frac{\frac{3}{4} + \frac{15}{4} - 6}{2 \times \frac{3}{4}} = 1 + \frac{18 - 24}{6} = 0$$

(c) $^5F_2$ Here $S = 2$, $L = 3$, $J = 2$

$$g = 1 + \frac{6 + 6 - 12}{2 \times 6} = 1$$

(d) $^5P_1$ Here $S = 2$, $L = 1$, $J = 1$

$$g = 1 + \frac{2 + 6 - 2}{2 \times 2} = \frac{5}{2}$$

(e) $^3P_0$. For states with $J = 0$, $L = S$ the g gactor is indeterminate.

6.145 (a) For the ^1F state $S = 0$, $L = 3 = J$

$$g = 1 + \frac{3 \times 4 - 3 \times 4}{2 \times 3 \times 4} = 1$$

Hence $$\mu = \sqrt{3 \times 4}\ \mu_B = 2\sqrt{3}\ \mu_B$$

(b) For the $^2D_{3/2}$ state $S = \frac{1}{2}$, $L = 2$, $J = \frac{3}{2}$

$$g = 1 + \frac{\frac{15}{4} + \frac{3}{4} - 6}{2 \times \frac{15}{4}} = 1 + \frac{18 - 24}{30} = \frac{4}{5}$$

Hence $$\mu = \frac{4}{5}\sqrt{15/4}\ \mu_B = \frac{2}{5}\sqrt{15}\ \mu_B = 2\sqrt{\frac{3}{5}}\ \mu_B.$$

(c) We have

$$\frac{4}{3} = 1 + \frac{J(J+1)+2-6}{2J(J+1)}$$

or
$$\frac{4}{3}J(J+1) = J(J+1)-4$$

or
$$J(J+1) = 12 \Rightarrow J = 3$$

Hence
$$\mu = \frac{4}{3}\sqrt{12}\,\mu_B = \frac{8}{\sqrt{3}}\,\mu_B.$$

6.146 The expression for the projection of the magnetic moment is

$$\mu_Z = g\, m_J\, \mu_B$$

where m_J is the projection of $\vec{J}$ on the Z-axis.

Maximum value of the m_J is J. Thus

$$gJ = 4$$

Since $J = 2$, we get $g = 2$. Now

$$2 = 1 + \frac{J(J+1)+S(S+1)-L(L+1)}{2J(J+1)}$$

$$= 1 + \frac{6+S(S+1)-6}{2\times 6}, \text{ as } L = 2$$

$$= 1 + \frac{S(S+1)}{12}$$

Hence
$$S(S+1) = 12 \quad \text{or} \quad S = 3$$

Thus
$$M_S = \hbar\sqrt{3\times 4} = 2\sqrt{3}\,\hbar$$

6.147 The angle between the angular momentum vector and the field direction is the least when the angular mommentum projection is maximum i.e. $J\hbar$.

Thus
$$J\hbar = \sqrt{J(J+1)}\,\hbar\cos 30°$$

or
$$\sqrt{\frac{J}{J+1}} = \frac{\sqrt{3}}{2}$$

Hence
$$J = 3$$

Then
$$g = 1 + \frac{3\times 4+1\times 2-2\times 3}{2\times 3\times 4} = 1+\frac{8}{24} = \frac{4}{3}$$

and
$$\mu = \frac{4}{3}\sqrt{3\times 4}\,\mu_B = \frac{8}{\sqrt{3}}\,\mu_B.$$

6.148 For a state with $n = 3$, $l = 2$. Thus the state with maximum angular momentum is

$$^2D_{5/2}$$

Then
$$g = 1 + \frac{\frac{5}{2}\times\frac{7}{2}+\frac{1}{2}\times\frac{3}{2}-2\times 3}{2\times\frac{5}{2}\times\frac{7}{2}}$$
$$= 1 + \frac{35+3-24}{70} = 1 + \frac{1}{5} = \frac{6}{5}.$$

Hence
$$\mu = \frac{6}{5}\sqrt{\frac{5}{2}\times\frac{7}{2}}\,\mu_B = 3\sqrt{\frac{7}{5}}\,\mu_B.$$

6.149 To get the greatest possible angular momentum we must have $S = S_{\max} = 1$
$$L = L_{\max} = 1+2 = 3 \text{ and } J = L+S = 4$$

Then
$$g = 1 + \frac{4\times 5+1\times 2-3\times 4}{2\times 4\times 5} = 1 + \frac{10}{40} = \frac{5}{4}$$

and
$$\mu = \frac{5}{4}\sqrt{4\times 5}\,\mu_B = \frac{5\sqrt{5}}{2}\,\mu_B.$$

6.150 Since $\mu = 0$ we must have either $J = 0$ or $g = 0$. But $J = 0$ is incompatible with $L = 2$ and $S = \frac{3}{2}$. Hence $g = 0$. Thus
$$0 = 1 + \frac{J(J+1)+\frac{3}{2}\times\frac{5}{2}-2\times 3}{2J(J+1)}$$

or
$$-3J(J+1) = \frac{15}{4} - 6 = -\frac{9}{4}$$

Hence
$$J = \frac{1}{2}$$

Thus
$$M = \hbar\sqrt{\frac{1}{2}\times\frac{3}{2}} = \frac{\hbar\sqrt{3}}{2}$$

6.151 From $M = \hbar\sqrt{J+1} = \sqrt{2}\hbar$
we find $J = 1$. From the zero value of the magnetic moment we find
$$g = 0$$

or
$$1 + \frac{1\times 2 L(L+1)+2\times 3}{2\times 1\times 2} = 0$$
$$1 + \frac{-L(L+1)+8}{4} = 0$$

or
$$12 = L(L+1)$$

Hence $L = 3$. The state is
$$^5F_1.$$

6.152 If $\vec{M}$ is the total angular momentum vector of the atom then there is a magnetic moment

$$\vec{\mu}_m = g\,\mu_B \vec{M}/\hbar$$

associated with it; here g is the Lande factor. In a magnetic field of induction $\vec{B}$, an energy

$$H' = -g\,\mu_B \vec{M}\cdot\vec{B}/\hbar$$

is associated with it. This interaction term corresponds to a presession of the angular momentum vector because if leads to an equation of motion of the angular momentum vector of the form

$$\frac{d\vec{M}}{dt} = \vec{\Omega}\times M$$

where $$\vec{\Omega} = \frac{g\,\mu_B \vec{B}}{\hbar}$$

Using Gaussian unit expression of μ_B $\mu_B = 0{\cdot}927\times 10^{-20}$ erg/gauss, $B = 10^3$ gauss $\hbar = 1{\cdot}054\times 10^{-27}$ erg sec and for the $^2P_{3/2}$ state

$$g = 1 + \frac{\frac{3}{2}\times\frac{5}{2}+\frac{1}{2}\times\frac{3}{2}-1\times 2}{2\times\frac{3}{2}\times\frac{5}{2}} = 1+\frac{1}{3} = \frac{4}{3}$$

and $$\Omega = 1{\cdot}17\times 10^{10}\ \text{rad/s}$$

The same formula is valid in MKS units also But $\mu_B = 0{\cdot}927\times 10^{-23}\ \text{A}\cdot\text{m}^2$, $B = 10^{-1}\,T$ and $\hbar = 1{\cdot}054\times 10^{-34}$ Joule sec. The answer is the same.

6.153 The force on an atom with magnetic moment $\vec{\mu}$ in a magnetic field of induction $\vec{B}$ is given by

$$\vec{F} = (\vec{\mu}\cdot\vec{\nabla})\vec{B}$$

In the present case, the maximum force arise when $\vec{\mu}$ is along the axis or close to it.

Then $$F_Z = (\mu_Z)_{\max}\frac{\partial B}{\partial Z}$$

Here $(\mu_Z)_{\max} = g\,\mu_B J$. The Lande factor g is for $^2P_{1/2}$

$$g = 1 + \frac{\frac{1}{2}\times\frac{3}{2}+\frac{1}{2}\times\frac{3}{2}-1\times 2}{2\frac{1}{2}\times\frac{3}{2}} = 1-\frac{1/2}{3/2} = \frac{2}{3}.$$

and $$J = \frac{1}{2}\ \text{so}\ (\mu_Z)_{\max} = \frac{1}{3}\mu_B.$$

The magnetic field is given by

$$B_Z = \frac{\mu_0}{4\pi}\cdot\frac{2I\pi r^2}{(r^2+Z^2)^{3/2}}$$

or
$$\frac{\partial B_Z}{\partial Z} = -\frac{\mu_0}{4\pi} 6 I \pi r^2 \frac{Z}{(r^2 + Z^2)^{5/2}} .$$

Thus
$$\left(\frac{\partial B_Z}{\partial Z}\right)_{Z = r} = \frac{\mu_0}{4\pi} \frac{3 I \pi}{\sqrt{8}\, r^2} .$$

Thus the maximum force is

$$F = \frac{1}{3} \mu_B \frac{\mu_0}{4\pi} \frac{3\pi}{\sqrt{8}} \frac{I}{r^2}$$

Substitution gives (using data in MKS units)

$$F = 4{\cdot}1 \times 10^{-27}\,\mathrm{N}$$

6.154 The magnetic field at a distance r from a long current carrying wire is mostly tangential and given by

$$B_\varphi = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0}{4\pi} \frac{2I}{r} .$$

The force on a magnetic dipole of moment $\vec{\mu}$ due to this magnetic field is also tangential and has a magnitude

$$(\vec{\mu} \cdot \nabla)\, B_\varphi$$

This force is nonvanishing only when the component of $\vec{\mu}$ along $\vec{r}$ non zero. Then

$$F = \mu_r \frac{\partial}{\partial r} B_\varphi = -\mu_r \frac{\mu_0}{4\pi} \frac{2I}{r^2}$$

Now the maximum value of $\mu_r = \pm\, \mu_B$. Thus the force is

$$F_{\max} = \mu_B \frac{\mu_0}{4\pi} \frac{2I}{r^2} = 2{\cdot}97 \times 10^{-26}\,\mathrm{N}$$

6.155 In the homogeneous magnetic field the atom experinces a force

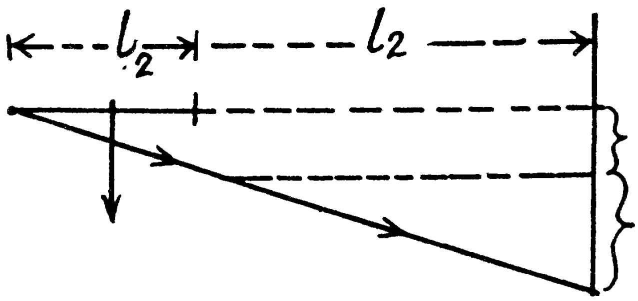
$$F = g J \mu_B \frac{\partial B}{\partial Z}$$

Depending on the sign of J, this can be either upward or downward. Suppose the latter is true. The atom then traverses first along a parabola inside the field and, once outside, in a straight line. The total distance between extreme lines on the screen will be

$$\delta = 2 g J \mu_B \frac{\partial B}{\partial Z} \left\{ \frac{1}{2} \left(\frac{l_1}{\mathrm{v}}\right)^2 + \frac{l_1}{\mathrm{v}} \cdot \frac{l_2}{\mathrm{v}} \right\} \Big/ \; m_V$$

Here m_V is the mass of the vanadium atom. (The first term is the displacement within the field and the second term is the displacement due to the transverse velocity acquired in the magnetic field).

Thus using $\frac{1}{2} m_V \mathrm{v}^2 = T$



we get $$\frac{\partial B}{\partial Z} = \frac{2T\delta}{g\mu_B J l_1 (l_1 + 2l_2)}$$

For vanadium atom in the ground state ${}^4F_{3/2}$.

$$g = 1 + \frac{\frac{3\times5}{4} + \frac{3\times5}{4} - 3\times4}{2\times\frac{3\times5}{4}} = 1 + \frac{30-48}{30} = 1 - \frac{18}{30} = \frac{2}{5}$$

$J = \frac{3}{2}$, using other data, and substituting

we get $$\frac{\partial B}{\partial Z} = 1{\cdot}45\times10^{13}\ \text{G/cm}$$

This value differs from the answer given in the book by almost a factor of 10^9. For neutral atoms in stern Gerlach experiments, the value $T = 22\,\text{MeV}$ is much too large. A more appropriate value will be $T = 22\,\text{meV}$ i.e. 10^9 times smaller. Then one gets the right answer.

6.156 (a) The term $3P_0$ does not split in weak magnetic field as it has zero total angular momentum.

(b) The term ${}^2F_{5/2}$ will split into $2\times\frac{5}{2}+1 = 6$ sublevels. The shift in each sublevel is given by

$$\Delta E = -g\mu_B M_Z B$$

where $M_J = -J(J-1), \ldots, J$ and g is the Landi factor

$$g = 1 + \frac{\frac{5\times7}{4} + \frac{1\times3}{4} - 3\times4}{2\times\frac{5\times7}{4}} = 1 + \frac{38-48}{70} = \frac{6}{7}$$

(c) In this case for the ${}^4D_{1/2}$ term

$$g = 1 + \frac{\frac{1\times3}{4} + \frac{3\times5}{4} - 2\times3}{2\times\frac{1\times3}{4}} = 1 + \frac{3+15-24}{6} = 1 - 1 = 0$$

Thus the energy differences vanish and the level does not split.

6.157 (a) For the ${}^1D_2$ term

$$g = 1 + \frac{2\times3+0-2\times3}{2\times2\times3} = 1$$

and $$\Delta E = -\mu_B M_J B$$

$M_J = -2, -1, 0, +1, +2$. Thus the splitting is

$$\delta E = 4\mu_B B$$

Substitution gives $\delta E = 57{\cdot}9\,\mu\,eV$

(b) For the $^3F_4$ term $g = 1 + \frac{4 \times 5 + 1 \times 2 - 3 \times 4}{2 \times 4 \times 5} = 1 + \frac{10}{40} = \frac{5}{4}$.

and $$\Delta = -\frac{5}{4}\mu_B B M_J$$

where $$M_J = -4 \text{ to } +4\text{. Thus}$$

$$\delta E = \frac{5}{4}\mu_B B \times 8 = 10\,\mu_B B\,(= 2gJ\mu_B)$$

Substitution gives $\delta E = 144{\cdot}7\,\mu\,\text{eV}$

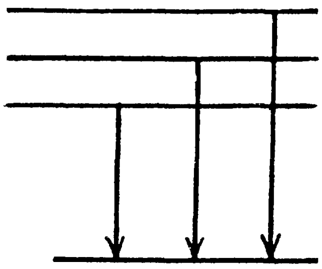
6.158 (a) The term $^1P_1$ splits into 3 lines with $M_Z = \pm 1,\ 0$ in accordance with the formula

$$\Delta E = -g\,\mu_B B M_Z$$

where $$g = 1 + \frac{1 \times 2 + 0 - 1 \times 2}{2 \times 1 \times 2} = 1$$

The term $^1S_0$ does not split in weak magnetic field. Thus the transitions between $^1P_1$ & $^1S_0$ will result in 3 lines i.e. a normal Zeeman triplet.

(b) The term $^2D_{5/2}$ will split ot in 6 terms in accordance with the formula



$$\Delta E = -g\,\mu_B M_Z$$

$$M_Z = \pm\frac{5}{2},\ \pm\frac{3}{2},\ \pm\frac{1}{2}, \text{ and}$$

$$g = 1 + \frac{5 \times 7 + 1 \times 3 - 4 \times 2 \times 3}{2 \times 5 \times 7} = \frac{6}{5}$$

Ther term $^2P_{3/2}$ will also split into 4 lines in accordance with the above formula with

$$M_Z = \pm\frac{3}{2},\ \pm\frac{1}{2} \text{ and } g = 1 + \frac{3 \times 5 + 1 \times 3 - 4 \times 1 \times 2}{2 \times 3 \times 5} = \frac{4}{3}$$

It is seen that the Z eeman splitting is auomalous as g factors are different.

(c) $^3D_1 \rightarrow {}^3P_0$

The term $^3D_1$ splits into 3 levels ($g = 5/2$)

The term $^3P_0$ does not split. Thus the Z eeman spectrum is normal.

(d) For the $5I_5$ term

$$g = 1 + \frac{5 \times 6 + 2 \times 3 - 6 \times 7}{2 \times 5 \times 6}$$

$$= 1 + \frac{36 - 42}{60} = 1 - \frac{1}{10} = \frac{9}{10}$$

For the $^5H_4$ term

$$g = 1 + \frac{4\times 5 + 2\times 3 - 5\times 6}{2\times 4\times 5} = 1 + \frac{26-30}{40} = \frac{9}{10}$$

We see that the splitting in the two levels given by $\Delta E = -g\,\mu_B B M_Z$ is the same though the number of levels is different (11 and 9). It is then easy to see that only the lines with following energies occur

$$\hbar\,\omega_0\,,\ \hbar\,\omega_0 \pm g\,\mu_B B\,.$$

The Zeeman pattern is normal

6.159 For a singlet term $S = 0$, $L = J$, $g = 1$

Then the total splitting is $\qquad \delta E = 2J\,\mu_B B$

Substitution gives $\qquad J = 3\,(= \delta E/2\,\mu_B B)$

The term is $^1F_3$.

6.160 As the spectral line is caused by transition between singlet terms, the Zeeman effect will be normal (since $g = 1$ for both terms). The energy difference between extreme components of the line will be $2\,\mu_B B$. This must equal

$$-\Delta\left(\frac{2\pi\hbar c}{\lambda}\right) = \frac{2\pi\hbar c\,\Delta\lambda}{\lambda^2}$$

Thus
$$\Delta\lambda = \frac{\mu_B B\lambda^2}{\pi\hbar c} = 35\text{ pm}\,.$$

6.161 From the previous problem, if the components are λ, $\lambda \pm \Delta\lambda$, then

$$\frac{\lambda}{\Delta\lambda} = \frac{2\pi\hbar c}{\mu_B B\lambda}$$

For resolution $\dfrac{\lambda}{\Delta\lambda} \le R = \dfrac{\lambda}{\delta\lambda}$ of the instrument.

Thus
$$\frac{2\pi\hbar c}{\mu_B B\lambda} \le R \quad\text{or}\quad B \ge \frac{2\pi\hbar c}{\mu_B\lambda R}$$

Hence the minimum megnetic induction is

$$B_{\min} = \frac{2\pi\hbar c}{\mu_B\lambda R} = 4\,kG = 0{\cdot}4\,\text{T}$$

6.162 The $^3P_0$ term does not split. The $^3D_1$ term splits into 3 lines corresponding to the shift.

$$\Delta E = -g\,\mu_B B M_Z$$

with $M_Z = \pm 1, 0$. The interval between neighbouring components is then given by

$$\hbar\,\Delta\omega = g\,\mu_B B$$

Hence
$$B = \frac{\hbar\,\Delta\omega}{g\,\mu_B}$$

Now for the $^3D_1$ term

$$g = 1 + \frac{1 \times 2 + 1 \times 2 - 2 \times 3}{2 \times 1 \times 2} = 1 + \frac{4-6}{4} = \frac{1}{2}.$$

Substitution gives $B = 3.00$ kG. $= 0.3$ T.

6.163 (a) For the $^2P_{3/2}$ term

$$g = 1 + \frac{\frac{3}{2} \times \frac{5}{2} + \frac{1}{2} \times \frac{3}{2} - 1 \times 2}{2 \times \frac{3}{2} \times \frac{5}{2}} = 1 + \frac{10}{30} = \frac{4}{3}$$

and the energy of the $^2P_{3/2}$ sublevels will be

$$E(M_Z) = E_0 - \frac{4}{3}\mu_B B M_Z$$

where $M_Z = \pm\frac{3}{2}, \pm\frac{1}{2}$. Thus, between neighbouring sublevels.

$$\delta E(^2P_{3/2}) = \frac{4}{3}\mu_B B$$

For the $^2P_{1/2}$ terms

$$g = 1 + \frac{\frac{1}{2} \times \frac{3}{2} + \frac{1}{2} \times \frac{3}{2} - 1 \times 2}{2 \times \frac{1}{2} \times \frac{3}{2}}$$

$$= 1 + \frac{6-8}{6} = 1 - \frac{1}{3} = \frac{2}{3}$$

and the separation between the two sublevels into which the $^2P_{1/2}$ term will split is

$$\delta E(^2P_{1/2}) = \frac{2}{3}\mu_B B$$

The ratio of the two splittings is 2 : 1.

(b) The interval between neighbouring Zeeman sublevels of the $^2P_{3/2}$ term is $\frac{4}{3}\mu_B B$. The energy separation between D_1 and D_2 lines is $\frac{2\pi\hbar c}{\lambda^2}\Delta\lambda$ (this is the natural separation of the ^2P them)

Thus
$$\frac{4}{3}\mu_B B = \frac{2\pi\hbar c\,\Delta\lambda}{\lambda^2\eta}$$

or
$$B = \frac{3\pi\hbar c\,\Delta\lambda}{2\mu_B\lambda^2\eta}$$

Substitution gives

$$B = 5{\cdot}46\ \mathrm{k\,G}$$

6.164 For the $^2P_{3/2}$ level $g = 4/3$ (see above) and the energies of sublevels are

$$E' = E'_0 - \frac{4}{3}\mu_B B M'_z$$

where $M'_z = \pm\frac{3}{2}, \pm\frac{1}{2}$ for the four sublevels

For the $^2S_{\frac{1}{2}}$ level, $g = 2$ (since $L = 0$) and

$$E = E_0 - 2\mu_B B M_z$$

where $$M_z = \pm\frac{1}{2}$$

Permitted transitions must have

$$\Delta M_z = 0, \pm 1$$

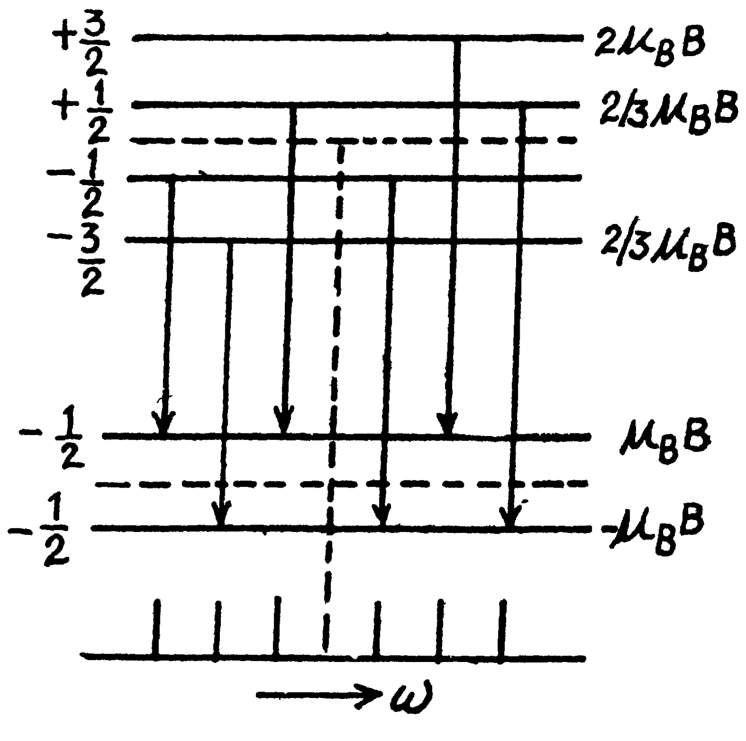
Thus only the following transitions occur

$$\left.\begin{array}{c}\frac{3}{2} \to \frac{1}{2} \\ -3/2 \to -1/2\end{array}\right\} \Delta\omega = \pm\mu_B B/\hbar = 3{\cdot}96 \times 10^{10} \text{ rad/s}$$

$$\left.\begin{array}{c}\frac{1}{2} \to \frac{1}{2} \\ -\frac{1}{2} \to -\frac{1}{2}\end{array}\right\} \Delta\omega = \pm\frac{1}{3}\mu_B B/\hbar = 1{\cdot}32 \times 10^{10} \text{ rad/s}$$

$$\left.\begin{array}{c}\frac{1}{2} \to -\frac{1}{2} \\ -\frac{1}{2} \to \frac{1}{2}\end{array}\right\} \Delta\omega = \pm\frac{5}{3}\frac{\mu_B B}{\hbar} = 6{\cdot}6 \times 10^{10} \text{ rad/s}$$

These six lines are shown below



6.165 The difference arises because of different selection rules in the two cases. In (1) the line is emitted perpendicular to the field. The selection rules are then

$$\Delta M_Z = 0, \pm 1$$

In (2) the light is emitted along the direction of the field. Then the selection rules are

$$\Delta M_Z = \pm 1$$

$$\Delta M_Z = 0 \text{ is forbidden.}$$

(a) In the transition $^2P_{3/2} \rightarrow {}^2S_{1/2}$

This has been considered above. In (1) we get all the six lines shown in the problem above

In (2) the line corresponding to $\frac{1}{2} \rightarrow \frac{1}{2}$ and $-\frac{1}{2} \rightarrow -\frac{1}{2}$ is forbidden.

Then we get four lines

(b) $^3P_2 \rightarrow {}^3S_1$

For the $^3P_2$ level, $g = 1 + \dfrac{2 \times 3 + 1 \times 2 - 1 \times 2}{2 \times 2 \times 3} = \dfrac{3}{2}$

so the energies of the sublevels are

$$E'(M'_Z) = E'_0 - \frac{3}{2} \mu_B B M'_Z$$

where $M'_Z = \pm 2, \pm 1, 0$

For the $^3S_1$ line, $g = 2$ and the energies of the sublevels are

$$E(M_Z) = E_0 - 2 \mu_B B M_Z$$

where $M_Z = \pm 1, 0$. The lines are

$$\Delta M_Z = M_Z - M_Z = +1 \quad : \quad -2 \rightarrow -1, -1 \rightarrow 0 \text{ and } 0 \rightarrow 1$$

$$\Delta M_Z = 0 \ - \ 1 \rightarrow -1, 0 \rightarrow 0, 1 \rightarrow 1$$

$$\Delta M_Z = 1, 2 \rightarrow 1, 1 \rightarrow 0, 0 \rightarrow -1$$

All energy differences are unequal because the two g values are unequal. There are then nine lines if viewed along (1) and Six lines if viewed along (2).

6.166 For the two levels

$$E'_0 = E_0 - g' \mu'_B M'_Z B$$

$$E_0 = E_0 - g \mu_B M_Z B$$

and hence the shift of the component is the value of

$$\Delta \omega = \frac{\mu_B B}{\hbar} \left[g' M'_Z - g M_Z \right]$$

subject to the selection rule $\Delta M_Z = 0, \pm 1$. For $^3D_3$

$$g' = 1 + \frac{3\times4+1\times2-2\times3}{2\times3\times4} = 1+\frac{8}{24} = \frac{4}{3}$$

For $^3P_2$,

$$g = 1 + \frac{2\times3+1\times2-1\times2}{2\times2\times3} = \frac{3}{2}$$

Thus
$$\Delta\omega = \frac{\mu_B B}{\hbar}\left|\frac{4}{3}M'_z - \frac{3}{2}M_z\right|$$

For the different transition we have the following table

	$M'_z g' - M_z g$		
$3 \to 2$	$\mu_B B$	$0 \to 1$	$-\frac{3}{2}\mu_B B$
$2 \to 2$	$-\frac{1}{3}\mu_B B$	$0 \to 0$	0
$2 \to 1$	$7/6\ \mu_B B$	$0 \to -1$	$3/2\ \mu_B B$
$1 \to 2$	$-5/3\ \mu_B B$	$-1 \to 0$	$-4/3\ \mu_B B$
$1 \to 1$	$-1/6\ \mu_B B$	$-1 \to -1$	$1/6\ \mu_B B$
$1 \to 0$	$4/3\ \mu_B B$	$-1 \to -2 \to$	$5/3\ \mu_B B$
		$-2 \to -1 \to$	$-7/6\ \mu_B B$
		$-2 \to -2 \to$	$1/3\ \mu_B B$
		$-3 \to -2 \to$	$-\mu_B B$

There are 15 lines in all.

The lines farthest out are $1 \to 2$ and $-1 \to -2$.

The splitting between them is the total splitting. It is

$$\Delta\omega = \frac{10}{3}\mu_B B/\hbar$$

Substitution gives $\Delta\omega = 7{\cdot}8 \times 10^{10}$ rad/sec.

6.4 MOLECULES AND CRYSTALS

6.167 In the first excited rotational level $J = 1$

so
$$E_J = 1 \times 2\frac{\hbar^2}{2I} = \frac{1}{2}I\omega^2 \text{ classically}$$

Thus
$$\omega = \sqrt{2}\,\frac{\hbar}{I}$$

Now
$$I = \Sigma m_i r_i^2 = \frac{m}{2}\frac{d^2}{4} + \frac{m}{2}\frac{d^2}{4} = m\frac{d^2}{4}$$

where m is the mass of the mole cub and r_i is the distance of the atom from the axis.

Thus
$$\omega = \frac{4\sqrt{2}\,\hbar}{m\,d^2} = 1{\cdot}56 \times 10^{11} \text{ rad/s}$$

6.168 The axis of rotation passes through the centre of mass of the HCl molecule. The distances of the two atoms from the centre of mass are

$$d_H = \frac{m_{Cl}}{m_{HCl}}d,\; d_{Cl} = \frac{m_H}{m_{HCl}}d$$

Thus I = moment of inertia about the axis

$$= \frac{4}{2}m_H d_H^2 + m_{Cl}d_{Cl}^2 = \frac{m_H m_{Cl}}{m_H + m_{Cl}}d^2$$

The energy difference between two neighbouring levels whose quantum numbers are J & $J-1$ is

$$\frac{\hbar^2}{2I}\cdot 2J = \frac{J\hbar^2}{I} = 7{\cdot}86 \text{ meV}$$

Hence $J = 3$ and the levels have quantum numbers 2 & 3.

6.169 The angular momentum is $\sqrt{2IE} = M$

Now $I = \frac{m\,d^2}{4}$ (m = mass of O_2 molecule) $= 1{\cdot}9584 \times 10^{-39}$ gm c^2

So
$$M = 3{\cdot}68 \times 10^{-27} \text{ erg sec.} = 3{\cdot}49\hbar$$
(This corresponds to $J = 3$)

6.170 From $E_J = \frac{\hbar^2}{2I}J(J+1)$

and the selection rule $\Delta J = 1$ or $J \rightarrow J-1$ for a pure rotational spectrum we get

$$\omega(J, J-1) = \frac{\hbar J}{I}$$

Thus transition lines are equispaced in frequency $\Delta\omega = \frac{\hbar}{I}$.

In the case of CH molecule

$$I = \frac{\hbar}{\Delta\omega} = 1{\cdot}93 \times 10^{-40}\,\text{gm cm}^2$$

Also
$$I = \frac{m_c\, m_H}{m_c + m_H} d^2$$

so
$$d = 1{\cdot}117 \times 10^{-8}\,\text{cm} = 111{\cdot}7\,\text{pm}$$

6.171 If the vibrational frequency is ω_0 the excitation energy of the first vibrational level will be $\hbar\,\omega_0$. Thus if there are J rotational levels contained in the band between the ground state and the first vibrational excitation, then

$$\hbar\,\omega_0 = \frac{J(J+1)\hbar^2}{2I}$$

where as stated in the problem we have ignored any coupling between the two. For HF molecule

$$I = \frac{m_H\, m_F}{m_H + m_F} d^2 = 1{\cdot}336 \times 10^{-4}\,\text{gm cm}^2$$

Then
$$J(J+1) = \frac{2I\omega_0}{\hbar} = 197{\cdot}4$$

For $J = 14$, $J(J+1) = 210$. For $J = 13$, $J(J+1) = 182$. Thus there lie 13 levels between the ground state and the first vibrational excitation.

6.172 We proceed as above. Calculating $\frac{2I\omega_0}{\hbar}$ we get

$$\frac{2I\omega_0}{\hbar} \cong 1118$$

Now this must equal $J(J+1) \approx \left(J + \frac{1}{2}\right)^2$

Taking the square root we get $J \approx 33$.

6.173 From the formula

$$J(J+1)\frac{\hbar^2}{2I} = E \quad \text{we get} \quad J(J+1) = 2IE/\hbar^2$$

or
$$\left(J + \frac{1}{2}\right)^2 - \frac{1}{4} = \frac{2IE}{\hbar^2}$$

Hence
$$J = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2IE}{\hbar^2}}$$

writing
$$J + 1 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2I}{\hbar^2}(E + \Delta E)}$$

we find $$1 = \sqrt{\frac{1}{4} + \frac{2I}{\hbar^2}E + \frac{2I}{\hbar^2}\Delta E} - \sqrt{\frac{1}{4} + \frac{2IE}{\hbar^2}}$$

$$= \sqrt{\frac{1}{4} + \frac{2I}{\hbar^2}E}\left[\left(1 + \frac{\Delta E}{E + \frac{\hbar^2}{8I}}\right)^{1/2} - 1\right]$$

$$= \sqrt{\frac{1}{4} + \frac{2I}{\hbar^2}E} \cdot \frac{\Delta E}{2\left(E + \frac{\hbar^2}{8I}\right)}$$

$$= \sqrt{\frac{2I}{\hbar^2}}\ \frac{\Delta E}{2\sqrt{E + \frac{\hbar^2}{8I}}}$$

The quantity $\frac{dN}{dE}$ is $\frac{1}{\Delta E}$. For large E it is

$$\frac{dN}{dE} = \sqrt{\frac{I}{2\hbar^2 E}}$$

For an iodine molecule

$$I = m_I d^2/2 = 7{\cdot}57 \times 10^{-38}\ \text{gm cm}^2$$

Thus for $J = 10$

$$\frac{dN}{dE} = \sqrt{\frac{I}{2\hbar^2 \cdot \frac{\hbar^2}{2I} J(J+1)}} = \frac{I}{\sqrt{J(J+1)}\,\hbar^2}$$

Substitution gives

$$\frac{dN}{dE} = 1{\cdot}04 \times 10^4 \text{ levels per eV}$$

6.174 For the first rotational level

$$E_{rot} = 2\frac{\hbar^2}{2I} = \frac{\hbar^2}{I} \text{ and}$$

for the first vibrational level $E_{vib} = \hbar\omega$

Thus $$\xi = \frac{E_{vib}}{E_{rot}} = \frac{I\omega}{\hbar}$$

Here ω = frequency of vibration. Now

$$I = \mu d^2 = \frac{m_1 m_2}{m_1 + m_2} d^2 .$$

(a) For H_2 molecule $I = 4{\cdot}58 \times 10^{-41}$ gm cm^2 and $\xi = 36$

(b) For HI molecule,

$$I = 4{\cdot}2\,47 \times 10^{-40} \text{ gm cm}^2 \text{ and } \xi = 175$$

(c) For I_2 molecule

$$I = 7{\cdot}57 \times 10^{-38} \text{ gm cm}^2 \text{ and } \xi = 2872$$

6.175 The energy of the molecule in the first rotational level will be $\frac{\hbar^2}{I}$. The ratio of the number of molecules at the first excited vibrational level to the number of molecules at the first excited rotational level is

$$\frac{e^{-\hbar\omega/kT}}{(2J+1)\, e^{-\hbar^2 J(J+1)/2IkT}}$$

$$= \frac{1}{3} e^{-\hbar\omega/kT} \times e^{-\hbar^2/IkT} = \frac{1}{3} e^{-\hbar(\omega - 2B)/kT}$$

where
$$B = \hbar/2I$$

For the hydrogen molecule $I = \frac{1}{2} m_H d^2$

$$= 4{\cdot}58 \times 10^{-41} \text{ gm cm}^2$$

Substitution gives $3{\cdot}04 \times 10^{-4}$

6.176 By definition

$$\langle E \rangle = \frac{\sum E_v e^{-E_v/kT}}{\sum \exp(-E_v/kT)} = \frac{\frac{\partial}{\partial \beta} \sum_{v=0}^{\infty} e^{-\beta E_v}}{\sum_{v=0}^{\infty} e^{-\beta E_v}}$$

$$= -\frac{\partial}{\partial \beta} \ln \sum_{v=0}^{\infty} e^{-\beta(v+1/2)\hbar\omega}, \quad \beta = \frac{1}{kT}$$

$$= -\frac{\partial}{\partial \beta} \ln e^{-1/2\beta\hbar\omega} \frac{1}{1 - e^{-\beta\hbar\omega}}$$

$$= -\frac{\partial}{\partial \beta} \left[-\frac{1}{2}\hbar\omega\beta - \ln\left(1 - e^{-\beta\hbar\omega}\right) \right]$$

$$= \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}.$$

Thus for one gm mole of diatomic gas

$$C_{V_{1\,vib}} = N\frac{\partial \langle E \rangle}{\partial T} = \frac{R\left(\frac{\hbar\omega}{kT}\right)^2 e^{\hbar\omega/kT}}{\left(e^{\hbar\omega/kT}-1\right)^2}$$

where $R = Nk$ is the gas constant.

In the present case $\qquad \dfrac{\hbar\omega}{KT} = 2{\cdot}7088$

and $\qquad C_{V_{1\,vib}} = 0{\cdot}56\,R$

6.177 In the rotation vibration band the main transition is due to change in vibrational quantum number $v \rightarrow v-1$. Together with this rotational quantum number may change. The "Zeroeth line" $0 \rightarrow 0$ is forbidden in this case so the neighbouring lines arise due to $1 \rightarrow 0$ or $0 \rightarrow 1$ in the rotational quantum number. Now

$$E = E_v + \frac{\hbar^2}{2I}J(J+1)$$

Thus
$$\hbar\omega = \hbar\omega_0 + \frac{\hbar^2}{2I}(\pm 2)$$

Hence
$$\Delta\omega = \frac{2\hbar}{I} = \frac{2\hbar}{\mu d^2}$$

so
$$d = \sqrt{\frac{2\hbar}{\mu\,\Delta\omega}}$$

Substitution gives $d = 0{\cdot}128$ nm.

6.178 If λ_R = wavelength of the red satellite
and λ_V = wavelength of the violet satellite

then
$$\frac{2\pi\hbar c}{\lambda_R} = \frac{2\pi\hbar c}{\lambda_0} - \hbar\omega$$

and
$$\frac{2\pi\hbar c}{\lambda_V} = \frac{2\pi\hbar c}{\lambda_0} + \hbar\omega$$

Substitution gives

$$\lambda_R = 424{\cdot}3 \text{ nm}$$
$$\lambda_V = 386{\cdot}8 \text{ nm}$$

The two formulas can be combined to give

$$\lambda = \frac{2\pi c}{\frac{2\pi c}{\lambda_0} \mp \omega} = \frac{\lambda_0}{1 \mp \frac{\lambda_0\omega}{2\pi c}}$$

6.179 As in the previous problem

$$\omega = \pi c \left(\frac{1}{\lambda_V} - \frac{1}{\lambda_R} \right) = \frac{\pi c (\lambda_R - \lambda_V)}{\lambda_R \lambda_V} = 1{\cdot}368 \times 10^{14} \text{ rad/s}$$

The force constant x is defined by

$$x = \mu \omega^2$$

where μ = reduced mass of the S_2 molecule.

Substitution gives $\qquad x = 5{\cdot}01$ N/cm

6.180 The violet satellite arises from the transition $1 \longrightarrow 0$ in the vibrational state of the scattering molecule while the red satellite arises from the transition $0 \longrightarrow 1$. The intensities of these two transitions are in the ratio of initial populations of the two states i.e. in the ratio

$$e^{-\hbar \omega / kT}$$

Thus

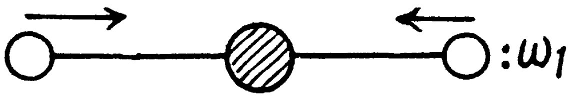
$$\frac{I_v}{I_r} = e^{-\hbar \omega / kT} = 0{\cdot}067$$

If the temperature is doubled, the rato increases to 0·259, an increase of 3·9 times.

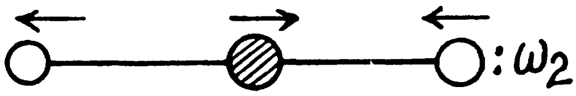
6.181 (a) $CO_2 (O - C - O)$

The molecule has 9 degrees of freedom 3 for each atom. This means that it can have up to nine frequencies. 3 degrees of freedom correspond to rigid translation, the frequency associated with this is zero as the potential energy of the system can not change under rigid translation. The P.E. will not change under rotations about axes passing through the C-atom and perpendicular to the $O - C - O$ line. Thus there can be at most four non zero frequencies. We must look for modes different from the above.

One mode is

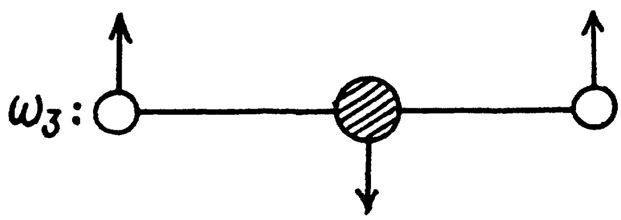


Another mode is



These are the only collinear modes.

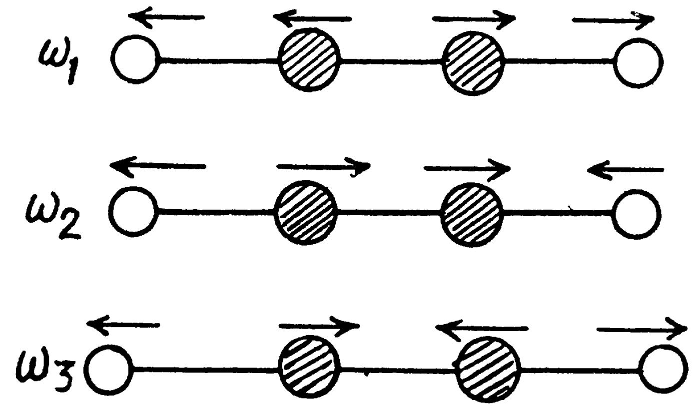
A third mode is doubly degenerate :



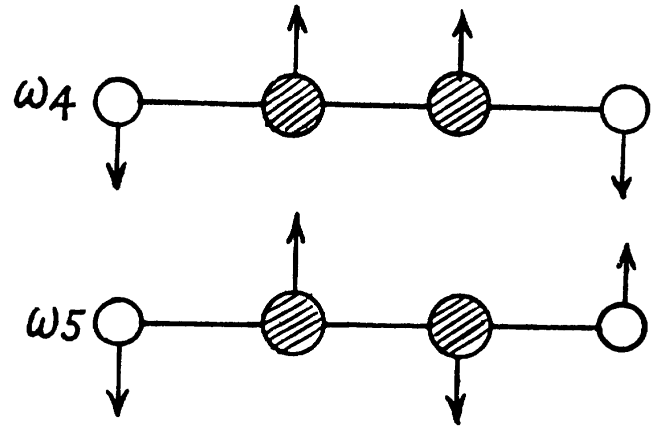
(vibration in & $\perp$ to the plane of paper).

(b) $C_2H_2 (H - C - C - H)$

There are $4 \times 3 - 3 - 2 = 7$ different vibrations. There are three collinear modes.



Two other doubly degenerate frequencies are



together with their counterparts in the plane $\perp^r$ to the paper.

6.182 Suppose the string is stretched along the x axis from $x = 0$ to $x = l$ with the end points fixed. Suppose $y(x, t)$ is the transverse displacement of the element at x at time t. Then $y(x, t)$ obeys

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

We look for a stationary wave solution of this equation

$$y(x, t) = A \sin \frac{\omega}{v} x \sin(\omega t + \delta)$$

where A & δ are constants.. In this from $y = 0$ at $x = 0$. The further condition

$$y = 0 \quad \text{at} \quad x = l$$

implies
$$\frac{\omega l}{v} = N\pi, \quad N > 0$$

or
$$N = \frac{l}{\pi v}\omega$$

N is the number of modes of frequency $\leq \omega$.

Thus
$$dN = \frac{l}{\pi v} d\omega$$

6.183 Let $\xi(x, y, t)$ be the displacement of the element at (x, y) at time t. Then it obeys the equation

$$\frac{\partial^2 \xi}{\partial t^2} = v^2 \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right)$$

where $\xi = 0$ at $x = 0,\ x = l,\ y = 0$ and $y = l$.

We look for a solution in the form

$$\xi = A \sin k_1 x \sin k_2 y \sin(\omega t + \delta)$$

Then
$$\omega^2 = v^2 (k_1^2 + k_2^2)$$

$$k_1 = \frac{n\pi}{l},\ k_2 = \frac{m\pi}{l}$$

we write this as

$$n^2 + m^2 = \left(\frac{l\omega}{\pi v} \right)^2$$

Here $n, m > 0$. Each pair (n, m) determines a mode. The total mumber of modes whose frequency is $\le \omega$ is the area of the quadrant of a circle of radius $\frac{l\omega}{\pi v}$ i.e.

$$N = \frac{\pi}{4} \left(\frac{l\omega}{\pi v} \right)^2$$

Then
$$dN = \frac{l^2}{2\pi v^2} \omega\, d\omega = \frac{S}{2\pi v^2} \omega\, d\omega.$$

where $S = l^2$ is the area of the membrane.

6.184 For transverse vibrations of a 3-dimensional continuum (in the form of a cube say) we have the equation

$$\frac{\partial^2 \vec{\xi}}{\partial t^2} = v^2 \vec{\nabla}^2 \vec{\xi},\ \text{div}\, \vec{\xi} = 0$$

Here $\vec{\xi} = \vec{\xi}(x, y, z, t)$. We look for solutions in the form

$$\vec{\xi} = \vec{A} \sin k_1 x,\ \sin k_2 y,\ \sin k_3 z,\ \sin(\omega t + \delta)$$

This requires $\omega^2 = v^2 (k_1^2 + k_2^2 + k_3^2)$

From the boundary condition that $\vec{\xi} = 0$ for $x = 0, x = l, y = 0, y = l, z = 0, z = l$, we get

$$k_1 = \frac{n_1 \pi}{l},\ k_2 = \frac{n_2 \pi}{l},\ k_3 = \frac{n_3 \pi}{l}$$

where n_1; n_2, n_3 are nonzero positive integers.

We then get
$$n_1^2 + n_2^2 + n_3^2 = \left(\frac{l\omega}{\pi v} \right)^2$$

Each triplet (n_1 , n_2 , n_3) determines a possible mode and the number of such modes whose frequency $\leq \omega$ is the volume of the all positive octant of a sphere of radius $\frac{l\,\omega}{\pi\,v}$. Considering also the fact that the subsidiary condition div $\vec{\xi} = 0$ implies two independent values of $\vec{A}$ for each choice of the wave vector (k_1 , k_2 , k_3)

we find

$$N(\omega) = \frac{1}{8}\cdot\frac{4\pi}{3}\left(\frac{l\,\omega}{\pi\,v}\right)^3 \cdot 2 = \frac{V\omega^3}{3\,\pi^2 v^3}$$

Thus
$$dN = \frac{V\omega^2}{\pi^2 v^3}\,d\omega .$$

6.185 To determine the Debye temperature we cut off the high frequency modes in such a way as to get the total number of modes correctly.

(a) In a linear crystal with $n_0\, l$ atoms, the number of modes of transverse vibrations in any given plane cannot exceed $n_0\, l$. Then

$$n_0\, l = \frac{l}{\pi\,v}\int_0^{\omega_0} d\omega = \frac{l}{\pi\,v}\,\omega_0$$

The cut off frequency ω_0 is related to the Debye temperature Θ by

$$\hbar\,\omega_0 = k\,\Theta$$

Thus
$$\Theta = \left(\frac{\hbar}{k}\right)\pi\, n_0\, v$$

(b) In a square lattice, the number of modes of transverse oscillations cannot exceed $n_0 S$. Thus

$$n_0 S = \frac{S}{2\,\pi\,v^2}\int_0^{\omega_0} \omega\, d\omega = \frac{S}{4\,\pi\,v^2}\,\omega_0^2$$

or
$$\Theta = \frac{\hbar}{k}\,\omega_0 = \left(\frac{\hbar}{k}\right)\left(\sqrt{4\,\pi\, n_0}\,\right) v$$

(c) In a cubic crystal, the maximum number of transverse waves must be $2\, n_0\, V$ (two for each atom). Thus

$$2\, n_0\, V = \frac{V}{\pi^2 v^3}\int_0^{\omega_0} \omega^2\, d\omega = \frac{V\omega_0^3}{3\,\pi^2 v^3}.$$

Thus
$$\Theta = \left(\frac{\hbar}{k}\right) v\,(6\,\pi^2 n_0)^{1/3}.$$

6.186 We proceed as in the previous example. The total number of modes must be $3\, n_0\, v$ (total transverse and one longitudinal per atom). On the other hand the number of transverse modes per unit frequency interval is given by

$$dN^{\perp} = \frac{V\omega^2}{\pi^2 v_{\perp}^3}\, d\omega$$

while the number of longitudinal modes ner unit frequency tnterval is given by

$$dN^{\parallel} = \frac{V\omega^2}{2\pi^2 v_{\parallel}^3}\, d\omega$$

The total number per unit frequency interval is

$$dN = \frac{V\omega^2}{2\pi^2}\left(\frac{2}{v_{\perp}^3} + \frac{1}{v_{\parallel}^3}\right) d\omega$$

If the high frequency cut off is at $\omega_0 = \frac{k\,\Theta}{\hbar}$, the total number of modes will be

$$3\, n_0\, V = \frac{V}{6\pi^2}\left(\frac{2}{v_{\perp}^3} + \frac{1}{v_{\parallel}^3}\right)\left(\frac{k\,\Theta}{\hbar}\right)^3$$

Here n_0 is the number of iron atoms per unit volume. Thus

$$\Theta = \frac{\hbar}{k}\left[18\,\pi^2 n_0 \bigg/ \left(\frac{2}{v_{\perp}^3} + \frac{1}{v_{\parallel}^3}\right)\right]^{1/3}$$

For iron

$$n_0 = N_A \bigg/ \frac{M}{\rho} = \frac{\rho N_A}{M}$$

(ρ = density, M = atomic weight of iron N_A = Avogadro number).

$$n_0 = 8{\cdot}389 \times 10^{22} \text{ per cc}$$

Substituting the data we get

$$\Theta = 469{\cdot}1\text{ K}$$

6.187 We apply the same formula but assume $v_{\parallel} \approx v_{\perp}$. Then

$$\Theta = \frac{\hbar}{k} v\,(6\,\pi^2 n_0)^{1/3}$$

or

$$v = k\,\Theta \bigg/ \left[\hbar\,(6\,\pi^2 n_0)^{1/3}\right]$$

For Al

$$n_0 = \frac{\rho N_A}{M} = 6{\cdot}023 \times 10^{22} \text{ per c.c}$$

Thus $$v = 3{\cdot}39\ \text{km/s}.$$

The tabulated values are $v_{\parallel} = 6{\cdot}3$ km/s

and $$v_{\perp} = 3{\cdot}1\ \text{km/s}.$$

6.188 In the Debye approximation the number of modes per unit frequency interval is given by

$$dN = \frac{l}{\pi v} d\omega \quad 0 \le \omega \le \frac{k\Theta}{\hbar}$$

But $$\frac{k\Theta}{\hbar} = \pi n_0 v$$

Thus $$dN = \frac{l}{\pi v} d\omega, \quad 0 \le \omega \le \pi n_0 v$$

The energy per mode is $$\langle E \rangle = \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}$$

Then the total interval energy of the chain is

$$U = \frac{l}{\pi v}\int_0^{\pi n_0 v} \frac{1}{2}\hbar\omega\, d\omega$$

$$+ \frac{l}{\pi v}\int_0^{\pi n_0 v} \frac{\hbar\omega}{e^{\hbar\omega/KT} - 1} d\omega = \frac{l\hbar}{4\pi v}(\pi n_0 v)^2 + \frac{l}{\pi v\hbar}(kT)^2 \int_0^{\Theta/T} \frac{x\,dx}{e^x - 1} = l n_0 k \cdot \frac{\hbar}{k}(\pi n_0 v)\cdot\frac{1}{4}$$

$$+ l n_0 k \frac{T^2}{(\pi n_0 v\hbar/k)} \int_0^{\Theta/T} \frac{x\,dx}{e^x - 1}$$

We put $l n_0 k = R$ for 1 mole of the chain.

Then $$U = R\Theta\left\{\frac{1}{4} + \left(\frac{T}{\Theta}\right)^2 \int_0^{\Theta/T} \frac{x\,dx}{e^x - 1}\right\}$$

Hence the molar heat capacity is by differentiation

$$C_V = \left(\frac{\partial U}{\partial T}\right)_\Theta = R\left[2\left(\frac{T}{\Theta}\right)\int_0^{\Theta/T} \frac{x\,dx}{e^x - 1} - \frac{\Theta/T}{e^{\Theta/T} - 1}\right]$$

when $$T \gg \Theta, \ C_v \approx R.$$

6.189 If the chain has N atoms, we can assume atom number 0 and $N+1$ held ficed. Then the displacement of the n^{th} atom has the form

$$u_n = A\left(\sin\frac{m\pi}{L}\cdot n\,a\right)\sin\omega t$$

Here $k = \dfrac{m\pi}{L}$. Allowed frequencies then have the form

$$\omega = \omega_{max}\sin\frac{k\,a}{2}$$

In our form only +ve k values are allowed.

The number of modes in a wave number range dk is

$$dN = \frac{L\,dk}{\pi} = \frac{L}{\pi}\frac{dk}{d\omega}d\omega$$

But
$$d\omega = \frac{a}{2}\omega_{max}\cos\frac{k\,a}{2}dk$$

Hence
$$\frac{d\omega}{dk} = \frac{a}{2}\sqrt{\omega_{max}^2-\omega^2}$$

So
$$dN = \frac{2L}{\pi a}\frac{d\omega}{\sqrt{\omega_{max}^2-\omega^2}}$$

(b) The total number of modes is

$$N = \int_0^{\omega_{max}}\frac{2L}{\pi a}\frac{d\omega}{\sqrt{\omega_{max}^2-\omega^2}} = \frac{2L}{\pi a}\cdot\frac{\pi}{2} = \frac{L}{a}.$$

i.e. the number of atoms in the chain.

6.190 Molar zero point energy is $\frac{9}{8}R\,\Theta$. The zero point energy per gm of copper is $\dfrac{9R\Theta}{8M_{Cu}}$, M_{Cu} is the atomic weight of the copper.
Substitution gives $48{\cdot}6$ J/gm.

6.191 (a) By Dulong and Petit's law, the classical heat capacity is $3R = 24{\cdot}94$ J/K – mole. Thus

$$\frac{C}{C_{Cl}} = 0{\cdot}6014$$

From the graph we see that this

value of $\dfrac{C}{C_{Cl}}$ corresponds to $\dfrac{T}{\Theta} = 0{\cdot}29$

Hence
$$\Theta = \frac{65}{0{\cdot}29} \approx 224\text{ K}$$

(b) 22·4 J/mole–K corresponds to $\frac{22 \cdot 4}{3 \times 8 \cdot 314} = 0 \cdot 898$. From the graph this corresponds to

$$\frac{T}{\Theta} \approx 0 \cdot 65 \text{. This gives } \Theta = \frac{250}{0 \cdot 65} \approx 385 \text{ K}$$

Then 80 K corresponds to $\frac{T}{\Theta} = 0 \cdot 208$

The corresponding value of $\frac{C}{C_{Cl}}$ is 0·42. Hence $C = 10 \cdot 5$ J/mole–K.

(c) We calculate Θ from the datum that $\frac{C}{C_{Cl}} = 0 \cdot 75$ at $T = 125$ K.

The x–coordinate corresponding to 0·75 is 0·40. Hence

$$\Theta = \frac{125}{0 \cdot 4} = 3125 \text{ K}$$

Now $$k\Theta = \hbar\, \omega_{max}$$

So $$\omega_{max} = 4 \cdot 09 \times 10^{13} \text{ rad/sec}$$

6.192 We use the formula (6.4d)

$$U = 9R\Theta\left[\frac{1}{8} + \left(\frac{T}{\Theta}\right)^4 \int_0^{\Theta/T} \frac{x^3\,dx}{e^x - 1}\right]$$

$$= 9R\Theta\left[\frac{1}{8} + \left\{\int_0^{\infty} \frac{x^3\,dx}{e^x - 1}\right\}\left(\frac{T}{\Theta}\right)^4 - \left(\frac{T}{\Theta}\right)^4 \int_{\Theta/T}^{\infty} \frac{x^3\,dx}{e^x - 1}\right]$$

In the limit $T << \Theta$, the third term in the bracket is exponentially small together with its derivatives.

Then we can drop the last term

$$U = \text{Const} + \frac{9R}{\Theta^3} T^4 \int_0^{\infty} \frac{x^3\,dx}{e^x - 1}$$

Thus $$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_\Theta = 36R\left(\frac{T}{\Theta}\right)^3 \int_0^{\infty} \frac{x^3\,dx}{e^x - 1}$$

Now from the table in the book

$$\int_0^{\infty} \frac{x^3\,dx}{e^x - 1} = \frac{\pi^4}{15}.$$

Thus $$C_V = \frac{12\pi^4}{5}\left(\frac{T}{\Theta}\right)^3$$

Note :- Call the 3rd term in the bracket above $-U_3$. Then

$$U_3 = \left(\frac{T}{\Theta}\right)^4 \int_{\Theta/T}^{\infty} \frac{x^3}{2\sinh(x/2)} \cdot e^{-x/2}\, dx$$

The maximum value of $\frac{x^3}{2\sinh\frac{x}{2}}$ is a finite +ve quantity C_0 for $0 \le x < \infty$. Thus

$$U_3 \le 2C_0\left(\frac{T}{\Theta}\right)^4 e^{-\Theta/2T}$$

we see that U_3 is exponentially small as $T \to 0$. So is $\frac{dU_3}{dT}$.

6.193 At low temperatures $C \propto T^3$. This is also a test of the "lowness" of the temperature We see that

$$\left(\frac{C_1}{C_2}\right)^{1/3} = 1{\cdot}4982 \approx 1{\cdot}5 = \frac{T_1}{T_2} = \frac{30}{20}$$

Thus T^3 law is obeyed and T_1, T_2 can we regarded low.

6.194

The total zero point energy of 1 mole of the solid is $\frac{9}{8}R\,\Theta$. Dividing this by the number of modes 3N we get the average zero point energy per mode. It is

$$\frac{3}{8}k\,\Theta.$$

6.195 In the Debye model

$$dN_\omega = A\,\omega^2,\ 0 \le \omega \le \omega_m$$

Then $$3N = \int_0^{\omega_m} dN_\omega = \frac{A\,\omega_m^3}{3}.\ \text{(Total no. of modes is 3N)}$$

Thus $$A = \frac{9N}{\omega_m^3}.$$

we get $$U = \frac{9N}{\omega_m^3}\int_0^{\omega_m} \frac{\omega^2 \cdot \hbar\omega}{e^{\hbar\omega/kT} - 1}\, d\omega \quad \text{ignoring zero point energy}$$

$$= 9N\hbar\omega_m \int_0^1 \frac{x^3\, dx}{e^{\hbar\omega_m x/kT} - 1}, \quad x = \frac{\omega}{\omega_m}$$

$$= 9R\Theta \int_0^1 \frac{x^3\, dx}{e^{x\Theta/T} - 1}, \quad \Theta = \hbar\omega_m/k$$

Thus $$\frac{1}{9R\Theta}\frac{dU(x)}{dx} = \frac{x^3}{e^{x\Theta/T} - 1} \quad \text{for} \quad 0 \le x \le 1$$

For $T = \Theta/2$, this is $\dfrac{x^3}{e^{2x} - 1}$; for

$T = \dfrac{\Theta}{4}$, it is $\dfrac{x^3}{e^{4x} - 1}$. Plotting then we get the figures given in the answer.

6.196 The maximum energy of the phonon is

$$\hbar\omega_m = k\Theta = 28{\cdot}4 \text{ meV}$$

On substituting $\Theta = 330$ K.

To get the corresponding value of the maximum momentum we must know the dispersion relation $\omega = \omega(\vec{k})$. For small $(\vec{k})$ we know $\omega = v|\vec{k}|$ where v is velocity of sound in the crystal. For an order of magnitude estimate we continue to use this result for high $|\vec{k}|$. Then we estimate v from the values of the modulus of elasticity and density

$$v \sim \sqrt{\frac{E}{\rho}}$$

We write $$E \sim 100 \text{ G Pa}, \ \rho = 8{\cdot}9 \times 10^3 \text{ kg/m}^3$$

Then $$v \sim 3 \times 10^3 \text{ m/s}$$

Hence $$\hbar|\vec{k}|_{max} \sim \frac{\hbar\omega_m}{v} \sim 1{\cdot}5 \times 10^{-19} \text{ gm cm s}^{-1}$$

6.197 (a) From the formula

$$dn = \frac{\sqrt{2}\, m^{3/2}}{\pi^2\hbar^3} E^{1/2}\, dE$$

the maximum value E_{max} of E is determined in terms of n by

$$n = \frac{\sqrt{2}\, m^{3/2}}{\pi^2 \hbar^3} \int_0^{E_{max}} E^{1/2}\, dE$$

$$= \frac{\sqrt{2}\, m^{3/2}}{\pi^2 \hbar^3} \frac{2}{3} E_{max}^{3/2}$$

or $$E_{max}^{3/2} = \left(\frac{\hbar^2}{2m}\right)^{3/2} (3\pi^2 n)$$

$$E_{max} = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}$$

(b) Mean K.E. $< E >$ is

$$< E > = \int_0^{E_{max}} E\, dn \Big/ \int_0^{E_{max}} dn$$

$$= \int_0^{E_{max}} E^{3/2} dE \Big/ \int_0^{E_{max}} E^{1/2} dE = \frac{2}{5} E_{max}^{5/2} \Big/ \frac{2}{3} E_{max}^{3/2} = \frac{3}{5} E_{max}$$

6.198 The fraction is

$$\eta = \int_{\frac{1}{2}E_{max}}^{E_{max}} E^{1/2} dE \Big/ \int_0^{E_{max}} E^{1/2} dE = 1 - 2^{-3/2} = 0{\cdot}646 \quad \text{or} \quad 64{\cdot}6\,\%$$

6.199 We calculate the concentration n of electron in the Na metal from

$$E_{max} = E_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}$$

we get from $$E_F = 3{\cdot}07 \text{ eV}$$

$$n = 2{\cdot}447 \times 10^{22} \text{ per c.c.}$$

From this we get the number of electrons per one Na atom as

$$\frac{n}{\rho} \cdot \frac{M}{N_A}$$

where ρ = density of Na, M = molar weight in gm of Na, N_A = Avogadro number

we get

0·963 elecrons per one Na atom.

6.200 The mean K.E. of electrons in a Fermi gas is $\frac{3}{5}E_F$. This must equal $\frac{3}{2}kT$. Thus

$$T = \frac{2E_F}{5k}$$

We calculate E_F first. For Cu

$$n = \frac{N_A}{M/\rho} = \frac{\rho N_A}{M} = \text{``}8{\cdot}442 \times 10^{22} \text{ per c.c.}$$

Then
$$E_F = 7{\cdot}01 \text{ eV}$$

and
$$T = 3{\cdot}25 \times 10^4 \text{ K}$$

6.201 We write the expression for the number of electrons as

$$dN = \frac{V\sqrt{2}\,m^{3/2}}{\pi^2\hbar^3} E^{1/2}\,dE$$

Hence if ΔE is the spacing between neighbouring levels near the Fermi level we must have

$$2 = \frac{V\sqrt{2}\,m^{3/2}}{\pi^2\hbar^3} E_F^{1/2}\,\Delta E$$

(2 on the RHS is to take care of both spins *f* electrons). Thus

$$\Delta E = \frac{\sqrt{2}\,\pi^2\hbar^3}{V m^{3/2} E_F^{1/2}}$$

But
$$E_F^{1/2} = \frac{\hbar}{\sqrt{2}\,m^{1/2}}(3\pi^2 n)^{1/3}$$

So
$$\Delta E = \frac{2\pi^2\hbar^2}{mV(3\pi^2 n)^{1/3}}$$

Substituting the data we get

$$\Delta E = 1{\cdot}79 \times 10^{-22} \text{ eV}$$

6.202 (a) From

$$dn(E) = \frac{\sqrt{2}\,m^{3/2}}{\pi^2\hbar^3} E^{1/2}\,dE$$

we get on using $E = \frac{1}{2}m\mathrm{v}^2$, $dn(E) = dn(\mathrm{v})$

$$dn(\mathrm{v}) = \frac{\sqrt{2}\,m^{3/2}}{\pi^2\hbar^3}\frac{1}{\sqrt{2}}m^{1/2}\,\mathrm{v}\,m\mathrm{v}\,d\mathrm{v} = \frac{m^3}{\pi^2\hbar^3}\mathrm{v}^2\,d\mathrm{v}$$

This holds for $0 < \mathrm{v} < \mathrm{v_F}$ where $\frac{1}{2}m\mathrm{v_F^2} = E_F$

and
$$dn(\mathrm{v}) = 0 \text{ for } \mathrm{v} > \mathrm{v_F}.$$

(b) Mean velocity is

$$< v > = \int_0^{v_F} v^3 \, dv \Big/ \int_0^{v_F} v^2 \, dv = \frac{3}{4} v_F$$

$$\therefore \frac{< v >}{v_F} = \frac{3}{4}.$$

6.203 Using the formula of the previous section

$$dn(v) = \frac{m^3}{\pi^2 \hbar^3} v^2 \, dv$$

We put $m v = \frac{2 \pi \hbar}{\lambda}$, where λ = de Broglie wavelength

$$m \, dv = -\frac{2 \pi \hbar}{\lambda^2} d\lambda$$

Taking account of the fact that λ decreases when v increases we write

$$dn(\lambda) = -dn(v) = \frac{(2\pi)^3 \, d\lambda}{\pi^2 \lambda^4}. = \frac{8\pi}{\lambda^4} d\lambda$$

6.204 From the kinetic theory of gasses we know

$$p = \frac{2}{3} \frac{U}{V}$$

Here U is the total interval energy of the gas. This result is applicable to Fermi gas also Now at $T = 0$

$$U = U_0 = N < E > = nV < E >$$

so
$$p = \frac{2}{3} n < E >$$

$$= \frac{2}{3} n \times \frac{3}{5} E_F = \frac{2}{5} n E_F$$

$$= \frac{\hbar^2}{5 m} (3 \pi^2)^{2/3} n^{5/3}$$

Substituting the values we get

$$p = 4{\cdot}92 \times 10^4 \text{ atmos}$$

6.205 From Richardson's equation

$$I = a T^2 e^{-A/kT}$$

where A is the work function in eV. When T increases by ΔT, I increases to $(1 + \eta) I$. Then

$$1+\eta = \left(\frac{T+\Delta T}{T}\right)^2 e^{-\frac{A}{kT}\left(\frac{T}{T+\Delta T}-1\right)} = \left(1+\frac{\Delta T}{T}\right)^2 e^{+\frac{A}{kT}\cdot\frac{\Delta T}{T+\Delta T}}$$

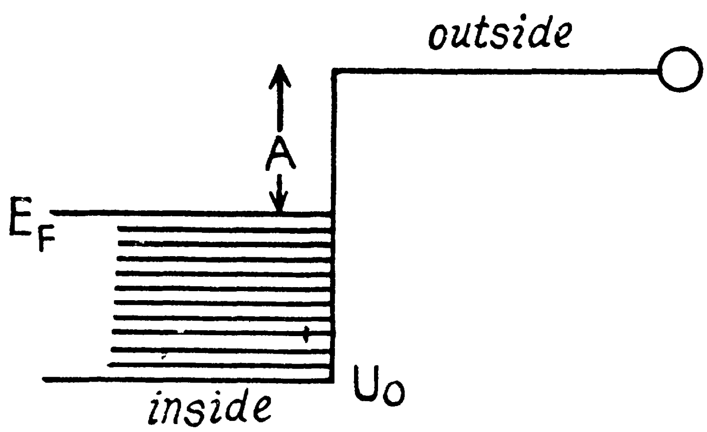
Expanding and neglecting higher powers of $\frac{\Delta T}{T}$ we get

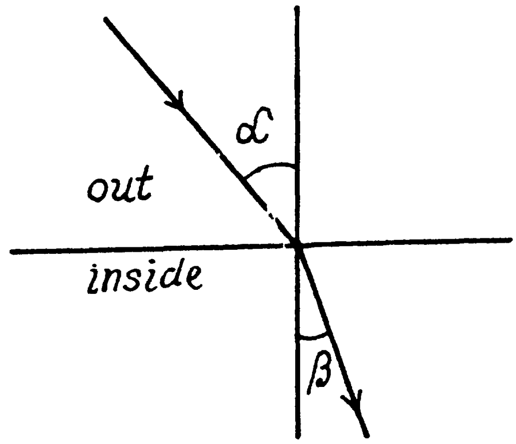
$$\eta = 2\frac{\Delta T}{T}+\frac{A}{kT^2}\Delta T$$

Thus
$$A = kT\left(\frac{\eta T}{\Delta T}-2\right)$$

Substituting we get
$$A = 4{\cdot}48 \text{ eV}$$

6.206





The potential energy inside the metal is $-U_0$ for the electron and it related to the work function A by

$$U_0 = E_F + A$$

If T is the K.E. of electrons outside the metal, its K.E. inside the metal will be $(E+U_0)$. On entering the metal electron connot experience any tangential force so the tangential component of momentum is unchanged. Then

$$\sqrt{2mT}\sin\alpha = \sqrt{2m(T+U_0)}\sin\beta$$

Hence

$$\frac{\sin\alpha}{\sin\beta} = \sqrt{1+\frac{U_0}{T}} = n \text{ by definition of refractive index.}$$

In sodium with one free electron per Na atom

$$n = 2{\cdot}54\times10^{22} \text{ per c.c.}$$
$$E_F = 3{\cdot}15 \text{ eV}$$
$$A = 2{\cdot}27 \text{ eV} \quad \text{(from table)}$$
$$U_0 = 5{\cdot}42 \text{ eV}$$
$$n = 1{\cdot}02$$

6.207 In a pure (intrinsic) semiconductor the conductivity is related to the temperature by the following formula very closely :

$$\sigma = \sigma_0 e^{-\Delta \varepsilon/2kT}$$

where $\Delta \varepsilon$ is the energy gap between the top of valence band and the bottom of conduction band; it is also the minimum energy required for the formation of electron-hole pair. The conductivity increases with temperature and we have

$$\eta = e^{+\frac{\Delta \varepsilon}{2k}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)}$$

or
$$\ln \eta = \frac{\Delta \varepsilon}{2k} \frac{T_2 - T_1}{T_1 T_2}$$

Hence
$$\Delta \varepsilon = \frac{2kT_1 T_2}{T_2 - T_1} \ln \eta$$

Substitution gives

$$\Delta \varepsilon = 0{\cdot}333 \text{ eV} = E_{min}$$

6.208 The photoelectric threshold determines the band gap $\Delta \varepsilon$ by

$$\Delta \varepsilon = \frac{2\pi\hbar c}{\lambda_{th}}$$

On the other hand the temperature coefficient of resistance is defined by (ρ is resistivity)

$$\alpha = \frac{1}{\rho}\frac{d\rho}{dT} = \frac{d}{dT}\ln \rho = -\frac{d}{dT}\ln \sigma$$

where σ is the conductivity. But

$$\ln \sigma = \ln \sigma_0 - \frac{\Delta \varepsilon}{2kT}$$

Then
$$\alpha = -\frac{\Delta \varepsilon}{2kT^2} = -\frac{\pi\hbar c}{kT^2 \lambda_{th}} = -0{\cdot}047 \text{ K}^{-1}$$

6.209 At high temperatures (small values of T^{-1}) most of the conductivity is infrinsic i.e. it is due to the transition of electrons from the upper levels of the valance band into the lower levels of conduction vands.

For this we can apply approximately the formula

$$\sigma = \sigma_0 \exp\left(-\frac{E_g}{2kT}\right)$$

or
$$\ln \sigma = \ln \sigma_0 - \frac{E_g}{2kt}$$

From this we get the band gap

$$E_g = -2k\frac{\Delta \ln \sigma}{\Delta(1/T)}$$

The slope must be calculated at small $\frac{1}{T}$. Evaluation gives $-\dfrac{\Delta \ln \sigma}{\Delta\left(\frac{1}{T}\right)} = 7000\text{ K}$

Hence $$E_g = 1{\cdot}21\text{ eV}$$

At low temperatures (high values of $\frac{1}{T}$) the conductance is mostly due to impurities. If E_0 is the ionization energy of donor levels then we can write the approximate formula (valid at low temperature)

$$\sigma' = \sigma'_0 \exp\left(-\frac{E_0}{2kT}\right)$$

So $$E_0 = -2kT\,\frac{\Delta \ln \sigma'}{\Delta\left(\frac{1}{T}\right)}$$

The slope must be calculated at low temperatures. Evaluation gives the slope

$$-\frac{\Delta \ln \sigma'}{\Delta\left(\frac{1}{T}\right)} = \frac{1}{3} \times 1000\text{ K}$$

Then $$E_0 \simeq 0{\cdot}057\text{ eV}$$

6.210 We write the conductivity of the sample as $\sigma = \sigma_i + \sigma_\gamma$ where σ_i = intrinsic conductivity and σ_γ is the photo conductivity. At $t = 0$, assuming saturation we have

$$\frac{1}{\rho_1} = \frac{1}{\rho} + \sigma_{\gamma_0} \quad \text{or} \quad \sigma_{\gamma_0} = \frac{1}{\rho_1} - \frac{1}{\rho}$$

Time t after light source is switched off

we have because of recombination of electron and holes in the sample

$$\sigma = \sigma_i + \sigma_{\gamma_0}\, e^{-t/T}$$

where T = mean lifetime of electrons and holes.

Thus $$\frac{1}{\rho_2} = \frac{1}{\rho} + \left(\frac{1}{\rho_1} - \frac{1}{\rho}\right) e^{-t/T}$$

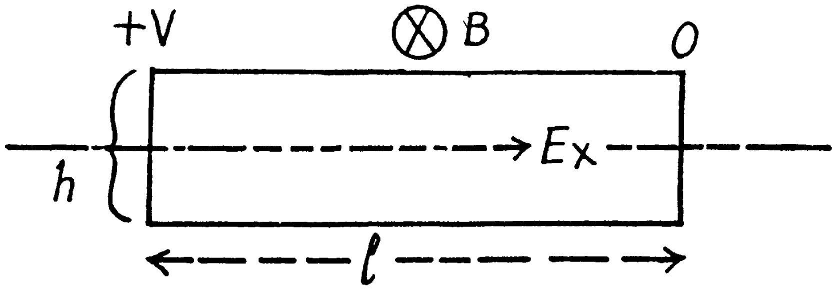
or $$\frac{1}{\rho_2} - \frac{1}{\rho} = \left(\frac{1}{\rho_1} - \frac{1}{\rho}\right) e^{-t/T}$$

or $$e^{t/T} = \frac{\frac{1}{\rho_1} - \frac{1}{\rho}}{\frac{1}{\rho_2} - \frac{1}{\rho}} = \frac{\rho_2(\rho - \rho_1)}{\rho_1(\rho - \rho_2)}$$

Hence $$T = t \Big/ \ln\left\{\frac{\rho_2(\rho - \rho_1)}{\rho_1(\rho - \rho_2)}\right\}$$

Substitution gives $T = 9{\cdot}87\text{ ms} \simeq 0{\cdot}01\text{ sec}$

6.211



We shall ignore minority carriers.

Drifting holes experience a sideways force in the magnetic field and react by setting up a Hall electric field E_y to counterbalance it. Thus

$$v_x B = E_y = \frac{V_H}{h}$$

If the concentration of carriers is n then

$$j_x = n e v_x$$

Hence
$$n = \frac{J_x}{e v_x} = \frac{\dfrac{j_x}{e V_H}}{h B} = \frac{j_x h B}{e V_H}$$

Also using
$$j_x = \sigma E_x = E_x / \rho = \frac{V}{\rho l}$$

we get
$$n = \frac{V h B}{e \rho l V_H}$$

Substituting the data (note that in MKS units $B = 5{\cdot}0\,\text{kG} = 0{\cdot}5\,\text{T}$)

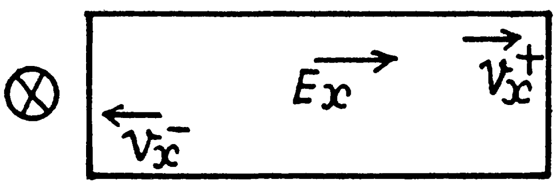
$$\rho = 2{\cdot}5 \times 10^{-2}\ \text{ohm–m}$$

we get
$$n = 4{\cdot}99 \times 10^{21}\ \text{m}^{-3}$$
$$= 4{\cdot}99 \times 10^{15}\ \text{per cm}^3$$

Also the mobility is
$$u_0 = \frac{v_x}{E_x} = \frac{V_H}{nB} \times \frac{l}{V} = \frac{V_H\, l}{h B V}$$

Substitution gives
$$u_0 = 0{\cdot}05\ \text{m}^2/\text{V} - \text{s}$$

6.212



If an electric field E_x is present in a sample containing equal amounts of both electrons and holes, the two drift in opposite directions.

In the presence of a magnetic field $B_z = B$ they set up Hall voltages in opposite directions.

The net Hall electric field is given by

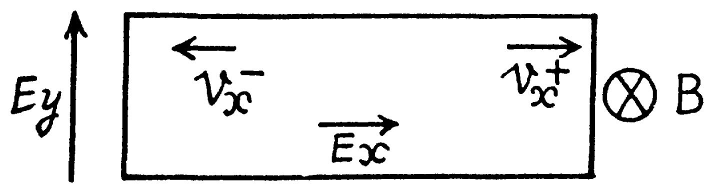
$$E_y = (v_x^+ - v_x^-) B$$
$$= (u_0^+ u_0^-) E_x B$$

But $$\frac{E_y}{E_x} = \frac{1}{\eta} \text{ Hence}$$

$$|u_0^+ - u_0^-| = \frac{1}{\eta B}$$

Substitution gives $|u_0^+ - u_0^-| = 0{\cdot}2\ \text{m}^2/\text{volt} - \text{sec}$

6.213



When the sample contains unequal number of carriers of both types whose mobilities are different, static equilibrium (i.e. no transverse movement of either electron or holes) is impossible in a magnetic field. The transverse electric field acts differently on electrons and holes. If the E_y that is set up is as shown, the net Lorentz force per unit charge (effective transverse electric field) on electrons is

$$E_y - v_x^- B$$

and on holes

$$E_y + v_x^+ B$$

(we are assuming $B = B_z$). There is then a transverse drift of electrons and holes and the net transverse current must vanish in equilibrium. Using mobility

$$u_0^- N_e e (E_y - u_0^- E_x B) + N_h e u_0^+ (E_y + u_0^+ E_x B) = 0$$

or $$E_y = \frac{N_e u_0^{-2} - N u_0^{+2}}{N_e u_0^- + N_h u_0^+} E_x B$$

On the other hand

$$j_x = (N_e u_0^- + N_h u_0^+) e E_x$$

Thus, the Hall coefficient is

$$R_H = \frac{E_y}{j_x B} = \frac{1}{e} \frac{N_e u_0^{-2} - N_h u_0^{+2}}{(N_e u_0^- + N_h u_0^+)^2}$$

We see that $R_H = 0$ when

$$\frac{N_e}{N_h} = \left(\frac{u_0^+}{u_0^-}\right)^2 = \frac{1}{\eta^2} = \frac{1}{4}$$

6.5 RADIOACTIVITY

6.214 (a) The probability of survival (i.e. not decaying) in time t is $e^{-\lambda t}$. Hence the probability of decay is $1 - e^{-\lambda t}$

(b) The probability that the particle decays in time dt around time t is the difference

$$e^{-\lambda t} - e^{-\lambda (t+dt)} = e^{-\lambda t}[1 - e^{-e\lambda dt}] = \lambda e^{-\lambda t} dt$$

Therefore the mean life time is

$$T = \int_0^\infty t\lambda e^{-\lambda t} dt \Big/ \int_0^\infty \lambda e^{-\lambda t} dt = \frac{1}{\lambda}\int_0^\infty x e^{-x} dx \Big/ \int_0^\infty e^{-x}\, dx = \frac{1}{\lambda}$$

6.215 We calculate λ first

$$\lambda = \frac{\ln 2}{T_{1/2}} = 9{\cdot}722 \times 10^{-3} \text{ per day}$$

Hence

fraction decaying in a month $= 1 - e^{-\lambda t} = 0{\cdot}253$

6.216 Here $N_0 \simeq \dfrac{1\,\mu g}{24\,g} \times 6{\cdot}023 \times 10^{23} = 2{\cdot}51 \times 10^{16}$

Also $\lambda = \dfrac{\ln 2}{T_{1/2}} = 0{\cdot}04621$ per hour

So the number of β rays emitted in one hour is

$$N_0\,(1 - e^{-\lambda}) = 1{\cdot}13 \times 10^{15}$$

6.217 If N_0 is the number of radionuclei present initially, then

$$N_1 = N_0\,(1 - e^{-t_1/\tau})$$

$$\eta N_1 = N_0\,(1 - e^{-t_2/\tau})$$

where $\eta = 2{\cdot}66$ and $t_2 = 3\,t_1$. Then

$$\eta = \frac{1 - e^{-t_2/\tau}}{1 - e^{-t_1/\tau}}$$

or $\eta - \eta\, e^{-t_1/\tau} = 1 - e^{-t_2/\tau}$

Substituting the values

$$1{\cdot}66 = 2{\cdot}66\, e^{-2/\tau} - e^{-6/\tau}$$

Put $e^{-2/\tau} = x$. Then

$$x^3 - 2{\cdot}66\,x + 1{\cdot}66 = 0$$

$$(x^2 - 1)\,x - 1{\cdot}66\,(x - 1) = 0$$

$$(x-1)(x^2+x-1{\cdot}66) = 0 \quad \text{or}$$

Now $\quad x \neq 1$ so $x^2+x-1{\cdot}66 = 0$

$$x = \frac{-1 \pm \sqrt{1+4\times 1{\cdot}66}}{2}$$

Negative sign has to be rejected as $x>0$.

Thus $\quad x = 0{\cdot}882$

This gives $\quad \tau = \dfrac{-2}{\ln 0{\cdot}882} = 15{\cdot}9$ sec.

6.218 If the half-life is T days

$$(2)^{-7/T} = \frac{1}{2{\cdot}5}$$

Hence $\quad \dfrac{7}{T} = \dfrac{\ln 2{\cdot}5}{\ln 2}$

or $\quad T = \dfrac{7 \ln 2}{\ln 2{\cdot}5} = 5{\cdot}30$ days.

6.219 The activity is proportional to the number of parent nuclei (assuming that the daughter is not radioactive). In half its half-life period, the number of parent nucli decreases by a factor

$$(2)^{-1/2} = \frac{1}{\sqrt{2}}$$

So activity decreases to $\quad \dfrac{650}{\sqrt{2}} = 460$ particles per minute.

6.220 If the decay constant $\left(\text{in (hour)}^{-1}\right)$ is λ, then the activity after one hour will decrease by a factor $e^{-\lambda}$. Hence

$$0{\cdot}96 = e^{-\lambda}$$

or $\quad \lambda = 1{\cdot}11\times 10^{-5}\,\text{s}^{-1} = 0{\cdot}0408$ per hour

The mean life time is 24·5 hour

6.221 Here $N_0 = \dfrac{1}{238}\times 6{\cdot}023\times 10^{23}$

$$= 2{\cdot}531\times 10^{21}$$

The activity is $\quad A = 1{\cdot}24\times 10^4$ dis/sec.

Then $\quad \lambda = \dfrac{A}{N_0} = 4{\cdot}90\times 10^{-18}$ per sec.

Hence the half life is

$$T_{\frac{1}{2}} = \frac{\ln 2}{\lambda} = 4{\cdot}49\times 10^9 \text{ years}$$

6.222 In old wooden atoms the number of C^{14} nuclei steadily decreases because of radioactive decay. (In live trees biological processes keep replenishing C^{14} nuclei maintaining a balance. This balance starts getting disrupted as soon as the tree is felled.)

If $T_{1/2}$ is the half life of C^{14} then $e^{-t \times \frac{\ln 2}{T_{1/2}}} = \frac{3}{5}$

Hence $$t = T_{1/2} \frac{\ln 5/3}{\ln 2} = 4105 \text{ years} \approx 4{\cdot}1 \times 10^3 \text{ years}$$

6.223 What this implies is that in the time since the ore was formed, $\frac{\eta}{1+\eta}$ U^{238} nuclei have remained undecayed. Thus

$$\frac{\eta}{1+\eta} = e^{-t \times \frac{\ln 2}{T_{1/2}}}$$

or $$t = T_{\frac{1}{2}} \frac{\ln \frac{1+\eta}{\eta}}{\ln 2}$$

Substituting $T_{\frac{1}{2}} = 4{\cdot}5 \times 10^9$ years, $\eta = 2{\cdot}8$

we get $$t = 1{\cdot}98 \times 10^9 \text{ years.}$$

6.224 The specific activity of Na^{24} is

$$\lambda \frac{N_A}{M} = \frac{N_A \ln 2}{M T_{\frac{1}{2}}} = 3{\cdot}22 \times 10^{17} \text{ dis/(gm.sec)}$$

Here $M =$ molar weight of $N_a^{24} = 24$ gm, N_A is Avogadro number & $T_{\frac{1}{2}}$ is the half-life of Na^{24}.

Similarly the specific activity of U^{235} is

$$\frac{6{\cdot}023 \times 10^{23} \times \ln 2}{235 \times 10^8 \times 365 \times 86400}$$
$$= 0{\cdot}793 \times 10^5 \text{ dis/(gm-s)}$$

6.225 Let $V =$ volume of blood in the body of the human being. Then the total activity of the blood is $A' V$. Assuming all this activity is due to the injected Na^{24} and taking account of the decay of this radionuclide, we get

$$V A' = A e^{-\lambda t}$$

Now $$\lambda = \frac{\ln 2}{15} \text{ per hour}, \quad t = 5 \text{ hour}$$

Thus $$V = \frac{A}{A'} e^{-\ln 2/3} = \frac{2{\cdot}0 \times 10^3}{(16/60)} e^{-\ln 2/3} \text{ cc} = 5{\cdot}95 \text{ litre}$$

6.226 We see that

Specific activity of the sample

$$= \frac{1}{M+M'} \{\text{Activity of } M \text{ gm of } Co^{58} \text{ in the sample}\}$$

Here M and M' are the masses of Co^{58} and Co^{59} in the sample. Now activity of M gm of Co^{58}

$$= \frac{M}{58} \times 6{\cdot}023 \times 10^{23} \times \frac{\ln 2}{71{\cdot}3 \times 86400} \text{ dis/sec}$$

$$= 1{\cdot}168 \times 10^{15} \text{ M}$$

Thus from the problem

$$1{\cdot}168 \times 10^{15} \frac{M}{M+M'} = 2{\cdot}2 \times 10^{12}$$

or
$$\frac{M}{M+M'} = 1{\cdot}88 \times 10^{-3} \quad \text{i.e. } 0{\cdot}188\ \%$$

6.227 Suppose N_1, N_2 are the initial number of component nuclei whose decay constants are λ_1, λ_2 $\left(\text{in (hour)}^{-1}\right)$

Then the activity at any instant is

$$A = \lambda_1 N_1 e^{-\lambda_1 t} + \lambda_2 N_2 e^{-\lambda_2 t}$$

The activity so defined is in units dis/hour. We assume that data ln A given is of its natural logarithm. The daughter nuclei are assumed nonradioactive.

We see from the data that at large t the change in ln A per hour of elapsed time is constant and equal to $-0{\cdot}07$. Thus

$$\lambda_2 = 0{\cdot}07 \text{ per hour}$$

We can then see that the best fit to data is obtained by

$$A(t) = 51{\cdot}1\, e^{-0.66\, t} + 10{\cdot}0\, e^{-0{\cdot}07\, t}$$

[To get the fit we calculate $A(t)\, e^{0{\cdot}07\, t}$. We see that it reaches the constant value 10·0 at $t = 7, 10, 14, 20$ very nearly. This fixes the second term. The first term is then obtained by subtracting out the constant value 10·0 from each value of $A(t)\, e^{0{\cdot}07\, t}$ in the data for small t.]

Thus we get $\lambda_1 = 0{\cdot}66$ per hour

$$\left.\begin{aligned} T_1 &= 1{\cdot}05 \text{ hour} \\ T_2 &= 9{\cdot}9 \text{ hours} \end{aligned}\right\} \text{half-lives}$$

Ratio
$$\frac{N_1}{N_2} = \frac{51{\cdot}1}{10{\cdot}0} \times \frac{\lambda_2}{\lambda_1} = 0{\cdot}54$$

The answer given in the book is misleading.

6.228 Production of the nucleus is governed by the equation

$$\frac{dN}{dt} = \underset{\substack{\uparrow \\ \text{supply}}}{g} - \underbrace{\lambda N}_{\text{decay}}$$

We see that N will approach a constant value $\frac{g}{\lambda}$. This can also be proved directly. Multiply by $e^{\lambda t}$ and write

$$\frac{dN}{dt}e^{\lambda t} + \lambda e^{\lambda t} N = g e^{\lambda t}$$

Then
$$\frac{d}{dt}(N e^{\lambda t}) = g e^{\lambda t}$$

or
$$N e^{\lambda t} = \frac{g}{\lambda} e^{\lambda t} + \text{const}$$

At $t = 0$ when the production is starteed, $N = 0$

$$0 = \frac{g}{\lambda} + \text{constant}$$

Hence
$$N = \frac{g}{\lambda}(1 - e^{-\lambda t})$$

Now the activity is

$$A = \lambda N = g(1 - e^{-\lambda t})$$

From the problem

$$\frac{1}{2 \cdot 7} = 1 - e^{-\lambda t}$$

This gives $\lambda t = 0 \cdot 463$

so
$$t = \frac{0 \cdot 463}{\lambda} = \frac{0 \cdot 463 \times T}{0 \cdot 693} = 9 \cdot 5 \text{ days}.$$

Algebraically
$$t = -\frac{T}{\ln 2} \ln\left(1 - \frac{A}{g}\right)$$

6.229 (a) Suppose N_1 and N_2 are the number of two radionuclides A_1, A_2 at time t. Then

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \quad (1)$$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 \quad (2)$$

From (1)

$$N_1 = N_{10} e^{-\lambda_1 t}$$

where N_{10} is the initial number of nuclides A_1 at time $t = 0$

From (2)

$$\left(\frac{dN_2}{dt} + \lambda_2 N_2\right) e^{\lambda_2 t} = \lambda_1 N_{10}\, e^{-(\lambda_1 - \lambda_2) t}$$

or $$(N_2 e^{\lambda_2 t}) = \text{const}\ \frac{\lambda_1 N_{10}}{\lambda_1 - \lambda_2} e^{-(\lambda_1 - \lambda_2) t}$$

since $$N_2 = 0 \quad \text{at} \quad t = 0$$

Constant $$N_2 = \frac{\lambda_1 N_{10}}{\lambda_1 - \lambda_2}$$

Thus $$= \frac{\lambda_1 N_{10}}{\lambda_1 - \lambda_2} (e^{-\lambda_2 t} - e^{-\lambda_1 t})$$

(b) The activity of nuclide A_2 is $\lambda_2 N_2$. This is maximum when N_2 is maximum. That happens when $\frac{dN_2}{dt} = 0$

This requires $$\lambda_2\, e^{-\lambda_2 t_m} = \lambda_1 e^{-\lambda_1 t_m}$$

or $$t_m = \frac{\ln (\lambda_1/\lambda_2)}{\lambda_1 - \lambda_2}$$

6.230 (a) This case can be obtained from the previous one on putting

$$\lambda_2 = \lambda_1 - \varepsilon$$

where ε is very small and letting $\varepsilon \to 0$ at the end. Then

$$N_2 = \frac{\lambda_1 N_{10}}{\varepsilon} (e^{\varepsilon t} - 1)\, e^{-\lambda_1 t} = \lambda_1 t\, e^{-\lambda_1 t} N_{10}$$

or dropping the subscript 1 as the two values are equal

$$N_2 = N_{10} \lambda t\, e^{-\lambda t}$$

(b) This is maximum when

$$\frac{dN_2}{dt} = 0 \quad \text{or} \quad t = \frac{1}{\lambda}$$

6.231 Here we have the equations

$$\frac{dN_1}{dt} = -\lambda_1 N_1$$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 \quad \text{and} \quad \frac{dN_3}{dt} = \lambda_2 N_2$$

From problem 229

$$N_1 = N_{10}\, e^{-\lambda_1 t}$$

$$N_2 = \frac{\lambda_1 N_{10}}{\lambda_1 - \lambda_2} (e^{-\lambda_2 t} - e^{-\lambda_1 t})$$

Then $$\frac{dN_3}{dt} = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} N_{10} (e^{-\lambda_2 t} - e^{-\lambda_1 t})$$

or $$N_3 = \text{Const} - \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}\left(\frac{e^{-\lambda_2 t}}{\lambda_2} - \frac{e^{-\lambda_1 t}}{\lambda_1}\right) N_{10}$$

since $N_3 = 0$ initially

$$\text{Const} = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} N_{10}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)$$

So $$N_3 = \frac{\lambda_1 \lambda_2 N_{10}}{\lambda_1 - \lambda_2}\left[\frac{1}{\lambda_2}(1 - e^{-\lambda_2 t}) - \frac{1}{\lambda_1}(1 - e^{-\lambda_1 t})\right]$$

$$= N_{10}\left[1 + \frac{\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1}\right]$$

6.232 We have the chain

$$B_i^{210} \xrightarrow[\lambda_1]{\beta} P_0^{210} \xrightarrow[\lambda_2]{\alpha} P_b^{206}$$

$$A_1 \qquad\qquad A_2 \qquad\qquad A_3$$

of the previous problem initially

$$N_{10} = \frac{10^{-3}}{210} \times 6{\cdot}023 \times 10^{23} = 2{\cdot}87 \times 10^{18}$$

A month after preparation

$$N_1 = 4{\cdot}54 \times 10^{16}$$

$$N_2 = 2{\cdot}52 \times 10^{18}$$

using the results of the previous problem.

Then $$A_\beta = \lambda_1 N_1 = 0{\cdot}725 \times 10^{11} \text{ dis/sec}$$

$$A_\alpha = \lambda_2 N_2 = 1{\cdot}46 \times 10^{11} \text{ dis/sec}$$

6.233 (a) Ra has $Z = 88$, $A = 226$

After 5 α emission and 4 β (electron) emission

$$A = 206$$

$$Z = 88 + 4 - 5 \times 2 = 82$$

The product is $^{82}Pb^{206}$

(b) We require

$$-\Delta Z = 10 = 2n - m$$

$$-\Delta A = 32 = n \times 4$$

Here n = no. of α emissions

m = no. of β emissions

Thus $n = 8$, $m = 6$

6.234 The momentum of the α–particle is

$\sqrt{2 M_\alpha T}$. This is also the recoil momentum of the daughter nuclear in opposite direction.

The recoil velocity of the daughter nucleus is

$$\frac{\sqrt{2 M_\alpha T}}{M_d} = \frac{2}{196}\sqrt{\frac{2T}{M_p}} = 3{\cdot}39 \times 10^5 \text{ m/s}$$

The energy of the daughter nucleus is $\frac{M_\alpha}{M_d} T$ and this represents a fraction

$$\frac{M_\alpha / M_d}{1 + \frac{M_\alpha}{M_d}} = \frac{M_\alpha}{M_\alpha + M_d} = \frac{4}{200} = \frac{1}{50} = 0{\cdot}02$$

of total energy. Here M_d is the mass of the daughter nucleus.

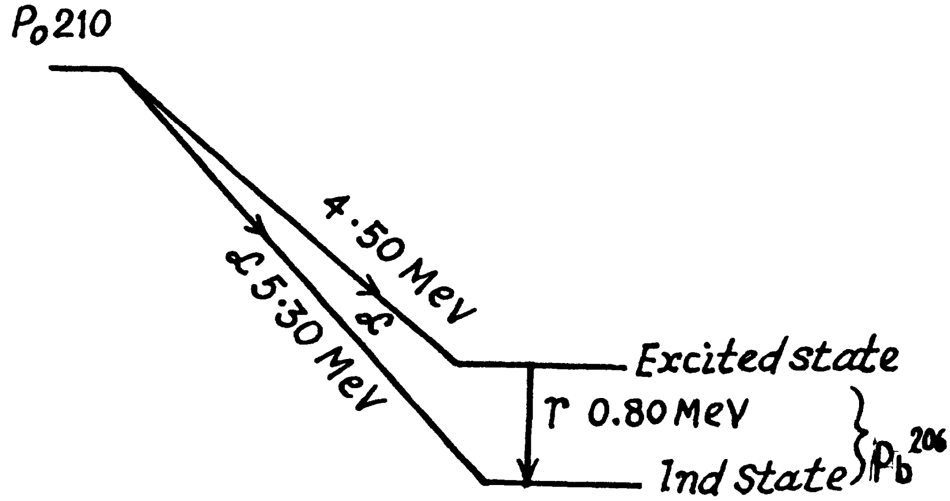
6.235 The number of nuclei initially present is

$$\frac{10^{-3}}{210} \times 6{\cdot}023 \times 10^{23} = 2{\cdot}87 \times 10^{18}$$

In the mean life time of these nuclei the number decaying is the fraction $1 - \frac{1}{e} = 0{\cdot}632$. Thus the energy released is

$2{\cdot}87 \times 10^{18} \times 0{\cdot}632 \times 5{\cdot}3 \times 1{\cdot}602 \times 10^{-13}$ J .= 1·54 M J

6.236 We neglect all recoil effects. Then the following diagram gives the energy of the gamma ray



6.237 (a) For an alpha particle with initial K.E. 7·0 MeV, the initial velocity is

$$v_0 = \sqrt{\frac{2T}{M_\alpha}}$$

$$= \sqrt{\frac{2 \times 7 \times 1{\cdot}602 \times 10^{-6}}{4 \times 1{\cdot}672 \times 10^{-24}}}$$

$$= 1{\cdot}83 \times 10^9 \text{ cm/sec}$$

Thus $R = 6{\cdot}02$ cm

(b) Over the whole path the number of ion pairs is

$$\frac{7\times10^6}{34} = 2{\cdot}06\times10^5$$

Over the first half of the path :- We write the formula for the mean path as $R \alpha E^{3/2}$ where E is the initial energy. Thus if the energy of the α–particle after traversing the first half of the path is E_1 *then*

$$R_0 E_1^{3/2} = \frac{1}{2} R_0 E_0^{3/2} \quad \text{or} \quad E_1 = 2^{-2/3} E_0$$

Hence number of ion pairs formed in the first half of the path length is

$$\frac{E_0 - E_1}{34\,e\,V} = (1 - 2^{-2/3})\times 2{\cdot}06\times10^5 = 0{\cdot}76\times10^5$$

6.238 In β^- decay

$$_Z X^A \rightarrow {}_{Z+1}Y^A + e^- + Q$$
$$Q = (M_x - M_y - m_e)\,c^2$$
$$= [(M_x + Z\,m_e) - (M_y + Z\,m_e + m_e)]\,c^2$$
$$= (M_p - M_d)\,c^2$$

since M_p, M_d are the masses of the atoms. The binding energy of the electrons in ignored.

In K capture

$$e_K^- + {}_Z X^A \rightarrow {}_{Z-1}Y^A + Q$$
$$Q = (M_X - M_Y)\,c^2 + m_e c^2$$
$$= (M_x^{c^2} + Z\,m_e c^2) - (M_Y c^2 + (Z-1)\,m_e c^2)$$
$$= c^2 (M_p - M_d)$$

In β^+ decay

$$_Z X^A \rightarrow {}_{Z-1}Y^A + e^+ + Q$$

Then

$$Q = (M_x - M_y - m_e)\,c^2$$
$$= [M_x + Z\,m_e]\,c^2 - [M_y + (Z-1)\,m_e]\,c^2 - 2\,m_e c^2$$
$$= (M_p - M_d - 2\,m_e)\,c^2$$

6.239 The reaction is $Be^{10} \rightarrow B^{10} + e^- + \bar{\nu}_e$

For maximum K.E. of electrons we can put the energy of $\bar{\nu}_e$ to be zero. The atomic masses are

$$Be^{10} = 10{\cdot}016711 \text{ amu}$$
$$B^{10} = 10{\cdot}016114 \text{ amu}$$

So the K.E. of electrons is (see previous problem)

$$597\times10^{-6} \text{ amu} \times c^2 = 0{\cdot}56 \text{ MeV}$$

The momentum of electrons with this K.E. is $0{\cdot}941 \frac{MeV}{c}$

and the recoil energy of the daughter is

$$\frac{(0{\cdot}941)^2}{2 \times M_d c^2} = \frac{(0{\cdot}941)^2}{2 \times 10 \times 938} \text{MeV} = 47{\cdot}2 \text{ eV}$$

6.240 The masses are

$$Na^{24} = 24 - 0{\cdot}00903 \text{ amu} \quad \text{and } Mg^{24} = 24 - 0{\cdot}01496 \text{ amu}$$

The reaction is

$$Na^{24} \rightarrow Mg^{24} + e^- + \bar{\nu}_e$$

The maximum K.E. of electrons is

$$0{\cdot}00593 \times 93 \text{ MeV} = 5{\cdot}52 \text{ MeV}$$

Average K.E. according to the problem is then $\frac{5{\cdot}52}{3} = 1{\cdot}84$ MeV

The initial number of Na^{24} is

$$\frac{10^{-3} \times 6{\cdot}023 \times 10^{23}}{24} = 2{\cdot}51 \times 10^{19}$$

The fraction decaying in a day is

$$1 - (2)^{-24/15} = 0{\cdot}67$$

Hence the heat produced in a day is

$0{\cdot}67 \times 2{\cdot}51 \times 10^{19} \times 1{\cdot}84 \times 1{\cdot}602 \times 10^{-13}$ Joul $= 4{\cdot}95$ MJ

6.241 We assume that the parent nucleus is at rest. Then since the daughter nucleus does not recoil, we have

$$\vec{p} = -\vec{p}_\nu$$

i.e. positron & ν mometum are equal and opposite. On the other hand

$\sqrt{c^2 p^2 + m_e^2 c^4} + c\,p = Q =$ total energy released. (Here we have used the fact that energy of the neutrino is $c\,|\vec{p}_\nu| = c\,p$)

Now $\quad Q = [\,(\text{Mass of } C^{11} \text{ nucleus}) - (\text{Mass of } B^{11} \text{nucleus})\,]\, c^2$

$\quad = [\,\text{Mass of } C^{11} \text{ atom} - \text{Mass of } B^{11} \text{ atom} - m_e\,]\, c^2$

$\quad = 0{\cdot}00213 \text{ amu} \times c^2 - m_e c^2$

$\quad = (0{\cdot}00213 \times 931 - 0{\cdot}511) \text{ MeV} = 1{\cdot}47 \text{ MeV}$

Then $\quad c^2 p^2 + (0{\cdot}511)^2 = (1{\cdot}47 - c\,p)^2 = (1{\cdot}47)^2 - 2{\cdot}94\, c\,p + c^2 p^2$

Thus $c\,p = 0{\cdot}646$ MeV = energy of neutrino

Also K.E. of electron $= 1{\cdot}47 - 0{\cdot}646 - 0{\cdot}511 = 0{\cdot}313$ MeV

6.242 The K.E. of the positron is maximum when the energy of neutrino is zero. Since the recoil energy of the nucleus is quite small, it can be calculated by successive approximation. The reaction is

$$N^{13} \rightarrow C^{13} + e^{+} + \nu_e .$$

The maximum energy available to the positron (including its rest energy) is

$$\begin{aligned} & c^2 \,(\text{Mass of } N^{13} \text{nucleus - Mass of } C^{13} \text{ nucleus}) \\ &= c^2 \,(\text{Mass of } N^{13} \text{ atom - Mass of } C^{13} \text{ atom } - m_e) \\ &= 0{\cdot}00239\, c^2 - m_e c^2 \\ &= (0{\cdot}00239 \times 931 - 0{\cdot}511)\ \text{MeV} \\ &= 1{\cdot}71\ \text{MeV} \end{aligned}$$

The momentum corresponding to this energy is $1{\cdot}636\ \text{MeV}/c$.

The recoil energy of the nucleus is then

$$E = \frac{p^2}{2M} = \frac{(1{\cdot}636)^2}{2 \times 13 \times 931} = 111\ \text{eV} = 0{\cdot}111\ \text{keV}$$

on using $Mc^2 = 13 \times 931$ MeV

6.243 The process is

$$e_k^- + Be^7 \rightarrow Li^7 + \nu$$

The energy available in the process is

$$\begin{aligned} Q &= c^2\,(\text{Mass of } Be^7 \text{ atom - Mass of } Li^7 \text{ atom}) \\ &= 0{\cdot}00092 \times 931\ \text{MeV} = 0{\cdot}86\ \text{MeV} \end{aligned}$$

The momentum of a K electron is negligible. So in the rest frame of the Be^7 atom, most of the energy is taken by neutrino whose momentum is very nearly $0{\cdot}86\ \text{MeV}/c$

The momentum of the recoiling nucleus is equal and opposite. The velocity of recoil is

$$\frac{0{\cdot}86\ \text{MeV}/c}{M_{Li}} = c \times \frac{0{\cdot}86}{7 \times 931} = 3{\cdot}96 \times 10^6\ \text{cm/s}$$

6.244 In internal conversion, the total energy is used to knock out K electrons. The K.E. of these electrons is energy available-B.E. of K electrons

$$= (87 - 26) = 61\ \text{keV}$$

The total energy including rest mass of electrons is $0{\cdot}511 + 0{\cdot}061 = 0{\cdot}572$ MeV

The momentum corresponding to this total energy is

$\sqrt{(0{\cdot}572)^2 - (0{\cdot}511)^2}\ /c = 0257\ \text{MeV}/c.$

The velocity is then $\dfrac{c^2 p}{E} = c \times \dfrac{0{\cdot}257}{0{\cdot}572} = 0{\cdot}449\, c$

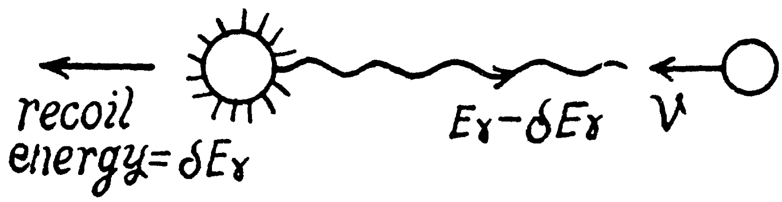
6.245 With recoil neglected, the γ–quantrum will have 129 keV energy. To a first approximation, its momentrum will be 129 keV/c and the energy of recoil will be

$$\frac{(0 \cdot 129)^2}{2 \times 191 \times 931} \text{MeV} = 4 \cdot 18 \times 10^{-8} \text{MeV}$$

In the next approximation we therefore write $E_\gamma \simeq \cdot 129 - 8 \cdot 2 \times 10^{-8}$ MeV

Therefore
$$\frac{\delta E_\gamma}{E_\gamma} = 3 \cdot 63 \times 10^{-7}$$

6.246 For maximum (resonant) absorption, the absorbing nucleus must be moving with enough speed to cancel the momentum of the oncoming photon and have just right energy ($\varepsilon = 129$ keV) available for transition to the excited state.



Since $\delta E_\gamma \simeq \frac{\varepsilon^2}{2Mc^2}$ and momentum of the photon is $\frac{\varepsilon}{c}$, these condition can be satisfied if the velocity of the nucleus is

$$\frac{\varepsilon}{Mc} = c\frac{\varepsilon}{Mc^2} = 218 \text{ m/s} = 0 \cdot 218 \text{ k m/s}$$

6.247 Because of the gravitational shift the frequency of the gamma ray at the location of the absorber is increased by

$$\frac{\delta \omega}{\omega} = \frac{gh}{c^2}$$

For this to be compensated by the Doppler shift (assuming that resonant absorption is possible in the absence of gravitational field) we must have

$$\frac{gh}{c^2} = \frac{v}{c} \quad \text{or} \quad v = \frac{gh}{c} = 0 \cdot 65 \ \mu \text{m/s}$$

6.248 The natural life time is

$$\Gamma = \frac{\hbar}{\tau} = 4 \cdot 7 \times 10^{-10} \text{eV}$$

Thus the condition $\delta E_\gamma \geq \Gamma$ implies $\frac{gh}{c^2} \geq \frac{\Gamma}{\varepsilon} = \frac{\hbar}{\tau \varepsilon}$

or
$$h \geq \frac{c^2 \hbar}{\tau \varepsilon g} = 4 \cdot 64 \text{ metre}$$

(h here is height of the place, not planck's constant.)

6.6 NUCLEAR REACTIONS

6.249 Initial momentum of the α particle is $\sqrt{2mT_\alpha}\ \hat{i}$ (where $\hat{i}$ is a unit vector in the incident direction). Final momenta are respectively $\vec{p}_\alpha$ and $\vec{p}_{Li}$. Conservation of momentum reads

$$\vec{p}_\alpha + \vec{p}_{Li} = \sqrt{2mT_\alpha}\ \hat{i}$$

Squaring $$p_\alpha^2 + p_{Li}^2 + 2p_\alpha p_{Li}\cos\Theta = 2mT_\alpha \tag{1}$$

where Θ is the angle between $\vec{p}_\alpha$ and $\vec{p}_{Li}$.

Also by energy conservation $$\frac{p_\alpha^2}{2m} + \frac{p_{L_i}^2}{2M} = T_\alpha$$

(m & M are respectively the masses of α particle and Li^6.) So

$$p_\alpha^2 + \frac{m}{M}p_{Li}^2 = 2mT_\alpha \tag{2}$$

Substracting (2) from (1) we see that

$$p_{Li}\left[\left(1-\frac{m}{M}\right)p_{Li} + 2p_\alpha\cos\Theta\right] = 0$$

Thus if $$p_{Li} \neq 0$$

$$p_\alpha = -\frac{1}{2}\left(1-\frac{m}{M}\right)p_{Li}\sec\Theta.$$

Since p_α, p_{Li} are both positive number (being magnitudes of vectors) we must have

$$-1 \le \cos\Theta < 0 \quad \text{if} \quad m < M.$$

This being understood, we write

$$\frac{p_{Li}^2}{2M}\left[1 + \frac{M}{4m}\left(1-\frac{m}{M}\right)^2 \sec^2\Theta\right] = T_\alpha$$

Hence the recoil energy of the L_i nucleus is

$$\frac{p_{Li}^2}{2M} = \frac{T_\alpha}{1 + \frac{(M-m)^2}{4mM}\sec^2\Theta}$$

As we pointed out above $\Theta \neq 60°$. If we take $\Theta = 120°$, we get recoil energy of $Li = 6$ MeV

6.250 (a) In a head on collision

$$\sqrt{2mT} = p_d + p_n$$

$$T = \frac{p_d^2}{2M} + \frac{p_n^2}{2m}$$

Where p_d and p_n are the momenta of deuteron and neutron after the collision. Squaring

$$p_d^2 + p_n^2 + 2p_d p_n = 2mT$$

$$p_n^2 + \frac{m}{M} p_d^2 = 2\,m\,T$$

or since $p_d \neq 0$ in a head on collisions

$$p_n = -\frac{1}{2}\left(1 - \frac{m}{M}\right) p_d .$$

Going back to energy conservation

$$\frac{p_d^2}{2M}\left[1 + \frac{M}{4m}\left(1 - \frac{m}{M}\right)^2\right] = T$$

So
$$\frac{p_d^2}{2M} = \frac{4\,m\,M}{(m+M)^2}\,T$$

This is the energy lost by neutron. So the fraction of energy lost is

$$\eta = \frac{4\,m\,M}{(m+M)^2} = \frac{8}{9}$$

(b) In this case neutron is scattered by 90°. Then we have from the diagram

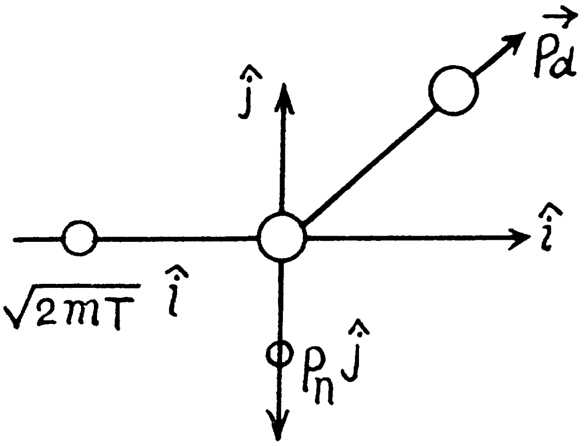
$\vec{p_d} = p_n\,\hat{j} + \sqrt{2\,m\,T}\,\hat{i}$

Then by energy conservation

$$\frac{p_n^2 + 2\,m\,T}{2M} + \frac{p_n^2}{2m} = T$$

or
$$\frac{p_n^2}{2m}\left(1 + \frac{m}{M}\right) = T\left(1 - \frac{m}{M}\right)$$

or
$$\frac{p_n^2}{2m} = \frac{M - m}{M + m}\cdot T$$



The energy lost by neutron in then

$$T - \frac{p_n^2}{2m} = \frac{2\,m}{M+m}\,T$$

or fraction of energy lost is $\eta = \dfrac{2\,m}{M+m} = \dfrac{2}{3}$

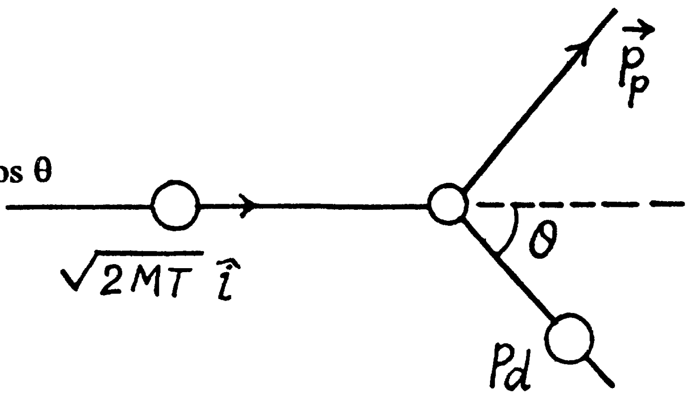
6.251 From conservation of momentum

$\sqrt{2\,M\,T}\,\hat{i} = \vec{p_d} + \vec{p_p}$

or $p_p^2 = 2\,M\,T + p_d^2 - 2\sqrt{2\,M\,T}\,p_d\cos\theta$

From energy conservation

$$T = \frac{p_d^2}{2M} + \frac{p_p^2}{2m}$$



(M = mass of denteron, m = mass of proton)

So $$p_p^2 = 2\,m\,T - \frac{m}{M}p_d^2\,.$$

Hence $$p_d^2\left(1+\frac{m}{M}\right) - 2\sqrt{2MT}\;p_d\cos\theta + 2\,(M-m)\,T = 0$$

For real roots $$4\,(2\,M\,T)\cos^2\theta - 4\times 2\,(M-m)\,T\left(1+\frac{m}{M}\right) \geq 0$$

$$\cos^2\theta \geq \left(1-\frac{m^2}{M^2}\right)$$

Hence $$\sin^2\theta \leq \frac{m^2}{M^2}$$

i.e. $$\theta \leq \sin^{-1}\frac{m}{M}$$

For deuteron-proton scaltering $\theta_{max} = 30°$.

6.252 This problem has a misprint. Actually the radius R of a nucleus is given by

$$R = 1{\cdot}3\;\sqrt[3]{A}\;fm$$

where $$fm = 10^{-15}\,\mathrm{m}\,.$$

Then the number of nucleous per unit volume is

$$\frac{A}{\frac{4\pi}{3}R^3} = \frac{3}{4\pi}\times(1{\cdot}3)^{-3}\times 10^{+39}\,\mathrm{cm}^{-3} = 1{\cdot}09\times 10^{38}\text{ per cc}$$

The corresponding mass density is

$(1{\cdot}09\times 10^{38}\times\text{mass of a nucleon})$ per cc $= 1{\cdot}82\times 10^{11}$ kg/cc

6.253 (a) The particle x must carry two nucleons and a unit of positive charge. The reaction is

$$B^{10}\,(d\,,\alpha)\,B_e^8$$

(b) The particle x must contain a proton in addition to the constituents of O^{17}. Thus the reaction is

$$O^{17}\,(d\,,n)\,F^{18}$$

(c) The particle x must carry nucleon number 4 and two units of +ve charge. Thus the particle must be $x = \alpha$ and the reaction is

$$Na^{23}\,(p\,,\alpha)\,Ne^{20}$$

(d) The particle x must carry mass number 37 and have one unit less of positive charge. Thus $x = Cl^{37}$ and the reaction is

$$Cl^{37}\,(p\,,n)\,Ar^{37}$$

6.254 From the basic formula

$$E_b = Z m_H + (A - Z) m_n - M$$

We define

$$\Delta_H = m_H - 1 \text{ amu}$$

$$\Delta_n = m_n - 1 \text{ amu}$$

$$\Delta = M - A \text{ amu}$$

Then clearly $E_b = Z\Delta_H + (A - Z)\Delta_n - \Delta$

6.255 The mass number of the given nucleus must be

$$27 \Big/ \left(\frac{3}{2}\right)^3 = 8$$

Thus the nucleus is Be^8. Then the binding energy is

$$E_b = 4 \times 0{\cdot}00867 + 4 + \times 0{\cdot}00783 - 0{\cdot}00531 \text{ amu}$$

$$= 0{\cdot}06069 \text{ amu} = 56{\cdot}5 \text{ MeV}$$

On using 1 amu = 931 MeV.

6.256 (a) Total binding energy of the O^{16} nucleus is

$$E_b = 8 \times {\cdot}00867 + 8 \times {\cdot}00783 + 0{\cdot}00509 \text{ amu}$$

$$= 0{\cdot}13709 \text{ amu} = 127{\cdot}6 \text{ MeV}$$

So B.E. per nucleon is 7·98 Mev/nucleon

(b) B.E. of neutron in B^{11}nucleus

$$= \text{B.E. of } B^{11} - \text{B.E. of } B^{10}$$

(since on removing a neutron from B^{11} we get B^{10})

$$= \Delta_n - \Delta_{B_{11}} + \Delta_{B_{10}} = {\cdot}00867 - {\cdot}00930 + {\cdot}01294$$

$$= 0{\cdot}01231 \text{ amu} = 11{\cdot}46 \text{ MeV}$$

B.E. of (an α–particle in B^{11})

$$= \text{B.E. of } B^1 - \text{B.E. of } Li^7 - \text{B.E. of } \alpha$$

(since on removing an α from B^{11} we get Li^7)

$$= -\Delta_{B_{11}} + \Delta_{Li_7} + \Delta_\alpha$$

$$= -0{\cdot}00930 + 0{\cdot}01601 + 0{\cdot}00260$$

$$= 0{\cdot}00931 \text{ amu} = 8{\cdot}67 \text{ MeV}$$

(c) This energy is

[B.E. of O^{16} + 4 (B.E. of α particles)]

$$= -\Delta_{O^{16}} + 4\Delta_\alpha$$

$$= 4 \times 0{\cdot}00260 + 0{\cdot}00509$$

$$= 0{\cdot}01549 \text{ amu} = 14{\cdot}42 \text{ MeV}$$

6.257 B.E. of a neutron in B^{11} – B.E. of a proton in B^{11}

$$= (\Delta_n - \Delta_{B^{11}} + \Delta_{B^{10}}) - (\Delta_p - \Delta_{B^{11}} + \Delta_{B_e^{10}})$$

$$= \Delta_n - \Delta_p + \Delta_{B^{10}} - \Delta_{B_e^{10}} = 0{\cdot}00867 - 0{\cdot}00783$$

$$+ 0{\cdot}01294 - 0{\cdot}01354 = 0{\cdot}00024 \text{ amu} = 0{\cdot}223 \text{ MeV}$$

The difference in binding energy is essentially due to the coulomb repulsion between the proton and the residual nucleus Be^{10} which together constitute B^{11}.

6.258 Required energy is simply the difference in total binding energies-

$$= \text{B.E. of } Ne^{20} - 2\,(\text{B.E. of } He^4) - \text{B.E. of } C^{12}$$

$$= 20\,\varepsilon_{Ne} - 8\,\varepsilon_\alpha - 12\,\varepsilon_C$$

(ε is binding energy per unit nucleon.)

Substitution gives $11{\cdot}88$ MeV.

6.259 (a) We have for Li^8

$$41{\cdot}3 \text{ MeV} = 0{\cdot}044361 \text{ amu} = 3\,\Delta_H + 5\,\Delta_n - \Delta$$

Hence $\Delta = 3 \times 0{\cdot}00783 + 5 \times 0{\cdot}00867 - 0{\cdot}09436 = 0{\cdot}02248$ amu

(b) For C^{10}

$$10 \times 6{\cdot}04 = 60{\cdot}4 \text{ MeV}$$
$$= 0{\cdot}06488 \text{ amu}$$
$$= 6\,\Delta_H + 4\,\Delta_n - \Delta$$

Hence $\Delta = 6 \times 0{\cdot}00783 + 4 \times 0{\cdot}00867 - 0{\cdot}06488 = 0{\cdot}01678$ amu

Hence the mass of C^{10} is $10{\cdot}01678$ amu

6.260 Suppose M_1, M_2, M_3, M_4 are the rest masses of the nuclei A_1, A_2, A_3 and A_4 perticipating in the reaction

$$A_1 + A_2 \rightarrow A_3 + A_4 + Q$$

Here Q is the energy released. Then by conservation of energy.

$$Q = c^2 (M_1 + M_2 - M_3 - M_4)$$

Now $M_1 c^2 = c^2 (Z_1 m_H + (A_1 - Z) m_n) - E_1$ etc. and

$$Z_1 + Z_2 = Z_3 + Z_4 \text{(conservation of change)}$$

$$A_1 + A_2 = A_3 + A_4 \text{ (conservation of heavy particles)}$$

Hence $Q = (E_3 + E_4) - (E_1 + E_2)$

6.261 (a) the energy liberated in the fission of 1 kg of U^{235} is

$$\frac{1000}{235} \times 6{\cdot}023 \times 10^{23} \times 200 \text{ MeV} = 8{\cdot}21 \times 10^{10} \text{ kJ}$$

The mass of coal with equivalent calorific value is

$$\frac{8{\cdot}21 \times 10^{10}}{30000} \text{ kg} = 2{\cdot}74 \times 10^6 \text{ kg}$$

(b) The required mass is

$$\frac{30 \times 10^9 \times 4{\cdot}1 \times 10^3}{200 \times 1{\cdot}602 \times 10^{-13} \times 6{\cdot}023 \times 10^{23}} \times \frac{235}{1000}\,\text{kg} = 1{\cdot}49\,\text{kg}$$

6.262 The reaction is (in effect).

$$H^2 + H^2 \rightarrow He^4 + Q$$

Then

$$Q = 2\,\Delta_{H^2} - \Delta_{He^4} + Q$$

$$= 0{\cdot}02820 - 0{\cdot}00260$$

$$= 0{\cdot}02560\ \text{amu} = 23{\cdot}8\ \text{MeV}$$

Hence the energy released in 1 gm of He^4 is

$$\frac{6{\cdot}023 \times 10^{23}}{4} \times 23{\cdot}8 \times 1\,6{\cdot}02 \times 10^{-13}\ \text{Joule} = 5{\cdot}75 \times 10^8\ \text{kJ}$$

This energy can be derived from

$$\frac{5{\cdot}75 \times 10^8}{30000}\,\text{kg} = 1{\cdot}9 \times 10^4\ \text{kg of Coal.}$$

6.263 The energy released in the reaction

$$Li^6 + H^2 \rightarrow 2\,He^4$$

is

$$\Delta_{Li^6} + \Delta_{H^2} - 2\,\Delta_{He^4}$$

$$= 0{\cdot}01513 + 0{\cdot}01410 - 2 \times 0{\cdot}00\,260\ \text{amu}$$

$$= 0{\cdot}02403\ \text{amu} = 22{\cdot}37\ \text{MeV}$$

(This result for change in B.E. is correct because the contribution of Δ_n & Δ_H cancels out by conservation law for protons & neutrons.)-

Energy per nucleon is then

$$\frac{22{\cdot}37}{8} = 2{\cdot}796\ \text{MeV/nucleon}\,.$$

This should be compared with the value $\frac{200}{235} = 0{\cdot}85$ MeV/nucleon

6.264 The energy of reaction

$$Li^7 + p \rightarrow 2\,He^4$$

is,

$$2 \times \text{B.E. of } He^4 - \text{B.E. of } Li^7$$

$$= 8\,\varepsilon_\alpha - 7\,\varepsilon_{Li} = 8 \times 7{\cdot}06 - 7 \times 5{\cdot}60 = 17{\cdot}3\ \text{MeV}$$

6.265 The reaction is $N^{14}(\alpha\,, p)\,O^{17}$.

It is given that (in the Lab frame where N^{14} is at rest) $T_\alpha = 4{\cdot}0$ MeV.

The momentum of incident α particle is

$$\sqrt{2\,m_\alpha\,T_\alpha}\ \hat{i} = \sqrt{2\,\eta_\alpha\,m_0\,T_\alpha}\ \hat{i}$$

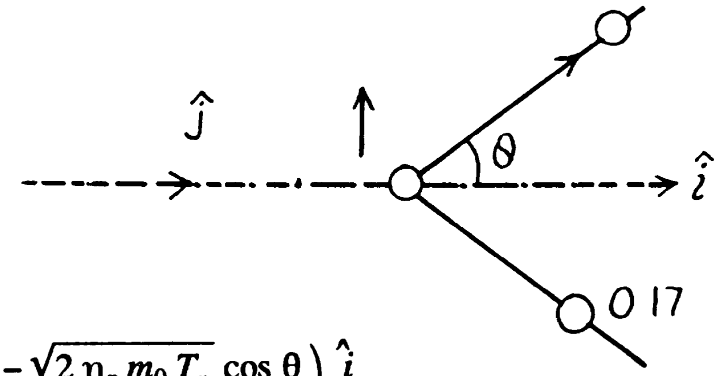
The momentum of outgoing proton is

$$\sqrt{2\, m_p\, T_p}\,(\cos\theta\ \hat{i} + \sin\theta\ \hat{j})$$

$$= \sqrt{2\, \eta_p\, m_0\, T_p}\,(\cos\theta\ \hat{i} + \sin\theta\ \hat{j})$$

where $\eta_p = \dfrac{m_p}{m_0}$, $\eta_\alpha = \dfrac{m_\alpha}{m_0}$,

and m_0 is the mass of O^{17}.



The momentum of O^{17} is

$$\left(\sqrt{2\, \eta_\alpha\, m_0\, T_\alpha} - \sqrt{2\, \eta_p\, m_0\, T_p}\,\cos\theta\right)\hat{i}$$
$$- \sqrt{2\, m_0\, \eta_p\, T_p}\,\sin\theta\ \hat{j}$$

By energy conservation (conservation of energy including rest mass energy and kinetic energy)

$$M_{14}\, c^2 + M_\alpha\, c^2 + T_\alpha$$
$$= M_p\, c^2 + T_p + M_{17}\, c^2$$
$$+ \left[\left(\sqrt{\eta_\alpha\, T_\alpha} - \sqrt{\eta_p\, T_p}\,\cos\theta\right)^2 + \eta_p\, T_p \sin^2\theta + \eta_p\, T_p \sin^2\theta\right]$$

Hence by definition of the Q of reaction

$$Q = M_{14}\, c^2 + M_\alpha\, c^2 - M_p\, c^2 - M_{17}\, c^2$$
$$= T_p + \eta_\alpha\, T_\alpha + \eta_p\, T_p - 2\sqrt{\eta_p\, \eta_\alpha\, T_\alpha\, T_p} \times \cos\theta - T_\alpha$$
$$= (1 + \eta_p)\, T_p + T_\alpha\,(1 - \eta_\alpha)$$
$$- 2\sqrt{\eta_p\, \eta_\alpha\, T_\alpha\, T_p}\,\cos\theta = -1{\cdot}19\ \text{MeV}$$

6.266 (a) The reaction is $Li^7\,(p\,,n)\,Be^7$ and the energy of reaction is

$$Q = (M_{Be^7} + M_{Li^7})\, c^2 + (M_p - M_n)\, c^2$$
$$= (\Delta_{Li_7} - \Delta_{Be^7})\, c^2 + \Delta_p - \Delta_n$$
$$= [0{\cdot}01601 + 0{\cdot}00783 - 0{\cdot}01693 - 0{\cdot}00867]\ \text{amu} \times c^2$$
$$= -1{\cdot}64\ \text{MeV}$$

(b) The reaction is $Be^9\,(n\,,\gamma)\,Be^{10}$.
Mass of γ is taken zero. Then

$$Q = (M_{Be^9} + M_n - M_{Be^{10}})\, c^2$$
$$= (\Delta_{Be^9} + \Delta_n - \Delta_{Be^{10}})\, c^2$$
$$= (0{\cdot}01219 + 0{\cdot}00867 - 0{\cdot}01354)\ \text{amu} \times c^2$$
$$= 6{\cdot}81\ \text{MeV}$$

(c) The reaction is $Li^7(\alpha, n)B^{10}$. The energy is

$$Q = (\Delta_{Li^7} + \Delta_\alpha - \Delta_n - \Delta_{B^{10}})c^2$$
$$= (0{\cdot}01601 + 0{\cdot}00260 - 0{\cdot}00867 - {\cdot}01294)\text{ amu} \times c^2$$
$$= -2{\cdot}79\text{ MeV}$$

(d) The reaction is $O^{16}(d, \alpha)N^{14}$. The energy of reaction is

$$Q = (\Delta_{O^{16}} + \Delta_d - \Delta_\alpha - \Delta_N^{14})c^2$$
$$= (-0{\cdot}00509 + 0{\cdot}01410 - 0{\cdot}00260 - 0{\cdot}00307)\text{ amu} \times c^2$$
$$= 3{\cdot}11\text{ MeV}$$

6.267 The reaction is $B^{10}(n, \alpha)Li^7$. The energy of the reaction is

$$Q = (\Delta_{B^{10}} + \Delta_n - \Delta_\alpha - \Delta_{Li^7})c^2$$
$$= (0{\cdot}01294 + 0{\cdot}00867 - 0{\cdot}00260 - 0{\cdot}01601)\text{ amu} \times c^2$$
$$= 2{\cdot}79\text{ MeV}$$

Since the incident neutron is very slow and B^{10} is stationary, the final total momentum must also be zero. So the reaction products must emerge in opposite directions. If their speeds are, repectively, v_α and v_{Li}

then
$$4v_\alpha = 7v_{Li}$$

and
$$\frac{1}{2}(4v_\alpha^2 + 7v_{Li}^2) \times 1{\cdot}672 \times 10^{-24} = 2{\cdot}79 \times 1{\cdot}602 \times 10^{-6}$$

So
$$\frac{1}{2} \times 4v_\alpha^2\left(1 + \frac{4}{7}\right) = 2{\cdot}70 \times 10^{18}\text{ cm}^2/\text{s}^2$$

or
$$v_\alpha = 9{\cdot}27 \times 10^6\text{ m/s}$$

Then
$$v_{Li} = 5{\cdot}3 \times 10^6\text{ m/s}$$

6.268 Q of this reaction ($Li^7(p, n)Be^7$) was calculated in problem 266 (a). If is $-1{\cdot}64$ MeV. We have by conservation of momentum and energy $p_p = p_{Be}$ (since initial Li and final neutron are both at rest)

$$\frac{p_p^2}{2m_p} = \frac{p_{Be}^2}{2m_{Li}} + 1{\cdot}64$$

Then
$$\frac{P_p^2}{2m_p}\left(1 - \frac{m_p}{m_{Be}}\right) = 1{\cdot}64$$

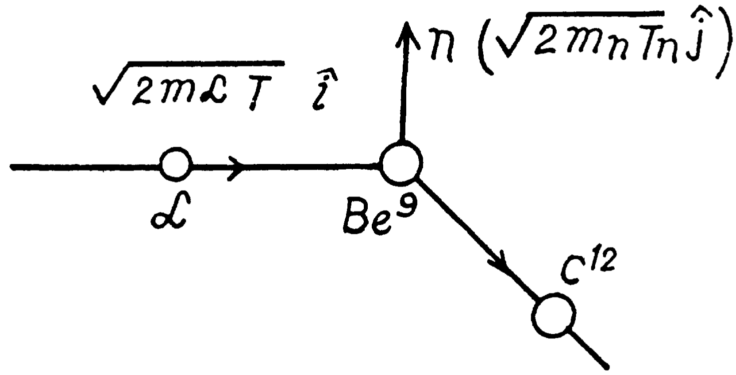
Hence
$$T_p = \frac{p_p^2}{2m_p} = \frac{7}{6} \times 1{\cdot}64\text{ MeV} = 1{\cdot}91\text{ MeV}$$

6.269 It is understood that Be^9 is initially at rest. The moment of the outgoing neutron is $\sqrt{2m_nT_n}\,\hat{j}$. The momenturm of C^{12} is

$\sqrt{2\,m_\alpha\,T}\;\hat{i} - \sqrt{2\,m_n\,T_n}\;\hat{j}$

Then by energy conservation

$$T + Q = T_n + \frac{2\,m_\alpha\,T + 2\,m_n\,T_n}{2\,m_{ci}}$$



(m_c is the mass of C^{12})

Thus $\quad T_n = \dfrac{m_c\,(T+Q) - m_\alpha\,T}{m_c + m_n}$

$$= \frac{(m_c - m_\alpha)\,T + m_c\,Q}{m_c + m_n} = \frac{Q + \left(1 - \dfrac{m_\alpha}{m_c}\right)T}{1 + \dfrac{m_n}{m_c}} = 8{\cdot}52\ \text{MeV}$$

6.270 The Q value of the reaction $Li^7\,(p\,,\alpha)\,He^4$ is

$$Q = (\Delta_{Li^7} + \Delta_H - 2\,\Delta_{He^4})\,c^2$$

$$= (0{\cdot}01601 + 0{\cdot}00783 - 0{\cdot}00520)\ \text{amu} \times c^2$$

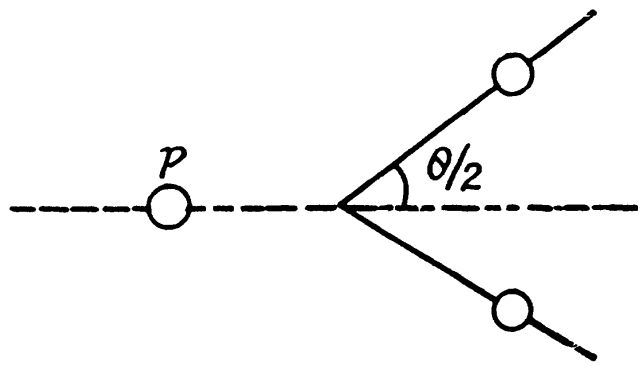
$$= 0{\cdot}01864\ \text{amu} \times c^2 = 17{\cdot}35\ \text{MeV}$$

Since the direction of He^4 nuclei is symmetrical, their momenta must also be equal. Let T be the K.E. of each He^4. Then

$$p_p = 2\sqrt{2\,m_{He}\,T}\,\cos\frac{\theta}{2}$$

(p_p is the momentum of proton). Also

$$\frac{p_p^2}{2\,m_p} + Q = 2\,T = T_p + Q$$



Hence $\qquad T_p + Q = 2\,\dfrac{p_p^2 \sec^2\dfrac{\theta}{2}}{8\,m_{He}}$

$$= T_p\,\frac{m_p}{2\,m_{He}}\sec^2\frac{\theta}{2}$$

Hence $\qquad \cos\dfrac{\theta}{2} = \sqrt{\dfrac{m_p}{2\,m_{He}}\,\dfrac{T_p}{T_p + Q}}$

Substitution gives

$$\theta = 170{\cdot}53°$$

Also $\qquad T = \dfrac{1}{2}(T_p + Q) = 9{\cdot}18\ \text{MeV}.$

6.271 Energy required is minimum when the reaction products all move in the direction of the incident particle with the same velocity (so that the combination is at rest in the centre of mass frame). We then have

$$\sqrt{2\,m\,T_{th}} = (m+M)\,v$$

(Total mass is constant in the nonrelativistic limit).

$$T_{th} - |Q| = \frac{1}{2}(m+M)\,v^2 = \frac{m\,T_{th}}{m+M}$$

or
$$T_{th}\frac{M}{m+M} = |Q|$$

Hence
$$T_{th} = \left(1+\frac{m}{M}\right)|Q|$$

6.272 The result of the previous problem applies and we find that energy required to split a deuteron is
$$T \geq \left(1+\frac{M_p}{M_d}\right)E_b = 3{\cdot}3\ \text{MeV}$$

6.273 Since the reaction $Li^7(p,n)Be^7$ ($Q = -1{\cdot}65$ MeV) is initiated, the incident proton energy must be

$$\geq \left(1+\frac{M_p}{M_{Li}}\right)\times 1{\cdot}65 = 1{\cdot}89\ \text{MeV}$$

since the reaction $Be^9(p,n)B^9$ ($Q = -1{\cdot}85$ MeV) is not initiated,

$$T \leq \left(1+\frac{M_p}{M_{Ve}}\right)\times 1{\cdot}85 = 2{\cdot}06\ \text{MeV} \quad \text{Thus} \quad 1{\cdot}89\ \text{MeV} \leq T_p \leq 2{\cdot}06\ \text{MeV}$$

6.274 We have $4{\cdot}0 = \left(1+\dfrac{m_n}{M_{B^{11}}}\right)|Q|$

or
$$Q = -\frac{11}{12}\times 4\ \text{MeV} = -3{\cdot}67\ \text{MeV}$$

6.275 The Q of the reaction $Li^7(p,n)Be^7$ was calculated in problem 266 (a). It is $-1{\cdot}64$ MeV Hence, the threshold K.E. of protons for initiating this reaction is

$$T_{th} = \left(1+\frac{m_p}{m_{Li}}\right)|Q| = \frac{8}{7}\times 1{\cdot}64 = 1{\cdot}87\ \text{MeV}$$

For the reaction $Li^7(p,d)Li^6$
we find
$$\begin{aligned} Q &= (\Delta_{Li^7} + \Delta_M - \Delta_d - \Delta_{Li^6})\,c^2 \\ &= (0{\cdot}01601 + 0{\cdot}00783 - 0{\cdot}01410 - 0{\cdot}01513)\ \text{amu}\times c^2 \\ &= -5{\cdot}02\ \text{MeV} \end{aligned}$$

The threshold proton energy for initiating this reaction is

$$T_{th} = \left(1+\frac{m_p}{m_{Li^7}}\right)\times|Q| = 5{\cdot}73\ \text{MeV}$$

6.276 The Q of $Li^7(\alpha, n)B^{10}$ was calculated in problem 266 (c). It is $Q = 2{\cdot}79$ MeV
Then the threshold energy of α-particle is

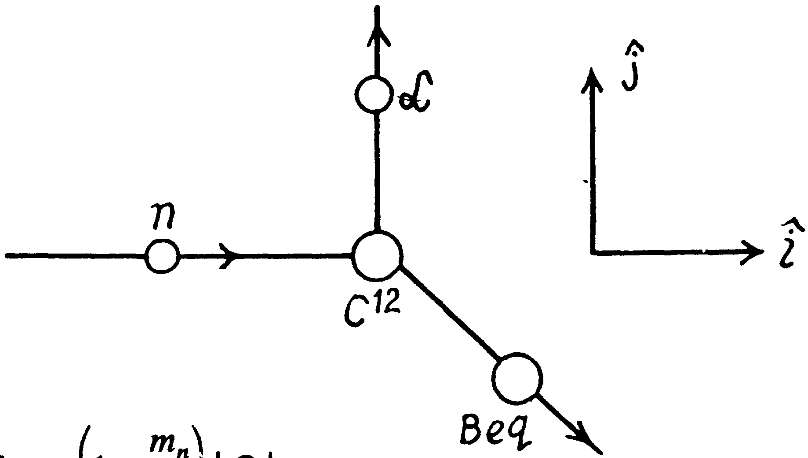
$$T_{th} = \left(1 + \frac{m_\alpha}{m_{Li}}\right)|Q| = \left(1 + \frac{4}{7}\right)2{\cdot}79 = 4{\cdot}38\text{MeV}$$

The velocity of B^{10} in this case is simply the volocity of centre of mass :-

$$v = \frac{\sqrt{2\,m_\alpha T_{th}}}{m_\alpha + m_{Li}} = \frac{1}{1 + \frac{m_{Li}}{m_\alpha}}\sqrt{\frac{2\,T_{th}}{m_\alpha}}$$

This is because both B^{10} and n are at rest in the CM frame at theshold.
Substituting the values of various quantities
we get $v = 5{\cdot}27 \times 10^6$ m/s

6.277 The momentum of incident neutron is $\sqrt{2\,m_n T}\,\hat{i}$, that of α particle is $\sqrt{2\,m_\alpha T_\alpha}\,\hat{j}$ and of Be^9 is
$-\sqrt{2\,m_\alpha T_\alpha}\,\hat{j} + \sqrt{2\,m_n T}\,\hat{i}$



By conservation of energy

$$T = T_\alpha + \frac{m_\alpha T_\alpha + m_n T}{M} + |Q|$$

(M is the mass of Be^9). Thus

$$T_\alpha = \left[T\left(1 - \frac{m_n}{M}\right) - |Q|\right]\frac{M}{M + m_\alpha}.$$

Using $T_{th} = \left(1 + \frac{m_n}{M}\right)|Q|$

we get $$T_\alpha = \frac{M}{M + m_\alpha}\left[\left(1 - \frac{m_n}{M}\right)T - \frac{T_{th}}{1 + \frac{m_n}{M}}\right]$$

M' is the mass of C^{12} nucleus.

or $$T_\alpha = \frac{1}{M + m_\alpha}\left[(M - m_n)\,T - \frac{M\,M'}{M' + m_n}T_{th}\right] = 2{\cdot}21\text{ MeV}$$

6.278 The formula of problem 6.271 does not apply here because the photon is always reletivistic. At threshold, the energy of the photon E_γ implies a momentum $\frac{E_\gamma}{c}$. The velocity of centre of mass with respect to the rest frame of initial H^2 is

$$\frac{E_\gamma}{(m_n + m_p)\,c}$$

Since both n & p are at rest in CM frame at threshold, we write

$$E_\gamma = \frac{E_\gamma^2}{2(m_n + m_p)c^2} + E_b$$

by conservation of energy. Since the first term is a small correction, we have

$$E_\gamma = E_b + \frac{E_b^2}{2(m_n + m_p)c^2}$$

Thus $$\frac{\delta E}{E_b} = \frac{E_b}{2(m_n + m_p)c^2} = \frac{2{\cdot}2}{2 \times 2 \times 938} = 5{\cdot}9 \times 10^{-4}$$

or nearly 0·06 %.

6.279 The reaction is

$$p + d \rightarrow He^3$$

Excitation energy of He^3 is just the energy available in centre of mass. The velocity of the centre of mass is

$$\frac{\sqrt{2\,m_p T_p}}{m_p + m_d} \approx \frac{1}{3}\sqrt{\frac{2\,T_p}{m_p}}$$

In the CM frame, the kinetic energy available is ($m_d \approx 2\,m_p$)

$$\frac{1}{2}m_p\left(\frac{2}{3}\sqrt{\frac{2\,T_p}{m_p}}\right)^2 + \frac{1}{2}2\,m_p\left(\frac{1}{3}\sqrt{\frac{2\,T}{m_p}}\right)^2 = \frac{2\,T}{3}$$

The total energy available is then $Q + \frac{2\,T}{3}$

where
$$\begin{aligned} Q &= c^2(\Delta_n + \Delta_d - \Delta_{He}^3) \\ &= c^2 \times (0{\cdot}00783 + 0{\cdot}01410 - 0{\cdot}01603)\ \text{amu} \\ &= 5{\cdot}49\ \text{MeV} \end{aligned}$$

Finally $$E = 6{\cdot}49\ \text{MeV}.$$

6.280 The reaction is

$$d + C^{13} \rightarrow N^{15*} \rightarrow n + N^{14}$$

Maxima of yields determine the energy levels of N^{15*}. As in the previous problem the excitation energy is

$$E_{exc} = Q + E_K$$

where E_K = available kinetic energy. This is found is as in the previous problem. The velocity of the centre of mass is

$$\frac{\sqrt{2\,m_d T_i}}{m_d + m_c} = \frac{m_d}{m_d + m_c}\sqrt{\frac{2\,T_i}{m_d}}$$

So $$E_K = \frac{1}{2}m_d\left(1-\frac{m_d}{m_d+m_c}\right)^2\frac{2\,T_i}{m_d} + \frac{1}{2}m_c\left(\frac{m_d}{m_d+m_c}\right)^2\frac{2\,T_i}{m_d} = \frac{m_c}{m_d+m_c}T_i$$

Q is the Q value for the ground state of N^{15} : We have

$$Q = c^2 \times (\Delta_d + \Delta_{C^{13}} - \Delta_{N^{15}})$$
$$= c^2 \times (0{\cdot}01410 + 0{\cdot}00335 - 0{\cdot}00011)\text{ amu}$$
$$= 16{\cdot}14\text{ MeV}$$

The excitation energies then are

$$16{\cdot}66\text{ MeV},\ 16{\cdot}92\text{ MeV}$$
$$17{\cdot}49\text{ MeV and } 17{\cdot}70\text{ MeV}.$$

6.281 We have the relation

$$\frac{1}{\eta} = e^{-n\sigma d}$$

Here $\frac{1}{\eta}$ = attenuation factor

n = no. of Cd nuclei per unit volume

σ = effective cross section

d = thickness of the plate

Now $$n = \frac{\rho N_A}{M}$$

(ρ = density, M = Molar weight of Cd, N_A = Avogadro number.)

Thus $$\sigma = \frac{M}{\rho N_A d}\ln\eta = 2{\cdot}53\text{ kb}$$

6.282 Here

$$\frac{1}{\eta} = e^{-(n_2\sigma_2 + n_1\sigma_1)d}$$

where 1 refers to O^1 and 2 to D nuclei

Using $n_2 = 2n$, $n_1 = n$ = concentration of O nuclei in heavy water we get

$$\frac{1}{\eta} = e^{-(2\sigma_2+\sigma_1)nd}$$

Now using the data for heavy water

$$n = \frac{1{\cdot}1 \times 6{\cdot}023 \times 10^{23}}{20} = 3{\cdot}313 \times 10^{22}\text{ per cc}$$

Thus substituting the values

$$\eta = 20{\cdot}4 = \frac{I_0}{I}.$$

6.283 In traversing a distance d the fraction which is either scattered or absorbed is clearly

$$1 - e^{-n(\sigma_s + \sigma_a)d}$$

by the usual definition of the attenuation factor. Of this, the fraction scattered is (by definition of scattering and absorption cross section)

$$w = \left\{1 - e^{-n(\sigma_s + \sigma_a)d}\right\} \frac{\sigma_s}{\sigma_s + \sigma_a}$$

In iron $$n = \frac{\rho \times N_A}{M} = 8{\cdot}39 \times 10^{22} \text{ per cc}$$

Substitution gives $$w = 0{\cdot}352$$

6.284 (a) Assuming of course, that each reaction produces a radio nuclide of the same type, the decay constant α of the radionuclide is k/w. Hence $T = \frac{\ln 2}{\lambda} = \frac{w}{k} \ln 2$

(b) number of bombarding particles is : $\frac{I t}{e}$

(e = charge on proton). Then the number of Be^7 produced is : $\frac{I t}{e} w$

If λ = decay constant of $Be^7 = \frac{\ln 2}{T}$, then the activity is $A = \frac{I t}{e} w \cdot \frac{\ln 2}{T}$

Hence $$w = \frac{e A T}{I t \ln 2} = 1{\cdot}98 \times 10^{-3}$$

6.285 (a) Suppose N_0 = no. of Au^{197} nuclei in the foil. Then the number of Au^{197} nuclei transformed in time t is

$$N_0 \cdot J \cdot \sigma \cdot t$$

For this to equal ηN_0, we must have

$$t = \eta / (J \cdot \sigma) = 323 \text{ years}$$

(b) Rate of formation of the Au^{198} nuclei is $N_0 \cdot J \cdot \sigma$ per sec

and rate of decay is λn, where n is the number of Au^{198} at any instant.

Thus $$\frac{dn}{dt} = n_0 \cdot J \cdot \sigma - \lambda n$$

The maximum number of Au^{198} is clearly

$$n_{\max} = \frac{N_0 J \sigma}{\lambda} = \frac{N_0 J \sigma T}{\ln 2}$$

because if n is smaller, $\frac{dn}{dt} > 0$ and n will increase further and if n is larger $\frac{dn}{dt} < 0$ and n will decrease. (Actually $n_{\max}$ is approached steadily as $t \to \infty$)

Substitution gives using $N_0 = 3{\cdot}057 \times 10^{19}$, $n_{\max} = 1{\cdot}01 \times 10^{13}$

6.286 Rate of formation of the radionuclide is $n{\cdot}J{\cdot}\sigma$ per unit area per sec. Rate of decay is λN. Thus

$$\frac{dN}{dt} = n{\cdot}J{\cdot}\sigma - \lambda N \text{ per unit area per second}$$

Then $$\left(\frac{dN}{dt} + \lambda N\right) e^{\lambda t} = n{\cdot}J{\cdot}\sigma\, e^{\lambda t} \quad \text{or} \quad \frac{d}{dt}(N e^{\lambda t}) = n{\cdot}J{\cdot}\sigma{\cdot}e^{\lambda t}$$

Hence $$N e^{\lambda t} = \text{Const} + \frac{n{\cdot}J{\cdot}\sigma}{\lambda} e^{\lambda t}$$

The number of radionuclide at $t = 0$ when the process starts is zero. So constant $= -\frac{n{\cdot}J{\cdot}\sigma}{\lambda}$

Then $$N = \frac{n{\cdot}J{\cdot}\sigma}{\lambda}(1 - e^{-\lambda t})$$

6.287 We apply the formula of the previous problem except that have
N = no. of radio nuclide and no. of host nuclei originally.

Here $$n = \frac{0{\cdot}2}{197} \times 6{\cdot}023 \times 10^{23} = 6{\cdot}115 \times 10^{20}$$

Then after time t $$N = \frac{n{\cdot}J{\cdot}\sigma{\cdot}T}{\ln 2}\left(1 - e^{-\frac{t \ln 2}{T}}\right)$$

T = half life of the radionuclide.

After the source of neutrons is cut off the activity after time T will be

$$A = \frac{n{\cdot}J{\cdot}\sigma{\cdot}T}{\ln 2}(1 - e^{-t \ln 2/T})\, e^{-\tau \ln 2/T} \times \frac{\ln 2}{T} = n{\cdot}J{\cdot}\sigma\,(1 - e^{-t \ln 2/T})\, e^{-\tau \ln 2/T}$$

Thus $$J = A\, e^{\tau \ln 2/T} \Big/ n\,\sigma\,(1 - e^{-t \ln 2/T}) = 5{\cdot}92 \times 10^{9} \text{ part/cm}^2 \cdot\text{s}$$

6.288 No. of nuclei in the first generation = No. of nuclei initially = N_0
N_0 in the second generation = $N_0 \times$ multiplication factor $= N_0 \cdot k$

N_0 in the the 3rd generation $= N_0 \cdot k \cdot k = N_0 k^2$

N_0 in the nth generation $= N_0 k^{n-1}$

Substitution gives $1{\cdot}25 \times 10^5$ neutrons

6.289 N_0 of fissions per unit time is clearly P/E. Hence no. of neutrons produced per unit time to $\frac{\nu P}{E}$. Substitution gives $7{\cdot}80 \times 10^{18}$ neutrons/sec

6.290 (a) This number is k^{n-1} where n = no. of generations in time $t = t/T$
Substitution gives 388.

(b) We write $$k^{n-1} = e^{\left(\frac{T}{\tau} - 1\right)\ln k} = e$$

or $$\frac{T}{\tau} - 1 = \frac{1}{\ln k} \quad \text{and} \quad T = \tau\left(1 + \frac{1}{\ln k}\right) = 10{\cdot}15 \text{ sec}$$

6.7 ELEMENTARY PARTICLES

6.291 The formula is

$$T = \sqrt{c^2 p^2 + m_0^2 c^4} - m_0 c^2$$

Thus $\quad T = 5{\cdot}3\,\text{MeV} \quad \text{for } p = 0{\cdot}10\,\frac{GeV}{c} = 5{\cdot}3 \times 10^{-3}\,\text{GeV}$

$$T = 0{\cdot}433\,\text{GeV} \quad \text{for } p = 1{\cdot}0\,\frac{GeV}{c}$$

$$T = 9{\cdot}106\,\text{GeV} \quad \text{for } p = 10\,\frac{GeV}{c}$$

Here we have used $m_0 c^2 = 0{\cdot}938\,\text{GeV}$

6.292 Energy of pions is $(1 + \eta)\, m_0 c^2$ so

$$(1 + \eta)\, m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - \beta^2}}$$

Hence $\quad \frac{1}{\sqrt{1 - \beta^2}} = 1 + \eta \quad \text{or} \quad \beta = \frac{\sqrt{\eta\,(2 + \eta)}}{1 + \eta}$

Here $\beta = \frac{v}{c}$ of pion. Hence time dilation factor is $1 + \eta$ and the distance traversed by the pion in its lifetime will be

$$\frac{c\,\beta\,\tau_0}{\sqrt{1 - \beta^2}} = c\,\tau_0 \sqrt{\eta\;(2 + \eta)} = 15{\cdot}0\,\text{metres}$$

on substituting the values of various quantities. (Note. The factor $\frac{1}{\sqrt{1 - \beta^2}}$ can be looked at as a time dilation effect in the laboratory frame or as length contraction factor brought to the other side in the proper frame of the pion).

6.293 From the previous problem $\quad l = c\,\tau_0 \sqrt{\eta\,(\eta + 2)}$

where $\eta = \frac{T}{m_\pi c^2}$, m_π is the rest mass of pions.

substitution gives $\quad \tau_0 = \frac{l}{c\sqrt{\eta\,(2 + \eta)}} = 2{\cdot}63\,\text{ns}$

$$= \frac{l\, m_\pi c}{\sqrt{T\,(T + 2\, m_\pi c^2)}}$$

where we have used $\eta = \frac{100}{139{\cdot}6} = 0{\cdot}716$

6.294 Here $\eta = \dfrac{T}{m c^2} = 1$ so the life time of the pion in the laboratory frame is

$$\eta = (1+\eta)\tau_0 = 2\tau_0$$

The law of radioactive decay implies that the flux decrease by the factor.

$$\frac{J}{J_0} = e^{-t/\tau} = e^{-l/v\tau} = e^{-l/c\tau_0 \sqrt{\eta(2+\eta)}}$$

$$= \exp\left(-\frac{m c l}{\tau_0 \sqrt{T(T+2 m c^2)}}\right) = 0{\cdot}221$$

6.295 Energy-momentum conservation implies

$$O = \vec{p_\mu} + \vec{p_\nu}$$

$$m_\pi c^2 = E_\mu + E_\nu \quad \text{or} \quad m_\pi c^2 - E_\nu = E_\mu$$

But $\qquad E_\nu = c\,|\vec{p_\nu}| = c\,|p_\mu|$. Thus

$$m_\pi^2 c^4 - 2 m_\pi c^2 \cdot c\,|\vec{p_\mu}| + c^2 p_\mu^2 = E_\mu^2 = c^2 p_\mu^2 + m_\mu^2 c^4$$

Hence $\qquad c\,|\vec{p_\mu}| = \dfrac{m_\pi^2 - m_\mu^2}{2 m_\pi} \cdot c^2$

So $\qquad T_\mu = \sqrt{c^2 p_\mu^2 + m_\mu^2 c^4} - m_\mu c^2 = \sqrt{\dfrac{(m_\pi^2 - m_\mu^2)^2}{4 m_\pi^2} + m_\mu^2} \cdot c^2 - m_\mu c^2$

$$= \frac{m_\pi^2 + m_\mu^2}{2 m_\pi} c^2 - m_\mu c^2 = \frac{(m_\pi - m_\mu)^2}{2 m_\pi} \cdot c^2$$

Substituting $\qquad m_\pi c^2 = 139{\cdot}6\ \text{MeV}$

$m_\mu c^2 = 105{\cdot}7\ \text{MeV}$ we get

$$T_\mu = 4{\cdot}12\ \text{MeV}$$

Also $\qquad E_\nu = \dfrac{m_\pi^2 - m_\mu^2}{2 m_\pi} c^2 = 29{\cdot}8\ \text{MeV}$

6.296 We have

$$O = \vec{p_n} + \vec{p_\pi} \tag{1}$$

$$m_\Sigma c^2 = E_n + E_\pi$$

or $\qquad (m_\Sigma c^2 - E_n)^2 = E_\pi^2$

or $\qquad m_\Sigma^2 c^4 - 2 m_\Sigma c^2 E_n = E_\pi^2 - E_n^2 = c^4 m_\pi^2 - c^4 m_n^2$

because (1) implies $\qquad E_\pi^2 - E_n^2 = m_\pi^2 c^4 - m_n^2 c^4$

Hence $\qquad E_n = \dfrac{m_\Sigma^2 + m_n^2 - m_\pi^2}{2 m_\Sigma} c^2$

and $$T_n = \left(\frac{m_\Sigma^2 + m_n^2 - m_\pi^2}{2\,m_\Sigma} - m_n\right)c^2 = \frac{(m_\Sigma - m_n)^2 - m_\pi^2}{2\,m_\Sigma}c^2 .$$

Substitution gives $$T_n = 19{\cdot}55 \text{ MeV}$$

6.297 The reaction is

$$\mu^+ \rightarrow e^+ + \bar{\nu}_e + \bar{\nu}_\mu$$

The neutrinoes are massless. The positron will carry largest momentum if both neutriones (ν_e & $\bar{\nu}_\mu$) move in the same direction in the rest frame of the nuon. Then the final product is effectively a two body system and we get from problem (295)

$$(T_{e^+})_{\max} = \frac{(m_\mu - m_e)^2}{2\,m_\mu}c^2$$

Substitution gives $$(T_{e^+})_{\max} = 52{\cdot}35 \text{ MeV}$$

6.298 By conservation of energy-momentum

$$M c^2 = E_p + E_\pi$$

$$O = \vec{p}_p + \vec{p}_\pi$$

Then $$m_\pi^2 c^4 = E_\pi^2 - \vec{p}_\pi^{\,2} c^2 = (M c^2 - E_p)^2 - c^2 \vec{p}_p^{\,2}$$

$$= M^2 c^4 - 2 M c^2 E_p + m_p^2 c^4$$

This is a quadratic equation in M

$$M^2 - 2\frac{E_p}{c^2}M + m_p^2 - m_\pi^2 = 0$$

or using $E_p = m_p c^2 + T$ and solving

$$\left(M - \frac{E_p}{c^2}\right)^2 = \frac{E_p^2}{c^4} - m_p^2 + m_\pi^2$$

Hence, $$M = \frac{E_p}{c^2} + \sqrt{\frac{E_p^2}{c^4} - m_p^2 + m_\pi^2}$$

taking the positive sign. Thus

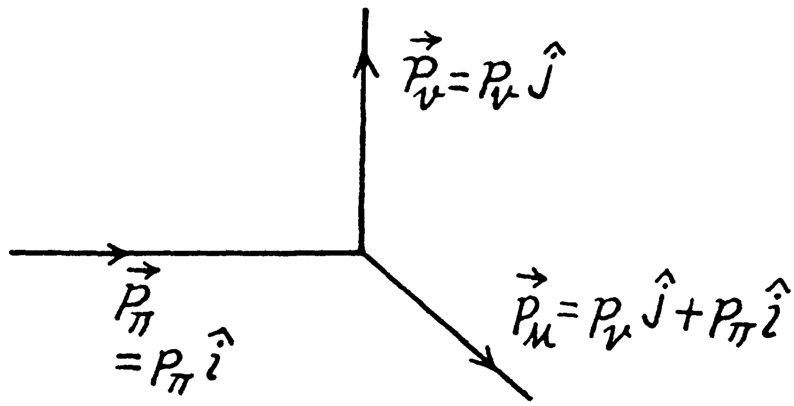
$$M = m_p + \frac{T}{c^2} + \sqrt{m_\pi^2 + \frac{T}{c^2}\left(2\,m_p + \frac{T}{c^2}\right)}$$

Substitution gives

$$M = 1115{\cdot}4\,\frac{\text{MeV}}{c^2}$$

From the table of masses we identify the particle as a Λ particle

6.299



See the diagram. By conservation of energy

$$\sqrt{m_\pi^2 c^4 + c^2 p_\pi^2} = c\, p_\nu + \sqrt{m_\mu^2 c^4 + p_\pi^2 c^2 + c^2 p_\nu^2}$$

or $$\left(\sqrt{m_\pi^2 c^4 + c^2 p_\pi^2} - c\, p_\nu\right)^2 = m_\mu^2 c^4 + c^2 p_\pi^2 + c^2 p_\nu^2$$

or $$m_\pi^2 c^4 - 2\, c\, p_\nu \sqrt{m_\pi^2 c^4 + c^2 p_\pi^2} = m_\mu^2 c^4$$

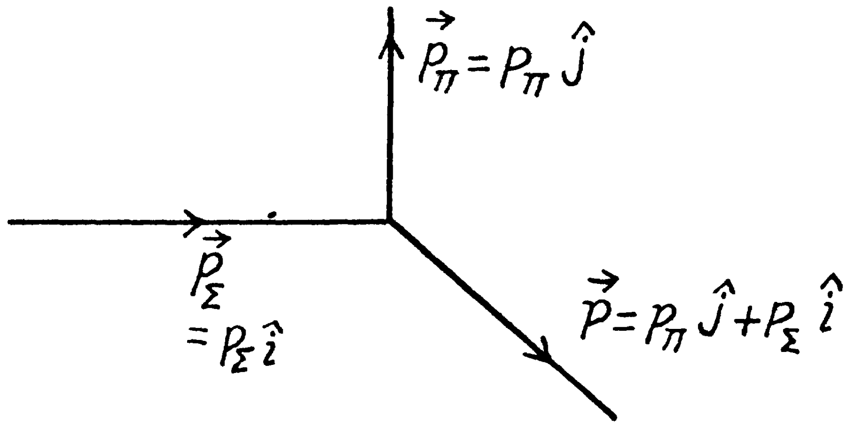
Hence the energy of the neutrino is

$$E_\nu = c\, p_\nu = \frac{m_\pi^2 c^4 - m_\mu^2 c^4}{2\,(m_\pi c^2 + T)}$$

on writing $$\sqrt{m_\pi^2 c^4 + c^2 p_\pi^2} = m_\pi c^2 + T$$

Substitution gives $$E_\nu = 21{\cdot}93 \text{ MeV}$$

6.300



By energy conservation

$$\sqrt{m_\Sigma^2 c^4 + c^2 p_\Sigma^2} = \sqrt{m_\pi^2 c^4 + c^2 p_\pi^2} + \sqrt{m_n^2 c^4 + c^2 p_\pi^2 + c^2 p_\Sigma^2}$$

or $$\left(\sqrt{m_\Sigma^2 c^4 + c^2 p_\Sigma^2} - \sqrt{m_\pi^2 c^4 + c^2 p_\pi^2}\right)^2 = m_n^2 c^4 + c^2 p_\pi^2 + c^2 p_\Sigma^2$$

or $$m_\Sigma^2 c^4 + c^2 p_\Sigma^2 + m_\pi^2 c^4 + c^2 p_\pi^2 - 2\sqrt{m_\pi^2 c^4 + c^2 p_\Sigma^2}\sqrt{m_\pi^2 c^2 + c^2 p_\pi^2}$$
$$= m_n^2 c^2 + c^2 p_\pi^2 + c^2 p_\Sigma^2$$

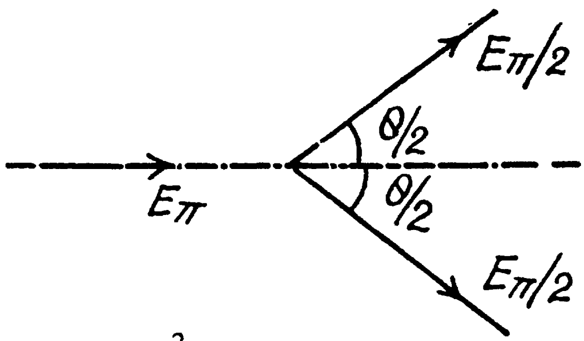
or using the K.E. of Σ & π

$$m_n^2 = m_\Sigma^2 + m_\pi^2 - 2\left(m_\Sigma + \frac{T_\Sigma}{c^2}\right)\left(m_\pi + \frac{T_\pi}{c^2}\right)$$

and
$$m_n = \sqrt{m_\Sigma^2 + m_\pi^2 - 2\left(m_\Sigma + \frac{T_\Sigma}{c^2}\right)\left(m_\pi + \frac{T_\pi}{c^2}\right)} = 0{\cdot}949\,\frac{\text{GeV}}{c^2}$$

6.301 Here by conservation of momentum

$$p_\pi = 2 \times \frac{E_\pi}{2c} \times \cos\frac{\theta}{2}$$



or $\quad c\,p_\pi = E_\pi \cos\dfrac{\theta}{2}$

Thus $\quad E_\pi^2 \cos^2\dfrac{\theta}{2} = E_n^2 - m_\pi^2 c^4$

or
$$E_\pi = \frac{m_\pi c^2}{\sin\dfrac{\theta}{2}}$$

and
$$T_\pi = m_\pi c^2\left(\operatorname{cosec}\frac{\theta}{2} - 1\right)$$

substitution gives $T_\pi = m_\pi c^2 = 135$ MeV for $\theta = 60°$.

Also
$$E_\gamma = \frac{m_\pi c^2 + T_\pi}{2} = \frac{m_\pi c^2}{2}\operatorname{cosec}\frac{\theta}{2}$$
$$= m_\pi c^2 \text{ in this case } (\theta = 60°)$$

6.302 With particle masses standing for the names of the particles, the reaction is

$$m + M \rightarrow m_1 + m_2 + \ldots$$

On R.H.S. let the energy momenta be $(E_1,\ c\vec{p_1})$, $(E_2,\ c\vec{p_2})$ etc. On the left the energy momentum of the particle m is $(E,\ c\vec{p})$ and that of the other particle is $(M c^2,\ \vec{0})$, where, ofcourse, the usual relations

$$E^2 - c^2\vec{p}^{\,2} = m^2 c^4 \text{ etc}$$

hold. From the conservation of energy momentum we see that

$$(E + M c^2)^2 - c^2 \vec{p}^{\,2} = (\Sigma E_i)^2 - (\Sigma\, c\vec{p_i})^2$$

Left hand side is

$$m^2 c^4 + M^2 c^4 + 2 M c^2 E$$

We evaluate the R.H.S. in the frame where $\Sigma \vec{p_i} = 0$ (CM frame of the decay product).

Then
$$R.H.S. = (\Sigma E_i)^2 \geq (\Sigma\, m_i c^2)^2$$

because all energies are +ve. Therefore we have the result

$$E \geq \frac{(\Sigma m_i)^2 - m^2 - M^2}{2M} c^2$$

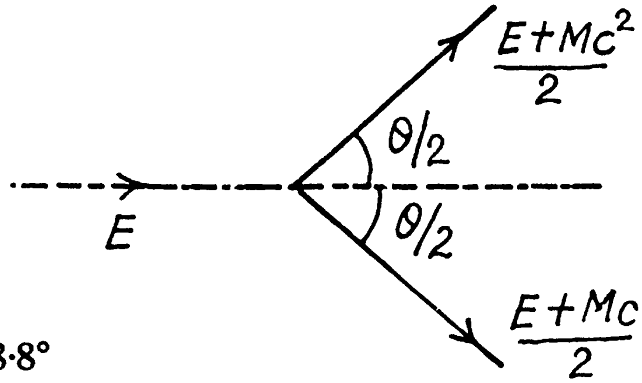
or since $E = mc^2 + T$, we see that $T \geq T_{th}$ where

$$T_{th} = \frac{(\Sigma m_i)^2 - (m+M)^2}{2M} c^2$$

6.303 By momentum conservation

$$\sqrt{E^2 - m_e^2 c^4} = 2\,\frac{E + m_e c^2}{2} \cos\frac{\theta}{2}$$

or $$\cos\frac{\theta}{2} = \sqrt{\frac{E - m_e c^2}{E + m_e c^2}} = \sqrt{\frac{T}{T + 2 m_e c^2}}$$



Substitution gives

$$\theta = 98{\cdot}8°$$

6.304 The formula of problem 3.02 gives

$$E_{th} = \frac{(\Sigma m_i)^2 - M^2}{2M} c^2$$

when the projectile is a photon

(a) For $\gamma + e^- \rightarrow e^- + e^- + e^+$

$$E_{th} = \frac{9 m_e^2 - m_e^2}{2 m_e} c^2 = 4 m_e c^2 = 2{\cdot}04 \text{ MeV}$$

(b) For $$\gamma + p \rightarrow p + \pi^+ \pi^-$$

$$E_{th} = \frac{(M_p + 2m_\pi)^2 - M_p^2}{2M_p} c^2 = \frac{4 m_\pi M_p + 4 m_\pi^2}{2 M_p} c^2 = 2\left(m_\pi + \frac{m_\pi^2}{M_p}\right) c^2 = 320{\cdot}8 \text{ MeV}$$

6.305 (a) For $p + p \rightarrow p + p + p + \bar{p}$

$$T \geq T_{th} = \frac{16 m_p^2 - 4 m_p^2}{2 m_p} c^2 = 6 m_p c^2 = 5{\cdot}63 \text{ GeV}$$

(b) For $p + p \rightarrow p + p + \pi^\circ$

$$T \geq T_{th} = \frac{(2 m_p + m_{\pi^\circ})^2 - 4 m_p^2}{2 m_p} c^2$$

$$= \left(2 m_\pi + \frac{m_{\pi^\circ}^2}{2 m_p}\right) c^2 = 0{\cdot}280 \text{ GeV}$$

6.306 (a) Here

$$T_{th} = \frac{(m_K + m_\Sigma)^2 - (m_\pi + m_p)^2}{2\,m_p} c^2$$

Substitution gives $T_{th} = 0{\cdot}904\,\text{GeV}$

(b) $T_{th} = \dfrac{(m_{K^+} + m_\Lambda)^2 - (m_{\pi^0} + m_p)^2}{2\,m_p} c^2$

Substitution gives $T_{th} = 0{\cdot}77\,\text{GeV}$.

6.307 From the Gell-Mann Nishijima formula

$$Q = T_Z + \frac{Y}{2}$$

we get $$O = \frac{1}{2} + \frac{Y}{2} \text{ or } Y = -1$$

Also $Y = B + S \Rightarrow S = -2$. Thus the particle is $=^\circ$ 0.

6.308 (1) The process $n \to p + e^- + \nu_e$ cannot occur as there are 2 more leptons (e^-, ν_e) on the right comopared to zero on the left.

(2) The process $\pi^+ \to \mu^+ + e^- + e^+$ is forbidden because this corresponds to a change of lepton number by, (0 on the left - 1 on the right)

(3) The process $\pi^- \to \mu^- + \nu_\mu$ is forbidden because μ^-, ν_μ being both leptons $\Delta L = 2$ hre.

(4), (5), (6) are allowed (except that one must distinguish between muon neutrinoes and electron neutrinoes). The correct names would be

(4) $p + e^- \to n + \nu_e$

(5) $\mu^+ \to e^+ + \nu_e + \tilde{\nu}_\mu$

(6) $K^- \to \mu^- + \tilde{\nu}_\mu$.

6.309 (1) $\pi^- + p \to \Sigma^- + K^+$

$$0 \quad 0 \quad -1 \quad 1$$

so $$\Delta S = 0. \text{ allowed}$$

(2) $\pi^- + p \to \Sigma^+ + K^-$

$$0 \quad 0 \quad -1 \quad 1$$

so $$\Delta S = -2. \text{ forbidden}$$

(3) $\pi^- + p \to K^- + K^+ + n$

$$0 \quad 0 \to -1 \quad 1 \quad 0$$

so $$\Delta S = 0, \text{ allowed.}$$

(4) $n + p \rightarrow \Lambda^\circ + \Sigma^+$

$$0 \quad 0 \quad -1 \quad -1$$

so $\Delta S = -2$. forbidden

(5) $\pi^- + n \rightarrow \Xi^- + K^+ + K^-$

$$0 \quad 0 \rightarrow -2 \quad 1 \quad -1$$

so $\Delta S = -2$. forbidden.

(6) $K^- + p \rightarrow \Omega^- + K^+ K^\circ$

$$-1 \quad 0 \quad -3 \quad +1 \quad +1$$

so $\Delta S = 0$. allowed.

6.310 (1) $\Sigma^- \rightarrow \Lambda^\circ + \pi^-$
is forbidden by energy conservation. The mass difference

$$M_{\Sigma^-} - M_{\Lambda^\circ} = 82 \frac{\text{MeV}}{c^2} < m_{\pi^-}$$

(The process $1 \rightarrow 2 + 3$ will be allowed only if $m_1 > m_2 + m_3$.)

(2) $\pi^- + p \rightarrow K^+ + K^-$
is disallowed by conservation of baryon number.

(3) $K^- + n \rightarrow \Omega^- + K^+ + K^\circ$
is forbidden by conservation of charge

(4) $n + p \rightarrow \Sigma^+ + \Lambda^\circ$
is forbidden by strangeness conservation.

(5) $\pi^- \rightarrow \mu^- + e^- + e^+$
is forbidden by conservation of muon number (or lepton number).

(6) $\mu^- \rightarrow e^- + \nu_e + \tilde{\nu}_\mu$
is forbidden by the separate conservation of muon number as well as lepton number.

☆☆☆